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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION*

L. E. BUSH, Kent State University

THE EIGHTEENTH ANNUAL COMPETITION, FEBRUARY 8, 1958

Problems. Part I

1. If $a_0, a_1, \cdots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

show that the equation $a_0+a_1x+a_2x^2+ \cdots +a_nx^n=0$ has at least one real root.

2. Two uniform solid spheres of equal radii are so placed that one is directly above the other. The bottom sphere is fixed, and the top sphere, initially at rest, rolls off. At what point will contact between the two spheres be "lost"? Assume the coefficient of fraction is such that no slipping occurs.
3. Real numbers are chosen at random from the interval $(0 \leqq x \leqq 1)$. If after choosing the nth number the sum of the numbers so chosen first exceeds 1, show that the expected or average value for n is e.
4. If $a_1, a_2, \cdots, a_n$ are complex numbers such that

$$|a_1| = |a_2| = \cdots = |a_n| = r \neq 0,$$

and if ${}_nT_s$ denotes the sum of all products of these n numbers taken s at a time, prove that $\left|{}_nT_s/{}_nT_{n-s}\right| = r^{2s-n}$ whenever the denominator of the left-hand side is different from zero.

5. Show that the integral equation

$$f(x, y) = 1 + \int_0^x \int_0^y f(u, v)dudv$$

has at most one solution continuous for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

6. What is the smallest amount that may be invested at interest rate i, compounded annually, in order that one may withdraw 1 dollar at the end of the first year, 4 dollars at the end of the second year, $\cdots$, n^2 dollars at the end of the nth year, in perpetuity?
7. Show that ten equal-sized squares cannot be placed on a plane in such a way that no two have an interior point in common and the first touches each of the others.

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* Reprints will be available about April 1, 1961, from Professor H. M. Gehman, Executive Director, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y. at 25¢ for single copies and 20¢ each for orders of five or more.

Problems. Part II

1. a) Given line segments A, B, C, D, with A the longest, construct a quadrilateral with these sides and with A and B parallel, when possible.
   b) Given any acute angled triangle ABC and one altitude AH, select any point D on AH, then draw BD and extend until it intersects AC in E. Draw CD and extend until it intersects AB in F. Prove angle AHE = angle AHF.
2. Prove that the product of 4 consecutive positive integers cannot be a perfect square or cube.
3. In a round-robin tournament with n players (each pair of players plays one game) in which there are no draws, the numbers of wins scored by the players are $s_1, s_2, \cdots, s_n$. Prove that a necessary and sufficient condition for the existence of 3 players A, B, C, such that A beat B, B beat C and C beat A is
$$s_1^2 + s_2^2 + \cdots + s_n^2 < (n-1)(n)(2n-1)/6.$$
4. What is the average straight line distance between two points on a sphere of radius 1?
5. Given an infinite number of points in a plane, prove that if all the distances determined between them are integers then the points are all in a straight line.
6. A projectile moves in a resisting medium. The resisting force is a function of the velocity and is directed along the velocity vector. The equation $x=f(t)$ gives the horizontal distance in terms of the time t. Show that the vertical distance y is given by
$$y = -gf(t)\int \frac{dt}{f'(t)} + g\int \frac{f(t)}{f'(t)}dt + Af(t) + B,$$
where A and B are constants and g is the acceleration due to gravity.
7. Prove that if $f(x)$ is continuous for $a \leqq x \leqq b$ and $\int_a^b x^n f(x)dx = 0$ for $n=0, 1, 2, \cdots$ then $f(x)$ is identically zero on $a \leqq x \leqq b$.

Solutions.* Part I

1. Let $f(x) = \sum_{i=0}^{n} (a_i x^{i+1})/(i+1)$. Then $f(0)=f(1)=0$ and, by the mean value theorem, $f'(\theta)=0$ for some real θ such that $0<\theta<1$. Hence $\sum_{i=0}^{n} a_i x^i$ has a real root.

2. Let M be the mass and r the radius of the rolling sphere. Taking the center of the stationary sphere as origin, let $R(\theta)$ be the position vector of the center of

* The solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

The solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. The published solutions are intended to be mere outlines. In order to be considered correct in a contestant's paper, more detail would have to be given.

the rolling sphere with θ as the angle between $R(\theta)$ and the upward vertical through the origin. Clearly $R(\theta)$ remains in a vertical plane as the motion takes place. The centrifugal force is $2Mr\dot{\theta}^2$, where $\dot{\theta}=d\theta/dt$. The component of the gravitational force which is directed toward the origin is $Mg\cos\theta$, and the spheres separate when $2rM\dot{\theta}^2$ just exceeds $Mg\cos\theta$, *i.e.*, the moment of separation occurs when $2r\dot{\theta}^2=g\cos\theta$. The loss of potential energy is $2rMg(1-\cos\theta)$ and must equal the total kinetic energy which is the translational energy $2Mr^2\dot{\theta}^2$ plus the rotational energy $\frac{4}{5}Mr^2\dot{\theta}^2$. Thus $\dot{\theta}^2=5g(1-\cos\theta)/(7r)$ and the separation occurs when $\cos\theta=10/17$.

3. Let $P_n(x)=\Pr(\sum_{i=1}^n x_i<x)$. Then

$$P_{n+1}(x)=\int_0^x (x-y)dP_n(y)=\int_0^x P_n(y)dy.$$

Since $P_1(x)=x$, it follows that $P_n(x)=x^n/n!$. The probability that the sum first exeeds 1 on the $(n+1)$th trial is

$$Q_{n+1}=\int_0^1 ydP_n(y)=\int_0^1 \frac{y^n}{(n-1)!}dy=\frac{n}{(n+1)!}\cdot$$

Hence $\bar{n}=\sum_{n=2}^\infty nQ_n=\sum_{n=2}^\infty 1/(n-2)!=e$.

4. Let $a_j=r\exp(\theta_j)$, $j=1,\cdots,n$. Then

$${}_nT_s=r^s\sum_{C_s}\exp\left(i(\theta_{j_1}+\cdots+\theta_{j_s})\right)$$

and the summation is over C_s, the set of all combinations of $1,\cdots,n$ taken s at a time. Similarly,

$${}_nT_{n-s}=r^{n-s}\sum_{C_{n-}}\exp\left(i(\theta_{j_1}+\cdots+\theta_{j_{n-s}})\right).$$

Let $\Omega=\sum_{j=1}^n\theta_j$. Then

$${}_nT_{n-s}=r^{n-s}\sum_{C_s}\exp\left(i(\Omega-\theta_{j_1}-\cdots-\theta_{j_s})\right),$$

$$\left|{}_nT_{n-s}\right|=r^{n-s}\left|\sum_{C_s}\exp\left(i(\theta_{j_1}+\cdots+\theta_{j_s})\right)\right|=r^{n-2s}\left|{}_nT_s\right|.$$

5. The difference of two solutions, say h, must satisfy the homogeneous equation $h(x,y)=\int_0^x\int_0^y h(u,v)dudv$. Let $M_p=\max h(x,y)$ for $0\leqq x,\ y\leqq p<1$. By the mean value theorem for integrals and the continuity of h, $M_p=h(x_p,y_p)\leqq M_px_py_p\leqq M_pp^2$. Thus $M_p\leqq 0$ for all p such that $0<p<1$. Hence $h(x,y)\leqq 0$ for $0\leqq x,\ y\leqq 1$ by the continuity of h. By a symmetrical argument, $h(x,y)\geqq 0$ and it follows that $h(x,y)=0$.

6. The present value of n^2 dollars n years from now at rate i per year is $n^2(1+i)^{-n}$. Thus the required sum is $\sum_{n=1}^\infty n^2(1+i)^{-n}$. Since $(1-x)^{-1}=\sum_{n=0}^\infty x^n$, it follows by differentiation that

$$x(1-x)^{-2}=\sum_{n=1}^{\infty} nx^n, \qquad (x+x^2)(1-x)^{-3}=\sum_{n=1}^{\infty} n^2x^n,$$

all these series being convergent for $-1<x<1$. Taking $x=(1+i)^{-1}$, the required sum is found to be $(1+i)(2+i)i^{-3}$.

7. Let Q be a square of side 1 and let Q^* be the square of side 2 having common center with Q and sides parallel to the corresponding sides of Q. The proof follows by showing that any square P of side 1 having only boundary points in common with Q must contain a portion of the perimeter of Q^* of length 1 or greater. The various cases can be classified as: (1) A corner of P is on a side of Q at a distance l from a nearest corner of Q and the smallest acute angle between the sides of Q and P is θ. (2) A side of P touches a corner of Q, the point of contact being distant l from the nearest vertex of P, and the acute angle between the sides of Q and P is θ. Each of (1) and (2) can be separated into two subcases according as a vertex of Q^* lies in P or not. All other cases are limiting cases of these. The discussion in each case is elementary.

Solutions. Part II

1. a) Construct $A-B$ and a triangle with sides $A-B$, C, and D, if this is possible. The condition for this is that any one of these segments be less than the sum of the other two. The quadrilateral is readily completed by extending side $A-B$ by a segment of length B and constructing a parallel to the nearer side of the triangle.

b) If one chooses a cartesian coordinate system with H as origin and with HA and CB as axes, the slopes of HE and HF are readily computed and one is the negative of the other.

2. Since $x(x+1)(x+2)(x+3)=(x^2+3x+1)^2-1$, the product of four consecutive positive integers differs by 1 from a perfect square and hence is not a square. Of four consecutive positive integers, x, $x+1$, $x+2$, $x+3$, either $x+1$ or $x+2$ is prime to the product of the other three. Thus, if the product of four consecutive positive integers is a perfect cube, one of $x+1$ or $x+2$ is a perfect cube also. Thus the product of the remaining three positive integers is a perfect cube. But this is absurd since $x(x+2)(x+3)$ and $x(x+1)(x+3)$ lie strictly between $(x+1)^3$ and $(x+2)^3$ if $x>1$. The case $x=1$ is tested directly.

3. If A beats B, B beats C, and C beats A, we shall say that these players form a "triangle." If two players have the same score, say A and B have score k, then a triangle must occur. Since A beats B or B beats A, $k>0$. Say that A beats B so that B beats k players other than A and B, and A only beats $k-1$ players other than A and B. Thus B beats some player who beats A. Conversely, if all scores are distinct, they must be $0, 1, \cdots, n-1$ and clearly no triangles are possible. Thus the scores $s_1, \cdots, s_n$ are $0, 1, \cdots, n-1$ if and only if no triangles occur. Also, $\sum_{i=1}^{n} s_i^2 \leqq \frac{1}{6}(n-1)(n)(2n-1)$, with equality only when the scores in order are $s_i=i-1$. This is true for $n=2, 3$ by inspection. Assuming the truth of the assertion for $n-1$ players, the result follows if $s_n=n-1$ and some

other $s_i \neq i-1$. If $s_n < n-1$, then some $s_i > i-1$. If s_n is increased by 1 and s_i decreased by 1, the sum of squares is increased by $2(s_n - s_i) + 2 > 0$. By a second finite induction, the case $s_n = n-1$ is reached and the result follows.

4. By symmetry it suffices to determine the average length of chords emanating from a single point, say the North Pole. Using spherical coordinates (with $\rho = 1$), the areal element may be taken as $2\pi \sin \phi d\phi$, where ϕ is the angle which the radius to the North Pole makes with the radius to the other end of the chord. The integral required is $(4\pi)^{-1}\int_0^\pi (2 \sin \frac{1}{2}\phi)(2\pi \sin \phi) d\phi = 4/3$.

5. Assume that P_1, P_2, P_3 are not collinear and are in the set. Any point P_4 in the set lies on the line P_iP_j or on one of the hyperbolas

$$| d(P, P_i) - d(P, P_j) | = 1, 2, \cdots, d(P_i, P_j) - 1,$$

where $i \neq j$, $i, j = 1, 2, 3$. Since any two such loci have at most four intersections, the possible positions for P_4 are finite in number.

6. Since $mx''(t) = Rx'(t)$ and $my''(t) = -mg + Ry'(t)$, it follows that $R = m\{f''(t)/f'(t)\}$. Hence $\{y'(t)/f'(t)\}' = -g/f'(t)$ and

$$y'(t) = \{y'(0)/f'(0)\}f'(t) - gf'(t)\int_0^t \{1/f'(\tau)\} d\tau.$$

Integrating once more,

$$y(t) = y(0) + \{y'(0)/f'(0)\}\{f(t) - f(0)\} - g\int_0^t f'(x)dx\int_0^x \{1/f'(\tau)\} d\tau.$$

7. Evidently $\int_a^b P(x)f(x)dx = 0$ for any polynomial $P(x)$. Given $\epsilon > 0$, $P(x)$ can be selected so that $|f(x) - P(x)| < \epsilon$, $a \leqq x \leqq b$. Hence

$$\int_a^b f^2(x)dx = \left|\int_a^b f(x)\{f(x) - P(x)\}dx\right| \leqq \epsilon(b-a)M,$$

where M is a bound for $|f(x)|$ on $a \leqq x \leqq b$. This implies that $\int_a^b f^2(x)dx = 0$ and since f is continuous it must vanish identically on the interval.

THE NINETEENTH ANNUAL COMPETITION, NOVEMBER 22, 1958

Problems. Part I

1. Let $f(m, 1) = f(1, n) = 1$ for $m \geqq 1, n \geqq 1$, and let $f(m, n) = f(m-1, n) + f(m, n-1) + f(m-1, n-1)$ for $m > 1$ and $n > 1$. Also let

$$S(n) = \sum_{a+b=n} f(a, b), \qquad a \geqq 1 \text{ and } b \geqq 1.$$

   Prove that $S(n+2) = S(n) + 2S(n+1)$ for $n \geqq 2$.
2. Let $R_1 = 1$, $R_{n+1} = 1 + n/R_n$, $n \geqq 1$. Show that for $n \geqq 1$, $\sqrt{n} \leqq R_n \leqq \sqrt{n} + 1$.
3. Under the assumption that the following set of relations has a unique solution for $u(t)$, determine it.

$$\frac{du(t)}{dt} = u(t) + \int_0^1 u(s)ds, \qquad u(0) = 1.$$

4. In assigning dormitory rooms, a college gives preference to pairs of students in this order:

$$AA,\ AB,\ AC,\ BB,\ BC,\ AD,\ CC,\ BD,\ CD,\ DD$$

in which AA means two seniors, AB means a senior and a junior, etc. Determine numerical values to assign to A, B, C, D so that the set of numbers $A+A$, $A+B$, $A+C$, $B+B$, etc., corresponding to the order above will be in descending magnitude. Find the general solution and the solution in least positive integers.

5. Show that the number of nonzero terms in the expansion of the nth order determinant having zeros in the main diagonal and ones elsewhere is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right].$$

6. Let $a(x)$ and $b(x)$ be continuous functions on $0 \leqq x \leqq 1$ and let $0 \leqq a(x) \leqq a < 1$ on that range. Under what other conditions (if any) is the solution of the equation for u, $u = \max_{0 \leqq x \leqq 1} [b(x) + a(x) \cdot u]$, given by

$$u = \max_{0 \leqq x \leqq 1} \left[\frac{b(x)}{1 - a(x)}\right]?$$

7. Let a and b be relatively prime positive integers, b even. For each positive integer q let $p = p(q)$ be chosen so that

$$\left|\frac{p}{q} - \frac{a}{b}\right|$$

is a minimum. Prove that

$$\lim_{n\to\infty} \sum_{q=1}^{n} \frac{q\left|\frac{p}{q} - \frac{a}{b}\right|}{n} = \frac{1}{4}.$$

Problems. Part II

1. Given

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^{-1}, \qquad n \geqq 1.$$

Prove that

$$b_n = \frac{n+1}{2n} b_{n-1} + 1,\ n \geqq 2, \text{ and hence, as a corollary, } \lim_{n\to\infty} b_n = 2.$$

2. Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.
3. If a square of unit side be partitioned into two sets, then the diameter (least upper bound of the distances between pairs of points) of one of the sets is not less than $\frac{1}{2}\sqrt{5}$. Show also that no larger number will do.
4. Let C be a real number, and let f be a function such that

$$\lim_{x\to\infty} f(x) = C, \qquad \lim_{x\to\infty} f'''(x) = 0.$$

Prove that

$$\lim_{x\to\infty} f'(x) = 0 \quad\text{and}\quad \lim_{x\to\infty} f''(x) = 0,$$

where superscripts denote derivatives.
5. The lengths of successive segments of a broken line are represented by the successive terms of the harmonic progression $1, 1/2, \cdots, 1/n, \cdots$. Each segment makes with the preceding segment a given angle θ. What is the distance and what is the direction of the limiting point (if there is one) from the initial point of the first segment?
6. Let a complete oriented graph on n points be given, *i.e.*, a set of n points $1, 2, \cdots, n$, and between any two points i and j a direction, $i\to j$. Show that there exists a permutation of the points, $[a_1, a_2, \cdots, a_n]$, such that $a_1\to a_2 \to \cdots \to a_n$.
7. Let $a_1, \cdots, a_n$ be a permutation of the integers $1, \cdots, n$. Call a_i a "big" integer if $a_i>a_j$ for all $j>i$. Find the mean number of "big" integers over all permutations on the first n positive integers.

Solutions. Part I

1. $$S(n+2)=\sum_{j=1}^{n+1} f(j,\, n+2-j)$$
$$=f(1,n+1)+\sum_{j=2}^{n}\{f(j-1,n+2-j)+f(j,n+1-j)+f(j-1,n+1-j)\}+f(n+1,1)$$
$$=\left[f(n,1)+\sum_{j=1}^{n-1}f(j,n+1-j)\right]+\left[\sum_{j=2}^{n}f(j,n+1-j)+f(1,n)\right]+\sum_{j=1}^{n-1}f(j,n-j)$$
$$=2S(n+1)+S(n).$$

2. For $n=1$, $\sqrt{n}\leqq R_n\leqq 1+\sqrt{n}$. Assume that the inequalities hold for $n=m\geqq 1$. Then

$$R_{m+1}=1+\frac{m}{R_m}\geqq 1+\frac{m}{1+\sqrt{m}}=\frac{1+m+\sqrt{m}}{1+\sqrt{m}}>\sqrt{(m+1)}$$

since $(1+m+\sqrt{m})^2>(1+m)(1+\sqrt{m})^2=(1+m)^2+2(1+m)\sqrt{m}$. Also R_{m+1}

$\leqq 1+m/\sqrt{m}<1+\sqrt{(m+1)}$. Thus, by induction, the inequality is valid for all positive integers n.

3. Let $\int_0^1 u(t)dt=a$. Then $du/dt=u(t)+a$ and hence $u=-a+ce^t$, where c is a constant. Then $a=\int_0^1 u(t)dt=-a+c(e-1)$ and thus $2a=c(e-1)$. Since $u(0)=-a+c=1$, the function u must be defined by $u(t)=1+2(e^t-1)/(3-e)$.

4. By assumption, $2A>A+B>A+C>2B>B+C>A+D>2C>B+D>C+D>2D$. It is clear that $A>B>C>D$ and we may set $C=D+h$, $B=C+k=D+h+k$; $A=B+m=D+h+k+m$. Since $2C>B+D$ and $A+C>2B$, we must have $h>k$, $m>k$. Thus $m=k+n$, $h=k+p$ and $A=D+3k+n+p$, $B=D+2k+p$, $C=D+k+p$ are necessary. Since $B+C>A+D>2C$, we must have $3k+2p>3k+n+p>2k+2p$ or $p>n$ and $k+n>p$. Taking $p=n+q$ we must have $k>q$ or $k=q+r$. Thus $A=D+4q+3r+2n$, $B=D+3q+2r+n$, $C=D+2q+n+r$ are necessary. These are readily found to be sufficient by substitution. Since all the quantities are to be positive integers, the minimum solution is found by taking $D=r=n=q=1$. This solution is $A=10$, $B=7$, $C=5$, $D=1$.

5. Let N_n be the number of nonzero terms in the expansion of the given nth order determinant. Then N_n is equal to the number of permutations $(j_1, \cdots, j_n)$ of the positive integers $1, \cdots, n$. Let $(j_1, \cdots, j_{n+1})$ be any permutation of the positive integers $1, \cdots, n+1, j_i \neq i, i=1, \cdots, n$. Suppose that $n+1$ occurs in the rth position. Interchanging $n+1$ and j_{n+1} results in another permutation of $1, \cdots, n+1$. Two cases arise according as $j_{n+1}=r$ or $j_{n+1}\neq r$. In the first case the positive integers form a permutation of $n-1$ objects with each object out of its natural position, and the number of such permutations is N_{n-1}. As r varies from 1 to n there is thus a total of nN_{n-1} possible permutations in the first case. In the second case, there are N_n permutations for a fixed r and thus nN_n permutations formed in this way. Thus $N_{n+1}=n(N_n+N_{n-1})$ and since $N_2=1$, $N_3=2$, an easy induction establishes that $N_n=n!\sum_{k=2}^{n}(-1)^k/k!$

6. If $u=\max[b(x)+ua(x)]$, $0\leqq x\leqq 1$, then $u\geqq b(x)+ua(x)$ for all x in the unit interval and equality holds for some x in the range. Since the inequality is equivalent to $u\geqq b(x)\{1-a(x)\}^{-1}$, it follows that $u=\max[b(x)\{1-a(x)\}^{-1}]$ for $0\leqq x\leqq 1$. Thus the solution is unique. Conversely, if $u=\max[b(x)\{1-a(x)\}^{-1}]$ for $0\leqq x\leqq 1$, then $u\geqq b(x)\{1-a(x)\}^{-1}$ for all x in the range, equality holding for one such x. Hence $u=\max[b(x)+ua(x)]$, $0\leqq x\leqq 1$.

7. It is clear that p must be chosen to minimize $|bp-aq|$ and that if $q_1\equiv q_2 \pmod{b}$, then the corresponding minima are equal. Thus we consider $q=1, \cdots, b$. Since aq is congruent (mod b) to one of the integers $-\frac{1}{2}b, 1-\frac{1}{2}b, \cdots, \frac{1}{2}b$, it follows that the minimizing value of p is such that $bp-aq$ belongs to the above class. Moreover, if $q_2=q_1+r$, $0<r<b$, then the minimizing values of p are such that $bp_1-aq_1\neq bp_2-aq_2$ since otherwise $ar=b(p_2-p_1)$ and this is impossible since a must divide p_2-p_1. Thus as q varies from 1 to b, inclusive, the minimizing value of p is such that it varies over the reduced residue class (mod b) exactly once, and $|bp-aq|$ takes on the values $1, \cdots, -1+\frac{1}{2}b$ twice and the values 0 and $\frac{1}{2}b$ just once. Thus $\sum_{q=1}^{b}|bp-aq|=\frac{1}{4}b^2$. If $n=kb+r$, $0\leqq r<b$, then

$$(*)\qquad \frac{1}{nb}\sum_{q=1}^{n}|bp-aq| = \frac{kb^2}{4b(kb+r)}+\frac{C_n}{nb},$$

where $C_n=\sum_{q=kb+1}^{n}|bp-aq|\leqq\frac{1}{2}b^2$. As $n\to\infty$ and hence $k\to\infty$, it is clear that the right member of (*) approaches $\frac{1}{4}$.

Solutions. Part II

1. By definition,

$$\begin{aligned} b_n &= \sum_{h=0}^{n}\frac{h!(n-h)!}{n!} = 1+\sum_{h=0}^{n-1}\frac{h!(n-1-h)!(n-h)}{n!} \\ &= 1+\sum_{h=0}^{n-1}\frac{(n-1-h)!h!(h+1)}{n!}\cdot \end{aligned}$$

Adding the last two expressions and multiplying by n, one obtains

$$2nb_n = 2n+\sum_{h=0}^{n-1}\frac{h!(n-1-h)!(n+1)}{(n-1)!},$$

whence $b_n=1+\{(n+1)/(2n)\}b_{n-1}$. This may be written

$$b_n-2=\frac{n+1}{2n}(b_{n-1}-2)+\frac{1}{n}\cdot$$

Since $b_5=2+9/15$ and for $n\geqq 5$, $2\leqq b_{n-1}\leqq 2+4/(n-1)$ implies that $2\leqq b_n\leqq 2+4/n$, it follows that $\lim_{n\to\infty} b_n=2$.

2. The statement is true for $n=1$ and we assume it true for n. Let k_i, $i=1,\cdots,n+2$, be positive integers such that $k_i<k_{i+1}\leqq 2n+2$, $i=1,\cdots,n+1$, and suppose that no one of them divides another. If $k_{n+2}=2n+2$, we may replace k_{n+2} by $n+1$ and the new set has the property that no element divides any other. Thus we may suppose $k_{n+2}<2n+2$. But then the set k_i, $i=1,\cdots,n+1$ contains no term exceeding $2n$ and thus contradicts the induction hypothesis.

3. Let A, B, C, D be consecutive vertices of the square. If F is the midpoint of segment AB and G the midpoint of segment CD, the partition with boundary FG shows that no larger number than $\frac{1}{2}\sqrt{5}$ is possible for the statement to be true. Now let S and T be the sets of any partitioning of the square. If opposite vertices belong to one of the sets, then the diameter of that set is $\sqrt{2}>\frac{1}{2}\sqrt{5}$. If this does not occur, we may assume, by relabeling vertices if necessary, that A and D belong to S and that C and D belong to T. Since F must be in S or in T, it follows that the diameter of one of these sets is at least $\frac{1}{2}\sqrt{5}$.

4. Given a positive real number a, we have

$$f(a\pm 1)=f(a)\pm f'(a)+\tfrac{1}{2}f''(a)\pm\tfrac{1}{6}f'''(a_\pm),$$

where $a<a_+<a+1$, $a-1<a_-<a$. Hence

$$f(a+1)-f(a-1) = 2f'(a) + \tfrac{1}{6}f'''(a_+) + \tfrac{1}{6}f'''(a_-)$$
$$f(a+1)+f(a-1) = 2f(a) + f''(a) + \tfrac{1}{6}f'''(a_+) - \tfrac{1}{6}f'''(a_-).$$

The result follows by taking limits as $a\to\infty$.

5. We may take the initial vector to be represented by the complex number 1 and then the nth vector is $n^{-1}\exp\{i\theta(n-1)\}$. If $r=\exp(i\theta)$, the sum of the first n vectors is $\sum_{j=0}^{n-1}\{r^j/(j+1)\}$. For $\theta\neq 0$, the corresponding infinite series converges by the Dirichlet test, and by Abel's theorem the sum is $-r^{-1}\log(1-r)$.

6. The result is trivial for $n=2$. An ordering such that $a_1\to a_2\to\cdots\to a_n$ will be called a chain. Assume that the result is true for $n<k$ and consider the case $n=k$. Let I be the set of those objects a_i such that $a_i\to a_k$ and let T be the set of those a_i such that $a_k\to a_i$. Both I and T may be ordered to form a chain, trivially if they have a single element or are empty and by the induction hypothesis in the contrary case. The elements of I, ordered as a chain followed by a_k and followed by the elements of T ordered as a chain, yield the chain for all elements.

7. Let S_n be the total number of "big" integers obtained from all the permutations of the positive integers $1,\cdots,n$. It is clear that S_n does not depend on the specific values but only on the ordering. Thus S_{n-1} is also the total number of "big" integers obtained from all permutations of the positive integers $2,\cdots.n$. Each such permutation gives rise to n permutations of $1,\cdots,n$ by inserting 1 in the n possible positions. Each permutation of $1,\cdots,n$ is obtained exactly once in this way. When 1 is annexed as the last element, the number of "big" numbers in that permutation is increased by 1, but when 1 is inserted in any of the other positions, the number of "big" numbers is not changed. Hence $S_n=nS_{n-1}+(n-1)!$ or

$$\frac{S_n}{n!}=\frac{S_{n-1}}{(n-1)!}+\frac{1}{n}.$$

Thus the average A_n of the "big" numbers satisfies the relation $A_n=A_{n-1}+n^{-1}$. Since $A_1=1$, it follows that $A_n=\sum_{j=1}^{n}j^{-1}$.

THE TWENTIETH ANNUAL COMPETITION, NOVEMBER 21, 1959

Problems. Part I

1. Let n be a positive integer. Prove that $x^n-(1/x^n)$ is expressible as a polynomial in $x-(1/x)$ with real coefficients if and only if n is odd.
2. Prove that if the points in the complex plane corresponding to two distinct complex numbers z_1 and z_2 are two vertices of an equilateral triangle, then the third vertex corresponds to $-\omega z_1-\omega^2 z_2$, where ω is an imaginary cube root of unity.
3. Find all complex-valued functions f of a complex variable such that $f(z)+zf(1-z)=1+z$ for all z.

4. If f and g are real-valued functions of one real variable, show that there exist numbers x and y such that $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and $|xy-f(x)-g(y)| \geqq \frac{1}{4}$.
5. A sparrow, flying horizontally in a straight line, is 50 feet directly below an eagle and 100 feet directly above a hawk. Both hawk and eagle fly directly toward the sparrow, reaching it simultaneously. The hawk flies twice as fast as the sparrow. How far does each bird fly? At what rate does the eagle fly?
6. Let m and n be integers greater than 1, and $a_1, \cdots, a_{m+1}$ real numbers. Prove that there exist real n by n matrices $A_1, \cdots, A_m$ such that (i) Det $(A_j)=a_j$ for $j=1, \cdots, m$ and (ii) Det $(A_1+\cdots+A_m)=a_{m+1}$.
7. If f is a real-valued function of one real variable which has a continuous derivative on the closed interval $[a, b]$ and for which there is no $x \in [a, b]$ such that $f(x)=f'(x)=0$, then prove that there is a function g with continuous first derivative on $[a, b]$ such that $fg'-f'g$ is positive on $[a, b]$.

Problems. Part II

1. Let each of m distinct points on the positive part of the X-axis be joined to n distinct points on the positive part of the Y-axis. Obtain a formula for the number of intersection points of these segments (exclusive of end points), assuming that no three of the segments are concurrent.
2. Let c be a positive real number. Prove that c can be expressed in infinitely many ways as a sum of infinitely many distinct terms selected from the sequence

$$1/10,\ 1/20,\ \cdots,\ 1/10n,\ \cdots.$$

3. Give an example of a continuous real-valued function f from $[0, 1]$ to $[0, 1]$ which takes on every value in $[0, 1]$ an infinite number of times.
4. Given the following matrix of 25 elements

$$\begin{pmatrix} 11 & 17 & 25 & 19 & 16 \\ 24 & 10 & 13 & 15 & 3 \\ 12 & 5 & 14 & 2 & 18 \\ 23 & 4 & 1 & 8 & 22 \\ 6 & 20 & 7 & 21 & 9 \end{pmatrix},$$

choose five of these elements, no two coming from the same row or column, in such a way that the minimum of these five elements is as large as possible. Prove that your answer is correct.
5. Find the equation of the smallest sphere which is tangent to both of the lines: (i) $x=t+1$, $y=2t+4$, $z=-3t+5$, and (ii) $x=4t-12$, $y=-t+8$, $z=t+17$.
6. Prove that if x and y are positive irrationals such that $1/x+1/y=1$, then the sequences $[x], [2x], \cdots, [nx], \cdots$ and $[y], [2y], \cdots, [ny], \cdots$ together include every positive integer exactly once. (The notation $[x]$ means the largest integer not exceeding x.)

7. For each positive integer n, let f_n be a real-valued symmetric function of n real variables. Suppose that for all n and for all real numbers $x_1, \cdots, x_{n+1}, y$, it is true that

1) $f_n(x_1 + y, \cdots, x_n + y) = f_n(x_1, \cdots, x_n) + y$,

2) $f_n(-x_1, \cdots, -x_n) = -f_n(x_1, \cdots, x_n)$,

3) $f_{n+1}(f_n(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n), x_{n+1}) = f_{n+1}(x_1, \cdots, x_{n+1})$.

Prove that $f_n(x_1, \cdots, x_n) = (x_1 + \cdots + x_n)/n$.

Solutions. Part I

1. Let P be a polynomial and suppose that $x^n - 1/x^n = P(x - 1/x)$. It is clear that the degree of P must be n and the leading coefficient must be $+1$ to produce the term x^n in the right member of the identity. But then $-1/x^n = (-1)^n/x^n$ and hence n is odd. Since

$$x^{n+2} - x^{-n-2} = (x - x^{-1})^2(x^n - x^{-n}) - (x^{n-2} - x^{-n+2}) + 2(x^n - x^{-n}),$$

it follows that $x^{n+2} - x^{-n-2}$ can be represented as a polynomial in $x - x^{-1}$ provided $x^n - x^{-n}$ and $x^{n-2} - x^{-n+2}$ have such a representation. The representations for $x - x^{-1}$ and $x^3 - x^{-3}$ are immediate, and the induction proves the result true for all odd n.

2. Let z_1, z_2, z_3 be any three distinct points in the plane. These points form an equilateral triangle if and only if $z_3 - z_1 = (z_2 - z_1)\exp(\pm\frac{1}{3}\pi)$, where the same notation is used for a point and its representation as a complex number. Hence, $z_3 = -[-1 + \exp(\pm\frac{1}{3}\pi)]z_1 - [-\exp(\pm\frac{1}{3}\pi)]z_2$. The bracketed quantities are clearly the two nonreal cube roots of 1. Conversely, if $z_3 = -\omega z_1 - \omega^2 z_2$, where ω is a nonreal cube root of 1, then ω and ω^2 can be identified with the brackets for one choice of sign and the result follows.

3. Since $f(z) + zf(1-z) = 1 + z$, it follows by substitution of $1 - z$ for z that $f(z) + z(2 - z - (1-z)f(z)) = 1 + z$. Hence $f(z)(1 - z + z^2) = 1 - z + z^2$ for all z. Hence $f(z) \equiv 1$ except possibly for $z = w_1, w_2$, where these are the zeros of $1 - z + z^2$. Let α be an arbitrary complex number and $f(w_1) = \alpha$. Then since $w_2 = 1 - w_1$, we must have $f(w_2) = 1 + w_2 - w_2\alpha$. Thus $f(w_1)$ and $f(w_2)$ are arbitrary except for the relation indicated.

4. If $|xy - f(x) - g(y)| < \frac{1}{4}$ for all x and y in the unit interval, then, in particular, $|f(0) + g(1)| < \frac{1}{4}$, $|f(0) + g(0)| < \frac{1}{4}$, and $|f(1) + g(0)| < \frac{1}{4}$. But then by the triangle inequality

$$\begin{aligned} |1 - f(1) - g(1)| &\geqq 1 - |f(1) + g(1)| \\ &\geqq 1 - |f(1) + g(0)| - |-g(0) - f(0)| - |g(1) + f(0)| > \tfrac{1}{4}. \end{aligned}$$

5. Let a pursuer and the prey be at the points $(h, 0)$ and $(0, 0)$, respectively, of the usual Cartesian plane at time $t = 0$. If the prey moves along the positive x-axis at a constant speed v, then the condition that the pursuer moves directly

toward the prey at all times is given by $x-tv=(dx/dy)y$, where y has been taken as the independent variable. If the pursuer moves with speed $s>v$, then

$$tv = \frac{v}{s}\int_y^h \sqrt{\{1 + (dx/dy)^2\}}\,dy.$$

Substituting for t in the first equation and then differentiating to eliminate the integral, the relation becomes

$$y\frac{d^2x}{dy^2} = \frac{v}{s}\sqrt{\{1 + (dx/dy)^2\}}\cdot$$

A first integral with the proper constant of integration is

$$\left(\frac{y}{h}\right)^{v/s} = \frac{dx}{dy} + \sqrt{\{1 + (dx/dy)^2\}}.$$

This last relation reduces to

$$2\frac{dx}{dy} = \left(\frac{y}{h}\right)^{v/s} - \left(\frac{y}{h}\right)^{-v/s}$$

and integrates to yield

$$2x - \frac{2(v/s)h}{1-(v^2/s^2)} = \frac{h(y/h)^{(v/s)+1}}{(v/s)+1} - \frac{h(y/h)^{-(v/s)+1}}{1-(v/s)}$$

with the proper initial conditions. Clearly the pursuit ends at

$$x = \frac{vsh}{s^2 - v^2}$$

at time $t=(sh)/(s^2-v^2)$ and the pursuer has gone a distance $(s^2h)/(s^2-v^2)$. In the case of the hawk and sparrow, $x=200/3$ ft. and $t=200/(3v)$. In the case of the eagle, $h=50$ ft. and his speed s is to be determined. The condition of the problem requires

$$\frac{50s}{s^2-v^2} = \frac{200}{v} \quad \text{or} \quad 4s^2 - 3sv - 4v^2 = 0.$$

Hence the eagle's speed is $(\frac{3}{8}+\frac{1}{8}\sqrt{73})v$.

6. Let a_{ij}^k denote the element in the ith row and jth column of the matrix A_k. Take

$$a_{ij}^{k} = 1,\ 1 < i \leqq n,\ 1 \leqq k \leqq m; \qquad a_{11}^{k} = a_k,\ 1 \leqq k \leqq m;$$

$$a_{21}^{2} = 1,\ a_{12}^{1} = m\sum_{j=1}^{m} a_j - m^{2-n} a_{m+1};$$

and all other elements not yet specified as zero. The matrices A_k have the desired property.

7. Let $S=\{x|f(x)=0\}$. If S were infinite it would have an accumulation point x_0, and it is readily verified that $f(x_0)=f'(x_0)=0$, a contradiction. Hence S is finite. Therefore there is a polynomial h such that if $x\in S$ then $f'(x)h(x)=-1$. It follows that there is a (relatively) open set $Q\supset S$ such that if $x\in Q$ then $f(x)h'(x)-f'(x)h(x)>0$. If $c>0$, let $g_c(x)=xf(x)+ch(x)$, $a\leqq x\leqq b$. Then

$$f(x)g_c'(x) - f'(x)g_c(x) = f^2(x) + c(f(x)h'(x) - f'(x)h(x)).$$

If $x\in Q$, this expression is positive. Since $fh'-f'h$ is a bounded function and $f^2(x)$ is bounded away from 0 on the complement of Q, $fg_c'-f'g_c$ is a positive function on the complement of Q for sufficiently small c.

Solutions. Part II

1. Each of the points of intersection interior to the first quadrant corresponds in a unique way to a choice of two points on the X-axis and of two points on the Y-axis, namely the endpoints of the two segments intersecting in the given point. Conversely each such choice of four points determines uniquely one intersection point interior to the first quadrant. Since there are ${}_mC_2$ ways of choosing two points on the X-axis, and ${}_nC_2$ ways of choosing two points on the Y-axis, there are ${}_mC_2\cdot{}_nC_2=\frac{1}{4}m(m-1)n(n-1)$ points of intersection interior to the first quadrant.

2. For a given positive number, c, let p be the least integer for which $1/(10p)<c$. Then we can construct a series whose sum is c beginning with $1/(10q)$ for every integer $q\geqq p$. Since these series all have different initial terms, they are all different. The construction of any series is as follows. Starting with $1/(10q)$, add terms in order from the given series, always omitting any term which would bring the partial sum to c or beyond c. The partial sums thus formed will be monotonically increasing and will all be less than c. The error will be less than the last term omitted from the original series at any stage, so that the error tends to zero, and the constructed series converges to c. (It is possible to construct the series so that no two of the infinitely many series constructed have any terms in common, but this is not required by the problem.)

3. An example can be constructed using the Peano space-filling curve. Consider the Cantor set, C, of those numbers x in $0\leqq x\leqq 1$ which can be written in the ternary scale using only 0's and 2's. This may require an infinite sequence of 2's. (*e.g.*, $\frac{1}{3}=0.0222\cdots$). For all x's in C, define $f(x)$ by $f(.a_1a_2a_3\cdots)$ $=.b_1b_3b_5\cdots$, where $b_i=0$ when $a_i=0$, and $b_i=1$ when $a_i=2$. Moreover, $f(x)$ is understood to be in the binary scale. The complement of the Cantor set is composed of intervals. Complete the definition of $f(x)$ in these intervals by requiring it to be linear and continuous. It is readily verified that $f(x)$ as defined is continuous. To show that it takes on every value infinitely many times, let y_0 be any number in $0\leqq y\leqq 1$. Then only the odd digits in the ternary expansion of the corresponding x are determined, and there are infinitely many different possible

values of x, formed from all choices of the even digits.

4. A possible attack is to consider possible least elements in allowable sets of five numbers. Since the largest element in the third row of the matrix is 18, no set of five can have a minimum greater than 18, since one element must be chosen from each row and each column. However, 18 is not possible as minimum, for 24 would have to be used in the second row, and then neither 22 nor 23 could be used from the fourth row, requiring the use of a number less than 18. Neither 16 nor 17 is a possible minimum, for if either is used, 25 can not be used, so that a number less than 16 or 17 would be required in the third column. However, 15 is a possible minimum. If 15 is the minimum, 25 must be used from the third column. Then in the second column, since 25 excludes 17, 20 must be used. In the first column, since 15 excludes 24, 23 must be used. Finally in the fifth column, 18 is the only element whose row is not yet occupied. With 15 as minimum we obtain the unique set 23, 20, 25, 15, 18. The above construction proves that the minimum of this set, 15, is as large as possible.

5. The given lines are not parallel, and it is readily verified that they do not intersect. Hence there is a unique line which is perpendicular to both of the given lines and intersects them both. The segment of this line between its two points of intersection is a diameter of the required sphere. To find the two points of intersection with the common perpendicular, minimize the distance between a pair of points, one from each line, *i.e.*, minimize $D^2=(t+1-4s+12)^2+(2t+4+s-8)^2+(-3t+5-s-17)^2$. The minimum is found to occur for $t=-\frac{782}{251}$, $s=\frac{657}{251}$. The corresponding points on the two lines are $(-\frac{531}{251}, -\frac{560}{251}, \frac{3601}{251})$ and $(-\frac{384}{251}, \frac{1351}{251}, \frac{4924}{251})$ respectively. The midpoint of the segment between these points is $(-\frac{915}{502}, \frac{791}{502}, \frac{8525}{502})$ and this is the center of the required sphere. The radius of the sphere is the distance from the center to either of the two points of intersection, and is $\frac{147}{502}\sqrt{251}$. The equation of the sphere can be written $(502x+915)^2+(502y-791)^2+(502z-8525)^2=5{,}423{,}859$.

6. Since x and y are both positive, $1/x$ and $1/y$ are both less than 1, so that both x and y are greater than 1. As a result, no two multiples of x have the same integral part, and no two multiples of y have the same integral part. Thus no integer appears more than one time in either of the two sequences. Suppose that an integer N appeared in both sequences. Then we could find integers p and q such that $N<px<N+1$ and $N<qy<N+1$. (No equality is possible since x and y are irrational.) Solving these inequalities for $1/x$ and $1/y$,we find

$$\frac{p}{N+1}<\frac{1}{x}<\frac{p}{N}, \qquad \frac{q}{N+1}<\frac{1}{y}<\frac{q}{N}.$$

Adding we find

$$\frac{p+q}{N+1}<1<\frac{p+q}{N} \quad \text{or} \quad N<p+q<N+1,$$

which is impossible since N, p, and q are all integers. Finally suppose that an

integer M is missing from both sequences. Then we can find integers p and q such that $px<M$, $(p+1)x>M+1$, $qy<M$, $(q+1)y>M+1$. Solving for $1/x$ and $1/y$ as before, we are led to $M-1<p+q<M$, which is again impossible. Therefore every integer is present in one sequence or the other, and each integer occurs exactly once.

7. The result will be proved by induction. From (2), $f_1(0)=f_1(-0)=-f_1(0)$, so that $f_1(0)=0$. Then from (1), $f_1(x)=f_1(0+x)=f_1(0)+x=0+x=x$. Hence the theorem is true for $n=1$. Now assume the theorem true through n. We first note that for any A, $f_{n+1}(0, \cdots, 0, A)=A/(n+1)$. To see that, start with

$$\begin{aligned} f_{n+1}(0, \cdots, -B, B) &= f_{n+1}(0, \cdots, B, -B) && \text{(by symmetry)} \\ &= -f_{n+1}(0, \cdots, -B, B) && \text{(by (2))}. \end{aligned}$$

Hence $f_{n+1}(0, \cdots, -B, B) = 0$. But, by the induction hypothesis, $f_n(0, \cdots, 0, -B)=-B/n$, so that by (3),

$$0 = f_{n+1}\left(-\frac{B}{n}, \cdots, -\frac{B}{n}, B\right) = f_{n+1}\left(0, \cdots, 0, \frac{n+1}{n} B\right) - \frac{B}{n}$$

by (1). Setting $B=(nA)/(n+1)$, we get $f_{n+1}(0, \cdots, 0, A)=A/(n+1)$. Now start with (3), writing $\bar{x}_n=(x_1+ \cdots +x_n)/n$:

$$\begin{aligned} f_{n+1}(x_1, \cdots, x_n, x_{n+1}) &= f_{n+1}(\bar{x}_n, \cdots, \bar{x}_n, x_{n+1}) && \text{(induction hypothesis)} \\ &= f_{n+1}(0, \cdots, 0, x_{n+1} - \bar{x}_n) + \bar{x}_n && \text{(by (1))} \\ &= \frac{x_{n+1}}{n+1} - \frac{\bar{x}_n}{n+1} + \bar{x}_n && \text{(by the lemma with } A = x_{n+1} - \bar{x}_n) \\ &= \frac{x_1 + \cdots + x_{n+1}}{n+1}. \end{aligned}$$

Hence the theorem is true for $n+1$ if it is true for n and, by induction, it is true for all n.

A CONFERENCE ON MATHEMATICS CURRICULA IN INSTITUTES

A Conference on Mathematics Curricula in NSF Institutes for High School Teachers, sponsored and planned by the MAA Committee on Institutes and financed by the National Science Foundation, was held in Washington, D. C., on September 10 and 11, 1960. This report is presented by the MAA Committee on Institutes as representing the consensus of the Conference. The organizations represented and the mathematicians in attendance are listed at the end of the report.

Background, purposes, and present status of institutes. From a modest beginning of two NSF summer institutes in 1953, this activity has increased to 396 NSF summer institutes in 1960, 33 academic year institutes in 1960–61, and 204 in-service institutes in 1960–61. The vast majority (89%) of these institutes are for high school teachers, and mathematics teachers are included in 212 summer institutes, in 32 academic year institutes, and in 136 in-service institutes. Approximately 13,000 mathematics teachers will attend an institute during the summer of 1960 or the academic year 1960–61.

The ultimate purpose of an institute for high school mathematics teachers is to improve the mathematical education of the students in our schools. Institutes aim to accomplish this purpose by increasing the mathematical competence and background of the teachers. Thus, courses and procedures in institutes should be relevant to the courses which the institute participants will teach in their classrooms.

NSF institutes constitute one of the most effective means of improving the teaching of high school mathematics. They have proved their efficacy in helping high school teachers do a better job with traditional courses and in enabling them to teach more modern courses. They seem to be the best hope of implementing the curricula changes being recommended by the various national agencies. Thus, the quality and effectiveness of mathematics courses in institutes is a matter of importance to the entire mathematical community.

It seems likely that institutes will be a heavy obligation on the mathematics profession for some years to come. Hence, it is essential that qualified small colleges, as well as large institutions, be engaged in the conduct of mathematics institutes, particularly in-service and summer institutes. As institutions and staffs with less experience undertake to conduct institutes, it becomes increasingly desirable to provide assistance to newcomers in institute work, to facilitate the exchange of information, and to encourage a certain degree of standardization of courses. The MAA Committee on Institutes can be of considerable service in this connection, and its functions should be continued and expanded. During the past year, the Committee has distributed to directors a list of possible mathematics instructors for institutes, it has called the attention of departmental chairmen to the need for more proposals for institutes of certain types, and it has conducted five regional conferences for mathematics lecturers in institutes. It was suggested that the Committee work in coordination with CUPM to prepare somewhat detailed guidelines for courses suitable to institutes.

In-service and academic year institutes have their own peculiar problems. While this report refers primarily to summer institutes, much of what it says applies to the other types, particularly the statements about courses.

Types and structure of institutes. Institutes should be planned for specified levels of teaching such as junior high school, senior high school, etc., and for specified levels of mathematical background—below-average, average, above-

average—on the part of teachers. The five-level teaching classification proposed by CUPM in the December 1960 MONTHLY might be adopted; in which case the recommendations of this report would be primarily applicable to levels II and III. In general, an institute should be planned for one teaching level, but in special cases institutes including participants from more than one level may have advantages. In any case relative homogeneity within a given course is highly desirable.

Sequential institutes, that is institutes planned to continue participants for two or more summer or in-service programs, are becoming frequent. They serve a good purpose in offering sustained training, which in some cases culminates in a master's degree. Such institutes have opportunities for courses at increasing levels of sophistication and can prepare teachers for teaching advanced twelfth grade courses such as the one in the Advanced Placement Program. An alternative plan, the success of which depends upon some standardization of courses, is for different colleges to operate institutes of different types and to encourage participants to attend institutes of progressively "higher" type in successive summers.

Procedures and materials. Courses should be conducted in such a fashion as to reassure the participants and make them feel at ease. High school teachers are usually rusty and frequently sensitive about their weaknesses. Generally they are hard-working conscientious people who respond as well as they can to encouragement and assistance.

Regular conference periods in which the participants work with the direct help of the lecturer or of skilled and knowledgeable assistants should be a prominent feature of an institute program. Also, ample opportunity for informal contact between the lecturers and the participants should be provided.

It is desirable that a text, a small group of texts, or prepared mimeographed notes be used in each course, and that these materials be available at the time of study in the course.

Staff. The quality of an institute's instructional staff is the most important factor in the success of the institute. This quality is measured by the instructor's knowledge of mathematics, his enthusiasm for mathematics at the level of the participants, his sensitivity to their reactions, and his ability to communicate effectively with them. He should be fully aware of the mathematics taught in high school and the relation his course bears to that high school mathematics.

Securing adequate staff is the most difficult and most important problem for institute directors. Continuing efforts should be made to seek out and interest appropriate people to teach institute courses.

Courses. Institute courses should be planned with careful regard to the background, ability, and needs of the participants. They should be designed to deepen understanding of the mathematics which the participants will teach. They should include both a study of the ideas underlying high school courses

and an indication of the nature of the mathematics being taught after high school.

Since institute participants generally will be teaching algebra or geometry or both, courses in these subjects should be included in the *first* institute experience of most participants.

Algebra. The first algebra course is perhaps the most basic of all the institute courses, since algebra constitutes such a large part of high school mathematics. Most teachers have no clear concept of algebraic structure or real understanding of the number systems of elementary algebra. A first institute course in algebra should aim to alleviate these deficiencies.

A few general principles concerning algebra courses were agreed upon. The content of the course should be chosen for its relevance to high school algebra. In the first algebra course number systems must be dealt with in some manner. The number systems furnish the motivation for the study of abstract systems, and the treatment of these last should be such as to throw light upon the ordinary number systems by putting the latter in the proper perspective. The course should not be too narrow, but, at the same time, a few concepts should be carefully developed. The language and symbolism of sets should be introduced and used. Algebra is appropriate material for teaching postulational methods and simple principles of logic. The course should contain some material which is new to the participant. Current experimental high school texts with the accompanying manuals and texts for teachers can be helpful in devising appropriate courses.

Geometry. The general outline of geometry courses is not as clear as in the case of algebra, nor are textbooks as readily available. Considerable reliance upon recently developed programs for high school geometry is perhaps indicated. Conceivably an institute course in geometry could be built upon one of the recently proposed high school courses with extensive insertions at the proper places. If a course is modeled upon some particular one of these programs, care should be taken to make clear that the course represents only one of several logically correct ways to develop Euclidean geometry.

There was much discussion of the intuitive approach *vs* deductive reasoning, particularly in the case of a course for junior high school teachers. The latter group perhaps needs a special course in which the intuitive aspect is emphasized. In the first course in geometry the language of sets should be used naturally, and, unless certain matters of logic are treated elsewhere, they must receive attention in this course.

In subsequent courses, various possibilities were suggested. There is much to be said for the study of one or more geometries other than Euclidean. There is a large variety of topics from higher geometries that can be fitted into courses. The big ideas of elementary geometry—measure, congruence, similarity, convexity, etc.—can be considered from an advanced point of view. A course stress-

ing applications and designed to enhance the participants' intuition is another possibility. A careful treatment of analytic geometry in which the postulates are those characterizing the real numbers and the geometric objects are defined entities is also worthy of consideration.

Analysis. A distinction should be made between a first course in analysis for the usual high school mathematics teacher and special courses serving to prepare for teaching the advanced placement course in calculus and analytics.

A first course in analysis for the generality of participants should present the essential concepts of calculus with more emphasis upon ideas than devices. A judicious selection of a small number of basic theorems should be made for careful proof. The course should give teachers some acquaintance with the content of mathematics courses beyond high school and the high school skills required for mastering the same. Moreover, it should illuminate parts of the high school mathematics program by increasing the participants' knowledge of the structure of the real number system, the function concept and properties of particular functions such as the logarithmic and exponential functions, limits of sequences and sums of infinite series, and areas bounded by curves.

Preparation for teaching the Advanced Placement Program course requires distinctly more than the introductory course discussed in the preceding paragraph. Such preparation is appropriately obtained from an academic year institute or from a sequence of appropriately graded summer institute courses.

A special type of institute for training teachers to teach the Advanced Placement course might include high school teachers who already have fair competence in elementary calculus, junior college teachers, and elementary calculus teachers in four-year colleges.

Probability and Statistics. It was agreed that courses in probability and statistics should be included in institutes *only after* algebra and geometry—and perhaps also analysis—are taken care of. There was also strongly held opinion on the part of some that further work in such branches of pure mathematics as number theory would be of greater value to participants. In any case, it was held that emphasis should be on probability rather than statistics, but there was objection to excluding statistics altogether. A number of texts are now available for a first course in probability. Adequate preparation to teach a course in probability in high school would require more than a single course.

Foundations and Logic. There was some division of opinion regarding the best way to handle questions of foundations and logic. Courses in formal logic were not considered appropriate, but it was recognized that some attention must be paid to elementary principles of logic. The suggestions were essentially of two kinds: to treat these principles as they arise naturally in mathematics courses, or to devote a few days in some one course to discussing them. The treatment of certain topics in algebra and geometry courses may be such as to illustrate adequately the basic logical principles.

In academic year institutes and sequential summer institutes, courses in other topics such as number theory, linear algebra, elementary topology, computation, etc., may well be appropriate. Cardinal principles should be to include the essential courses first and to tailor all courses to the level of the participants.

Academic activities other than courses. Seminars on high school curricula have proved valuable in acquainting the participants with various current curriculum enterprises, and in relating the material being taught in the institute courses to high school mathematics courses.

If a demonstration class is part of an institute program, arrangements should be made to supplement meetings of the class with introductory or follow-up discussions between the teacher of the class and the participant observers.

Visiting lecturers for short periods can add interest to an institute, but can easily be overdone. The participants need plenty of time to study, and features outside of courses should not be allowed to interfere. In any case, visiting lecturers should be carefully chosen and requested to talk about something in the range of comprehension of the participants, preferably something which fits in with the overall plan of the institute.

Academic credit and degrees. Most teachers attending institutes want graduate credit for the courses taken. This desire is due to certification requirements and conditions governing salary increases. Many institutions are handling this matter by awarding graduate credit for institute courses which can count only toward a special type of master's degree, such as a Master of Arts in the Teaching of Mathematics. Such degrees, which allow most of the required courses to be in mathematics with only a minimum requirement in education, have been inaugurated recently in a large number of institutions.

Appendix. Organizations represented at the Conference: MAA Committee on Institutes, MAA Committee on the Undergraduate Program, School Mathematics Study Group, AAAS Cooperative Committee on the Teaching of Science and Mathematics, National Council of Teachers of Mathematics, National Association of State Directors of Teacher Education and Certification.

Persons attending the Conference: C. B. Allendoerfer, R. D. Anderson, H. M. Bacon, E. G. Begle, S. J. Bezuszka, W. E. Briggs, C. F. Brumfiel, E. A. Cameron, J. W. Cell, Roy Dubisch, W. L. Duren, W. T. Guy, Jr., S. P. Hughart, P. S. Jones, Joseph Landin, N. H. McCoy, W. H. L. Meyer, E. E. Moise, H. T. Muhly, W. W. Osborn, C. R. Phelps, G. B. Price, H. W. Syer, G. B. Thomas, Marie S. Wilcox, N. B. Winters, R. J. Wisner.

Members of the MAA Committee on Institutes: E. A. Cameron, Chairman; E. G. Begle, W. T. Guy, Jr., Kenneth May, W. H. L. Meyer.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

INFINITE PRODUCTS FOR πe AND π/e

Z. A. MELZAK,* Bell Telephone Laboratories

Let S_n be a closed solid n-dimensional sphere and let C_n be the solid n-dimensional cylinder of largest volume that can be inscribed into S_n; $V(S_n)$ and $V(C_n)$ will denote the volumes of the respective solids. It is shown first that

$$\lim_{n\to\infty} V(C_n)/V(S_n) = (2/\pi e)^{1/2}, \tag{1}$$

and next that

$$\pi/2e = \prod_{n=1}^{\infty} (1 + 2/n)^{(-1)^{n+1}n} \tag{2}$$

and

$$6/\pi e = \prod_{n=2}^{\infty} (1 + 2/n)^{(-1)^{n}n}. \tag{3}$$

The derivation of (2) and (3) from (1) parallels closely Wallis' derivation of the infinite product for π.

Let E^n be the n-dimensional Euclidean space equipped with a Cartesian coordinate system. Use will be made of the vector notation: $x=(x_1, x_2, \cdots, x_n)$ will denote both the point and the corresponding vector (from the origin); $x\cdot y$ is the scalar product, and so on. A closed solid sphere S_n is any set of points x, such that $(x-a)\cdot(x-a)\leqq r^2$; here r is the radius and the point a is the center of S_n. A hyperplane H is any set of points x, such that $a\cdot x=b$ and $a\cdot a=1$. The vector a is the direction of H. Two hyperplanes H_1 and H_2 are parallel if their equations are $a\cdot x=b_1$, and $a\cdot x=b_2$, $b_1\neq b_2$. The distance d between H_1 and H_2 is then $d=|b_1-b_2|$.

Let H_1 and H_2 be two parallel hyperplanes in the direction a and let K be any closed set contained in H_1. Let $[xy]$ denote the closed straight segment with the end points x and y. A closed solid cylinder C is any set of the following form: C consists of all points z such that z lies on a segment $[xy]$ where x is in K, y is in H_2, and the vector $x-y$ is in the direction a. The hyperplanes H_1 and H_2 will be called the bases of the cylinder C, the distance d between H_1 and H_2 is the height of C, and the set K is the directrix of C. A cylinder C is called spherical if its directrix is a sphere (of some dimension). (The above definition reduces for $n=3$ to the usual definition of a right cylinder.)

* Now at the University of British Columbia.

Let C be the cylinder of largest volume, inscribable into a sphere S_n of unit radius. Strictly speaking, one should prove first that this largest cylinder exists. Such a proof could be easily deduced from Blaschke's selection principle [1], [2], but it will not be given here. Let H_1 and H_2 be the bases of C and let K_1 and K_2 be their intersections with S_n. Let $d=2h$ be the distance between H_1 and H_2. Now C must lie within the spherical cylinder C_1 with the directrix K_1 and height d, and C must also lie within the spherical cylinder C_2 with the directrix K_2 and height d. The bases of C_1 and C_2 are H_1 and H_2 and their directrices are the $(n-1)$-dimensional spheres K_1 and K_2 of radii r_1 and r_2 respectively.

If follows that C lies in the intersection, C_3, of C_1 and C_2. Since C is maximal and C_3 is a cylinder inside S_n, it follows that $C=C_3$, that $r_1=r_2=r$, and that the center o of S_n lies half-way between H_1 and H_2.

It is shown in any text on analysis, *e.g.*, [3], that the volume V of a closed solid n-dimensional sphere of radius r is

$$V = r^n\pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1). \tag{4}$$

The volume of a spherical cylinder is the product of its height and the volume of its spherical base. Hence for the sphere S_n of unit radius and for its inscribed maximal cylinder C_n one gets

$$V(S_n) = \pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1), \tag{5}$$

$$V(C_n) = 2h\pi^{(n-1)/2}r^{n-1}/\Gamma(\tfrac{1}{2}(n - 1) + 1). \tag{6}$$

The quantities r and h are not independent and in fact

$$r^2 + h^2 = 1, \tag{7}$$

since the radius of S_n is 1.

To make sure that $V(C_n) =$ maximum, one has to maximize in (6) the product hr^{n-1} under the side-condition (7). This is easily accomplished and the maximum of hr^{n-1} is found to be

$$n^{-1/2}(1 - 1/n)^{(n-1)/2}. \tag{8}$$

Now let

$$\rho_n = V(C_n)/V(S_n); \tag{9}$$

by (5), (6), and (8)

$$\rho_n = 2(\pi n)^{-1/2}(1 - 1/n)^{(n-1)/2}\Gamma(\tfrac{1}{2}n + 1)/\Gamma(\tfrac{1}{2}n + \tfrac{1}{2}). \tag{10}$$

Observing that

$$\lim_{x\to\infty} \Gamma(x + \alpha)/x^\alpha\Gamma(x) = 1,$$

([4], p. 254) and recalling the definition of e, one gets from (10) by passing to the limit

$$\rho = \lim_{n\to\infty} \rho_n = (2/\pi e)^{1/2}, \tag{11}$$

which proves (1). To prove (2) and (3) one calculates first $\sigma_n = \rho_{n+2}/\rho_n$. This expression is used in order to apply the difference equation of the Γ-function. One has by (10)

$$\sigma_n = [n/(n+2)]^{n/2}[(n+1)/(n-1)]^{(n-1)/2}. \tag{12}$$

Simple calculation shows that

$$\rho_2 = 2/\pi, \qquad \rho_3 = 3^{-1/2}. \tag{13}$$

Now one has

$$\rho = \rho_2 \prod_{n=1}^{\infty} \sigma_{2n}, \qquad \rho = \rho_3 \prod_{n=1}^{\infty} \sigma_{2n+1}. \tag{14}$$

Hence by (11), (12) and (13)

$$(2/\pi e)^{1/2} = 2/\pi \prod_{n=1}^{\infty} [n/(n+1)]^n[(2n+1)/(2n-1)]^{(2n-1)/2}, \tag{15}$$

$$(2/\pi e)^{1/2} = 3^{-1/2} \prod_{n=1}^{\infty} [(2n+1)/(2n+3)]^{(2n+1)/2}[(n+1)/n]^n. \tag{16}$$

Squaring and simplifying (15) and (16) yields (2) and (3). Convergence proofs for the products (2) and (3) are easily deduced from the standard criteria for infinite products, [4]. New infinite products for e^2 and π^2 are obtained on multiplying the expressions in (2) and (3). The first four partial products in (2) are: .75, .686 $\cdots$, .657 $\cdots$, .639 $\cdots$; as might be expected, convergence to $\pi/2e = .57786 \cdots$ is rather slow.

Two conjectures will be made in conclusion:

(A) *π/e is irrational and the irrationality can be proved by starting with* (2).

(B) *Let S be a closed convex bounded n-dimensional set with nonempty interior, then there is in S a convex n-dimensional cylinder C, such that $V(C)/V(S) \geqq k > 0$ and the constant k does not depend on the dimension n. By considering the case of a cone it is easy to show that $k \leqq 1/e$.*

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A POLYNOMIAL IN EACH VARIABLE SEPARATELY IS A POLYNOMIAL

F. W. CARROLL, Purdue University

The result in this note is probably not new, although I have been unable to find it in the literature. Perhaps the most surprising aspect of the problem is that the proof, while elementary, is not trivial.

THEOREM. *Let $f\colon R\times R\to R$ have the property that for each fixed value of x (respectively, y), $f(x, y)$ assumes the values of a polynomial $P_x(y)$ (respectively, of a polynomial $P_y(x)$). Then f is a polynomial on $R\times R$.*

Proof. It follows from the hypothesis that

$$f(x, y) = \sum_{n=0}^{\infty} a_n(y)x^n, \tag{1}$$

where, for each y, $a_n(y)=0$ for all but finitely many n. Since R is not a countable union of finite sets, there exists an integer N such that the set $F=\{y\colon a_n(y)=0$ for all $n>N\}$ is infinite. Denoting by f_1 the restriction of f to $R\times F$, we have

$$f_1(x, y) = \sum_{n=0}^{N} a_n(y)x^n, \qquad (x, y) \in R \times F. \tag{2}$$

Choosing $N+1$ distinct values $x_0, x_1, \cdots, x_N$, substituting them for x in (2), and solving the resulting system of equations, we obtain

$$a_n(y) = \sum_{j=0}^{N} c_{jn} f_1(x_j, y) \qquad (y \in F, n = 0, \cdots, N), \tag{3}$$

where the c_{jn} are real constants. Thus the function g, defined on $R\times R$ by

$$g(x, y) = \sum_{n=0}^{N}\sum_{j=0}^{N} c_{jn} f(x_j, y)x^n, \tag{4}$$

is a polynomial. Moreover, (2) and (3) show that, for each $x'\in R$, the polynomial $f(x', y)-g(x', y)$ has a zero at each point of F, hence is equal to zero for all y.

A NOTE ON FACTORIZABLE GROUPS

OSSIE HUVAL, University of Southwestern Louisiana

In the classification of groups the question whether or not a group can be written as a product of two of its subgroups often arises. In this note we will prove with the aid of a lemma of complementation theory of groups that a certain torsion group is completely factorizable.

DEFINITION. *A group G is factorizable if $G=AB$ for subgroups A and B in G. If in addition $A\cap B=1$, the identity of G, G is completely factorizable.*

DEFINITION. *Let π be a given set of primes. The group H is a π-group if the prime divisors of the orders of the elements of H belong to π.*

DEFINITION. *A normal subgroup A of a group G is complemented in G if there exists a subgroup F of G such that $G=AF$ and $A\cap F=1$.*

In the following lemma π is a given set of primes, and $(n, \pi)=1$ means $(n, p)=1$ for all p in π.

LEMMA. *Suppose A is a normal, solvable, torsion, π-subgroup of finite index n in the group G. If $(n, \pi)=1$, then A is complemented in G.*

Proof. We will prove this lemma by induction on the length k of the solvable series of A. If $k=1$, then A is an abelian group, and the proof for finite groups given in [1], pages 201–202, is applicable. Suppose the lemma is true for groups satisfying the conditions of the lemma and for normal subgroups whose solvable series are of length k, and suppose the length of the solvable series of A is $k+1$. The commutator subgroup A' of A is normal in G since it is characteristic in A and A is normal in G. The group A/A' is an abelian, torsion, π' subgroup of index n in G/A', where π' is a subset of π. Since the lemma has been proved for $k=1$, there exists $\overline{H}\subset G/A'$ such that $G/A'=\overline{H}(A/A')$ and $\overline{H}\cap(A/A')=1$. Let H be the complete inverse image of $\overline{H}$. By the isomorphism theorem for groups, HA/A is isomorphic to $H/H\cap A$. Thus, the index of $H\cap A$ in H is n which is prime to π by hypothesis. Since $\overline{H}\cap(A/A')=1$, $H\cap A=A'$. The length of the solvable series of A' is k. By the induction hypothesis there exists a subgroup $F\subset H$ such that $H=A'F$ and $A'\cap F=1$. Now $G=HA=FA'A=FA$. Since $(n, \pi)=1$ and the order of F is n, $F\cap A=1$.

Suppose G is the group of permutations on the symbols 1, 2, 3, and 4. The commutator subgroup G' of G is the alternating group. The group G contains 3 Sylow 2-subgroups of order 8 which we will designate by P_1, P_2, and P_3. The intersection $P_*=P_1\cap P_2\cap P_3$ is the group K whose elements are (12) (34), (13) (24), (14) (23), and the identity. Now $P_1\cap G'=K=P_*\cap G'$. Hence, $P_1\cap G'\subset P_*\cap G'$. Thus the group G satisfies the hypotheses of the following theorem.

THEOREM. *Suppose the torsion group G contains a solvable Sylow π-subgroup P of finite index in G. If $P_*\cap G'\supset P\cap G'$, where P_* is the intersection of the complete set of conjugates of P, then G is completely factorizable.*

Proof. Under the natural homomorphism of G onto G/G' the image of P, $\overline{P}$, is a Sylow η-subgroup of G/G' where η is a subset of π. Since G/G' is torsion and abelian and since $\overline{P}$ is a Sylow η-subgroup of G/G', G/G' can be expressed as the direct product $\overline{P}\otimes\overline{Q}$ where the orders of the elements of $\overline{Q}$ are divisible by primes which are not in η. Let H be the complete inverse image of $\overline{P}$. The group H is generated by P and G' since H is generated by a set of representatives of the cosets of $\overline{P}$ and G' and the set of representatives can be taken in P. Since G' is normal in G, G' is normal in H, and $H=PG'$. Let Q be the complete inverse

image of $\bar{Q}$. Since Q is normal in G and since P has a finite number of conjugates, by ([1], p. 161 and 163), $P\cap Q$ is a Sylow λ-subgroup of Q where λ is a subset of π.

Now $P\cap Q\supset P\cap G'$. Suppose $P\cap G'\not\supset P\cap Q$. Then there is an element x in $P\cap Q$ which is not in $P\cap G'$, and therefore, not in G'. The image of x under the natural homomorphism is in $\bar{Q}$, and it is not the identity since x is not in G'. Since x is in $P\cap Q$, the order of x is divisible by primes in π. Hence the order of the image of x is divisible by primes in η. But $\bar{Q}$ contains no element whose order is divisible by primes in η. Hence $P\cap Q=P\cap G'$, and since $P_*\cap G'\supset P\cap G'$, $P_*\cap G'\supset P\cap Q$. Certainly $P\cap Q\supset P_*\cap G'$. Therefore $P\cap Q=P_*\cap G'$.

Since P is of finite index in G and since it has a finite number of conjugates, P_* is of finite index in G. Now $[Q\colon Q\cap P_*]$ is finite since it is equal to $[P_*Q\colon P_*]$. Since $[Q\colon Q\cap P_*]=[Q\colon Q\cap P][Q\cap P\colon Q\cap P_*]$ and since $[Q\colon Q\cap P_*]$ is finite, $[Q\colon Q\cap P]$ is finite. Now P_* is normal in G since the intersection of a complete set of conjugate subgroups in a group is normal in the group. Since $P\cap Q=P_*\cap G'$, $P\cap Q$ is normal in G, and hence in Q. Since $P\cap Q$ is a Sylow λ-subgroup of Q, $([Q\colon Q\cap P],\lambda)=1$. By the above lemma there exists a subgroup $F\subset Q$ such that $Q=(P\cap Q)F$ and $(P\cap Q)\cap F=1$. Now $G=HQ=PG'Q=PQ=P(P\cap Q)F=PF$. Since $F\subset Q$, $1=(P\cap Q)\cap F=P\cap F$.

Reference

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A DECOMPOSITION OF CONTINUITY IN TOPOLOGICAL SPACES

NORMAN LEVINE,* University of Pittsburgh

0. Introduction. In the literature there are many "weakened" forms of continuity, *e.g.*, upper and lower semicontinuity, continuity almost everywhere, approximate continuity, sequential continuity and no doubt many others. In this note we (1) introduce two new weakened forms of continuity, (2) show they are independent, and (3) show that taken together, they characterize continuity in the most general cases. One of the weakened forms is characterized in terms of inverses of open sets. In addition, examples are given to illustrate the various connections between the two weakened forms of continuity, sequential continuity, and continuity.

I. Let $f\colon X\to X^*$ be a single valued function (not necessarily continuous), X and X^* being topological spaces.

DEFINITION 1. *$f\colon X\to X^*$ will be termed weakly continuous at $x\in X$ if and only if for $f(x)\in 0^*$ open in X^*, there is an open set 0 in X such that $x\in 0$ and $f(0)\subset c^*0^*$, c and c^* denoting the closure operators in X and X^* respectively. $f\colon X\to X^*$ will be termed weakly continuous (denoted henceforth as w.c.) if and only if $f\colon X\to X^*$ is weakly continuous at each of the points in X.*

* Now at The Ohio State University.

THEOREM 1. *Let* $f: X \to X^*$ *be single valued,* X *and* X^* *being topological spaces. Then* $f: X \to X^*$ *is w.c. if and only if for* 0^* *open in* X^*, $f^{-1}(0^*) \subset \operatorname{Int} f^{-1}c^*0^*$, Int *denoting the interior operator.*

Proof. Sufficiency. Let $x \in X$ and $f(x) \in 0^*$. Then $x \in f^{-1}(0^*) \subset \operatorname{Int} f^{-1}c^*0^*$. Let $0 = \operatorname{Int} f^{-1}c^*0^*$. $f(0) = f \operatorname{Int} f^{-1}c^*0^* \subset ff^{-1}c^*0^* \subset c^*0^*$.

Necessity. Let $x \in f^{-1}0^*$. Then there exists an open set 0 such that $x \in 0$ and $f(0) \subset c^*0^*$. Hence $x \in 0 \subset f^{-1}c^*0^*$ and $x \in \operatorname{Int} f^{-1}c^*0^*$.

THEOREM 2. *Let* $f: X \to X^*$ *be single valued and* X^* *regular. Then* $f: X \to X^*$ *is w.c. if and only if* $f: X \to X^*$ *is continuous.*

Proof. The sufficiency is clear. Necessity. Let $x \in X$ and $f(x) \in 0^*$. Then $f(x) \in 0^\# \subset c^*0^\# \subset 0^*$. There exists an open set 0 such that $x \in 0$ and $f(0) \subset c^*0^\# \subset 0^*$. Thus $f: X \to X^*$ is continuous.

Example 1. w.c. does not imply sequential continuity. Let J denote the unit interval with the usual topology. Let S denote the unit interval with the following topology: a set A in S is open if and only if either $A = \phi$ or $\mathcal{C}A$ is countable, $\mathcal{C}$ denoting the complement operator. Let $I: J \to S$ be the identity mapping. The reader will easily verify that $I: J \to S$ is w.c. but not sequentially continuous.

Example 2. Sequential continuity does not imply w.c. Let $I: S \to J$ be the identity mapping, S and J being the spaces in Example 1. The reader will easily verify that $I: S \to J$ is sequentially continuous but not w.c.

Example 3. w.c. plus sequential continuity does not imply continuity. Let S be the space in Example 1 and $S^* \equiv \{a^*, b^*\}$ where the open sets are ϕ, $\{a^*\}$, and S^*. Let $f: S \to S^*$ as follows:

$$f(x) \equiv \begin{cases} a^* & \text{if } x \text{ is rational} \\ b^* & \text{if } x \text{ is irrational.} \end{cases}$$

Then $f: S \to S^*$ is not continuous since $f^{-1}(a^*)$ is not open in S, but is sequentially continuous as well as w.c. We leave the details to the reader.

II. We now introduce a complementary form of weak continuity.

DEFINITION 2. $f: X \to X^*$ *will be termed w*.c. if and only if when* 0^* *is open in* X^*, *then* f^{-1} fr 0^* *is closed in* X, *where* fr *denotes the frontier operator.*

Example 4. w.c. does not imply w*.c. Let $S \equiv \{a, b\}$ with open sets ϕ, $\{a\}$, and S and $S^* \equiv \{a^*, b^*\}$ with open sets ϕ, $\{b^*\}$, and S^* and let $f(x) \equiv x^*$ on S. Then $f: S \to S^*$ is clearly w.c. but not w*.c. since fr $b^* = c^*b^* \cap c^*a^* = a^*$. But $f^{-1}a^* = a$ and $\{a\}$ is not closed in X.

Example 5. w*.c. does not imply w.c. Let $S \equiv \{a, b\}$ with open sets ϕ and S and $S^* \equiv \{a^*, b^*\}$ with ϕ, $\{a^*\}$, $\{b^*\}$, and S^* as the open sets. Let $f(x) \equiv x^*$. Then $f: S \to S^*$ is not w.c. for $f(a) = a^* \in \{a^*\} = c^*a^*$. But the only open set con-

taining a is S and $f(S) \not\subset c^*a^*$. $f: S \to S^*$ is w*.c. for fr $a^* = a^* \cap b^* = \phi$ and $f^{-1}(\phi)$ is closed.

Example 6. Sequential continuity plus w*.c. does not impy w.c. Let S be the space in Example 1 and $S^* \equiv \{a^*, b^*\}$ with ϕ, $\{a^*\}$, $\{b^*\}$, and S^* the open sets. Define

$$f(x) \equiv \begin{cases} a^* & \text{if } x \text{ is rational} \\ b^* & \text{if } x \text{ is irrational.} \end{cases}$$

$f: S \to S^*$ is clearly sequentially continuous. Since fr $a^* = \phi =$ fr b^*, f^{-1} fr 0^* is closed for all 0^* open in S^* and thus $f: S \to S^*$ is w*.c. But $f: S \to S^*$ is not w.c. since $f(\frac{1}{2}) = a^*$ and a^* is open in S^*. $c^*a^* = a^*$ and there is no open set 0 containing $\frac{1}{2}$ such that $f(0) = a^*$.

Example 7. Sequential continuity plus w.c. does not imply w*.c. See Example 3.

III. THEOREM 3. *Let $f: X \to X^*$ be single valued, X and X^* being topological spaces. Then $f: X \to X^*$ is continuous if and only if it is both* w.c. *and* w*.c.

Proof. The necessity is clear. Sufficiency. Let $f(x) \in 0^*$. Since $f: X \to X^*$ is w.c. there exists an open set 0 containing x and $f(0) \subset c^*0^*$. Now fr $0^* = c^*0^* - 0^*$ and thus $f(x) \notin$ fr 0^*. Hence $x \notin f^{-1}$ fr 0^* and therefore $x \in 0 - f^{-1}$ fr 0^*, an open set since the transformation is w*.c. The proof will be complete when we show $f\{0 - f^{-1} \text{ fr } 0^*\} \subset 0^*$. To this end let $y \in 0 - f^{-1}$ fr 0^*. Then $y \in 0$ and hence $f(y) \in c^*0^*$. But $y \notin f^{-1}$ fr 0^* and thus $f(y) \notin$ fr $0^* = c^*0^* - 0^*$ which implies that $f(y) \in 0^*$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

LINEAR PROGRAMMING AND APPROXIMATION PROBLEMS

L. E. WARD, JR., University of Oregon and Naval Ordnance Test Station, China Lake, California

1. Introduction. In this note we attempt to propagandize in favor of the teaching of linear programming techniques in undergraduate numerical analysis courses. Although linear programming has been the instrument of a near-incredible tidal wave of large-scale industrial problem solving, and is admitted, even by its detractors, as a tool of undoubted merit in many applications,

notably in the areas of resource allocation and transportation, it is mentioned only fleetingly in most undergraduate curricula. The reasons for this are fairly transparent; not only is the subject new (and hence the victim of the customary lag between professional utility and curricular acceptance), but it has the added handicap that practically all except trivial examples are not easily susceptible to pencil-and-paper solution. Linear programming is the child of the young age of the high-speed, large-memory electronic digital computer; the algorithms for solution of linear programming problems have been designed with computer utility in mind. The simplex method and its invariants do not lend themselves to efficient pencil-and-paper use, and the instructor whose institution does not possess a moderate-size computer is hard put to find problems which suitably dramatize linear programming without involving a tedious, back-breaking mountain of unenlightening computation.

With the passage of time more colleges and universities will acquire computers and this limitation will grow less critical. But there is another barrier to the recognition and development of linear programming in the undergraduate curriculum. The contexts in which its applications have been most fruitful are unfamiliar to most teachers of mathematics. Though these applications are legion, they have for the most part arisen directly from nonmathematical disciplines in which mathematical techniques have seldom been used in a sophisticated way. (The only exception which comes readily to mind is economics, and even here the spirit of classical mathematical economics has little in common with linear programming.)

We suggest, therefore, that an application in a familiar mathematical setting deserves publicity. Indeed, we offer here an application in polynomial approximation which is admirably suited not only to introduce linear programming in a numerical analysis course, but to solve an important type of problem efficiently.

2. Polynomial approximation. Suppose, from experimental data, one has a finite set of observed points (x_i, y_i) and it is desired to find the polynomial $P_m(x)$ of degree m which is a best fit to these data in some specified sense. The most popular technique is the so-called method of least squares in which $P_m(x)$ is termed a best approximation if it minimizes the expression

$$\sum_i (P_m(x_i) - y_i)^2.$$

Apparently the popularity of the least squares method is due almost entirely to its painless submission to analytic treatment. But in many situations it is obvious that the least squares criterion for best fit is not a good one. Suppose the observed points (x_i, y_i) have the quality that the independent variable x can be measured with great accuracy while the dependent variable is subject to significant not-necessarily-uniform error. It is desired to choose the coefficients of the polynomial $P_m(x)$ so that the quantities $|y_i - P_m(x_i)|$ stay within the region of error. More specifically, one may ask these two questions.

QUESTION 1. *Given* $\epsilon_i > 0$ *and* (x_i, y_i) *for* $i = 1, \cdots, n$, *and nonnegative integers* $k_0 < k_1 < \cdots < k_N$, *does there exist a polynomial*

$$P(N, k, x) = \sum_{j=0}^{N} a_j x^{k_j}$$

such that $|y_i - P(N, k, x_i)| < \epsilon_i$ *for each* i?

QUESTION 2. *Given* ϵ_i *and* (x_i, y_i) *as above, what is the least* N *such that Question* 1 *has an affirmative answer for the polynomial*

$$Q(N, x) = \sum_{j=0}^{N} a_j x^j.$$

Such problems* can be reduced to finding an approximation which is best in the Tchebycheff sense, *i.e.*, which minimizes the maximum error. This is an old problem whose treatment by classical methods is not as succinct as the least squares problem. Among recent contributions we cite the papers of Kelley [3], Motzkin [4] and Selfridge [5]. Particularly relevant to this discussion is Kelley's paper which characterizes the problem of optimum curve fitting to a finite set of points in linear programming terms, apparently for the first time.

It is not our present intent to claim any priority for the discussion which follows, but we are not aware of its existence in the literature. Although Kelley treats Question 2 only for the case of uniform error (all ϵ_i equal) in a manner particularly suited to problems where the number of observed points is very large, the extension of his method to the case of nonuniform error offers no particular difficulty. However, the treatment given below is simpler and more readily accessible to any mathematics major at the upper division level.

3. Linear programming. Let us digress for a moment to discuss the central mathematical problem of linear programming. Consider a linear form

$$z = c_1 x_1 + \cdots + c_n x_n$$

where the variables x_j are constrained by inequalities of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\geqq b_1, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\geqq b_m, \end{aligned} \qquad (\text{all } x_j \geqq 0).$$

For what values of $x_1, \cdots, x_n$ is z minimized? Since minimizing z is equivalent to maximizing $-z$, there is an equivalent maximization problem. Thus the linear programming problem can be capsulized in the following way: to extremize a linear form subject to linear inequality constraints. Provided the number

* These problems were first brought to my attention by Dr. R. B. Leipnik. I am further indebted to Mr. William Clelland III of the Data Computation Branch, Naval Ordnance Test Station, for computational assistance.

of constraint equations is not too large ($\leqq 255$) the general problem can be solved with present day computing equipment by the simplex method of Dantzig (see [1] and [2]).

4. A linear program for the curve fitting problem. It is sufficient then to reduce the curve fitting problem to an exercise in minimizing a linear form subject to linear constraints. Referring to Question 1, we are given observed points (x_i, y_i) and positive numbers ϵ_i for each $i=1, \cdots, n$. If $k_0<k_1< \cdots <k_N$ are nonnegative integers we wish to know whether there exist coefficients $a_0, a_1, \cdots, a_N$ such that

$$E_i = \eta_i \left| y_i - \sum_{j=0}^{N} a_j x_i^{k_j} \right| < 1 \tag{1}$$

for each i, where $\eta_i = 1/\epsilon_i$. If we can compute

$$z = \min_{a} \max_{1 \leqq i \leqq n} E_i \tag{2}$$

where the min is taken over all $(N+1)$-tuples $a=(a_0, \cdots, a_N)$, then the existence problem is answered by comparing z and 1.

In order to interpret this as a linear programming problem we define some new variables. For each i and j let u_i, v_i, u_{n+j+1}, v_{n+j+1} be new quantities constrained by

$$\begin{aligned} &u_i - v_i = \eta_i\left(y_i - \sum_{j=0}^{N} a_j x_i^{k_j}\right) && (i = 1, \cdots, n), \\ &u_{n+j+1} - v_{n+j+1} = a_j && (j = 0, \cdots, N), \\ &u_i \geqq 0,\ v_i \geqq 0,\ u_{n+j+1} \geqq 0,\ v_{n+j+1} \geqq 0, \end{aligned} \tag{3}$$

and consider the new problem of determining

$$z' = \min_{a} \max_{1 \leqq i \leqq n} (u_i + v_i) \tag{4}$$

subject to

$$\eta_i y_i = u_i - v_i + \sum_{j=0}^{N} \eta_i (u_{n+j+1} - v_{n+j+1}) x_i^{k_j}. \tag{5}$$

Suppose z' is computed and that for some i both u_i and v_i take positive values, say $u_i = \delta_i + c_i$, $v_i = \delta_i$, where $c_i \geqq 0$. It is clear that z' cannot be increased if the new values $u_i = c_i$, $v_i = 0$ are assigned, but the constraint (5) is still satisfied. A similar argument applies if $u_i < v_i$ so that a solution to (4) can always be found with $u_i = 0$ or $v_i = 0$. But in this case $u_i + v_i = E_i$ and we infer that $z' = z$.

Our problem now is to minimize a quantity which is no less than each of n distinct linear forms (of the type $u_i + v_i$) where the variables are constrained by (3) and (5). Introducing a fictitious slack variable t, we can express the prob-

lem in the following way. Minimize the linear form $w=u_1+v_1+t$, subject to

$$u_1+v_1-u_i-v_i+t \geqq 0 \qquad (i=2,\cdots,n),$$

$$u_i-v_i+\sum_{j=0}^{N}\eta_i x_i^{k_j}(u_{n+j+1}-v_{n+j+1})=\eta_i y_i \qquad (i=1,\cdots,n),$$

$$u_i \geqq 0,\ v_i \geqq 0,\ t \geqq 0.$$

This is a straightforward linear programming problem and the minimized value of w is precisely the z sought in (2). If $z<1$ then Question 1 has an affirmative answer. To answer Question 2 it suffices merely to repeat the above procedure for $Q(N, x)$ for increasing values of N, commencing with a value so low that a negative answer to Question 1 is assured. However, it seems likely that this brute-force approach is unnecessarily laborious; by appropriately altering the final simplex tableau for a given value of N one may expect to reduce significantly the necessary computation for the next value of N.

5. An alternative formulation. Returning to Section 2, it is sufficient, in order to answer Question 1, to find $\gamma>0$ for which

$$(6) \qquad |y_i-P(N,k,x_i)| \leqq \epsilon_i-\gamma. \qquad (i=1,\cdots,n)$$

An equivalent linear program is to maximize γ subject to

$$(7) \qquad \begin{aligned} \gamma-\sum_{j=0}^{N}a_j x_i^{k_j} &\leqq \epsilon_i-y_i,\\ \gamma+\sum_{j=0}^{N}a_j x_i^{k_j} &\leqq \epsilon_i+y_i, \end{aligned}$$

for $i=1,\cdots,n$. If $\max\gamma>0$ then Question 1 has an affirmative answer.

Every linear programming problem has a dual problem associated with it which is often more readily solved. (See, for example, Chapter V of [2].) The dual of this problem is to minimize μ where

$$\mu=\sum_{i=1}^{n}s_i(\epsilon_i-y_i)+t_i(\epsilon_i+y_i)$$

subject to

$$\sum_{i=1}^{n}(s_i+t_i)=1,$$

$$\sum_{i=1}^{n}(s_i-t_i)x_i^{k_j}=0, \qquad (j=0,\cdots,N)$$

$$s_i \geqq 0,\quad t_i \geqq 0.$$

Since $\min\mu=\max\gamma$, the solution of the dual problem produces the answer to

Question 1, and since most linear programming codes will produce the optimum values of the variables of both the primal and dual problems, nothing has been lost. In addition, the final formulation above involves only $N+2$ equations in $2n$ unknowns, so that the dual problem has a smaller matrix, resulting in less computing time. Finally, the number of observed points (x_i, y_i) is now quite unrestricted, for the number of constraint equations depends only on the degree of the approximating polynomial.

6. Numerical examples. We give two examples by way of illustration. For expository simplicity the method of Section 4 is employed in each case.

First, let us find a quadratic expression $a_0+a_1x+a_2x^2$ which is a best fit to the points (1, 2), (2, 5), (3, 3) and (4, 2). Since we seek an optimum solution without regard to prescribed error requirements it is sufficient to take $\epsilon_i=1$. Thus, according to the foregoing, we wish to determine the minimum of

$$w = u_1 + v_1 + t$$

subject to

$$\begin{aligned}
u_1 + v_1 - u_2 - v_2 + t &\geqq 0,\\
u_1 + v_1 - u_3 - v_3 + t &\geqq 0,\\
u_1 + v_1 - u_4 - v_4 + t &\geqq 0,\\
u_1 - v_1 + u_5 - v_5 + u_6 - v_6 + u_7 - v_7 &= 2,\\
u_2 - v_2 + u_5 - v_5 + 2u_6 - 2v_6 + 4u_7 - 4v_7 &= 5,\\
u_3 - v_3 + u_5 - v_5 + 3u_6 - 3v_6 + 9u_7 - 9v_7 &= 3,\\
u_4 - v_4 + u_5 - v_5 + 4u_6 - 4v_6 + 16u_7 - 16v_7 &= 2,
\end{aligned}$$

with all variables nonnegative. A problem of this magnitude is certainly not beyond pencil-and-paper solution. The calculation was performed in short order on an IBM type 704 computer with the following results:

$$\begin{aligned}
w &= 0.75, & u_4 &= 0.75,\\
v_1 &= 0.75, & v_5 &= 0.75,\\
u_2 &= 0.75, & u_6 &= 4.50,\\
v_3 &= 0.75, & v_7 &= 1.00,
\end{aligned}$$

all other variables being zero. In particular, if $\epsilon_i > 0.75$ for $i=1, 2, 3, 4$, then the second degree polynomial $P(x) = -0.75+4.5x-x^2$ satisfies $|y_i - P(x_i)| < \epsilon_i$. Moreover, if each $\epsilon_i \leqq 1$ then it is clear by inspection that no polynomial of lower degree is satisfactory.

As a second example we seek a polynomial of the form $a+bx+cx^3$ which is a best fit for the data $(-4, 2)$, $(-2, 1)$, $(1, 0)$ and $(5, 2)$. Again we take $\epsilon_i=1$ so that, as above, we are to minimize $w=u_1+v_1+t$ in nonnegative variables subject to

$$u_1 + v_1 - u_2 - v_2 + t \geqq 0,$$
$$u_1 + v_1 - u_3 - v_3 + t \geqq 0,$$
$$u_1 + v_1 - u_4 - v_4 + t \geqq 0,$$
$$u_1 - v_1 + u_5 - v_5 - 4u_6 + 4v_6 - 64u_7 + 64v_7 = 2,$$
$$u_2 - v_2 + u_5 - v_5 - 2u_6 + 2v_6 - 8u_7 + 8v_7 = 1,$$
$$u_3 - v_3 + u_5 - v_5 + u_6 - v_6 + u_7 - v_7 = 0,$$
$$u_4 - v_4 + u_5 - v_5 + 5u_6 - 5v_6 + 125u_7 - 125v_7 = 2.$$

Another calculation gave the results

$$\begin{aligned} w &= .6296296, & u_4 &= .6296296, \\ u_1 &= .6296296, & u_5 &= 1.000000, \\ v_2 &= .6296296, & v_6 &= .3888888, \\ v_3 &= .6296296, & u_7 &= .0185185, \end{aligned}$$

all other variables being zero. Thus an optimum solution is $P(x) = 1 - .389x + .0185x^3$.

7. Concluding remarks. Linear programming is no universal balm, a fact not always apparent in some discussions of the subject. Consider this intriguing generalization of the original problem posed in Section 2. Suppose, in addition to the error quantity ϵ_i attributed to the dependent variable, that a significant error δ_i may occur in measuring the independent variable. The analogous problem in this case is to determine whether a polynomial of degree N can be found whose graph intersects each of a collection of similarly oriented rectangles of dimensions $2\delta_i \times 2\epsilon_i$. Here (compare with (1) and (2)) the problem is to compute

$$\min_{a} \min_{|x-x_i| \leqq \delta_i} \max_{1 \leqq i \leqq n} \eta_i \left| y_i - \sum_{j=0}^{N} a_j x^{k_j} \right|.$$

The treatments of Sections 4–5 are now altogether inadequate, for this problem reduces to minimizing a linear form subject to constraints which are exceedingly nonlinear.

The reader will note, however, that the methods developed above permit wider application. Instead of determining max γ in (6) and (7) or z of (2), the terms $x_i^{k_j}$ can be replaced by $f_j(x_i)$ where the functions f_j are completely unrestricted.

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THE BOOLE TABLE GENERALIZED

WALTER E. STUERMANN, University of Tulsa

In Classroom Notes of this MONTHLY (vol. 67, 1960, pp. 170–172), the writer presented a graphical device for analyzing Boolean functions which were displayed in disjunctive or conjunctive normal form. This device was called a Boole table. The referee indicated how the Boole table could be used to analyze Boolean functions constructed from the operations of product ($\cap$), sum ($\cup$), and complement ($'$), no matter how complexly the elements (x, y, z, $\cdots$) are overlaid by stacks of these operations. The present note briefly describes this more general use of the Boole table.

We shall represent the columns of T's and F's in the conventional truth table of the propositional calculus by bars and the absence of bars. Thus, for three propositional variables, p, q, and r, the conventional truth table and the graphical version are given in Figures 1 and 2.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

FIG. 1

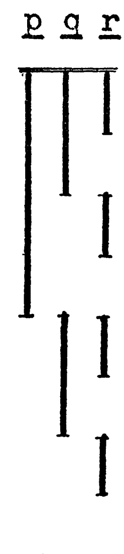

FIG. 2

Due to the isomorphism between the propositional calculus and Boolean algebra, all of the conventional truth table operations can be transformed into equivalent Boolean operations. Logical conjunction ($\cdot$) becomes the Boolean product ($\cap$); logical disjunction ($\vee$) becomes the Boolean sum ($\cup$); logical negation ($\sim$) becomes the Boolean complement ($'$); and so on. The isomorphism

for the graphical version of the truth table is the Boole table.

Let us illustrate the use of the Boole table by analyzing the function, $F=\{[(x\cap y)\cup z']'\cup(x'\cap y)\}$. First, display the Boole table and the function, F, as shown in Figure 3—the reference framework for the Boole table is in a column on the left and the function is written with wide spacing in a row at the

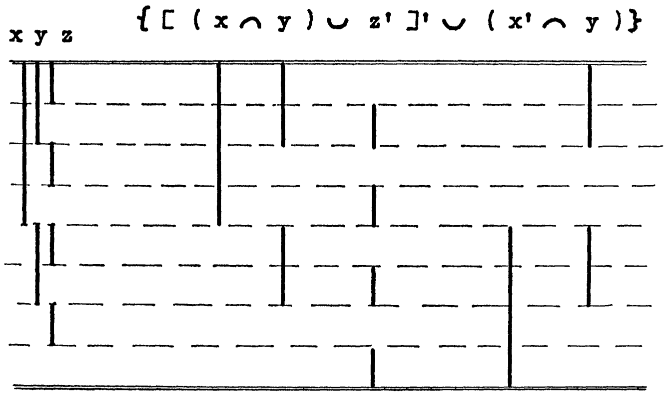

FIG. 3

top. Now one can proceed in this manner. (1) Enter bars for the memberships of sets denoted by single variables. (2) Enter bars for the complements (if any) of (1)-entries. (In Figure 3, steps (1) and (2) have been performed.) (3) Enter bars for every binary operation conjoining preceding entries. (4) Enter bars for the complements of the formulas designated under (3). Continue in this manner until a final column of bars is entered which represents the function as a whole. Figure 4 gives the complete analysis of F, where the final column of bars

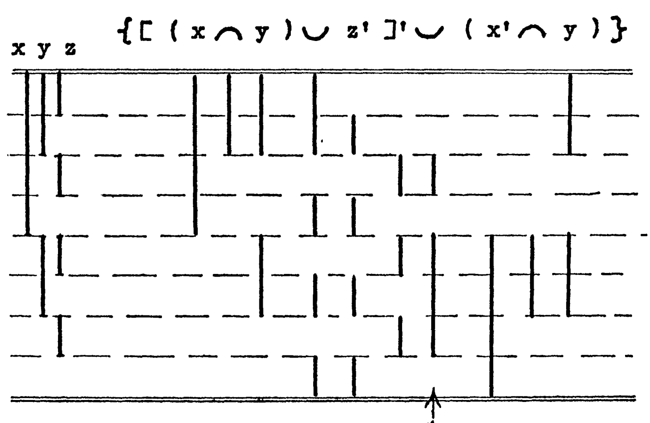

FIG. 4

above the asterisk (*) indicates the membership of the set denoted by the whole

function. The Boole table shows that the function, F, reduces to

$$[(x' \cap y) \cup (y' \cap z)].$$

As a second example, we give the analysis of $G=\{[(x' \cup y')' \cap z]' \cup (x \cup w)'\}'$. It is shown in Figure 5. The function, G, is equivalent to $(x \cap y \cap z)$.

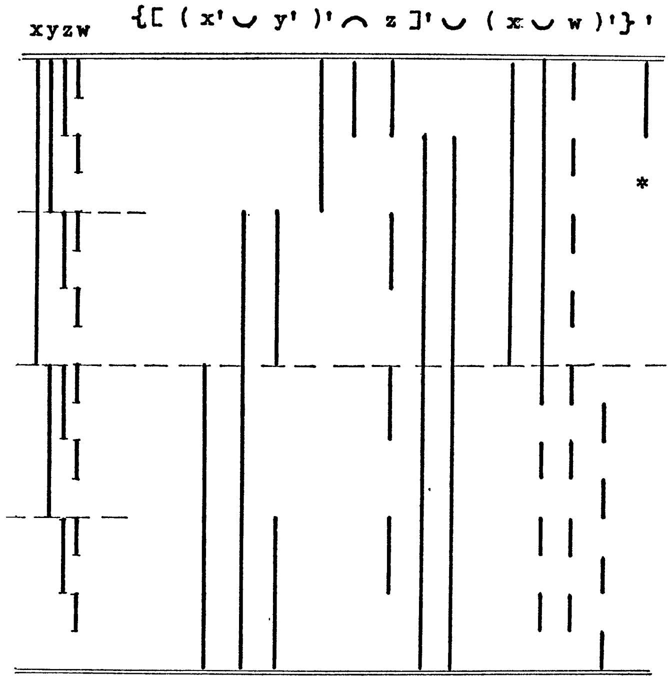

FIG. 5

Where the column of bars indicating the membership of the set designated by the function as a whole is continuous, the function denotes the universe class. In the case where the final column is devoid of bars, the function designates the null class. The function, $H=\{[(x \cup y')' \cap (z' \cup x')']\cap[(x \cup z) \cup y]'\}$, is an example. Otherwise, the function is "contingent." The functions F and G above illustrate this case.

Turning attention to the isomorphism in the propositional calculus, a brief comment will be sufficient. Take a logical formula expressing an argument. Let it be

$$\{[r \supset (t \vee s)]\cdot[n \supset (s \vee b)]\cdot[\sim s]\} \supset [(\sim t \cdot \sim b) \supset (\sim r \cdot \sim n)],$$

which is the form of the argument in Exercise 24, page 52, of I. M. Copi's *Symbolic Logic* (New York, 1954), where $\cdot$ is logical conjunction, $\vee$ is logical disjunction, $\sim$ is logical negation, and $\supset$ is material implication. In any such

formula, we replace $\supset$ (and $\equiv$, if it occurs) by its definition in terms of conjunction, disjunction, and negation. We can then use the graphical version of the truth table (Fig. 2) and operations analogous to those described above to determine the validity of the argument. Validity requires, of course, a continuous bar down the truth table in the final column for the logical function.

FUBINI IMPLIES LEIBNIZ IMPLIES $F_{yx}=F_{xy}$

R. T. SEELEY, Harvey Mudd College

Three of the most common and important theorems on the interchange of limit operations are the rules for interchanging the order of integration in a repeated integral (Fubini's theorem), differentiating under the integral sign (Leibniz's rule), and taking mixed partial derivatives in either order. This note points out that the first of these leads easily to the other two. This has advantages for the course which aims to prove everything, by replacing two rather difficult arguments with what seem to be simpler ones; and for the course which does not want to handle all complications, by reducing the number of unproved statements. Moreover, the connections between these results seem interesting in themselves, and provide good results for Lebesgue integrals.

The note first assumes a simple form of Fubini's theorem for Riemann integrals (A), uses this to prove Leibniz's rule (B), and uses this in turn to prove one of the stronger forms of the theorem on mixed partials (C). It then states the corresponding results for Lebesgue integrals; the proofs remain the same. R denotes the rectangle $a\leqq x\leqq b$, $c\leqq y\leqq d$.

(A) *If $f(x, y)$ is continuous on R, then $\int_c^d f(x, y)dy$ is a continuous function of x, $\int_a^b f(x, y)dx$ is a continuous function of y, and*

$$\int_a^b\left[\int_c^d f(x, y)dy\right]dx = \int_c^d\left[\int_a^b f(x,y)dx\right]dy.$$

(B) *If $g(x, y)$ and its partial derivative $g_x(x, y)$ are continuous on R, then $\int_c^d g(x, y)dy$ has a derivative on $a<x<b$, which equals $\int_c^d g_x(x, y)dy$.*

Proof. By the fundamental theorem of calculus (FTC) and Theorem (A),

$$\begin{aligned}\int_c^d g(x, y)dy &= \int_c^d\left[\int_a^x g_x(s, y)ds + g(a, y)\right]dy \\ &= \int_a^x\left[\int_c^d g_x(s, y)dy\right]ds + \int_c^d g(a, y)dy.\end{aligned} \tag{1}$$

By (A), $\int_c^d g_x(s, y)dy$ is continuous, so that the FTC asserts that the expression (1) has the derivative $\int_c^d g_x(x, y)dy$.

(C) *If f_y and f_{yx} are continuous on R, and $f_x(x, c)$ exists for $a<x<b$, then f_x and f_{xy} exist in the interior of R and f_{xy} equals f_{yx}. (Here f_{yx} is $\partial f_y/\partial x$, and f_{xy} is $\partial f_x/\partial y$.)*

Proof. By the FTC, $f(x, y)=\int_c^y f_y(x, t)dt+f(x, c)$. By Theorem (B)

$$f_x(x, y) = \int_c^y f_{yx}(x, t)dt + f_x(x, c). \tag{2}$$

Again by the FTC, the expression on the right in (2) has a derivative with respect to y which equals $f_{yx}(x, y)$.

For the Lebesgue integral, we have the following formulation.

(A′) *If f is integrable on R, then $\int_c^d[\int_a^b f(x, y)dx]dy=\int_a^b[\int_c^d f(x, y)dy]dx$, both repeated integrals existing.*

(FTC′) *If f is integrable on $a<x<b$, then $F(x)=constant+\int_a^x f(t)dt$ has a derivative $F'(x)=f(x)$ almost everywhere. Any function F of this form is called absolutely continuous.*

(B′) *If g is integrable on R and $G(x, y)=\int_a^x g(t, y)dt+G(a, y)$ for almost every y in $c\leqq y\leqq d$, and $\int_c^d G(a, y)dy$ exists, then $\int_c^d G(x, y)dy$ has a derivative equal to $\int_c^d g(x, y)dy$ for almost every x in $a\leqq x\leqq b$.*

(C′) *If $f(x, y)$, defined on R, is absolutely continuous in y for almost every x, and $f(x, c)$ is absolutely continuous, and f_y is absolutely continuous in x for almost every y, and f_{yx} is integrable on R, then f_x is absolutely continuous in y for almost every x, and $f_{xy}=f_{yx}$ almost everywhere.*

In conclusion, we point out that (B) can be proved by (FTC) and (C), through the device of considering $f(x, y)=\int_c^y g(x, t)dt$; and ($A$) can be gotten in part from (B) by showing that the derivatives of $\int_a^x[\int_c^y f(s, t)dt]ds$ equal those of $\int_c^y[\int_a^x f(s, t)ds]dt$. But this sequence appears to be less useful than the one presented.

The proof of (B) given here can also be found in Kaplan, *Advanced Calculus.*

LINE INTEGRALS OF EXACT DIFFERENTIALS

WILLIAM J. FIREY, Washington State University

In this note we prove the following theorem:

If the coefficients $M(x, y)$, $N(x, y)$ of the differential form $Mdx+Ndy$ satisfy $\partial M/\partial y=\partial N/\partial x$ throughout a simply-connected region A, then the line integral

$$\int_{(a_0,b_0)}^{(a_1,b_1)} Mdx + Ndy$$

has the same value for each continuously differentiable arc in A joining (a_0, b_0) to (a_1, b_1).

The usual proof employs Gauss' theorem:

Therefore

$$\frac{dJ(C_\vartheta)}{d\vartheta}=\int_0^1\left\{\left[\frac{\partial M}{\partial x}\frac{\partial x_\vartheta}{\partial\vartheta}+\frac{\partial M}{\partial y}\frac{\partial y_\vartheta}{\partial\vartheta}\right]\frac{dx_\vartheta}{dt}+\left[\frac{\partial N}{\partial x}\frac{\partial x_\vartheta}{\partial\vartheta}+\frac{\partial N}{\partial y}\frac{\partial y_\vartheta}{\partial\vartheta}\right]\frac{dy_\vartheta}{dt}\right.$$
$$\left.+M\left[\frac{dx_1}{dt}-\frac{dx_0}{dt}\right]+N\left[\frac{dy_1}{dt}-\frac{dy_0}{dt}\right]\right\}dt.$$

From (1):

$$\frac{\partial x_\vartheta}{\partial\vartheta}=x_1-x_0,\qquad \frac{\partial y_\vartheta}{\partial\vartheta}=y_1-y_0.$$

Using this and the exactness condition $\partial M/\partial y=\partial N/\partial x$ we obtain

$$\frac{dJ(C_\vartheta)}{d\vartheta}=\int_0^1\left\{\left[\frac{\partial M}{\partial x}\frac{dx_\vartheta}{dt}+\frac{\partial M}{\partial y}\frac{dy_\vartheta}{dt}\right](x_1-x_0)+M\left[\frac{dx_1}{dt}-\frac{dx_0}{dt}\right]\right.$$
$$\left.+\left[\frac{\partial N}{\partial x}\frac{dx_\vartheta}{dt}+\frac{\partial N}{\partial y}\frac{dy_\vartheta}{dt}\right](y_1-y_0)+N\left[\frac{dy_1}{dt}-\frac{dy_0}{dt}\right]\right\}dt$$
$$=\int_0^1\frac{d}{dt}\left[(x_1-x_0)M+(y_1-y_0)N\right]dt$$
$$=\left[(x_1-x_0)M+(y_1-y_0)N\right]_{t=0}^{t=1}.$$

At the upper and lower limits $x_1(t)-x_0(t)$ and $y_1(t)-y_0(t)$ are zero. Hence $dJ(C_\vartheta)/d\vartheta=0$ and the theorem is proved.

In an elementary course in differential equations, the theorem furnishes a general formula for constructing solutions of exact equations. The foregoing proof allows one to treat this matter without a preliminary discussion of Gauss' theorem.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

OKLAHOMA STATE COMMITTEE FOR THE IMPROVEMENT OF MATHEMATICS INSTRUCTION

JAMES H. ZANT, Oklahoma State University

Introduction. This committee, appointed in 1958 by the Oklahoma Curriculum Improvement Commission of the State Department of Education, consists of teachers of mathematics and mathematics education at all levels, supervisors

and administrators. It is advisory in character and has sponsored meetings, workshops, symposia and experimental teaching in the state. Its primary function is the improvement of mathematics teaching.

Accomplishments. The activities of this Committee have created in the state an interest and concern among both teachers and administrators about a more modern program in mathematics at all levels. Experimental programs have been organized at a number of places over the state. Chief among these experimental programs were the seven centers for teaching the School Mathematics Study Group textbooks in Grades 9, 10 and 11 during 1959–60. This activity, supported by SMSG and supervised by the Committee, involved 23 school systems, and 84 classes with 2,700 high school students. Forty-one high school teachers and 8 college mathematicians worked in the program. A similar program of about the same magnitude has been organized for 1960–61 using SMSG textbooks for Grades 4, 5 and 6. Fifty-two school districts in Oklahoma have bought 13,500 copies of SMSG textbooks Grades 7–12 for use during 1960–61.

The Committee expects to continue its work next year with the primary objective of informing teachers and administrators of promising developments now taking place in the mathematical community with reference to improving mathematics at the school level and encouraging ways of making the necessary knowledge of mathematics available to teachers so that they can teach this exciting material with confidence and enthusiasm. It is expected that a preliminary report will be published soon for use in the state. A report *Tentative Recommendations of the State Mathematics Committee* was published December 1960 and is available from the Oklahoma State Department of Education, Oklahoma City.

UNIVERSITY OF MARYLAND MATHEMATICS PROJECT (JUNIOR HIGH SCHOOL)

The University of Maryland Mathematics Project (Junior High School) is starting its fourth year with special emphasis on inservice programs for elementary teachers and psychological studies in the learning of mathematics. The Maryland experimental texts for grades 7 and 8 are being used by approximately 15,000 junior high school students in all parts of the country during this school year. There will be no further attention given this year to a revision of the texts based on teaching experiences.

An experimental program of inservice education for elementary teachers of mathematics is being conducted by Helen M. Garstens, Associate Director of the project. This program consists of two parts, one of which is a course in mathematics for elementary teachers being offered to 25 teachers selected by the five major school systems in the Washington area. A second part of the program involves a small class in mathematics for elementary school supervisors from seven school systems who attend the class on late Monday afternoons and then offer a somewhat similar course for elementary school teachers in their school systems on another night during the week.

The learning studies in mathematics are to be conducted under the direction of Professor Robert Gagné, Department of Psychology, Princeton University. A full-time psychologist, Noel Paradise, has been employed to work on the Maryland campus. The studies will be based in considerable part on topics selected from the University of Mary-

land experimental courses and will be conducted at the 6th to 8th grade levels. These studies will be related to some of the research being carried on by Professor Gagné at Princeton University.

KENTUCKY CONFERENCE OF COLLEGE SCIENCE AND MATHEMATICS STAFF MEMBERS

The first of the state conferences held as a part of the NASDTEC Teacher Preparation Certification Study (see article by G. S. Young, this MONTHLY, vol. 67, 1960, pp. 792–797) was held in Louisville, Kentucky, on September 23–24. Approximately 150 staff members of the science and mathematics departments of the 34 colleges of the state offering teacher programs in science and mathematics attended. Among the purposes of the conference was to plan how the programs recommended by the NASDTEC regional conferences could become a basis for approval of teacher education programs in Kentucky. A major part of the conference provided for reports from the various curriculum studies in the sciences. Among the consultants were Gerald W. Zacharias, Massachusetts Institute of Technology; E. G. Begle, Yale University; and Arnold Grobman, University of Colorado.

ONTARIO MATHEMATICS COMMISSION

The Ontario Department of Education, together with a number of professional groups, including the Ontario Teachers' Federation, and the Ontario Association of Teachers of Mathematics and Physics, have recently established the Ontario Mathematics Commission. The objectives of the Commission, as stated in the constitution are:

To promote excellence in the teaching of mathematics in the province of Ontario by:

(a) Undertaking curriculum research aimed at keeping the province abreast of the best contemporary practice and circulating such information to interested bodies;
(b) Encouraging the production of experimental teaching material in mathematics, and seeking the cooperation of the Department of Education in testing such material in the schools of the province;
(c) Cooperating with the universities, the Ontario Teachers' Federation, and other appropriate bodies, in providing courses which will enable teachers to improve their qualifications and keep abreast of current experimentation in curriculum changes and in teaching techniques;
(d) Establishing scholarships and prizes for teachers and students of mathematics;
(e) Maintaining communication with similar groups in other provinces and countries; and
(f) Such other means as the Commission may deem necessary.

An article, *Reforming the high school mathematics curriculum*, by A. J. Coleman appears in the Canad. Math. Bull., vol. 3, 1960. The article reviews some of the considerations which led to the formation of the Commission and also gives brief outlines for mathematics courses, grades 9 through 12. It is indicated in the article that anyone wishing to keep abreast of developments in Ontario may write to the Ontario Mathematics Commission, 1260 Bay Street, Toronto 5, Ontario, and ask to be put on the mailing list.

Careers in Mathematics

A short pamphlet on choosing a career in mathematics has been prepared under the auspices of the Conference Board of the Mathematical Sciences. The pamphlet is a reprint from the *Mathematics Teacher*, May 1960. One section is on teaching careers in mathematics and the second section on industrial and government careers in mathematics. Copies of the reprint may be obtained by writing to Dr. G. Baley Price, Executive Secretary, Conference Board of the Mathematical Sciences, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1446. *Proposed by David Bickerstaff, University of Mississippi*

"How about telling me confidentially the secret order of the five beauties to be featured in this year's Annual?" I proposed to the editor. She, of course, refused, but agreed to pass judgment on my guess. "Is it A-B-C-D-E?" I asked.

"You are most skillful at being wrong," she chided. "You not only got each person out of her true position but, furthermore, not one in your ranking followed correctly her immediate predecessor."

"Well, then, is it D-A-E-C-B?" I asked.

"Now you are improving," she encouraged cautiously. "You have two in proper position and you have two following correctly their immediate predecessors."

After a little figuring I then told her the correct order, and she swore me to secrecy. What is the correct order?

E 1447. *Proposed by Walter Bluger, Dominion Bureau of Statistics, Ottawa, Canada*

Construct a triangle given R, r, h_a.

E 1448. *Proposed by C. B. Grosch, General Mills, Inc., Minneapolis, Minnesota*

Show that any plane section of an oblate spheroid, not perpendicular to the axis of the spheroid, is an ellipse with major axis parallel to the equatorial plane of the spheroid and with minor axis (or minor axis extended) intersecting the axis of the spheroid.

E 1449. *Proposed by C. S. Patlak, Department of Health, Education, and Welfare, Bethesda, Maryland*

Assume that (1) A_i, B_i, C_i, D_i $(i=1, \cdots, n)$ are all positive, (2) $\sum A_i \geqq \sum C_i$, (3) $A_i - C_i = B_i - D_i$ $(i = 1, \cdots, n)$. $P_i = A_iB_i/C_iD_i$ and $P=\max(P_1, \cdots, P_n)$. Prove that $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i) \leqq P$.

E 1450. *Proposed by Lawrence Shepp, Princeton University*

If a_n, $b_n>0$, $a_n \downarrow 0$, then $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j = \sum_{n=1}^{\infty} a_n b_n$.

SOLUTIONS

Sum of Two Squares

E 1416 [1960, 474]. *Proposed by Leonard Cohen, City College of New York*

Prove that $x^2+y^2=z^n$, where n is a positive integer, always has nonzero integral solutions.

I. *Solution by I. D. Ruggles, San Jose State College.* If n is odd, let $n=2k+1$, $k=0, 1, 2, \cdots$; then $(2^k)^2+(2^k)^2=2^{2k+1}$. If n is even, let $n=2k+2$, $k=0, 1, 2, \cdots$; then $(4\cdot 5^k)^2+(3\cdot 5^k)^2=5^{2k+2}$.

II. *Solution by J. W. Ellis, Louisiana State University in New Orleans.* If (a, b, c) is any Pythagorean triple, then $(ac^{n-1})^2+(bc^{n-1})^2=(c^2)^n$.

III. *Solution by Sidney Kravitz, Dover, N. J.* An infinite number of solutions exist for both $n=1$ and $n=2$. For every solution $x^2+y^2=z^n$, there exists a solution $(zx)^2+(zy)^2=z^{n+2}$.

IV. *Solution by William Becker, New York, N. Y.* Let $z=a^2+b^2$. The conclusion follows from the fact that the product of the sum of two squares is again the sum of two squares.

V. *Solution by L. R. Ford, Charlottesville, Va.* Let u be any complex integer whose nth power, $x+iy$, is neither real nor pure imaginary, and let $u\bar{u}=z$; then $u^n\bar{u}^n=x^2+y^2=z^n$.

VI. *Solution by J. R. Trollope and Walter Zayachkowski, University of Alberta.* Choose for z a prime $p\equiv 1 \pmod 4$. Since every integer which is the product of such primes can be expressed as the sum of two integral squares, it follows that $x^2+y^2=z^n$ always has nonzero integral solutions.

Also solved by Ray Authement, Leon Bankoff, Robert Bart, D. A. Breault, Brother Joseph Heisler, J. L. Brown, Jr., James Burling, Marcus Charles, P. R. Chernoff, Richard Cottle, C. H. Cunkle, Monte Dernham, Gus Di Antonio, F. J. Duarte, G. W. Erwin, Jr., J. C. Ferguson, R. D. Freeman, Jr., B. E. Fristedt, Michael Goldberg, L. D. Goldstone, R. Gramann, Bernard Greenspan, Corinne Hattan, Margaret Herzog, Vern Hoggatt, J. E. Homer, Jr., A. S. Howard, J. A. H. Hunter, Ronald Jacobowitz, Gerald Janusz, J. Jordan, Ray Jurgensen, Irving Katz, P. G. Kirmser, M. S. Klamkin, Donald Knuth, J. D. E. Konhauser, Robert Kruse, Harry Langman, A. T. Lauria, Aaron Lieberman and Sheldon Weinberg (jointly), J. R. Lux, C. R. MacCluer, Peter Marks, D. C. B. Marsh, N. S. Mendelsohn, M. V. Mielke, J. B. Muskat, Judith Ng, Thomas O'Brien, D. J. Persico, R. A. Phillips, C. F. Pinzka, T. J. Robinson, Jonathan Robison, Ray Rogers, Norman Schaumberger, Allen Shields, Jack Silver, D. L. Silverman, M. L. Slater, T. H. Slook, Samuel Stern, E. E. Strock, Eric Sturley, J. R. Sullivan, Wu Ta-Sun, L. Thomas, Allan Trojan, Chih-yi Wang, W. C. Waterhouse, J. S. White, and the proposer. Late solution by Guy Torchinelli.

Editorial Note. Solution II leads to the two-parameter solution $x=2uv(u^2+v^2)^{n-1}$, $y=(u^2-v^2)\times(u^2+v^2)^{n-1}$, $z=(u^2+v^2)^2$; Solution I shows that this does not include all solutions. Solution IV is based on the identity $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$. The basic fact used in Solution IV is found in most texts on the theory of numbers, see, *e.g.*, Jones, *The Theory of Numbers*, Th. 6.6a, p. 133. Th. 6.6b, p. 135 of this reference, solves Problem E 1416 by giving necessary and sufficient conditions for a number to be the sum of two squares.

A Curious Sequence

E 1417 [1960, 474]. *Proposed by Robert Hartop, Los Angeles, California*

Given a unit circle with point P on its circumference and a distance d_1, $0<d_1<2$. With center at P, construct the circle of radius d_1 to cut the given circle in two points. If d_2 is the distance between these points, construct a new circle of radius d_2 with center at P and so obtain $d_3, d_4, \cdots, d_n, \cdots$. Find $\lim_{n\to\infty} d_n$.

Solution by D. C. B. Marsh, Colorado School of Mines. Recursively, $d_n = d_{n-1}(4-d_{n-1}^2)^{1/2}$, which transforms into $|2 \sin \phi_n| = |2 \sin 2\phi_{n-1}|$ by setting $d_j = |2 \sin \phi_j|$ for all j. The latter implies $\phi_n = k_n\pi \pm 2\phi_{n-1}$ (with k_n an arbitrary integer) and has as solution $\phi_n = M\pi \pm 2^{n-1}\phi_1$ (M an arbitrary integer). If a limit exists ($d_n \to d$), then $d=0$ or $\sqrt{3}$, so that $\phi_n \to \phi = k\pi$ or $k\pi \pm \pi/3$ (which may be written as $K\pi/3$) for all $n \geqq$ some positive integer p. The converse also holds, so that $\lim_{n\to\infty} d_n$ exists if and only if d_1 is of the form $d_1 = |2 \sin(K\pi/3\cdot 2^p)|$ for some positive integral p and K, in which case the limiting value is $|2 \sin(K\pi/3)|$.

Also solved by Robert Bart, Frederick Cunningham, Jr., J. A. Faucher, Michael Goldberg, R. Gramann, R. E. Greenwood, J. O. Herzog, Vern Hoggatt and I. D. Ruggles (jointly), A. R. Hyde, Gerald Janusz, Ray Jurgensen, L. M. Kaplan, P. G. Kirmser, J. D. E. Konhauser, William Lopez, Peter Marks, D. A. Moran, D. E. Robison and E. M. Scheuer (jointly), G. B. Robison, Ray Rogers, L. L. Sleizer, L. Thomas, J. S. White, and the proposer. Most of these solutions were incomplete.

Editorial Note. If S and T are the sets of values of d_1 which lead to the limits 0 and $\sqrt{3}$ respectively, it can be shown that S and T are each denumerable and dense.

Interesting is the case where d_1 is chosen as a side of a regular n-gon. If $n=2^k$, then $\lim d_n=0$; if $n=3\cdot 2^k$, then $\lim d_n=\sqrt{3}$; in all other cases the sequence $\{d_n\}$ becomes periodic.

Weakened Hypotheses

E 1418 [1960, 474]. *Proposed by R. C. Buck, Institute for Defense Analyses, Princeton, New Jersey*

In *Comptes Rend.* (1945) vol. 62, pp. 95–97, there appears the theorem: If $f(x, y)$ has partial derivatives of first and second orders, and if $f(a, b)+f(b, c) = f(a, c)$ for all a, b, c, then there is a function ϕ such that $f(x, y) = \phi(x) - \phi(y)$. Can these hypotheses be weakened?

Solution by W. C. Waterhouse, Harvard University. We need only $f(a, b) + f(b, k) = f(a, k)$ for a fixed k, as can be seen by putting $\phi(x) = f(x, k)$.

Also solved by Alan Beal, Robert Breusch, P. R. Chernoff, H. E. Chrestenson, R. B. Deal, N. J. Fine, Don Freeman and Fred Gilman (jointly), Michael Goldberg, Gerald Janusz, A. F. Kaupe, Jr., P. G. Krimser, M. S. Klamkin, C. R. MacCluer, Peter Marks, D. C. B. Marsh, E. J. Mickle, Ray Rogers, Jack Silver, Allan Trojan, J. S. White, Albert Wilansky, J. E. Wilkins, Jr., R. J. Wisner, and the proposer.

Deal showed that if S is any nonempty set and f is a function on $S \times S$ to a group G such that for all a, b and one particular c in S, $f(a, b)f(b, c) = f(a, c)$, then there exists a function ϕ on S to G such that $f(x, y) = \phi(x)[\phi(y)]^{-1}$.

An Oversight in the Collected Papers of Ramanujan

E 1419 [1960, 474]. *Proposed by C. C. Yalavigi, Government College, Mercara, India*

In the *Collected Papers of Srinivasa Ramanujan*, edited by G. H. Hardy, Seshu Aiyar, and B. M. Wilson (1928), appears (p. 334, Question 1076 (XI, 199)) the following:

Show that

$$\text{(i)} \qquad [7(20)^{1/3} - 19]^{1/8} = (5/3)^{1/3} - (2/3)^{1/3},$$

$$\text{(ii)} \qquad [4(2/3)^{1/3} - 5(1/3)^{1/3}]^{1/6} = (4/9)^{1/3} - (2/9)^{1/3} + (1/9)^{1/3}.$$

Show that these relations are incorrect, and that to correct them we must interchange the exponents 1/8 and 1/6 appearing in the left-hand sides.

Solution by Robert Bart, Michigan College of Mining and Technology. (i) Set $x=(5/3)^{1/3}$, $y=(2/3)^{1/3}$. Then

$$\begin{aligned}(x-y)^6 &= (x^6 - 20x^3y^3 + y^6) + x^2y(15y^3 - 6x^3) + xy^2(15x^3 - 6y^3) \\ &= -19 + 21xy^2 = -19 + 7(20)^{1/3}.\end{aligned}$$

Therefore $[7(20)^{1/6} - 19]^{1/6} = x - y = (5/3)^{1/3} - (2/3)^{1/3}$.
(ii) Set $x = (2/3)^{1/3}$, $y={}_11/3)^{1/3}$. Then

$$\begin{aligned}(x+y)^8 &= x^2(x^6 + 56x^3y^3 + 28y^6) + xy(8x^6 + 70x^3y^3 + 8y^6) \\ &\quad + y^2(28x^6 + 56x^3y^3 + y^6) \\ &= 16x^2 + 20xy + 25y^2,\end{aligned}$$

whence $(x+y)^8(4x-5y)=(4x)^3-(5y)^3=128/3-125/3=1=x^3+y^3$.
Therefore

$$\begin{aligned}[4(2/3)^{1/3} - 5(1/3)^{1/3}]^{1/8} &= (4x - 5y)^{1/8} = (x^3 + y^3)/(x + y) \\ &= x^2 - xy + y^2 = (4/9)^{1/3} - (2/9)^{1/3} + (1/9)^{1/3}.\end{aligned}$$

Also solved by A. N. Aheart, Leon Bankoff, D. A. Breault, Robert Breusch, Marcus Charles, F. J. Duarte, J. A. Faucher, Michael Goldberg, L. D. Goldstone, Corinne Hattan, J. E. Homer, Jr., Gerald Janusz, Sidney Kravitz, Peter Marks, D. C. B. Marsh, D. J. Persico, D. L. Silverman, C. L. Sterling, W. B. Stovall, Jr., G. C. Thompson, Walter Zayachkowski, and the proposer.

A Property of the Nine-Point Center

E 1420 [1960, 474]. *Proposed by the late Victor Thêbault, Tennie, Sarthe, France*

Let A', B', C' (A'', B'', C'') be the centers of squares described exteriorly (interiorly) on the sides BC, CA, AB of a triangle ABC. Show that the radical center of the circles $A(A')$, $B(B')$, $C(C')$ ($A(A'')$, $B(B'')$, $C(C'')$) coincides with the nine-point center of triangle ABC.

Solution by Leon Bankoff, Los Angeles, Calif. Let N denote the nine-point

center, P the orthogonal projection of N upon BC, Q the midpoint of BC, and R the foot of the altitude from A. It is sufficient to show that the powers of N with respect to any pair of circles of the same triad are equal; for example, that $(NB)^2-(NC)^2=(BB')^2-(CC')^2=(BB'')^2-(CC'')^2$. Now

$$\begin{aligned}(NB)^2-(NC)^2 &= (BP)^2-(PC)^2=(BP+PC)(BP-PC)\\ &= a(BQ-RC)=a^2/2-ab\cos C,\\ (BB')^2-(CC')^2 &= [a^2+b^2/2-ab\sqrt{2}\cos(C+45^\circ)]\\ &\quad-[a^2+c^2/2-ac\sqrt{2}\cos(B+45^\circ)]\\ &= ac\cos B-ab\cos C+(b^2-c^2)/2.\end{aligned}$$

Also

$$\begin{aligned}(BB'')^2-(CC'')^2 &= [a^2+b^2/2-ab\sqrt{2}\cos(C-45^\circ)]\\ &\quad-[a^2+c^2/2-ac\sqrt{2}\cos(B-45^\circ)]\\ &= ac\cos B-ab\cos C+(b^2-c^2)/2.\end{aligned}$$

Then, by virtue of the cosine law, we have

$$(NB)^2-(NC)^2=(BB')^2-(CC')^2=(BB'')^2-(CC'')^2.$$

Consequently N coincides with the radical center common to each triad of circles.

Also solved by L. D. Goldstone, Gerald Janusz, Peter Marks, D. C. B. Marsh, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4941. *Proposed by Leonard Carlitz, Duke University*

Bateman's polynomial $F_n(z)$ is defined by

$$F_n(z)={}_3F_2\left[\begin{matrix}-n, & n+1, & \frac{1}{2}(1+z);\\ & 1, & 1\end{matrix}\right]$$

in the usual notation for generalized hypergeometric functions. Now if $p=2n+1$ is prime, show that

$$F_n(0) = \begin{cases} 0 \pmod{p} & (n \text{ odd}), \\ 4a^2 \pmod{p} & (n \text{ even}), \end{cases}$$

where a^2 is the odd square in the decomposition $p=a^2+b^2$.

This result is analogous to that of problem 4628 [1956, 348].

4942. *Proposed by R. H. Bruck, University of Wisconsin*

Let p be an arbitrary but fixed rational prime. For each positive integer n define the permutation T_n to be the following product of p^{n-1} disjoint cycles of length p:

$$T_n = \prod_i (i, i + p^{n-1}, i + 2p^{n-1}, \cdots, i + (p-1)p^{n-1}), \qquad i = 1, 2, \cdots, p^{n-1}.$$

Let $G(n)$ be the permutation group generated by $T_1, T_2, \cdots, T_n$. Also define G to be the union of all the groups $G(n)$. Prove:

(a) $G(n)$ is a maximal p-subgroup of the symmetric group on $1, 2, \cdots, p^n$.

(b) Every finite p-group is isomorphic to at least one subgroup of G.

(c) $G(n)$ is nilpotent of class p^{n-1},

(d) G is a locally finite p-group such that $(G, G)=((G, G), G)$.

(e) G is a maximal p-subgroup of the group of all "finite" permutations of the positive integers.

4943. *Proposed by D. J. Newman, Yeshiva University*

Let A and B be two convex plane sets, both symmetric about the origin. Suppose that $z\in A$, $\zeta\in A$ implies that either $z+\zeta\in B$ or $z-\zeta\in B$. Show that this persists for n vectors, *i.e.*, $z_i\in A$, $i=1, 2, \cdots, n$ implies $z_1 \pm z_2 \pm \cdots \pm z_n \in B$ for some choice of $\pm$ signs.

4944. *Proposed by Morris Newman, National Bureau of Standards*

Prove that $p(n)$, the total number of partitions of n, is odd for infinitely many values of n and even for infinitely many.

4945. *Proposed by N. R. Riesenberg, Brooklyn College*

If $F_n(x)$ is the coefficient of z^n in the expansion of $2hze^{xz}/(e^{hz}-e^{-hz})$ in ascending powers of z so that $F_0(x)=1$, $F_1(x)=x$, $F_2(x)=(3x^2-h^2)/6$, etc., show that:

(1) $F_n(x)$ is a homogeneous polynomial of degree n in x and h.

(2) $dF_n(x)/dx=F_{n-1}(x) \quad n\geqq 1$.

(3) $\int_{-h}^{h} F_n(x)dx=0 \quad n\geqq 1$.

(4) If $y=a_0F_0(x)+a_1F_1(x)+a_2F_2(x)+\cdots$, where a_i are real constants, then the mean value of d^ry/dx^r in the interval $-h\leqq x\leqq h$ is a_r.

4946. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Let $S_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n$. Sum $\sum_1^\infty S_n/n!$

SOLUTIONS

A Function Regular Everywhere but at the Composite Integers

4555 [1953, 555]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

If $\tanh u = \tanh^2(\pi\sqrt{z}\,\sin\theta)$ and $\tanh v = \tan^2(\pi\sqrt{z}\,\cos\theta)$, then the function* $F(z) = \int_0^\pi \coth(u+v)d\theta + 4\cot\pi z$ has simple poles at the composite integers and is regular everywhere else.

Solution by the proposer. The expansion

$$f(z) = \pi z \cot \pi z - 1 = \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2} \tag{1}$$

yields $f(z/m) = \sum_{n=1}^{\infty} 2z^2/(z^2 - m^2n^2)$. If we sum on m we find that the denominator $z^2 - N^2$ occurs once for each divisor of N. That is,

$$g(z) = \sum_{m=1}^{\infty} f(z/m) = \sum_{n=1}^{\infty} \frac{2z^2 d(n)}{z^2 - n^2},$$

where $d(n)$ is the number of divisors of n, counting n and 1, but $d(1) = 1$.

From (1) it follows that

$$f(z) = -2\sum_{k=1}^{\infty} H_{2k} z^{2k}, \quad \text{where} \quad H_{2k} = \sum_{n=1}^{\infty} n^{-2k}. \tag{2}$$

(See Franklin, *Treatise on Advanced Calculus*, p. 474.) Therefore, by inspection,

$$g(z) = -2\sum_{k=1}^{\infty} H_{2k}^2 z^{2k}. \tag{3}$$

If $h(z) = \sum a_n z^n$, then for small $|z|$

$$\frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta}\sqrt{z})h(e^{-i\theta}\sqrt{z})d\theta = \sum a_n^2 z^n.$$

Applying this to $h(z) = \pi z \cot \pi z$ in (2) we get

$$1 + \sum_{k=1}^{\infty} (2H_{2k})^2 z^{2k} = \frac{\pi z}{2}\int_0^{2\pi} \frac{1 + \tan^2 c \tanh^2 s}{\tan^2 c + \tanh^2 s}\,d\theta, \tag{4}$$

where $c = \pi z^{1/2}\cos\theta$, $s = \pi z^{1/2}\sin\theta$. Since

$$\coth(u + v) = \frac{1 + \tanh u \tanh v}{\tanh u + \tanh v},$$

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* The coefficient 4 was misprinted in the original proposal.

the choice $\tanh u = \tanh^2 s$, $\tanh v = \tan^2 c$ in (4) gives

$$2g(z) = 1 - \pi z \int_0^\pi \coth(u+v)d\theta$$

when we observe that the integrand is even. This result and (1) give the explicit formula

$$\int_0^\pi \coth(u+v)d\theta + 4\cot \pi z = \frac{5}{\pi z} + \frac{4z}{\pi}\sum_{n=1}^{\infty} \frac{d(n)-2}{n^2-z^2},$$

and, hence, the result follows. (We must take 0 and 1 as composite, and not discriminate against the negatives of composite integers.)

Derivatives of a Composite Function

4782 [1958, 212]. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

Given a composite function $F(x) \equiv f[g(x)]$. Denoting the nth derivative of $f(g)$ by $D^n f$, and the derivatives of $g(x)$ by $g', g'', \cdots, g^{(n)}$, show that

$$(1) \qquad F^{(n)}(x) = \begin{vmatrix} g' & g'' & g''' & g^{iv} & \cdots & g^{(n)} \\ -1 & g'D & 2g''D & 3g'''D & \cdots & \binom{n-1}{1} g^{(n-1)}D \\ 0 & -1 & g'D & 3g''D & \cdots & \binom{n-1}{2} g^{(n-2)}D \\ 0 & 0 & -1 & g'D & \cdots & \binom{n-1}{3} g^{(n-3)}D \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots -1 & g'D \end{vmatrix} Df.$$

Solution by Frank Schmittroth, Oregon State College. Expand the determinant by minors, using elements of the last row:

$$F^{(n)}(x) = g'DF^{(n-1)}(x) + \begin{vmatrix} g' & g'' & \cdots & g^{(n-2)} & g^{(n)} \\ -1 & g'D & \cdots & \binom{n-3}{1} g^{(n-3)}D & \binom{n-1}{1} g^{(n-1)}D \\ 0 & -1 & \cdots & \binom{n-3}{2} g^{(n-4)}D & \binom{n-1}{2} g^{(n-2)}D \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & -1 & \binom{n-1}{n-2} g''D \end{vmatrix} Df.$$

Continuing in like manner:

$$F^{(n)}(x) = g'DF^{(n-1)}(x) + \binom{n-1}{n-2}g''DF^{(n-2)}(x) + \cdots$$
$$+ \binom{n-1}{1}g^{(n-1)}DF'(x) + g^{(n)}Df,$$

where $F^{(n-1)}(x)$, $F^{(n-2)}(x)$, $\cdots$, $F'(x)$ can be written as determinants. From $F(x)=f[g(x)]$ obtain at once $F'(x) = (df/dg)g'(x)=g'Df$, which agrees with (2) for $n=1$.

The argument now proceeds by induction. Assume (2) to be true for all $n\leqq N$. Replace n in (2) by N, differentiate both sides with respect to x, and group like terms:

$$\begin{aligned}F^{(N+1)}(x) = {} & g'DF^{(N)}(x) + \binom{N-1}{N-2}g''DF^{(N-1)}(x) + \binom{N-1}{N-3}g'''DF^{(N-2)}(x) \\ & + g''DF^{(N-1)}(x) + \binom{N-1}{N-2}g'''DF^{(N-2)}(x) \\ & + \cdots + \binom{N-1}{1}g^{(N-1)}DF''(x) + \binom{N-1}{0}g^{(N)}DF'(x) \\ & + \cdots + \binom{N-1}{2}g^{(N-1)}DF''(x) + \binom{N-1}{1}g^{(N)}DF'(x) + g^{(N+1)}Df.\end{aligned} \tag{3}$$

Making use of the fact,

$$\binom{N-1}{N-k+1} + \binom{N-1}{N-k} = \binom{N}{N-k+1},$$

and the fact (easily proved) that the order of differentiation with respect to g and differentiation with respect to x may be reversed, equation (3) becomes

$$F^{(N+1)}(x) = g'DF^{(N)}(x) + \binom{N}{N-1}g''DF^{(N-1)}(x) + \binom{N}{N-2}g'''DF^{(N-2)}(x)$$
$$+ \cdots + \binom{N}{2}g^{(N-1)}DF''(x) + \binom{N}{1}g^{(N)}DF'(x) + g^{(N+1)}Df,$$

which is the same as (2) with n replaced by $N+1$. Thus (2) is proved and the equivalent relation (1) is established.

Other formulas for the nth derivative of a compound function are given in: Arnold Dresden, [1943, 9]; M. McKiernan, [1956, 331]; also E. P. Adams, *Smithsonian Misc. Collection*, 74 (1922), No. 1, p. 157; I. M. Ryzhik, *Tables of Series, Products and Integrals*, 3d ed., Berlin (1957), p. 20.

Diophantine Equations

4832 [1959, 147; 1959, 924]. *Proposed by A. Oppenheim, University of Malaya in Kuala Lumpur*

Find all integral solutions of the following Diophantine equations

(A) $$ax^2 + ay^2 + z^2 - 2axyz - 1 = 0 \qquad (a = 1, 2, \cdots),$$

(B) $$9x^2 + 25y^2 + 49z^2 - 210xyz - 1 = 0,$$

(C) $$9x^2 + 25y^2 + 4z^2 - 60xyz - 1 = 0.$$

II. *Solution by the proposer.* The equation (A) for the case $a=1$ was solved in the proposer's note, *On the Diophantine equation* $x^2+y^2+z^2+2xyz=1$. (This MONTHLY, LXIV (1957), 101–103.) A similar method applies here, and we have the

THEOREM. All integral solutions (x, y, z) of (A) (except the trivial solutions $(0, 0, \pm 1)$) are found uniquely by the following rule: (i) Take arbitrary non-negative integers p, q, r, with greatest common divisor unity, with p, q odd, and such that one of them is equal to the sum of the other two; (ii) define θ by $\cosh\theta = a^{\frac{1}{2}}u$, where u is any positive integer; then the numbers x, y, z are defined by

$$a^{1/2}x = \pm \cosh p\theta, \qquad a^{1/2}y = \pm \cosh q\theta, \qquad z = \pm \cosh r\theta,$$

where the product of the ambiguous signs is $+1$.

That z is an integer is clear since $\cosh r\theta$ is an integral polynomial in even powers of $a^{\frac{1}{2}}u = \cosh\theta$; and x, y are integers since each of $\cosh p\theta$, $\cosh q\theta$ is $a^{\frac{1}{2}}u$ times an integral polynomial in even powers of $a^{\frac{1}{2}}u = \cosh\theta$. For example, $p=5$, $q=3$, $r=2$ give the solutions:

$$x = 16a^2u^5 - 20au^3 + 5u, \qquad y = 4au^3 - 3u, \qquad z = 2au^2 - 1.$$

The proof of the theorem may be obtained by the method of descent used in the note cited above. Suppose a solution exists such that $\min(|x|, |y|, |z|) > 1$. We develop a triad with a smaller minimum and, hence, after a finite number of steps, a triad with $\min = 1$ or 0.

If (x, y, z) is a solution, so also are (x', y, z), (x, y', z) and (x, y, z') (called associated solutions), where

$$x' = 2yz - x, \qquad y' = 2zx - y, \qquad z' = 2axy - z.$$

For if $\phi(x) = 0$, where $\phi(X) = aX^2 + ay^2 + z^2 - 2aXyz - 1$, so also is $\phi(x') = 0$, since $x + x' = 2yz$. Similarly for y' and z'.

Without loss of generality we take $|x| \geqq |y|$. Suppose $|x| \geqq |y| \geqq |z| > 1$. Then the associated solution (x', y, z) is such that $|y| \geqq |z| \geqq |x'|$, as is easily shown. Similarly in case $|x| \geqq |z| \geqq |y| > 1$ or $|z| \geqq |x| \geqq |y| > 1$. In all cases, therefore, after a finite number of steps we reach a solution in which $\min(|x|, |y|, |z|) = 1$ or 0.

If x or $y=0$, we get simply $(0, 0, \pm 1)$. If $z=0$, there is no solution. If $x=\pm 1$ (or $y=\pm 1$), descent yields $x=\pm 1$, $y=\pm 1$, $z=\pm 1$. Finally, if $z=\pm 1$, then $x=\pm y(|x|\geqq 1)$.

Induction on the basic triads $(u, u, 1)$, $(u, -u, 1)$, with u a positive integer, leads to the theorem as stated.

A similar method applies to (B) and to other equations of the form

$$(*) \qquad a^2x^2 + b^2y^2 + c^2z^2 - 2abcxyz - 1 = 0,$$

where a, b, c, are integers with greatest common divisor unity. In fact, putting $X=ax$, $Y=by$, $Z=cz$ we have the case already discussed. The previous solution then applies except that we must have finally X, Y, Z divisible by a, b, c, respectively.

In particular for equation (B) it is not difficult to find the conditions under which the divisibility requirements on X, Y, Z are met and thus determine that all solutions are given by the rule: (i) p, q are odd integers, $r\equiv 2 \pmod 4$; coprime; one equal to the sum of the other two; (ii) $\cosh\theta=105t\pm 30$, with t integral; then the solutions are x, y, z where

$$3x = \pm \cosh p\theta, \qquad 5y = \pm \cosh q\theta, \qquad 7z = \pm \cosh r\theta$$

with product of ambiguous signs equal to $+1$.

For the more general case, it is not difficult to prove the following result: The necessary and sufficient condition that solutions (in fact, infinitely many solutions) of (*) exist is that at least one of a, b, c is of the form $8k\pm 1$.

Thus there are no integers satisfying (C), as has already been shown in I.

Bounded Functions

4879 [1959, 921]. *Proposed by L. J. Wallen, Massachusetts Institute of Technology*

Let f be a measurable real function defined for all real x, and let G be a continuous real function of two variables. Show that if

$$|f(x+y)| \leqq G(f(x), f(y))$$

for all x and y, then f is bounded on bounded sets.

Solution by D. J. Newman, Brown University. Since f is measurable, the function h defined by $h(x)=|f(x)|+|f(-x)|$ is measurable. Thus there exists some set S of positive measure on which h is bounded. Thus the hypotheses imply that $f(x-y)$ is bounded for all x, y in S. Now it is well known, since S has positive measure, that the set of all $x-y$ with x and y in S is a neighborhood of the origin. Hence $f(z)$ is bounded for all z in some neighborhood U of the origin. Thus, by applying induction to the hypotheses, it follows that, for an arbitrary positive integer n, $f(nz)$ is bounded for all z in U. That is, f is bounded on any bounded set.

Also solved by the proposer.

Integral Equation

4885 [1960, 87]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass., and D. J. Newman, Brown University*

Determine the unique solution of the integral equation

$$F(x_1, x_2, \cdots, x_n) = 1 + \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} F(y_1, y_2, \cdots, y_n)dy_1dy_2\cdots dy_n.$$

(The uniqueness when $n=2$ was one of the problems in the 1958 Putnam competition.)

Solution by P. G. Rooney, University of Toronto. Let $u_0=1$, and

$$u_{r+1}(x_1, \cdots, x_n) = 1 + \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} u_r(y_1, \cdots, y_n)dy_1\cdots dy_n.$$

Then, by induction,

$$u_r(x_1, \cdots, x_n) = \sum_{m=0}^{r} (x_1x_2\cdots x_n)^m/(m!)^n.$$

Clearly $u(x_1, \cdots, x_n) = \lim u_r(x_1, \cdots, x_n)$ exists uniformly in any bounded region of n-space, and satisfies the integral equation. Thus a solution is

$$u(x_1, \cdots, x_n) = \sum_{m=0}^{\infty} (x_1x_2\cdots x_n)^m/(m!)^n.$$

Now, if u and v are two bounded measurable solutions and $w=u-v$, then

$$w(x_1, \cdots, x_n) = \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} w(y_1, \cdots, y_n)dy_1\cdots dy_n.$$

Hence, if M is a bound for $|w|$, then by induction

$$|w(x_1, \cdots, x_n)| \leqq M|x_1x_2\cdots x_n|^m/(m!)^n \to 0$$

as $m\to\infty$, and u is unique.

Also solved by P. G. Engstrom, N. J. Fine, George Glauberman, Immanuel Marx, Chih-yi Wang, Robert Weinstock, and the proposers.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Theory of Storage. By P. A. P. Moran. Wiley, New York, 1960. 111 pp. $2.50.

The author, a pioneer in the application of probability theory to problems of water storage in dams, presents a concise and lucid account of the present status of the subject. In the first two chapters the mathematical preliminaries (including Markov processes) and inventory problems are briefly summarized. The core of the book is in the remaining four chapters: Dams—Discrete Time; Dams—Continuous Time; Monte Carlo and Other Statistical Methods; The Programming of Storage Systems. A bibliography consisting chiefly of post-1950 items bears witness to the novelty of the topic. This book can be recommended not only to the designer concerned with problems of dam construction but to any one interested in applications of recently developed techniques in probability theory.

H. KAUFMAN
McGill University

Classical Mathematics, A Concise History of the Classical Era in Mathematics. By Joseph Ehrenfried Hofmann. Philosophical Library, New York, 1959. 159 pp. $4.75.

This is a translation of Parts II and III of the author's *Geschichte der Mathematik* that appeared in Sammlung Göschen, Nos. 875 and 882. (Part I of this *Geschichte* had been translated earlier under the title *History of Mathematics.*) It presents a compact review of concepts and techniques that were discovered or developed during the periods known as the High and Late Baroque and the Age of Enlightenment. (*ca.* 1625–*ca.* 1790). The main characters in the grand drama of the discovery (or should it be "invention," "Erfindung" in the German original?) of the calculus and the development of power series are illumined succinctly and eruditely. Considerable space is given to details of the uncivil Newton-Leibniz War, Hofmann stating categorically that "the discovery of the calculus is exclusively to the credit of G. W. Leibniz." The book is clearly not for the novice. The initiate, however, will find many an interesting item in this concise history.

It is to be greatly regretted that the highly valuable bibliographies (covering over 80 pp.) have been curtailed in the translation. The price differential, too, is noteworthy; the German three booklets cost D.M. 7.20 (*ca.* $2.00); the English two volumes—$9.50.

PINCUS SCHUB
University of Pennsylvania

Complex Variables and Applications. By R. V. Churchill. McGraw-Hill, New York, 1960. $6.75.

The second edition of R. V. Churchill's *Complex Variables and Applications* preserves the virtues of the earlier edition. It is still the most suitable text for a one-semester course in complex variables, primarily intended for engineers and physicists, or for any other group which aims to reach the level of applications as rapidly as possible and spend only as much time on theory as is necessary to obtain a sound foundation for subsequent applications. Like most mathematicians, when the reviewer is trying to decide the merit of a complex variables book, his reaction is to study the section on integration. From this point of view the new edition seems to be a substantial improvement. There is a wide selection of applications in this book. In addition to the applications discussed in the first edition, we now have a chapter dealing with integral formulas of the Poisson type.

F. HAAS
Wayne State University

Vector Analysis with Applications to Geometry and Physics. By Manuel Schwartz, Simon Green, and W. A. Rutledge. Harper, New York, 1960. xii+556 pp. $7.50.

The book might better be entitled "Topics from Geometry and Physics Done with Vector Analysis." Indeed, the authors regard vector analysis as a language of physics and attempt to train the reader to think in this language. This is done by interspersing applications with mathematical development. A large number of illustrative examples and exercises are supplied. Any reader brave enough to do the work called for will certainly emerge with considerable facility in the uses of vector analysis, and will learn a lot of geometry and physics along the way. It is also valuable as a reference work.

The chapter headings are worth quoting to illustrate the spirit of the book. They are: Vector algebra, Statics, Differentiation of vectors, Kinematics, Vector integral calculus, Dynamics, General coordinates, Differential geometry, Harmonic functions, Electrostatics, Magnetism and electrodynamics, Linear vector functions. An index is included.

Many difficult mathematical concepts are treated only cursorily, but a competent instructor can elaborate on them. The degree of mathematical rigor is quite variable. In many places theorems are stated precisely and proofs or references to same are supplied. In other places, even a precise statement of theorems is lacking. The reader must make additional differentiability assumptions to do many of the exercises. In short, sometimes it reads like a mathematics book and sometimes like a run-of-the-mill physics text. Despite this, the level of mathematical precision is, over-all, higher than that of most physics books on vector analysis.

MELVIN HENRIKSEN
Wayne State University

A Survey of Basic Mathematics. By H. G. Apostle. Little, Brown, Boston, 1960. xv+464 pp. \$6.00.

Professor Apostle's book gives a year's terminal course for liberal arts students, though a semester course could be selected from it. It seems to the reviewer that so little is presupposed, only a bit more than elementary arithmetical operations, that in algebra and plane geometry there is too much review for many college students. Topics are chosen from logic, arithmetic, algebra, geometry, statistics, plane trigonometry, vector analysis, plane and solid analytic geometry and calculus. Brief appendices deal very well with equivalent systems, the number of primes and cardinal numbers. There are tables, as well as answers to odd-numbered problems.

Many of the exercises are well chosen and decidedly thought-provoking. The author is very careful about insisting on "checks" for problems, but not so careful about stating that a factor like $x-a$ may be taken from numerator and denominator only if nonzero, though he does give a good comment on this on page 124. The reviewer likes especially Professor Apostle's discussion of meaningless answers to algebraic problems, definition of cylinder with circular cylinder as special case, chapter on non-Euclidean geometries, presentation of probability and functions, choice of material on statistics, particularly comments on high coefficient of correlation vs. causation, discussion of vectors, good form for illustrative problems on progressions and amusing mnemonic devices for letters giving elements of *A. P.* and *G. P.*, applications of conics, manner of introducing calculus and habit of defining new concepts in terms of old ones.

The part of the book on analytic geometry and calculus calls them "higher mathematics." These topics seem to the reviewer *elementary* college mathematics, and they are sometimes taught in preparatory school. In trigonometry why spend so much time on solving triangles, when many present books are turning away from that topic? Careful correction of slight errors should be done for a second edition.

MARION E. STARK
Wellesley College

Théorie des graphes et ses applications. By Claude Berge. Dunod, Paris, 1958. viii+277 pp.

The tentacles of the theory of graphs steadily grow more numerous and penetrate more deeply into many phases of mathematics. The jargon of the theory includes such words and phrases as: arc, edge, path, chain, Hamiltonian circuit, tree, node, function of Grundy, latin square, incidence matrix, totally unimodular matrix, chromatic class, cyclomatic number, semi-factor, capacity, coupling, network. Applications of the theory spread over not only geometry, topology, and algebra, but also games, economics, military affairs, social structures, physics, statistics, communications, dynamic programming, operations research. Many famous questions which have helped to motivate the develop-

ment of the broad theory of graphs are seemingly unrelated. To illustrate their diversity, we mention: the problem of the eight queens on the chessboard; the promenade of the fifteen school girls; generalizations of Nim; the boat ferrying the wolf, goat, and cabbage across the river one at a time; the transportation problem; the excursion problem (where m families with $r_1, \cdots, r_m$ members respectively travel in n vehicles with $s_1, \cdots, s_n$ available seats respectively, and no family has two of its members in the same conveyance); the assignment of personnel; the path of a knight over every square on the chessboard; the design of a tournament so that the results will be "fair"; the traveling salesman problem; the seven bridges of Koenigsberg; the longest circular sequence composed of zeroes and ones with no repeated portion of k consecutive digits; the number of days in a conference where eleven ministers sit at a round table but no two ministers sit beside each other more than one day; the most effective bombardment of communication channels; the noncrossing of supply lines for the three public utilities to the three cities; the dissection of a rectangular region into pairwise incongruent square regions; the four-color problem. Of the several appendices, one lists fourteen questions still awaiting solution. All in all, in this book we have an up-to-date exposition, by one of the developers himself, of an intriguing theory capable of handling a fascinating potpourri of situations.

R. A. GOOD
University of Maryland and University of Oklahoma

Advances in Applied Mechanics, Vol. 6. Edited by H. L. Dryden et al. Academic Press, New York, 1960. x+294 pp. $9.00.

The founding editors (Von Mises and Von Kármán) of this series of volumes have explained clearly what the reader may expect to find here: "the principal aim of the *Advances* is to give surveys of the present state of research work in various fields of applied mechanics" (vol. 2, 1951); the volumes are "intended for students, scholars and engineers who are familiar with the contents of textbooks and handbooks and who are unable to follow up the research papers currently published" (vol. 1, 1948).

All the surveys in the present volume relate to fluid dynamics. Two surveys study boundary layers, the first dealing with unsteady laminar layers and the second with the phenomena of ionization and dissociation which are manifested in flows with very high velocities. The third studies shock waves in ducts of varying cross-section and the fifth examines theoretical and experimental work on Kármán vortex streets since 1953; interest in this topic has revived since it is applicable to technical processes, e.g. fuel sprays.

The fourth article is a detailed study (119 pp.) of similarity and equivalence in compressible flow.

The book has an international flavor; two of the contributors are in British universities and three in German research institutes.

JOHN MCNAMEE
University of Alberta

Introduction to Matrices and Linear Transformations. By Daniel T. Finkbeiner, II. Freeman, San Francisco, 1960. vii+248 pp. $6.50.

After giving in Chapter 1 a rather thorough introduction into *Abstract Systems*, the author continues in Chapter 2 with *Vector Spaces* to lay the foundation for the treatment of the subject. The remaining chapters are: 3. *Linear Transformations*, 4. *Matrices*, 5. *Linear Equations and Determinants*, 6. *Equivalence Relations on Matrices*, 7. *Characteristics of a Matrix*, 8. *A Canonical Form for Linear Transformations*, 9. *Metric Concepts*, and 10. *Functions of Matrices*. To supplement the discussion of abstract systems the first appendix formulates the necessary *Algebraic Concepts*. The second appendix, *Combinatorial Equivalence*, gives a thorough discussion of the pivot operations on matrices.

Throughout the book the wealth of exercises enhances the value of this excellent book as a text for a course. Both the notation and the printing format are very excellent. The book is extremely free from errors and is written in a very readable style.

ALBERT NEWHOUSE
University of Houston

Fonctions Hypergéométriques Confluentes. By F. G. Tricomi. (Mémorial des Sciences Mathématiques, No. 140) Gauthier-Villars, Paris, 1960. 86 pp. 20 NF (U. S. $4.27).

In this abridged version of his monograph *Funzioni Ipergeometriche Confluenti* (Cremonese, Rome, 1954; reviewed in Bull. Amer. Math. Soc., vol. 61, 1955, pp. 456–460), the author surveys the more important properties of confluent hypergeometric functions and illustrates through a few well-chosen examples the widespread applicability of these functions. By means of skillful organization, he is able to include all the dominant features of the parent Italian volume and to bind them together into a cohesive unit. For detailed proofs and for especially involved formulae, however, the reader is often referred to the parent volume or to other easily accessible literature. Not written to be an introduction to the subject (the author has already written an excellent one), nor at the other extreme to be a handbook (an excellent one by Herbert Buchholz is available), this little book very effectively summarizes the current state of knowledge of the subject.

C. A. SWANSON
University of British Columbia

Confluent Hypergeometric Functions. By L. J. Slater. Cambridge University Press, London-New York, 1960. ix+247 pp. $12.50.

This superbly produced volume contains in its first part the main properties of Kummer's and Whittaker's confluent hypergeometric functions, and in its second part the most extensive numerical tables of Kummer's function ${}_1F_1[a; b; x]$ which have yet been published. It assumes particular importance as a work of

reference because of the multitude of problems occurring in mathematics and physics that can be solved in terms of confluent hypergeometric functions. This is the first book in English to devote more than a chapter or two to the subject (although four books in Italian, German, and French have appeared since 1952; see this MONTHLY, present issue, for another review). It is written rather concisely, and contains a wealth of formulae and diagrams. About one half of the book comprises tables, computed by the author on EDSAC I in the Cambridge University Mathematical Laboratory.

An interesting feature is a very detailed chapter on asymptotic expansions. After producing various elementary results, the author proceeds to derive asymptotic expansions when both a and x in Kummer's functions become large together by appealing to some recent theorems of F. W. J. Olver. The same type of results were obtained in 1957 by A. Erdélyi and the reviewer (Memoirs Amer. Math. Soc., No. 25). The author could have increased the practical value of the asymptotic formulae by including statements of them in the important Coulomb wave function notation.

Finally, the extreme usefulness of this book as a permanent reference volume is accentuated by the inclusion of a good general index as well as symbolic index, and by the excellent printing and fine quality of the materials used.

C. A. SWANSON
University of British Columbia

Elementary Statistics. By Sidney F. Mack. Holt, Rinehart and Winston, New York, 1960. ix+198 pp. $4.50.

This text is written for the standard one-semester course in elementary statistics given in a mathematics department but largely to nonmathematics majors. As such, the contents are quite standard—summarization of data, fundamentals of probability and probability distributions, binomial and normal distributions, sampling distributions, confidence limits, testing hypotheses, including the use of Student's distribution and chi-square, and linear correlation and regression. An interesting addition is that of a refresher chapter going as far back as the arithmetic of fractions, which may be a useful addition for the type of audience that such courses attract. Also noteworthy is the brevity of the chapter on descriptive statistics; the obsolete concepts that frequently clutter up such chapters are here omitted. However, this reviewer deplores the obsolete definition of the variance with divisor n rather than $n-1$.

The probability treatment is based on the relative frequency definition of probability; several simple theorems on relative frequencies are proved "rigorously." There is also a sketchy section on empirical probability. The instructor may have a hard time with this and with the brief casual transition from discrete to (normal) continuous probabilities. Presumably he will have to take out some time to amplify the statement made on page 40, "we therefore refer to an event whose probability is 0 as an impossible event."

General definitions are frequently omitted, *e.g.*, population, confidence limits—though these concepts are illustrated or defined for special cases. Also point estimation is omitted and as a result it is necessary to introduce the least squares regression line simply as "the 'nearest' line to all the points." The careful setting out of the theorems is commendable even though most of these must be given without proof in a text at this level. However, it is questionable whether statistical rules of thumb should be glorified as theorems. Several of these were noted (Th. 6.3, 7.2, 9.1). The topics of confidence intervals and hypothesis testing appear to be well done with a modern (pre-decision theory) point of view within the framework established. In particular, the interpretations are carefully and correctly worded.

The reviewer regrets the absence of any approach to abstractions or generality but recognizes that it is debatable how much of this can be done with the students for whom this text is designed. The teacher who wishes to convey some statistical ideas to mathematically weak students, and who does not wish any abstract concepts, should give this book serious consideration among its numerous competitors.

DOUGLAS G. CHAPMAN
University of Washington

BRIEF MENTION

Unvollständigkeit und Unentscheidbarkeit. By Wolfgang Stegmüller. Springer-Verlag, Wien, Germany, 1959. 114 pp. $4.70.

This discussion, in German, of the results of Gödel, Church, Kleene, Rosser etc. of basic logic should be called to the attention of logicians among our readers. Presumably, it will be reviewed more extensively in the Journal of Symbolic Logic.

Nonlinear Problems in Random Theory. By Norbert Wiener. Wiley, New York, 1958. ix+131 pp. $4.50.

This book, which covers an amazing variety of topics, will be reviewed more fully in the *Bulletin* of the American Mathematical Society.

Ramification Theoretic Methods in Algebraic Geometry. By Shreeram Abhyankar. Princeton University Press, Princeton, N. J., 1959. vi+96 pp. $2.75.

This, too, will be reviewed in the *Bulletin* of the American Mathematical Society.

Automatic Data-Processing Systems. By Robert H. Gregory and Richard L. Van Horn. Wadsworth, San Francisco, 1960. xii+705 pp. $8.75.

Not a mathematics book, but a management-oriented book on data processing using high speed computers.

Instruction in Arithmetic. National Council of Teachers of Mathematics 25th Yearbook. NCTM, Washington, D. C., 1960. viii+366 pp. $4.50. To members of the Council $3.50.

Brief Course in Analytics (3rd ed.). By M. A. Hill, Jr. and J. B. Linker. Holt, New York, 1960. viii+232 pp. $3.90.

Analytic Geometry (3rd ed.). By John W. Cell. Wiley, New York, 1960. xii+330 pp. $4.95.

Differential Equations (2nd ed.). By Alfred L. Nelson, Karl W. Folley and Max Coral. Heath and Co., Boston, Mass., 1960. x+308 pp. $5.25.

Elementary Differential Equations (5th ed.). By Lyman M. Kells. McGraw-Hill, New York, 1960. x+318 pp. $6.25.

Calculus and Analytic Geometry (3rd ed.). By George B. Thomas, Jr. Addison-Wesley, Reading, Mass., 1960. xii+1010 pp. $10.75.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montizingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. A. Rosenbaum, Wesleyan University, represented the Association at the Silver Convocation honoring Dr. A. N. Jorgensen, President of the University of Connecticut, on November 12, 1960.

California Institute of Technology: Professor H. Wielandt, University of Tubingen, Tubingen, Germany, has been appointed Visiting Professor for the academic year 1960–61; Dr. G. D. Chakerian, University of California, Berkeley, has been appointed Instructor; Dr. E. C. Dade, Princeton University, has been appointed Bateman Research Fellow in Mathematics; Dr. R. D. Ryan has been appointed Research Fellow in Mathematics; Assistant Professors W. A. J. Luxemburg and C. H. Wilcox have been promoted to Associate Professors; Dr. R. E. Block has been promoted to Assistant Professor.

Carleton College: Associate Professor John Dyer-Bennet, Purdue University, has been appointed Associate Professor; Dr. N. W. Johnson, University of Toronto, and Mr. Joseph Woolfson, Brooklyn College, have been appointed Instructors; Mr. P. S. Jorgensen is on leave at the University of Copenhagen, Denmark, for the academic year 1960–61; Assistant Professor F. L. Wolf has been promoted to Associate Professor and is on leave at the University of California, Berkeley, for the year 1960–61.

Eastern Michigan University: Mr. D. D. Heikkinen and Mr. G. D. Anderson have been appointed Assistant Professors; Mr. W. M. Fitzgerald has been appointed Instructor.

Indiana University: Professor P. R. Masani, Brown University, has been appointed Professor; Professor L. C. Young, University of Wisconsin, has been appointed Visiting Professor; Dr. R. J. Troyer has been appointed Instructor.

Montana State College: Dr. C. P. Quesenberry, Virginia Polytechnic Institute, and Dr. H. R. Fischer, Swiss National Foundation Research Fellow, have been appointed Assistant Professors; Mr. J. L. Simpson has been promoted to Assistant Professor; Assistant Professor C. J. Mode has been promoted to Associate Professor.

Trenton State College: Mr. L. B. Sklar, Rutgers University, has been appointed Assistant Professor; Assistant Professor John McIlroy has been promoted to Associate Professor.

University of Arizona: Visiting Professor Berthold Schweizer, University of California, Los Angeles, Dr. Paul Slepian, Hughes Research Laboratories, Culver City, California, and Assistant Professor H. M. Lieberstein, University of Wisconsin, have been appointed Associate Professors.

University of Colorado: Dr. Wolfgang Schmidt, University of Vienna, Vienna, Austria, has been appointed Assistant Professor; Dr. Marguerite Dunton has been appointed Acting Assistant Professor; Assistant Professors Arne Magnus, B. C. Meyer, and Robert McKelvey have been promoted to Associate Professors; Professor Sarvadaman Chowla is on leave as Visiting Professor at the University of Notre Dame; Professor Robert McKelvey is on leave at the University of California, Los Angeles.

University of Georgia: Assistant Professor L. W. Anderson, University of Oregon, and Dr. R. W. Heath, Woman's College of the University of North Carolina, have been appointed Assistant Professors.

University of Minnesota: Dr. H. F. Weinberger, University of Maryland, has been appointed Associate Professor; Dr. S. A. Gal, Yale University, has been appointed Visiting Associate Professor; Assistant Professor Walter Littman, University of Wisconsin, and Dr. E. R. Rodemich, Massachusetts Institute of Technology, have been appointed Assistant Professors; Associate Professors Eugenio Calabi, Lawrence Markus, J. C. C. Nitsche and J. B. Serrin, have been promoted to Professors; Drs. W. A. Harris, Jr., and Richard Juberg have been promoted to Assistant Professors.

University of Utah: Assistant Professor R. C. Bzoch, University of Minnesota, has been appointed Assistant Professor; Associate Professor Edward Nelson, University of North Dakota, has been appointed Assistant Professor; Assistant Professors W. J. Coles, E. A. Davis, E. E. Kohlbecker, and D. V. V. Wend, have been promoted to Associate Professors.

University of Wisconsin: Assistant Professor Fred Brauer, University of British Columbia, Drs. M. I. Knopp and F. A. Raymond, Institute for Advanced Study, and Dr. J. E. Ohm, University of California, have been appointed Assistant Professors; Drs. A. N. Feldzamen, University of Chicago, P. E. Miles, Yale University, and S. H. Coleman, University of Virginia, have been appointed Instructors.

University of Wisconsin—Mathematics Research Center: Drs. Creighton Buck, Millard Johnson, Walter Rudin, Hans Schneider, and Kennan Smith, University of Wisconsin, have joined the staff as part-time members. Drs. Donald Greenspan and Johan Kemperman, Purdue University, Robert Moore, recently returned from study in Germany, Sam Saunders, Boeing Airplane Company, Seattle, Washington, Jose Gonzalez-Fernandez, Institute of Mathematical Sciences, J. Schroder, Hamburg, Germany, Peter Werner, Aachen, Germany, Edwardo Zarantonello, Argentina, and Professor O. Bjorgum, Norway, have joined the staff for this year.

University of Wisconsin—Milwaukee: Drs. F. W. Caroll, University of Amsterdam, Amsterdam, Netherlands, H. C. Howard, University of Wisconsin, Madison, Morris Katz, and Togo Nishiura, have been appointed Assistant Professors; Mr. R. H. Black and Mr. J. E. McAdam have been appointed Instructors.

University of Wyoming: Professor J. R. Hanna, University of Wichita, has been appointed Associate Professor; Mr. Kenneth Batker, University of Colorado, and Mr. Richard Finley have been appointed Instructors.

Washington University: Professor K. A. Hirsch, Queen Mary College, University of London, London, England, has been appointed Visiting Professor; Assistant Professor R. H. McDowell, Rutgers University, has been appointed Assistant Professor; Assistant Professor F. J. Schnitzer, Wayne State University, has been appointed Research Associate and Visiting Assistant Professor; Associate Professor Guido Weiss, DePaul University, has been appointed Associate Professor and granted leave to accept a National Science Foundation Fellowship for research in Europe for the academic year 1960–61;

Associate Professor Allen Devinatz is on leave for one year to accept a Senior National Science Foundation Postdoctoral Fellowship to undertake research at the Institute for Advanced Study; Professor I. I. Hirschman, Jr., is on leave for one year to do research in Zurich, Switzerland, under an Air Force contract.

Wesleyan University: Drs. S. L. Salas, Yale University, and H. J. Arnold, University of Western Ontario, London, Ontario, Canada, have been appointed Assistant Professors; Dr. G. P. Johnson, Standard Oil Company of California, San Francisco, California, has been appointed Associate Professor; Associate Professor Hing Tong has been promoted to Professor.

Wheaton College (Massachusetts): Associate Professor Barbara J. Beechler, Wilson College, has been appointed Associate Professor; Associate Professor Anne F. O'Neill has been promoted to Professor; Professor C. A. Garabedian retired June, 1960, with the title Professor Emeritus.

Miss Martha Anderson, Wisconsin State College, has been appointed Assistant Instructor at the University of Kansas.

Mr. J. A. Brown, Montana State College, has been appointed Research Worker at Johns Hopkins University Operations Research.

Miss Joyce C. M. Cimelus, Butler University, has been appointed Instructor at the University of Bridgeport.

Mr. B. F. Edwards, Jr., Chance Vought Aircraft, Dallas, Texas, has been appointed Assistant Professor at Stephen F. Austin State College.

Professor Tomlinson Fort, University of Georgia, has been appointed Professor at the University of Miami.

Research Professor B. E. Gatewood, Air Force Institute of Technology, has been appointed Professor at The Ohio State University.

Assistant Professor Artur Grigori, St. Bonaventure University, has been appointed Assistant Professor at the University of Redlands.

Dr. John Gurland, Iowa State University, has been appointed Professor at the Mathematics Research Center, U. S. Army, University of Wisconsin.

Assistant Professor B. W. Helton, University of Utah, has been appointed Associate Professor at Southwest State College.

Mr. R. R. Korfhage, University of Michigan, has been appointed Assistant Professor at North Carolina State College.

Mr. Knox Millsaps, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, has accepted the position of Executive Director at the Air Force Office of Scientific Research, Washington, D. C.

Mr. J. S. Moore, Jr., Florida Christian College, has been appointed Director of the Division of Terminal Electronics. He continues his position as Head of the Division of Physics and Mathematics.

Professor Seymour Parter, Indiana University, has been appointed Professor at Cornell University.

Mr. R. E. Rundus, Park College, has been appointed Assistant Professor at Northern Illinois University.

Mr. R. F. Rutschow, Pennsylvania State University, has been appointed Instructor at the Virginia Military Institute.

Mr. B. F. Ryder, Balboa High School, Balboa, Canal Zone, has been appointed Teacher at Weymouth High School, East Weymouth, Massachusetts.

Sister Maris Stella Schrot, University of Notre Dame, has been appointed Teacher at St. Bernadine High School, San Bernardino, California.

Dr. H. R. Stevens, Duke University, has received a Fulbright Award for study at the University of Hamburg, Hamburg, Germany, during the year 1960–61.

Mr. J. E. Strout, University of Illinois, has been appointed Instructor at Indiana State Teachers College.

Dr. L. H. Turner, Space Technology Laboratories, Los Angeles, California, has been appointed Assistant Professor at the University of Minnesota.

Assistant Professor M. J. Walsh, University of Wyoming, has accepted a position with the Martin Company, San Diego, California.

Mr. R. J. Winterbottom, III, Ursinus College, has accepted a position as Service Engineer with the Spinco Division of Beckman Instruments, Palo Alto, California.

Mr. Walter Zayachkowski, University of Alberta, Edmonton, Alberta, Canada, has been appointed Lecturer at Essex College, Windsor, Ontario, Canada.

Professor Alfred Errera, Uccle, Belgium, died September 18, 1960. He was a member of the Association for twenty-one years.

Professor B. M. Ingersoll, Arizona State University, died October 3, 1960. He was a member of the Association for seven years.

Professor G. B. Lang, University of Florida, died July 21, 1960. He was a member of the Association for twenty-five years.

Professor G. E. Moore, Eastern Michigan University, died during the summer, 1960. He was a member of the Association for thirty-seven years.

Professor W. L. Fields, Hampton Institute, died May 29, 1960. He was a member of the Association for nineteen years.

Sister Mary Clementia, S.S.F., St. Mary's Academy, New Orleans, Louisiana, died March 7, 1960.

GRADUATE LABORATORY DEVELOPMENT PROGRAM

The National Science Foundation announces that March 1, 1961, is the next closing date for receipt of proposals in the Graduate Laboratory Development Program. Proposals received after that date will be reviewed following the next closing date, September 1, 1961. This program requires at least 50% participation by the institution with funds derived from non-Federal sources.

The purpose of the grants is to aid institutions of higher education in modernizing, renovating, or expanding graduate-level basic research laboratories used by staff members and graduate students. Only those departments having an on-going graduate training program leading to the doctoral degree in science at the time of the proposal submission are eligible at present.

Proposals, as well as requests for additional information, should be addressed to: Office of Institutional Programs, National Science Foundation, Washington 25, D. C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FILMS BY MCSHANE AND HENKIN

The Mathematical Association of America announces the availability of two mathematical films for classroom use.

For high school seniors and college freshmen and sophomores:

MATHEMATICAL INDUCTION by Professor Leon Henkin, University of California, Berkeley. (Two reels, each 30 minutes.)

For college juniors and seniors and beginning graduate students:

THE THEORY OF LIMITS by Professor E. J. McShane, University of Virginia. (Four reels, averaging 15 minutes each.)

These films are accompanied by a booklet containing the complete written script, supplementary material for class discussion, and problems for the students. It is suggested that some class discussion of the material presented in the film follow each reel.

These films are being distributed by the Association on a loan basis without charge except for postage. Requests for films should be directed to: Committee on Production of Films, Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Anyone interested in purchasing these films should also write to the above address.

A STUDY OF THE DESIGN OF FACILITIES FOR MATHEMATICS

As its first project, the Conference Board* will conduct a study of the design of buildings and facilities for mathematics; Educational Facilities Laboratories, established by the Ford Foundation in New York City, has agreed to provide the necessary funds.

There are many reasons why it is desirable to make a study of the design of facilities for the mathematical sciences at the present time. In the first place, mathematics has been very poorly housed in the past. In the second place, enrollments are now expanding rapidly. Many colleges and universities have five times as many majors in mathematics as they had only four or five years ago, and the great increases in enrollments for the nation as a whole are still to come.

In the third place, a study of the design of mathematics facilities is appropriate because of the many changes that have taken place in the mathematical sciences. The project will undertake a study of the design of facilities to support the total activities of the mathematical sciences. These activities include research and instruction in pure mathematics, applied mathematics, and statistics; preparation of the manuscripts of research papers; preparation of the manuscripts of textbooks and expository manuscripts for instructional purposes; teacher training; instruction in the operation of desk calculators and electronic digital computers; and the operation of summer and academic year institutes. Modern facilities for the mathematical sciences must provide headquarters space; classrooms; seminar rooms; offices for the staff; library space; a statistics laboratory with desk calculators; a computation center for the electronic digital computer; facilities for the use of films, television, and other teaching aids; and a common room.

The construction of appropriately designed facilities for mathematics is important for a special reason at this time. There is a great shortage of mathematics teachers, and it is probable that this shortage will continue for many years. Under these conditions it is imperative that we make the teachers we do have more efficient than they have been in the past. Some universities are teaching elementary courses in sections of one to two hundred students; others would like to do so, but lecture rooms for classes of this size are not available. In many cases several staff members—even senior staff members—are crowded into one office. When one has a visitor, the others stop work. There are important universities that have never been able to provide one chair and desk per staff member. Better classrooms, better offices, and better facilities of all kinds will certainly make our mathematics staffs more efficient—will enable our mathematics teachers to teach more students and to teach them better.

In the study of the design of facilities for mathematics it will be kept in mind that mathematics is a peculiarly human and personal activity. Some academic subjects involve work with laboratory equipment or machines, but the typical mathematician sits

* This MONTHLY, vol. 67, 1960, pp. 1056–1057.

at a desk and works with pencil and paper. If he is to be productive of ideas, he must be in pleasant, comfortable, and congenial surroundings.

The project will begin by collecting information about good features of buildings already in existence, both in the United States and abroad. It will be necessary to hold a small conference at the beginning of the project to consider the nature of the activities to be conducted in a center for the mathematical sciences, and to explore the design of facilities. The end result of the project will be a report which can be published commercially. The staff will consist of a mathematician and a secretary, and architectural services will be provided by an architectural firm in Washington, D. C. The project, expected to be completed in one year, will begin as soon as the staff has been organized.

The project will include a study of mathematics facilities for high schools, and the final report will include a section or chapter on mathematics classrooms and other facilities for high schools. This part of the report will be published separately so that it will be easily available to high school administrators and others interested only in the high school field.

G. BALEY PRICE

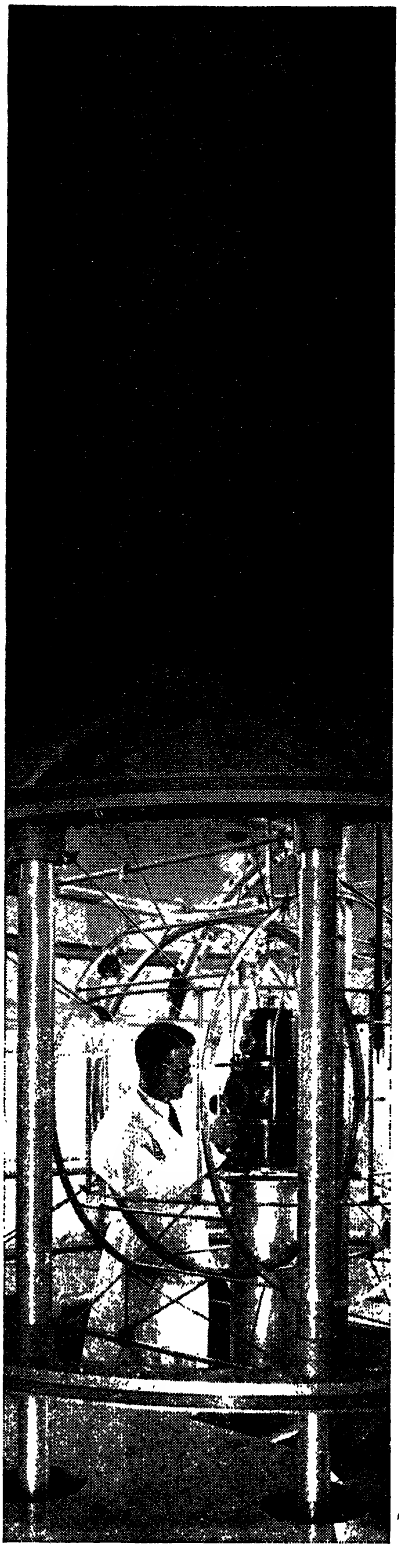
CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.
ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.
INDIANA, Rose Polytechnic Institute, Terre Haute, May 6, 1961.
IOWA, Simpson College, Indianola, April 14, 1961.
KANSAS, Ottawa University, April 15, 1961.
KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN, Wayne State University, Detroit, March 25, 1961.
MINNESOTA
MISSOURI, University of Missouri, Columbia, April 22, 1961.
NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.
NEW JERSEY
NORTHEASTERN
NORTHERN CALIFORNIA
OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.
OKLAHOMA
PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.
PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.
ROCKY MOUNTAIN, University of Colorado, Boulder, April 28–29, 1961.
SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.
SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.
SOUTHWESTERN, University of Arizona, Tucson, April, 1961.
TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.
UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.
WISCONSIN, University of Wisconsin, Madison, May 13, 1961.



One of a series.

Why we eliminated the earth's magnetic field . . . almost

In an isolated laboratory in southwestern Ohio, GM Research scientists have reduced the earth's magnetic field to one ten-thousandth of its usual strength. This is about as weak as the interplanetary field detected by the Pioneer V solar satellite.

Why neutralize the earth's field? To perform with precision one of the more fundamental experiments in magnetism—measuring the Einstein-DeHaas effect. The measurement is simple in concept, experimentally difficult because of the tiny forces involved. It is made by suspending a ferromagnetic rod in a nearly field-free environment . . . magnetizing the rod . . . then measuring the effect (how much the rod rotates) when this known magnetization is reversed.

The beauty of the experiment is that the resulting values can be related directly to the motions of electrons in the rod. The values indicate the large portion of magnetization due to the *spin* of electrons . . . and the slight, but theoretically important, remaining portion due to *orbital* motion of electrons.

These measured values are helping scientists form a better understanding of the perplexing phenomenon—ferromagnetism. Currently being pursued in cooperation with the Charles F. Kettering Foundation, this long-standing project is one of the ventures in basic research of the General Motors Research Laboratories.

General Motors Research Laboratories
Warren, Michigan

System of Helmholtz coils used to neutralize earth's magnetic field.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

A CONTEMPORARY APPROACH TO CLASSICAL GEOMETRY

By

WALTER PRENOWITZ

Professor of Mathematics

Brooklyn College

The Ninth

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

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VOLUME 68 | NUMBER 1

PART II

A Contemporary Approach to Classical Geometry

WALTER PRENOWITZ

Number 9
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

JANUARY 1961

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

RALPH D. JAMES, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. JAMES, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y

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PREFACE

This paper presents an approach to classical geometry that attempts to be modern in spirit—broad in scope, conceptually oriented, synthetic in treatment in that it deals with the subject matter directly—but well grounded in geometric intuition. It aims to be abstract and rigorous but tries to avoid getting mired in the esoteric subtleties of the foundations of geometry as conventionally studied. The treatment is centered on a study of the join operation applied to two figures, for example, the construction of a pyramid or cone by joining a point to a set of points. The burden is put on the simplest case of this operation, namely that of joining two points to form a segment.

The treatment is an outgrowth of my study of classical geometries and their generalizations (projective geometries, ordered and partially ordered linear geometries, spherical geometries) in terms of an operation of joining two points to form a basic connective (line, segment, circular arc). It is similar in spirit to the lattice-theoretic approach to geometry of Birkhoff and of Menger: it is operation-centered rather than configuration-centered and deals with geometrical material (of arbitrary dimension) directly, not through the intervention of coordinate representation. It differs from their approaches in that the basic operation applies initially only to points, not higher dimensional linear spaces. This makes possible the framing of postulates that are directly verifiable in simple diagrams. Since the treatment is based on a single binary operation (joining) it bears closer analogies to group theory than to lattice theory. These become rather strong as the development progresses.

The paper assumes no *formal* knowledge of geometry and is essentially complete in itself but the reader will not find that mathematical maturity hampers his comprehension. To expedite development of the theory much material giving important examples and counterexamples has been placed in an appendix, which however should be considered an integral part of the paper, as it adds richness to the theory and should aid in its comprehension. The major part of the appendix will be understandable after Section 9 of the paper is finished (some of it earlier) and the reader is encouraged to examine it as he reads the paper.

I wish to express my gratitude to the many students and associates who read and criticized a preliminary version of a portion of the manuscript. In particular my warm thanks are extended to Meyer Jordan, Abraham Karrass and Donald Solitar for their devotion in reading and criticizing the manuscript and making suggestions for its improvement. Some of the material presented here was originally tried out in various forms in my classes, and I must express deep satisfaction with my students at Brooklyn College for their stimulation and appreciation of a teacher's efforts to rethink his subject.

W. P.

I. THE CONCEPT OF JOIN SYSTEM AND ITS BASIC PROPERTIES

1. Introduction.* If Euclid returned to earth today he would find our basic approach to geometry and our way of conceiving it not very different from his own. I venture the guess that if Euclid were to examine the famous work of Hilbert on the foundations of geometry [5], he would have more trouble with the German than the geometry. It is true that Hilbert's treatment is more precise and more rigorous than Euclid's, but the basic spirit remains unchanged. The propositions still tend to be direct transliterations of raw visual data, and are kept in mind mainly through visual intuition. The proofs usually are pictorially motivated and remembered. There are no general concepts and methods which serve to unify the material, such as we find in modern axiomatic algebra or modern analysis. As a result, we frequently encounter special or degenerate cases, so that to give a complete proof of an important theorem often is a triumph of character rather than intellect.

In the Euclid-Hilbert treatment of geometry it is hard to find an *intrinsic* motivation for the choice of objects studied or the choice of postulates. For example why study a triangle as a broken line, rather than a triangular region? Why an angle, rather than the figure formed by any two rays? Why apart from naive visual experience assume that a line is separated into three parts by two of its points?

Let me make the point in a somewhat different way: The tremendous outpouring of geometrical knowledge in the 19th century involving projective geometry, non-Euclidean geometry and n-dimensional geometry has not affected our basic approach to classical geometry. The new subjects form a sort of historical addition to the edifice—it has not been rebuilt in the light of them. Higher geometry is not an outgrowth of basic geometry, but a more or less related subject, studied primarily by *analytic* methods. Is it any wonder that many students find their experience in geometry unsatisfying and lacking the elegance, rigor and conceptual unity of modern algebra and analysis? Is it any wonder that geometry seems uninviting to many students and teachers, and seems almost to be disappearing from the college curriculum?

Surely, by the middle of the 20th century, a serious attempt should be made to rethink basic geometry in the light of the great 19th century advances, in order to achieve at least a minimum of conceptual attractiveness. If we default or fail in this attempt, geometry may disappear as an autonomous branch of mathematics and be reduced to a graphical way of describing certain results in the algebra of n variables.

My object is to call this problem to your attention and to suggest a way of dealing with part of it.

Following Hilbert [5] we may take as the basic divisions of classical geom-

* This paper is an amplification of a lecture which was presented by invitation to the Mathematical Association of America at Rochester, N. Y., December 29, 1956.

etry the theories of incidence, of order, and of congruence. I am going to indicate a treatment of the theories of incidence and order which is completely independent of dimension, and so is applicable to geometrical systems of arbitrary dimension, finite or infinite. To achieve this, the Euclid-Hilbert visual-centered concepts, bound down as they are to the line and the plane, must be replaced by dimensionally neutral ideas. Such ideas are well known in geometry and have received recent attention in the algebra of linear inequalities, functional analysis, game theory and linear programming. I have in mind the notion *convex set* and its important specializations, *linear space* and *half-space.* Surely these should be central in a 20th century formulation of basic geometry.*

2. The operations of joining and extending. How can we organize the material around the idea of convex set? By definition, a set is convex if it contains the segment joining each pair of its points—that is, if it is closed under the operation of joining two points to form a segment. But this is precisely the basic operation in Euclid. My point of departure is very simple, it is just this: *To take the operation of joining two points to form a segment as fundamental, and to throw the burden of unifying our material on the consistent and relentless exploitation of this operation.*

Let me show you the concrete geometric basis for the procedure. We take the join of two distinct points a, b to be the open interval or segment ab joining a and b, considered as a set of points. Now if our operation is not to be artificially restricted, it is essential that it apply equally well whether $a \neq b$ or $a = b$. Thus we must clarify the idea of the join of a and a—this we naturally take to consist just of a.

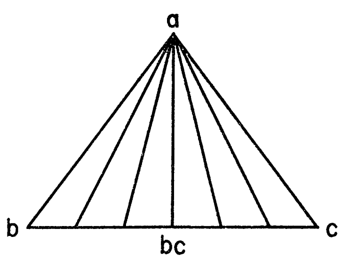


FIG. 1

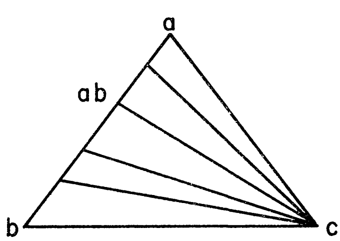


FIG. 2

Can we form the join of more than two points? For example consider three points a, b, c, say the vertices of a triangle. To get their join we first join b to c getting segment bc. Then we join a to each point of bc and aggregate all the joins formed. Clearly this yields the interior of $\triangle abc$ (Fig. 1). You may observe it is just as reasonable to "obtain" the join of a, b, c by a different process: Join each point of ab to c and aggregate the joins formed. Clearly this also yields the in-

* For an axiomatization of ordered linear geometry based on the notion of convex set see W. Prenowitz, Total lattices of convex sets and linear spaces, Ann. of Math., vol. 49, 1948, pp. 659–688.

terior of $\triangle abc$ (Fig. 2). If we describe the respective results of the two processes as the join of a to bc and the join of ab to c, we may assert the equality of these results as a sort of associative law for the operation "join."

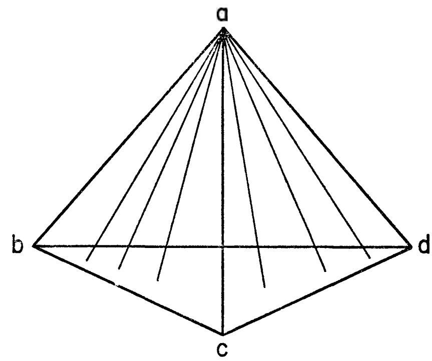


FIG. 3

Similarly let a, b, c, d be the vertices of a tetrahedron. To get their join we first form the join of b, c, d. This of course is the interior of $\triangle bcd$. Then we join a to each point of this and aggregate the joins formed, thus obtaining the interior of tetrahedron $abcd$ (Fig. 3).

Observe that we can not merely construct the join of two points, or of a point and a set, but also the join of two sets of points. For example take segments ab, cd in the last illustration. We get their join by forming the set union of all joins obtained by joining a point of ab and a point of cd. This also is seen to be the interior of tetrahedron $abcd$.

We now formalize the preceding discussion. In a Euclidean geometry we define the *join* of *points* a, $b (a \neq b)$ to be the segment whose endpoints are a, b—that is, the set of points lying strictly between a and b and so *excluding* a and b. The *join* of point a and itself we define to consist just of a. It is essential to extend our operation from points to sets of points, in order to be able to apply it to several terms and to general geometric figures. Thus we define the *join* of the *sets of points* A, B to be the set formed by joining each point of A to each point of B and aggregating all joins formed in this way. If A consists of a single point a we call the result simply the join of a and B; similarly if B reduces to point b, we refer to the join of A and b.

We now state the fundamental properties of the operation join in Euclidean geometry. They are easily verified pictorially and can be deduced formally from the order postulates of Euclidean geometry.* Letters a, b, c refer to arbitrary points.

(A) (CLOSURE LAW). *The join of a and b is a nonempty set of points.*

* For a formulation of these postulates see Veblen ([13], [14]), Forder ([4], pp. 42–48) or Coxeter ([2], pp. 161–162).

(B) (COMMUTATIVE LAW). *The join of a and b is identical to the join of b and a.*

(C) (ASSOCIATIVE LAW). *The join of a with the join of b and c is identical to the join with c of the join of a and b.*

(D) (IDEMPOTENT LAW). *The join of a and a consists of a.*

Although the basic operation in Euclid is to join points to form segments, it is not the only important operation. Next in importance we must consider the operation of extending or prolonging segments to form half-lines or rays. We can easily characterize *extend* in terms of *join*. Formally we define the *extension of a from b* to be the set of points x whose join to b contains a. If $a \neq b$ this is the *open* half-line or ray which emanates from a and is directed away from b (Fig. 4). If $a=b$ (and our definition is significant for this case also) it consists solely of a. The basic properties of extension are:

FIG. 4

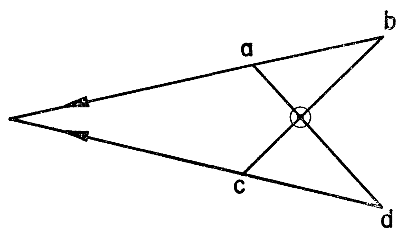


FIG. 5

(E) (CLOSURE LAW). *The extension of a from b is a nonempty set of points.*

(F) (TRANSPOSITION LAW). *Let the extension of a from b meet the extension of c from d. Then the join of a and d meets the join of b and c* (Fig. 5).

(G) (IDEMPOTENT LAW). *The extension of a from a consists of a.*

We conclude with some observations on properties (A), $\cdots$, (G). It cannot be emphasized too strongly that these properties—unlike most current formulations of geometrical principles—are completely general and hold for all degenerate or "limiting" cases. For example (C), the associative law for join, holds if a, b, c are distinct and collinear or if $a=b$ or even if $a=b=c$. Similarly for (F), the transposition principle.

The transposition principle is very important since it gives a formal relation between join, the basic operation, and extend, the derived one. It is a generalization in "join language" of a triangle postulate used by Peano which may be

stated: *Segments which join two vertices of a triangle to respective points of their opposite sides intersect.*

Property (G), the idempotent law for extension, is a direct consequence of the fact that a segment does not contain its endpoints. For certainly a is a point of the extension of a from a—since the join of a to a is a by property (D). Suppose a point x distinct from a is contained in the extension of a from a. Then by definition, the join of x to a contains a. That is, the segment whose endpoints are x and a contains a. This is impossible. Hence the extension of a from a contains only point a.

It may seem preferable to define the join of two distinct points in Euclidean geometry to be a closed interval rather than an (open) segment. This is not justified for several reasons. First, if this were done, we would lose the formal simplicity of the idempotent law for extension—in fact the extension of a from a would contain all points. Second, it is formally simpler, as a set theoretic operation, to adjoin its endpoints to a segment to form a closed interval, rather than the reverse—it is easier to study unions, rather than differences, of sets. Finally, open geometric figures (for example, segments, rays, triangle interiors) are in a sense simpler to study than closed ones, since they are "homogeneous"—all points are "interior" points.

Our properties (A), $\cdots$, (G) are too weak to characterize Euclidean geometry. Much has been omitted. We have left out (1) a parallel postulate; (2) any reference to congruence; (3) the basic incidence property, "two distinct points belong to a unique line." Moreover (A), $\cdots$, (G) do not imply that the points on a line* are "fully ordered." In join language this may be stated: *If three distinct points colline, one of them is in the join of the other two.* Finally note that we have not included any dimensionality restriction. Thus (A), $\cdots$, (G) hold for Euclidean geometries of arbitrary dimension.

3. Abstract geometry based on join. It seems desirable to formulate an abstract theory of geometry based on properties (A), $\cdots$, (G) of the join operation. This will have several advantages: (1) It will enhance rigor of argument by removing any possible pictorial basis for verification of theorems. It will make our treatment logically independent of, though still applicable to, school geometry. (2) It will broaden greatly the scope of interpretation of the theory, since (A), $\cdots$, (G) are not a full complement of postulates for Euclidean geometry. (3) It will afford a deeper insight into the relations of algebra and geometry than is otherwise obtainable, since the geometric theory, like algebra, will be operationally based.

Before beginning the formal treatment we explain the notational conventions which we shall adopt. As is customary, we use $\cup$, $\cap$ for union and intersection of two sets, and $\bigcup$ for union of an arbitrary system of sets. $\emptyset$ denotes the empty or void set. The finite set whose elements (not necessarily distinct) are $a_1, \cdots, a_n$ is denoted $(a_1, \cdots, a_n)$.

* The formal sense in which we use the term "line" is stated in Section 9 below.

It will be very helpful to make certain other less-conventional simplifications of notation. In our theory, unlike algebra, we must be concerned with sets from the beginning, since the join of two elements will be a *set*. Thus we must study the set containment relations $\supset$, $\subset$ as well as element containment, $\in$. Furthermore, relations involving arbitrary sets will have valid degenerate cases in which the sets "reduce" to single elements. As a trivial example suppose $A \supset B$ and $B=(b)$, the set whose only element is b. In this case we write $A \supset b$ or $b \subset A$ instead of $A \supset (b)$. That is, we shall not formally distinguish between set (b) and element b. This permits us to dispense with the $\in$ relation altogether, and as we shall see yields other simplifications.

Furthermore, again unlike algebra, we shall constantly have to study the important geometric *relation*, intersection. That is, we say set A *intersects* or *meets* set B if $A \cap B \neq \varnothing$. This is awkward to deal with symbolically. Thus we adopt the simple form $A \approx B$ for A meets B. This turns out to be very useful. If $A=a$ and $B=b$ the relation $A \approx B$ reduces to the equality $a=b$. In general the relation $\approx$ will behave as a sort of weak equality relation—it is reflexive and symmetric but not transitive. Also it covers "element containment" relations—for example $A \supset b$, where b is an element, is equivalent to $A \approx (b)$ which we write simply $A \approx b$.

Now to proceed formally, consider an abstract system $(G, \cdot)$ in which G is a set of elements $a, b, c, \cdots$ and $\cdot$ is a 2-term operation which associates to each pair of elements a, b of G a uniquely determined set called the *product* or *join* of a and b, and denoted $a \cdot b$ or ab. Our system satisfies the following postulates.

(J1) (CLOSURE LAW). $\varnothing \neq ab \subset G$.

Thus ab is a uniquely determined nonvoid subset of G.

(J2) (COMMUTATIVE LAW). $ab=ba$.

In order to state the associative law and in general to extend our operation to several terms we must define the join AB of the subsets A, B of G. This we define to be the set of elements obtained by joining each element of A to each element of B and aggregating the results. Formally stated:

$$AB = \bigcup_{a \subset A,\, b \subset B} (ab).$$

In simple terms this means: $x \subset AB$ if and only if there exist a, b such that $x \subset ab$ and $a \subset A$, $b \subset B$. If A (or B) reduces to a single element a (or b) we write AB simply as aB (or Ab) in view of our agreement to "identify" a unit set with its element. Then we can state

(J3) (ASSOCIATIVE LAW). $(ab)c=a(bc)$.

Can our operation join be inverted? By analogy with ordinary algebra we might be tempted to say a/b is the unique solution x of $bx=a$. However bx is

not in general an element but a set of many elements—thus it is more natural to consider the relation $bx \supset a$. In this relation a is so to speak one of the values of bx, and there may be many x for which the relation holds. Thus we define a/b (read a divided by b or a over b) to be the set of all solutions x of the relation $bx \supset a$. We call a/b the *extension* of a from b.*

(J4) (CLOSURE LAW). $\varnothing \neq a/b \subset G$.

(J5) (TRANSPOSITION LAW). *If* $a/b \approx c/d$ *then* $ad \approx bc$.

(J6) (IDEMPOTENT LAWS). $aa = a = a/a$.

We call a system $(G, \cdot)$ which satisfies (J1), $\cdots$, (J6) a *join system* or a *join geometry*. Sometimes if there is no ambiguity in the specification of the operation $\cdot$, we simply refer to the join system G.

The application to Euclidean geometry is immediate. Interpret G to be the set of points of a Euclidean geometry and $a \cdot b$ to be the join of points a, b as defined in Section 2. Then a/b is the extension of a from b as defined there and (J1), $\cdots$, (J6) become properties (A), $\cdots$, (G). Consequently $(G, \cdot)$ is a model or interpretation of postulate system (J1), $\cdots$, (J6) or as we say a *join system*.

Observe in this model the intuitive-geometric relation between join and extension: segment $a \cdot b$ is described by moving from a toward b, ray a/b by moving from a directly away from b (Fig. 6). Moreover the important decomposition of

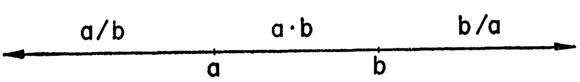


FIG. 6

a Euclidean line effected by its distinct points a, b (Fig. 6) is neatly expressed in terms of join and extension: line $ab = a/b \cup a \cdot b \cup b/a \cup a \cup b$.†

Our postulates merit a few remarks. (J1), $\cdots$, (J5) bear striking analogies with school algebra. Join—in contrast to multiplication in school algebra—may be described as a *many*-valued operation, since it associates with a pair of elements a set of elements rather than a single one. With this in mind (J1), $\cdots$, (J4) obviously correspond to familiar properties of multiplication and division in school algebra (or for that matter in an abelian group). (J5) is a generalization of the principle of cross-multiplication in a proportion (of school algebra or abelian group theory) to which it reduces when the four expressions involved reduce to single elements. This is a rather unexpected analogy between algebra

* It is interesting that Coxeter [2] uses this quotient notation to denote ray in a classical geometrical framework.

† Here we adopt the notational convention that in formulas involving $\cdot$, $/$, $\cup$, portions separated by $\cup$ signs are to be considered enclosed in parentheses.

and geometry. Although $aa=a$, the idempotent law for join, is well known in modern logic and Boolean algebra, (J6) is in a certain sense the most "geometric" of the postulates, since it is a combinatory property which holds for points and fails for numbers.

On the basis of (J1), $\cdots$, (J6) we shall develop a formal theory with many analogies to classical multiplicative algebra. The relations to classical algebra are interesting and stimulating. But a slavish imitation of classical algebra might distract us from our basic purpose: to study geometry by intrinsic methods. Our treatment should be judged primarily I think by its effectiveness in organizing, developing and generalizing classical geometrical material.

4. **Formal properties of join.** To prepare for the study of the basic geometric ideas, convex set, linear space and half-space, we develop the formalism of join and its inverse, extension, in this and the next section.

In the remainder of the paper, unless the contrary is indicated, we study an arbitrary system $(G, \cdot)$ which satisfies (J1), $\cdots$, (J6); we use $a, b, c, \cdots$ to denote elements of G and $A, B, C, \cdots$ subsets of G. Usually $(G, \cdot)$ is not explicitly mentioned in the theorems.*

We begin with a very simple "monotonic law."

THEOREM 1. *$A \subset B$ implies $AC \subset BC$ and $CA \subset CB$.*

Proof. We suppose $x \subset AC$ and deduce $x \subset BC$. By definition of join of sets (Sec. 3) $x \subset AC$ implies the existence of a, c satisfying

$$x \subset ac, \qquad a \subset A,\ c \subset C.$$

By hypothesis $a \subset A$ implies $a \subset B$. Thus $x \subset ac$, $a \subset B$, $c \subset C$. This implies by definition of join of sets $x \subset BC$. We infer $AC \subset BC$. Similarly we show $CA \subset CB$.

This says in effect that both sides of the "inequality" $A \subset B$ may be multiplied by C. We extend this to the multiplication of "inequalities" in

COROLLARY 1. $A' \subset A$, $B' \subset B$ imply $A'B' \subset AB$.

Proof. Multiplying $A' \subset A$ by B' and $B' \subset B$ by A (on the left) we obtain $A'B' \subset AB'$, $AB' \subset AB$, so that $A'B' \subset AB$.

In Corollary 1, let A' consist of a single element a and B' consist of b. Then we have

COROLLARY 2. *$a \subset A$, $b \subset B$ imply $ab \subset AB$.*†

As we have indicated earlier it is essential to operate with sets from the beginning. Thus we extend the associative law (J3) to sets.

* We assume and employ without specific reference the familiar intrinsic properties of equality and in addition, $a=b$ and $c=d$ imply $ac=bd$, and its analogue for sets.

† This is a direct consequence of the definition of AB; it is inserted here for convenience of formal presentation.

THEOREM 2. (ASSOCIATIVE LAW). $(AB)C=A(BC)$.

Proof. We show every element of each member is in the other. Suppose $x\subset(AB)C$. By the definition of join of sets (Sec. 3) this implies

$$x \subset yc, \tag{4.1}$$

where $y\subset AB$, $c\subset C$. Similarly we obtain

$$y \subset ab, \tag{4.2}$$

where $a\subset A$, $b\subset B$. By Theorem 1 we may multiply both members in (4.2) by "set" c, getting

$$yc \subset (ab)c = a(bc). \tag{4.3}$$

Then (4.1), (4.3) imply

$$x \subset a(bc). \tag{4.4}$$

By Corollary 2 above, $b\subset B$, $c\subset C$ imply $bc\subset BC$. This with $a\subset A$ implies by Corollary 1 above

$$a(bc) \subset A(BC). \tag{4.5}$$

Relations (4.4), (4.5) imply $x\subset A(BC)$. Similarly we show $x\subset A(BC)$ implies $x\subset(AB)C$ so that $(AB)C=A(BC)$.

As an immediate consequence of the theorem we have

$$a(b(cd)) = (ab)(cd)$$

taking a, b, cd as the three sets. This justifies formally our observation in Section 2 that the interior of a tetrahedron is the join of two opposite edges.

The commutative law for sets is easily proved and is left as an exercise.

THEOREM 3. $AB=BA$.

We do not want to restrict the application of join to two or three terms. So as in classical algebra we extend our operation to an arbitrary finite number of elements inductively:

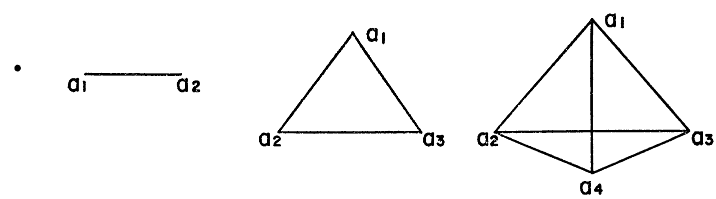
$$a_1 \cdots a_{n+1} = (a_1 \cdots a_n)a_{n+1}.$$

Similarly we extend the operation to a finite number of sets. As in ordinary algebra the associative and commutative laws can be extended by induction to joins of n elements, but the proofs have to be modified somewhat since the join of two elements is not an element but a set. However it is easier to deal with sets rather than elements. For we have just seen that the basic associative and commutative laws do hold for sets, and certainly the join of two sets is a set. Thus we can apply the usual algebraic proofs (see, for example, [7], pp. 20–21) to obtain the associative and commutative laws for the join of n sets. Taking the

case where the sets reduce to single elements, we obtain the associative and commutative laws for the join of n elements.

This extension of join and its basic properties to n terms may seem dull and trivial. But consider for a moment the usual treatment of geometry in this regard. In classical plane and solid geometry the basic figures such as segments, triangles and tetrahedrons are studied on a visually motivated basis. When we study higher dimensional geometry it is hard to see how to generalize this treatment, and it is assumed conventionally that the only feasible procedure is to dispense with a direct (that is, conceptual) treatment of geometric objects and to assume the validity of coordinate or vectorial representation. The advantage is precisely that we can use n coordinates or combine n vectors essentially as easily as 2 or 3. So in our treatment the trivial-seeming extendability of the basic concept to n terms is of the essence: it fosters the direct study of geometric material of arbitrary dimension, while helping to liberate us from dependence on visual intuition. To make the point concrete, consider the following table which indicates the typical or nondegenerate Euclidean interpretation of some joins listed:

a_1	a_1a_2	$a_1a_2a_3$	$a_1a_2a_3a_4$	$a_1a_2a_3a_4a_5$



We have appended no diagram for the typical join of 5 elements! It is represented by the interior of the 4-dimensional analogue of a triangle or a tetrahedron, called a 4-dimensional simplex. The graphical representation in the second row breaks down at this stage—but the symbolic expression in the first row can be studied formally and abstractly, independently of the number of factors and so of dimension.*

We conclude this section with a remark on the application of the join operation to $\varnothing$, the void set. Although this is not of geometric importance, it is permitted since the definition of $A \cdot B$ involves no restriction on the sets A, B. The basic criterion of Section 3, $x \subset A \cdot B$ if and only if there exist a, b such that $x \subset ab$ and $a \subset A$, $b \subset B$, implies that $x \subset A \cdot \varnothing$ is always false, since $b \subset \varnothing$ is always false. Thus $A \cdot \varnothing = \varnothing$. Similarly $\varnothing \cdot B = \varnothing$.

5. The inverse operation. We are now ready to study the properties of extension, the operation inverse to join. Following a familiar pattern we extend this operation to sets.

* This suggests a questionable piece of humor to make a serious point: 4-dimensional space is only 25% harder to study than 3-dimensional space.

DEFINITION. $A/B = \bigcup_{a\subset A,\, b\subset B} (a/b)$. Thus $x\subset A/B$ if and only if $x\subset a/b$, where $a\subset A$, $b\subset B$.

It follows from this definition that $A/\varnothing = \varnothing = \varnothing/A$ by the argument of the last paragraph of Section 4.

In Euclidean geometry, A/B will be the set of points each of which is in an extension, a/b, of a point of A from a point of B. For example if A is a single point not in the plane of circle B, A/B is a one-sheeted conical surface with vertex A, which is open in the sense of not containing A (Fig. 7).

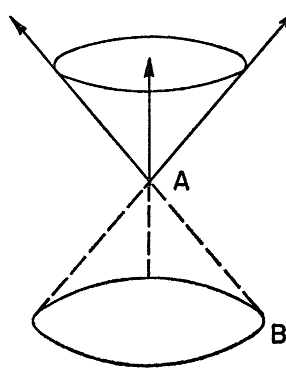


FIG. 7

The proof of Theorem 1, the monotonic law for join, is valid for the inverse operation as well and we may assert

THEOREM 4. *$A\subset B$ implies $A/C\subset B/C$ and $C/A\subset C/B$.*

COROLLARY 1. *$A'\subset A$, $B'\subset B$ imply $A'/B'\subset A/B$.*

COROLLARY 2. *$a\subset A$, $b\subset B$ imply $a/b\subset A/B$.*

Our most important technical procedure involves a simple calculus of intersection relations. Given $A\approx B$, an intersection relation between two sets, we can immediately infer other intersection relations, for example if $X\neq\varnothing$, $AX\approx BX$ or $A/X\approx B/X$ or even $X/A\approx X/B$. These correspond to simple geometric properties of intersecting sets. Moreover the relation between join and its inverse, extension, can be expressed by a very simple and convenient intersection algorithm.

THEOREM 5. *$A\approx BC$ if and only if $A/B\approx C$.*

Proof. Suppose $A\approx BC$. This means that there exists a such that $a\subset A$, $a\subset BC$. The latter implies $a\subset bc$ where $b\subset B$, $c\subset C$. Thus $bc\supset a$ and by definition of a/b we have $c\subset a/b$. This with $a\subset A$, $b\subset B$ implies, by definition of A/B, that $c\subset A/B$. Since we already know $c\subset C$ we conclude $A/B\approx C$.

For the converse we reason similarly. Suppose $A/B\approx C$. Then $c\subset A/B$,

$c \subset C$ for some c. Thus $c \subset a/b$ where $a \subset A$, $b \subset B$. By definition of a/b we have $bc \supset a$. Thus $BC \supset a$ and since $A \supset a$ we conclude $A \approx BC$.

Permitting A, B, C to reduce to single elements, we have the

COROLLARY. *$a \approx bc$ if and only if $a/b \approx c$.*

Note here the algorithmic advantage of using $\approx$, which is suggestive of equality—without it we would have a sort of inequality principle: $a \subset bc$ if and only if $a/b \supset c$.

The transposition law (J5) is easily extended to sets.

THEOREM 6. *$A/B \approx C/D$ implies $AD \approx BC$.*

Proof. By hypothesis there exists x such that

$$x \subset A/B, \qquad x \subset C/D.$$

Hence by definition

$$x \subset a/b, \qquad x \subset c/d,$$

where a, b, c, $d \subset A$, B, C, D, respectively. Thus $a/b \approx c/d$ and (J5) yields $ad \approx bc$. By Theorem 1, Corollary 2,

$$AD \supset ad, \qquad BC \supset bc$$

and we infer $AD \approx BC$.

We continue with somewhat deeper properties of extension or division. Glancing back at (J1), $\cdots$, (J6) we see that we have not postulated the existence of an identity element or of inverses of elements. Consequently we cannot hope to *reduce* division to multiplication in the ordinary way, $a/b = a(1/b)$. To compensate for this we *relate* division to multiplication by "mixed associative principles" like the following:

THEOREM 7. *$a/bc = (a/b)/c$.**

Proof. We show every element of each member is in the other. Suppose $x \subset a/bc$, that is, $x \approx a/bc$. Then

$$\begin{aligned} x(bc) &\approx a && \text{(Th. 5)},\\ (xc)b &\approx a && \text{(J2), (J3)},\\ xc &\approx a/b && \text{(Th. 5)},\\ x &\approx (a/b)/c && \text{(Th. 5)}. \end{aligned}$$

Conversely we suppose $x \approx (a/b)/c$ and retrace the steps to complete the proof.

* Here a/bc stands for $a/(bc)$. In general in formulas involving $\cdot$, $/$, we eliminate excess parentheses by the agreement that portions separated by / signs are to be considered enclosed in parentheses. For example $(A/B)C/DE$ denotes $((A/B)C)/(DE)$.

Using the method employed in Theorem 2 to extend the associative law from elements to sets, we easily get the

COROLLARY. $A/BC = (A/B)/C$.

You might expect that the other laws of school algebra which link multiplication and division

$$a/(b/c) = ac/b, \qquad a(b/c) = ab/c$$

hold here also. This is not so. It would be too boring if there were no divergencies between ordinary algebra and our geometric algebra! However, generalizations of these laws, replacing equality by containment, do hold and can be derived by the method used above:

THEOREM 8. (a) $a/(b/c) \subset ac/b$; (b) $a(b/c) \subset ab/c$.

These can be extended as usual to sets:

COROLLARY. (a) $A/(B/C) \subset AC/B$; (b) $A(B/C) \subset AB/C$.

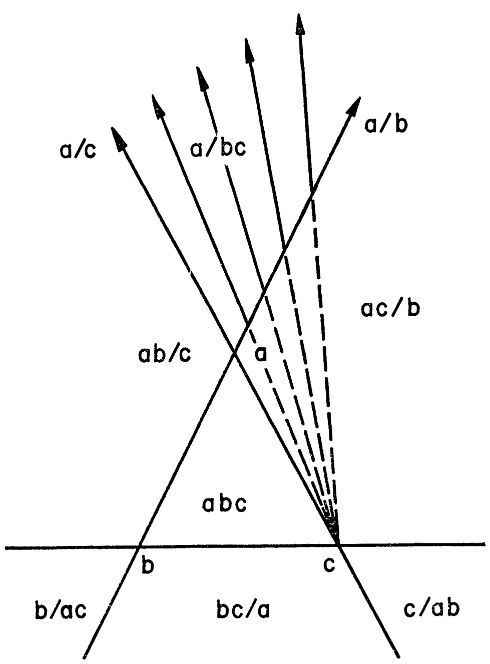


FIG. 8

You must be curious about the concrete geometric significance of Theorems 7, 8. Let a, b, c be the vertices of a triangle in a Euclidean plane (Fig. 8). Observe that the seven regions into which the sidelines of $\triangle abc$ separate the plane are

represented as certain fractional expressions involving a, b, c. For example, a/bc is the interior of the angle formed by rays a/b, a/c. Now note that this region is covered by the rays which emanate from the points of a/b and are directed away from c. Thus a/bc and $(a/b)/c$ are the same region and Theorem 7 is verified. Theorem 8 is easily verified in the same example.

6. Convex sets. Finally we are prepared to study convex sets. We formulate the concept in join language as follows:

DEFINITION. *A is* ***convex*** *or* ***multiplicatively closed*** *if $A \supset x, y$ implies $A \supset xy$.*

Observe that G, the basic set, is convex and that each element a forms a convex set since by (J6), $a = aa \supset aa$. Note that $\emptyset$ also is convex since the definition imposes no restriction on it.

The definition is easily reformulated in terms of join of sets to yield

THEOREM 9. *A is convex if and only if* (a) $A \supset AA$; *or* (b) $A = AA$.

Proof. (a) Suppose A convex. Then $A \supset xy$ for $x \subset A$, $y \subset A$. Thus $A \supset AA$, which is the union of all such sets xy. Conversely if $A \supset AA$ certainly $A \supset xy$, for $x \subset A$, $y \subset A$, and is convex.

(b) Observe for any set A that $A \subset AA$, for if $x \subset A$ then by (J6), $x = xx \subset AA$. Thus conditions (a), (b) are equivalent and the theorem holds.

COROLLARY 1. *$a_1a_2 \cdots a_n$ is convex.*

Proof. By the generalized associative and commutative laws and (J6)

$$(a_1a_2 \cdots a_n)(a_1a_2 \cdots a_n) = (a_1a_1)(a_2a_2) \cdots (a_na_n) = a_1a_2 \cdots a_n$$

and the result follows by (b) of the theorem.

In Euclidean geometry this justifies the convexity of the interior of a triangle or a tetrahedron or in general of any n-simplex.

By definition, convexity is closure under join of elements—it implies closure under join of sets:

COROLLARY 2. *If A is convex and $A \supset X$, Y then $A \supset XY$.*

Proof. $A \supset X$, $A \supset Y$ can be combined to yield $AA \supset XY$ or $A \supset XY$.

We naturally are concerned about the application of familiar operations to convex sets.

THEOREM 10. *If A, B are convex then $A \cap B$, AB, and A/B are convex.*

Proof. The result for $A \cap B$ is immediate since $A \supset x$, y and $B \supset x$, y imply A, $B \supset xy$ so that $A \cap B \supset xy$.

For AB we have using Theorem 9(b) $(AB)(AB) = (AA)(BB) = AB$ so that AB is convex by Theorem 9(b).

For A/B it is a bit more complex and interesting. We have

$$
\begin{aligned}
(A/B)(A/B) &\subset (A/B)A/B && \text{(Th. 8, Corol. (b))}\\
&= A(A/B)/B && \text{(Th. 3)}\\
&\subset (AA/B)/B && \text{(Th. 8, Corol. (b); Th. 4)}\\
&= AA/BB && \text{(Th. 7, Corol.)}\\
&= A/B && \text{(Th. 9(b))}
\end{aligned}
$$

and A/B is convex by Theorem 9(a).

An illustration of the last principle: In Euclidean geometry let A be a point and B the interior of an ellipse whose plane does not contain A, then A/B is the interior of an elliptic cone with vertex A and is convex.

7. Convexifying sets. Suppose we have a set which is not convex. It is natural then to try to *convexify* it—that is to make it convex in the simplest possible way. For example, in Euclidean geometry consider the set (a, b) composed of the distinct points a, b. If we adjoin to (a, b) each point of ab we obtain the closed interval with endpoints a, b which is convex. Clearly this is the simplest way to convert (a, b) into a convex set, since any convex set containing a

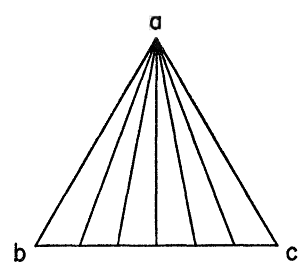


FIG. 9

and b must by definition contain ab. Similarly, if a, b, c are the vertices of a Euclidean triangle the simplest way to make (a, b, c) convex is to adjoin to it each point of ab, bc, ac and abc, thus obtaining the closed triangular region with vertices a, b, c (Fig. 9).

These illustrations are typical of the general theory as we show now.

THEOREM 11. *Consider the finite set* $(a_1, \cdots, a_n)$. *Let* S *be the union of joins of* $a_1, \cdots, a_n$ *taken one or more at a time*:
$S = a_1 \cup a_2 \cup \cdots \cup a_n \cup a_1a_2 \cup a_1a_3 \cup \cdots \cup a_{n-1}a_n \cup \cdots \cup a_1a_2 \cdots a_n.$

Then S *is the only set which satisfies the following properties*:

(a) S *is convex*;
(b) $S \supset (a_1, \cdots, a_n)$;
(c) *If* X *is convex and* $X \supset (a_1, \cdots, a_n)$ *then* $X \supset S$.

Proof. (a) Suppose $x, y \subset S$. Then x, y belong to joins of the a's. For example, suppose $x \subset a_1a_2a_5$, $y \subset a_2a_6a_7$. Combining these as inequalities (Th. 1, Corol. 2),

$$xy \subset (a_1a_2a_5)(a_2a_6a_7) = a_1a_2a_5a_6a_7.$$

Thus $xy \subset S$ and S is convex by definition.

(b) This is trivial by definition of S.

(c) Suppose X convex and $X \supset (a_1, \cdots, a_n)$. By Theorem 9, Corollary 2, used repeatedly, X contains any join of the a's and so $X \supset S$.

To prove uniqueness, suppose S' satisfies (a), (b), (c). Letting $X = S'$ in (c) we have $S' \supset S$. Similarly $S \supset S'$ so that $S' = S$.

Thus in constructing S we have converted the finite set $(a_1, \cdots, a_n)$ into a convex set in the simplest possible way, since by (c) any other convex set containing $(a_1, \cdots, a_n)$ must be "larger than" S. This suggests a precise formulation of the concept of simplest convex set containing a given (finite or infinite) set:

DEFINITION. *Let set A be given. Let S be the only set which satisfies the following properties*:

(a) *S is convex*;

(b) $S \supset A$;

(c) *If X is convex and $X \supset A$ then $X \supset S$.*

Then we call S the ***least convex set*** *containing A, the* ***convex set generated*** *or* ***spanned by*** *A or the* ***convex closure*** *of A and denote it by $[A]$. We write $[(a_1, \cdots, a_n)]$ simply $[a_1, \cdots, a_n]$.*

Observe that $[A] = A$ if A is convex; in particular $[a] = a$. Restating Theorem 11 we have the

COROLLARY. $[a_1, \cdots, a_n] = a_1 \cup \cdots \cup a_n \cup a_1a_2 \cup \cdots \cup a_{n-1}a_n \cup \cdots \cup a_1a_2 \cdots a_n$.

For any set A, not necessarily finite, it is not hard to show a similar result, that $[A]$ exists and is the union of all joins $a_1 \cdots a_r$, where the a's are in A. Thus not merely are finite joins convex (Th. 9, Corol. 1) but they are the building blocks of which all convex sets are composed.

In Euclidean geometry the notion convex closure covers the ideas of closed interval, closed convex polygon and closed convex polyhedron. For example, if a, b are distinct points, $[a, b]$ is the closed interval with endpoints a, b. Further if a, b, c, d are points of a Euclidean plane no one of which is in the join of the other three, $[a, b, c, d]$ is the closed convex polygonal region whose vertices are a, b, c, d (Fig. 10(a)). If however one of the points is in the join of the others, say $d \subset abc$, then $[a, b, c, d]$ reduces to $[a, b, c]$ which in general is a closed triangular region (Fig. 10(b)). Observe that if a, b, c, d are in *general* position, that is do not coplane, $[a, b, c, d]$ is a 3-dimensional simplex and the terms in

its expansion given in the last corollary are disjoint. This is not true if a, b, c, d coplane.

Finally let me remark that the concept of triangle, which is visually inspired and historically grounded in practical problems of measurement, might in the present context have been discovered by a blind student. It forces itself on our attention as one of the simplest kinds of convex set. In effect a visual-pragmatic motivation of the idea is replaced by a conceptual one.

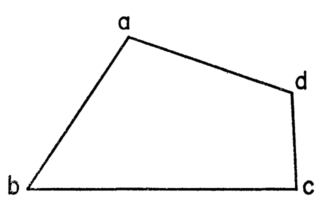


FIG. 10(a)

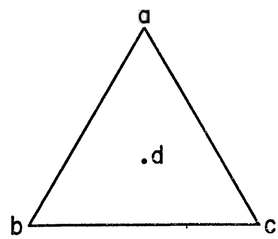


FIG. 10(b)

8. **Linear sets.** Linear set—or linear space as it is usually called—is a generic idea designed to cover the basic "linear" (or uncurved) figures: endless line, plane, 3-space, etc. It is defined in Euclidean geometry (or in other linear geometries) as a set of points containing the (endless) line joining each two of its points.

This puts us face to face with a disturbing question: How in an abstract join system can we introduce the notion line? We have "join" to correspond to segment, "quotient" to correspond to ray, but it is hard to see how to give a simple and natural construction for line. The conventional definition in the foundations of Euclidean geometry ([2], p. 161) expressed in join language is

$$\text{line } ab = ab \cup a/b \cup b/a \cup a \cup b \qquad (a \neq b). \tag{8.1}$$

But this is precisely what we are trying to avoid: namely to assume, in the guise of a definition, an apparently restrictive form—in effect another postulate—without intrinsic motivation, merely on the basis of visual intuition or practical necessity.

So we lay aside for the moment the question of line, and try to define *linear set* in terms of the ideas at our disposal. This is easier! The clue lies in the Euclidean situation. Clearly here a point set containing a and $b (a \neq b)$ will contain line ab if and only if it contains segment ab and rays a/b, b/a. Thus a Euclidean point set is linear if and only if it contains the join and the extension of any two of its points, or equivalently, is closed under the operations join and extension. Thus we adopt formally the

DEFINITION. *A is **linear** (or a **linear space** or a **linear subspace** of G) if $A \supset x, y$ implies $A \supset xy, x/y$.*

A linear set is a convex set which is so to speak "fully extended." Observe that each individual element is linear, since $a=aa=a/a$ by (J6). Also G and $\varnothing$ are linear.

We begin with an analogue of Theorem 9, which is proved by the method of Theorem 9.

THEOREM 12. *A is linear if and only if it is convex and* (a) $A\supset A/A$, *or* (b) $A=A/A$.

Corresponding to Corollary 2 of Theorem 9 we have the

COROLLARY. *If A is linear and $A\supset X$, Y then $A\supset XY$, X/Y.*

Corresponding to Theorem 10 we have merely

THEOREM 13. *If A, B are linear then $A\cap B$ is linear.*

If A and B are linear, AB and A/B in general are not linear—for example, in Euclidean geometry let A, B be parallel lines, or A a line and B a point not in A. Evidently it is harder to construct linear sets than convex ones—however the following theorem gives an important method.

THEOREM 14. *If A is convex A/A is linear.*

Proof. A/A is convex by Theorem 10. Further we have

$$\begin{aligned}
(A/A)/(A/A) &\subset (A/A)A/A && \text{(Th. 8, Corol. (a))}\\
&= A(A/A)/A && \text{(Th. 3)}\\
&\subset (AA/A)/A && \text{(Th. 8, Corol. (b); Th. 4)}\\
&= AA/AA && \text{(Th. 7, Corol.)}\\
&= A/A. && \text{(Th. 9)}
\end{aligned}$$

Thus A/A is linear by Theorem 12(a).

Note the similarity between the argument above and that of the last part of Theorem 10. To illustrate the result let A be the interior of a Euclidean sphere. Then A/A contains all rays which are prolongations of segments in A and clearly is the 3-space containing A.

In general A/A might be called the "complete extension" of A, since it is the aggregate of all extensions a_1/a_2 for a_1, $a_2\subset A$. Then Theorem 14 asserts that the complete extension of a convex set is linear.

The most important special case of the theorem occurs when $A=a_1\cdots a_n$, which we know is convex by Theorem 9, Corollary 1. Thus we have the

COROLLARY. *$a_1\cdots a_n/a_1\cdots a_n$ is linear.*

This enables us to construct a linear set from any finite set. For example in higher-dimensional Euclidean geometry if a_1, a_2, a_3, a_4, a_5 are not in a 3-space then $a_1a_2a_3a_4a_5/a_1a_2a_3a_4a_5$ is a 4-space.

We saw in the paragraph following Theorem 13, that if A, B are linear A/B need not be so. However A/B is linear in the face of the additional condition $A \approx B$.

THEOREM 15. *If A, B are linear and $A \approx B$ then A/B is linear.*

We shall dispense with the proof which is somewhat similar to, but more difficult than, those of Theorems 10 and 14.* The result is very important since it supplements Theorem 14 as a method of constructing linear sets. Despite its

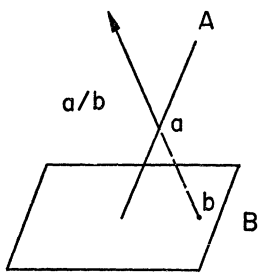


FIG. 11

generality (it involves no dimensional restriction) it has strong intuitive geometric content. For example in a Euclidean 3-space G, let line A meet plane B in a single point. Observe how the extensions a/b for $a \subset A$, $b \subset B$ cover G (Fig. 11).

We conclude this section with an important theoretical point involving linear sets. Suppose S is a linear set of join system $(G, \cdot)$. Since S is closed under the operation $\cdot$ it is natural to consider the system $(S, \cdot)$. Is it a join system? Clearly $(S, \cdot)$ satisfies (J1). The formal laws (J2), (J3) and the first part of (J6) hold in $(G, \cdot)$ and so certainly in $(S, \cdot)$. To consider the remaining postulates we must define extension in $(S, \cdot)$. But S is closed under /, the extension operation in $(G, \cdot)$. Hence the extension operation in $(S, \cdot)$ must be / applied merely to elements of S. It is now not hard to see that (J4), (J5), (J6) hold in $(S, \cdot)$. Thus $(S, \cdot)$ is indeed a join system, which we signalize by saying $(S, \cdot)$ is a *subsystem* of $(G, \cdot)$ or simply S is a *subsystem* of G. To summarize: *Any linear set S may be considered to be a join system in its own right to which our theory can be applied.* This of course is analogous to the subsystem concept in modern algebra, as exemplified by subgroups of a group, subfields of a field, etc.

9. Linearizing sets. Suppose we have a set which is not linear. It is natural then to try to *linearize* it—that is to make it linear in the simplest possible way. In Euclidean geometry, when we form the line determined by points a, b or

* For proof see [10], p. 352, Theorem 5, which covers Theorem 15. Note that [10] adopts an additive notation for join rather than the multiplicative one used here, and a different definition for joins involving the void set.

the plane determined by a, b, c, we are in effect linearizing set (a, b) or (a, b, c). In these cases (and in Euclidean geometry in general) an analogy with the process of convexification of Section 7 holds. Thus to linearize (a, b) we merely adjoin to it ab and a/b and b/a (see Fig. 6). To linearize (a, b, c) we adjoin the joins ab, bc, ac, abc, the extensions a/b, b/a, b/c, c/b, a/c, c/a and the six "mixed" expressions a/bc, b/ac, c/ab, ab/c, bc/a, ac/b (see Fig. 8).

This process of "linearizing by adjunction" is not valid in our abstract join theory. The corollary to Theorem 14 provides a clue to the correct result:

THEOREM 16. *Consider the finite set* $(a_1, \cdots, a_n)$. *Let* $S = a_1 \cdots a_n/a_1 \cdots a_n$. *Then S is the only set which satisfies the following properties*:

(a) *S is linear*;

(b) $S \supset (a_1, \cdots, a_n)$;

(c) *If X is linear and* $X \supset (a_1, \cdots, a_n)$ *then* $X \supset S$.

Proof. (a) Theorem 14 Corollary.

(b) We have to show $a_i \approx a_1 \cdots a_n/a_1 \cdots a_n$, $(1 \leqq i \leqq n)$. This is valid since it is equivalent by Theorem 5 to $a_i(a_1 \cdots a_n) \approx a_1 \cdots a_n$, which is equivalent by J6 to the truism $a_1 \cdots a_n \approx a_1 \cdots a_n$.

(c) Let X be linear, $X \supset (a_1, \cdots, a_n)$. By repeated application of Theorem 9, Corollary 2, $X \supset a_1 \cdots a_n$ and by Theorem 12, Corollary, $X \supset a_1 \cdots a_n /a_1 \cdots a_n$.

Uniqueness follows as in Theorem 11 and the proof is complete.

Thus we have linearized any *finite* set. Naturally we adopt the

DEFINITION. *Let set A be given. Let S be the only set which satisfies the following properties*:

(a) *S is linear*;

(b) $S \supset A$;

(c) *If X is linear and* $X \supset A$ *then* $X \supset S$.

Then we call S the **least linear set** *containing A, the linear set* **generated or determined by** *A or the* **linear closure** *of A, and denote it by* $\{A\}$. *We write* $\{(a_1, \cdots, a_n)\}$ *as* $\{a_1, \cdots, a_n\}$.

Note $\{A\} = A$ if A is linear, in particular $\{a\} = a$, $\{\varnothing\} = \varnothing$.

COROLLARY. $\{a_1, \cdots, a_n\} = a_1 \cdots a_n/a_1 \cdots a_n$.

Now we can easily answer the question: What should a line be? Clearly line ab should contain the distinct elements a and b, should be linear, and should be the least or simplest set with these properties. Thus we adopt the

DEFINITION. *If* $a \neq b$, *line ab is* $\{a, b\}$.

By the corollary above we have the formula

$$\text{line } ab = ab/ab.$$

Observe that by definition $\{a, b\}$ contains a, b and so ab, a/b, b/a. This gives

$$\{a, b\} \supset ab \cup a/b \cup b/a \cup a \cup b, \tag{9.1}$$

a simple but important relation which connects, when $a \neq b$, the conventional definition of line (8.1), and the one we have adopted.*

Does $\{A\}$ exist for an arbitrary set A, not necessarily finite? To answer this suppose X linear and $X \supset A$. Then X is convex, and $X \supset [A]$ by definition of $[A]$. Further $X \supset [A]/[A]$ by Theorem 12, Corollary. $[A]/[A]$ is linear by Theorem 14. By (J6) and the definition of $[A]$, $[A]/[A] \supset [A] \supset A$. Thus $[A]/[A]$ satisfies (a), (b), (c) of the definition of $\{A\}$. Uniqueness follows easily (as in the proof of Theorem 11) and we infer

THEOREM 17. $\{A\} = [A]/[A]$.

That is, to linearize A we merely convexify it and take the "complete extension" of the result.

The notion of linear closure can be extended to several sets. The importance of this is indicated by the use in school geometry of phrases such as "the plane determined by a line and a point," or "the plane determined by two lines." In higher-dimensional Euclidean geometry we should like to refer, for example, to the linear space determined by two planes which intersect in a single point—it would of course be a 4-space. Thus for two sets A, B we define $\{A, B\}$, the linear set *generated* or *determined by* A and B or the *linear closure* of A and B, to be the unique set S which satisfies

(a) S is linear;

(b) $S \supset A, B$;

(c) *If X is linear and $X \supset A, B$ then $X \supset S$.*

The particular case where A, B are linear and $A \approx B$ is quite important. We recall (Th. 15) that in this case A/B is linear. Further we can show $A/B \supset A, B$.† Clearly if linear set $X \supset A, B$ then $X \supset A/B$. Uniqueness follows as usual and we infer

THEOREM 18. *If A, B are linear and $A \approx B$ then $\{A, B\} = A/B$.*‡

For arbitrary A, B we can obtain a formula for $\{A, B\}$ (see, for example, [10], p. 351, Th. 4) by suitably generalizing the argument of Theorem 17. However the *existence* of $\{A, B\}$ is easily shown by a kind of argument which is familiar in modern algebra. Let S be the intersection of all linear sets X such that $X \supset A, B$. Then S is linear by the argument of Theorem 13. $S \supset A, B$; and if linear $X \supset A, B$, clearly $X \supset S$. We infer $S = \{A, B\}$. The same kind of argu-

* The conventional definition of line is not equivalent to ours, see Appendix, Section E for an example of a join system in which (9.1) is not an equality.

† For $A/B \supset B$ see proof of [10], p. 352, Theorem 5. For $A/B \supset A$ it suffices to show $A/a \supset A$ if $a \subset A$, which follows from the convexity of A.

‡ See [10], p. 352, Theorem 5.

ment could have been used to prove the existence of $[A]$ and $\{A\}$ (see, for example, [7], p. 30, Sec. 11).

10. Half-spaces and congruence relations. We consider now the theory of half-spaces: rays or half-lines, half-planes, half-3-spaces, etc. These form a subclass of the convex sets which is next in importance to the linear sets. As the name suggests, half-spaces are often defined in terms of separation of linear spaces. Thus in Euclidean geometry a half-plane is either of the two sets into which a line separates a plane, (see [4], p. 68, Def. 2.1; or [2], p. 163). This characterization is not appropriate here, since our postulates are too weak to imply such separation—even in Euclidean geometry it would force us to prove a rather complicated separation theorem ([4], p. 68) before discussing the relatively simple notion of half-space. However, an analysis of the Euclidean situation yields a simple "congruence" or equivalence relation, which has immediate geometric content and leads directly to the notion of half-space.

First we consider a specific Euclidean example. Let M be a line in a Euclidean 3-space and a a point not in M (Fig. 12). We know that there is a unique half-plane with edge M containing a, but instead of trying to construct this we ask:

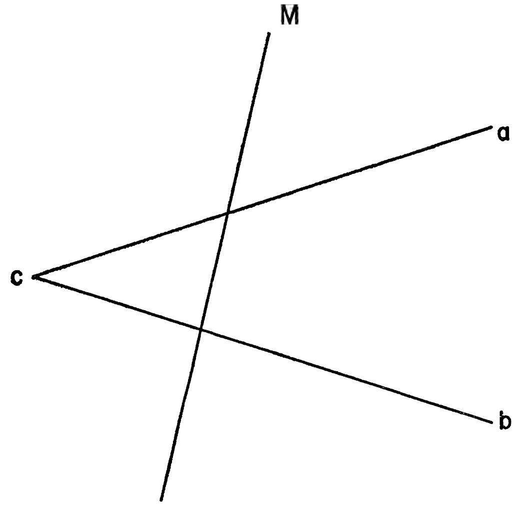


FIG. 12

When are points a and b in the same half-plane with edge M, or simply when are a and b on the same side of M? More precisely stated, we seek a criterion, in join terminology, for the relation of a and b to M if a and b are on the same side of M.

Suppose then a, b are on the same side of M. Instead of trying to relate a and b directly, we use the principle that if a and b are on the same side of M they are both on the opposite side of M from some point c. Thus there exists a

point c (see Fig. 12) such that the joins of c to a and b meet M; that is, we have

(10.1) $$ac \approx M, \qquad bc \approx M.$$

Solving the relations in (10.1) for c we get

(10.2) $$c \approx M/a, \qquad c \approx M/b,$$

so that $M/a \approx M/b$. Applying the transposition law for sets (Th. 6) we get

(10.3) $$Ma \approx Mb.$$

Conversely suppose (10.3) and a, $b \not\subset M$. Then for some point d, $d \approx Ma$, $d \approx Mb$ (see Fig. 13), so that $d \approx m_1 a$, $d \approx m_2 b$, where m_1, $m_2 \subset M$. We infer that a and b are on the same side of M, since each is on the same side of M as d. To summarize: Suppose a, $b \not\subset M$. Then a, b are on the same side of M if and only if they satisfy (10.3).

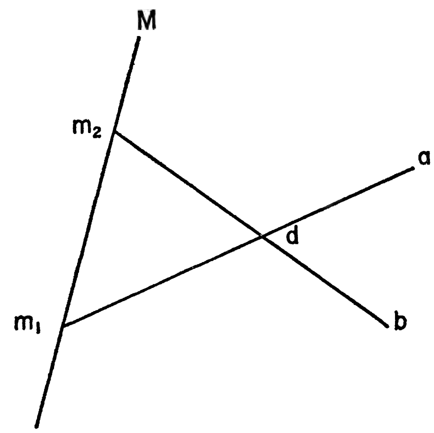


FIG. 13

This suggests the importance, in our abstract join theory, of studying (10.3) as a relation between a and b when M is a given linear set. Since the relation depends on the choice of M, and a, b are equivalent in a certain sense relative to M, I adopt the algebraic terminology of "congruence with respect to a modulus" to describe it.

DEFINITION. *Let M be a nonvoid linear set. Then* $a \equiv b(\text{mod } M)$, *read a* ***is congruent to*** *b* ***modulo*** *M, means $aM \approx bM$.*

Although the relation $a \equiv b(\text{mod } M)$ is suggested by the Euclidean relation, a and b are on the same side of linear space M, it is not exactly equivalent to this, since it also covers the "degenerate" case where a and b are in M. Precisely, $a \equiv b(\text{mod } M)$ holds in a Euclidean geometry if and only if (a) a and b are on the same side of M or (b) a and b are in M. We show this for the case M is a

line. We saw above that (a) is equivalent to $a \equiv b(\text{mod } M)$ provided $a,b \not\subset M$. Suppose then one of a, b, say a, is in M. Let $a \equiv b(\text{mod } M)$. Then $aM \approx bM$ which implies $b \approx aM/M \subset M$, and both a and b are in M. Conversely suppose a, $b \subset M$. Then aM, $bM \supset ab$ so that $aM \approx bM$ and $a \equiv b(\text{mod } M)$. Thus our statement is justified in the given case.

We proceed to study the properties of congruence mod M in an abstract join system. It is not surprising that congruence mod M is an equivalence relation:

THEOREM 19. (a) $a \equiv a(\text{mod } M)$; (b) $a \equiv b(\text{mod } M)$ implies $b \equiv a(\text{mod } M)$; (c) $a \equiv b(\text{mod } M)$, $b \equiv c(\text{mod } M)$ imply $a \equiv c(\text{mod } M)$.

Proof. (a), (b) are immediate from the definition.

To prove (c): We are given $aM \approx bM$, $bM \approx cM$. Solving for b, $b \approx aM/M$, $b \approx cM/M$; eliminating b, $aM/M \approx cM/M$. Applying the transposition law for sets (Th. 6), $aMM \approx cMM$. Since M is linear it is convex. Thus $MM = M$ by Theorem 9(b) so that $aM \approx cM$ and $a \equiv c(\text{mod } M)$.

We know in the arithmetic of integers that the congruences $a \equiv b(\text{mod } m)$, $c \equiv d(\text{mod } m)$ can be added to yield $a + c \equiv b + d(\text{mod } m)$. Can geometric congruences be combined similarly under the operation join or multiplication? This would require us to find an appropriate extension of the notion congruence from elements to *sets*. The answer is implicit in the following

LEMMA. *Suppose $a \equiv b(\text{mod } M)$ and c is any element. Then each element of ca is congruent modulo M to some element of cb.*

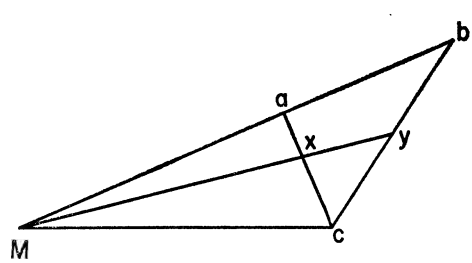


FIG. 14

Proof. Let $x \subset ca$. By hypothesis $aM \approx bM$. (See Fig. 14 which illustrates the Euclidean case where M is a point.) Solving the last relation for a we get $a \subset bM/M$. Multiplying both sides by c and applying Theorem 8, Corollary (b),

$$ca \subset c(bM/M) \subset cbM/M.$$

Thus $x \approx cbM/M$ so that $xM \approx cbM$, $xM/M \approx cb$. Consequently $xM/M \approx y$ and $y \approx cb$ for some y. Transposing M in the former relation yields $xM \approx yM$ so that $x \equiv y(\text{mod } M)$ and the Lemma holds.

To answer our question on multiplication of congruences, we consider first

the special case: Does $a \equiv b(\text{mod } M)$ imply, in a suitable sense, $ca \equiv cb(\text{mod } M)$? Suppose $a \equiv b(\text{mod } M)$. Then by the lemma each element of ca is congruent mod M to some element of cb. By symmetry each element of cb is congruent mod M to some element of ca. This rather close relation of the sets ca and cb we symbolize by: $ca \equiv cb(\text{mod } M)$. In general we adopt the

DEFINITION. *Let M be a nonvoid linear set. Then $A \equiv B(\text{mod } M)$ means that for each $x \subset A$ there is a $y \subset B$ such that $x \equiv y(\text{mod } M)$ and vice versa.*

Observe that Theorem 19 is valid for congruence of sets. That is, the relation congruence mod M is an equivalence when applied to sets just as well as to elements—this tends to confirm the appropriateness of our definition.

Our discussion is summarized in

THEOREM 20. *$a \equiv b(\text{mod } M)$ implies $ca \equiv cb(\text{mod } M)$.*

This easily yields the basic result on combining congruences:

COROLLARY. *$a \equiv b(\text{mod } M)$, $c \equiv d(\text{mod } M)$ imply $ac \equiv bd(\text{mod } M)$.*

Proof. Multiplying the given congruences by c, b respectively we have

$$ac \equiv bc(\text{mod } M), \qquad bc \equiv bd(\text{mod } M),$$

and the result follows since congruence mod M is an equivalence for sets.

Every arithmetic congruence relation has an *additive identity element*, since $a + m \equiv a(\text{mod } m)$. We have an analogue:

THEOREM 21. *$am \equiv a(\text{mod } M)$ provided linear set $M \supset m$.*

Proof. Suppose $x \subset am$. Then

$$xM \subset amM = a(mM) \subset aM.$$

Certainly $xM \approx aM$, so that $x \equiv a(\text{mod } M)$. The last definition yields $am \equiv a(\text{mod } M)$.

COROLLARY. *Let M be linear and $m \subset M$. Then $x \equiv m(\text{mod } M)$ if and only if $x \subset M$.*

Proof. $x \equiv m(\text{mod } M)$ implies $xM \approx mM$ and $x \approx mM/M \subset M$. Conversely suppose $x \subset M$. The theorem implies

$$xm \equiv x(\text{mod } M), \qquad xm \equiv m(\text{mod } M)$$

so that $x \equiv m(\text{mod } M)$.

Since congruence modulo M is an equivalence relation (Th. 19), we naturally introduce its associated *equivalence classes* ([1], pp. 155, 156).

DEFINITION. *The set of x which satisfies $x \equiv a(\text{mod } M)$ is called the* ***congruence set*** *mod M* ***determined by*** *a and is denoted by $(a)_M$. Briefly we call $(a)_M$ the* ***coset of M determined by*** *a.*

We employ the term coset not merely as an abbreviation of congruence set, but to signalize the striking analogies of this concept to the notion coset in classical group theory ([1] p. 142). For geometrical suggestiveness and for application to Euclidean (or any linearly ordered) geometry we call $(a)_M$ the *half-space with edge M determined by a* provided $a \not\subset M$.*

We continue with properties of cosets. Directly from Theorem 19 we have ([1], p. 156, Lemma 2):

THEOREM 22. *The cosets of linear set M form a partition of G.*

It is important to have a formula for a coset:

THEOREM 23. $(a)_M = aM/M$.

Proof. $x \subset (a)_M$ is equivalent to $x \equiv a (\text{mod } M)$. By definition this is equivalent to $xM \approx aM$ and so to $x \approx aM/M$.

COROLLARY. *$(a)_M$ is convex.*

Proof. By Theorem 10, aM/M is convex since a, M are.

Restating Theorem 21, Corollary, in coset language we have:

THEOREM 24. *Let M be linear and $m \subset M$. Then $(m)_M = M$.*

Thus the "modulus" M is a coset.

Just as a Euclidean half-space (say a half-line or half-plane) has an opposite half-space, so every coset has an "opposite" coset:

DEFINITION. *Cosets A, B of M are* **opposite** *(or A is* **opposite** *to B) if there exist $a \subset A$, $b \subset B$ such that $ab \approx M$.*

The defining property extends to each pair of elements chosen from the respective cosets:

THEOREM 25. *Let A, B be opposite cosets of M. Then $x \subset A$, $y \subset B$ imply $xy \approx M$.*

Proof. By definition $A \supset a$, $B \supset b$ and $ab \approx M$. Then A and $(a)_M$ have a in common and $A = (a)_M$ by Theorem 22. Similarly $B = (b)_M$. Consequently $x \subset (a)_M$, $y \subset (b)_M$ so that

$$x \equiv a (\text{mod } M), \qquad y \equiv b (\text{mod } M).$$

Combining these congruences (Th. 20, Corol.), $xy \equiv ab (\text{mod } M)$. Let $m \subset ab$, M. By definition of congruent sets, $m \subset ab$ implies the existence of $m' \subset xy$ such that $m' \equiv m (\text{mod } M)$. By Theorem 21, Corollary, $m' \subset M$. Thus $xy \approx M$ and the proof is complete.

COROLLARY 1. *Let A, B be opposite cosets of M. Then $A \cap B = \emptyset$ or $A = B = M$.*

* This is to some extent a misnomer, since it suggests a separation theorem which does not hold in all join systems, see Appendix Section G.

Proof. Suppose $A \cap B \neq \emptyset$. Then $A = B$ (Th. 22). Let $p \subset A$. Then $p \subset B$ and the theorem implies $p = pp \approx M$. Thus $A = B = M$ by Theorem 22.

COROLLARY 2. *M is the only coset of M which is its own opposite.*

Proof. M is opposite to M by definition, since if $m \subset M$ then $mm = m \subset M$. Uniqueness follows from Corollary 1.

THEOREM 26. *A coset $(a)_M$ of M has a unique opposite coset of M, namely M/a.*

Proof. Choose a' such that $a'a \approx M$. Then $(a')_M$ is opposite to $(a)_M$ by definition.

To prove uniqueness, we suppose $(b)_M$ opposite to $(a)_M$ and show $(b)_M = M/a$. Suppose $x \subset (b)_M$. By Theorem 25, $xa \approx M$, so that $x \approx M/a$. Thus $(b)_M \subset M/a$. Conversely suppose $x \approx M/a$. Then $xa \approx M$. By Theorem 25, $ab \approx M$. We eliminate a between the last two relations and solve for x:

$$a \approx M/x, \qquad a \approx M/b,$$
$$M/b \approx M/x,$$
$$Mx \approx Mb \qquad \text{(Transposition Law)},$$
$$x \approx Mb/M = (b)_M \qquad \text{(Th. 23)}.$$

Thus $(b)_M = M/a$ and the theorem is proved.

COROLLARY. *Let M be a nonvoid linear set. Then any coset A of M is expressible in the form M/b; conversely M/b always is a coset of M.*

Proof. Let $(b)_M$ be opposite to A. Then A is opposite to $(b)_M$ and is M/b by the theorem. Conversely let M/b be given. By the theorem, M/b is a coset, namely that opposite to $(b)_M$.

This with Theorem 23 gives us two formulas for a coset, namely $(a)_M = aM/M = M/b$. The first, although more complex, expresses $(a)_M$ directly in terms of its element a, the second is simpler but involves the choice of an element b in the coset opposite to $(a)_M$.

As a simple illustration of our theory of congruence modulo a linear set, let G be a Euclidean 3-space and M a fixed point of G. From our observation above in the paragraph following the definition of congruence of elements, $x \equiv a \pmod{M}$ is equivalent to: x is in M (that is, in this case $x = M$) or x and a are on the same side of M. The latter holds if and only if x is in the half-line or ray $\overrightarrow{Ma}$ emanating from M which contains a. Thus the cosets $(a)_M$ are the rays of G which emanate from M, together with point M itself as a sort of degenerate or null ray.* Consequently the decomposition of G determined by the relation congruence mod M corresponds to the important geometrical fact that there is in G a unique ray from M containing each point of G distinct from M. The geometrical significance of congruence of *sets* is now easily given. For simplicity suppose $A, B \not\supset M$.

* Compare the notion of null vector in the geometrical theory of vectors.

Then $A \equiv B (\text{mod } M)$ holds, if and only if each ray from M which passes through a point of A also passes through a point of B, and vice versa (Fig. 15). That is, $A \equiv B (\text{mod } M)$ if and only if the cones with vertex M and "bases" A, B are identical.

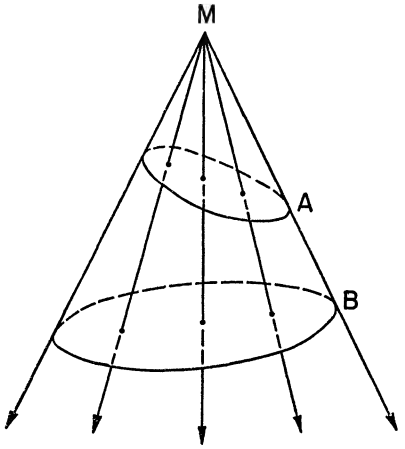


FIG. 15

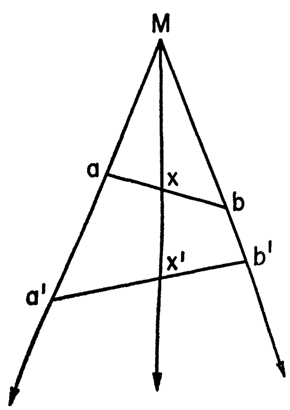


FIG. 16

Furthermore consider the principle: $a \equiv a' (\text{mod } M)$, $b \equiv b' (\text{mod } M)$ imply $ab \equiv a'b' (\text{mod } M)$. As a typical (that is nondegenerate) case suppose a, b, M are noncollinear. Then the rays $\overrightarrow{Ma}$, $\overrightarrow{Mb}$ are distinct and form an angle, namely $\measuredangle aMb$. Moreover a', b' are in the respective sides $\overrightarrow{Ma}$, $\overrightarrow{Mb}$ of the angle. What is the significance of $ab \equiv a'b' (\text{mod } M)$? By our observation above on congruence of sets it asserts that each point x of segment ab (Fig. 16) lies in the same ray from M as some point x' of segment $a'b'$ and vice versa. It implies that any ray from M, the vertex of the angle, which meets one segment joining the sides of the

angle, meets all such segments. This is a very subtle and important property of angles which is intimately related to the notion of angle interior ([4], p. 70, Th. 7). Finally note that opposite cosets will appear as opposite rays from M. You may find it interesting to interpret our other results geometrically, in particular Theorems 21, 23, 25, 26.

Corresponding results hold when M is a linear set in a Euclidean space G of arbitrary dimension. For example, if M is a line and G a 3-space, the cosets are the half-planes of G with edge M together with M as a "degenerate" half-plane, and the theory is applicable to dihedral angles with edge M. Similarly if M is a plane in 4-space G, the nondegenerate cosets are half-3-spaces bounded by M and we have an application to hyperangles formed by two such cosets.

11. Join systems compared with abelian groups. We digress at this point to compare our theory of join systems with abelian group theory. This section is intended to give a deeper insight into the theory of join systems but is actually dispensable for its further development.

Let $(G', \cdot)$ be an abelian group multiplicatively expressed. Then the operations join and extension in a join system $(G, \cdot)$ correspond to multiplication and division in group $(G', \cdot)$, and we can translate the concepts and theorems of our theory into corresponding concepts and statements of abelian group theory. Of course this translation process need not yield important concepts or valid theorems—however I think you will be surprised, at first view, by the frequency with which this does happen.

First, corresponding to the notion of a convex set in G we have the idea of a set closed under multiplication in G'. Linear sets which are nonvoid correspond to nonvoid sets closed under multiplication *and* division—that is to *subgroups* of group G'. The notion of linear closure of set S or linear set generated by S corresponds to the important algebraic idea of subgroup generated by set S ([7], p. 30).

Continuing, we translate into group theory the notion of congruence. This yields $x \equiv y(\text{mod } M)$, where M is a subgroup of G', in the sense that xM and yM have a common element. Thus

$$xm_1 = ym_2, \qquad x = ym_2m_1^{-1},$$

where $m_1, m_2 \in M$. Hence $x = ym$, $m \in M$. This is the familiar definition of $x \equiv y(\text{mod } M)$ in group theory ([7], p. 37). Further, our formula for the coset of M determined by a, is in group G'

$$aM/M = a(M/M) = aM,$$

which is the familiar form of a coset in abelian group theory. Thus cosets in the theory of join systems have a more than nominal relation to cosets in abelian groups.

It is interesting that opposite cosets correspond in G' to cosets determined by mutually inverse elements, a and a^{-1}. To show this let A, B be "opposite"

cosets in G'—that is there exist a, b such that $a \in A$, $b \in B$ and $ab \approx M$. We have $ab = m \in M$ and $b = a^{-1}m$. Then $A = aM$ and

$$B = bM = a^{-1}mM = a^{-1}M,$$

justifying our statement. Further note that B, the coset "opposite" to A, is expressible as M/a which is precisely our formula (Th. 26) for the opposite of $(a)_M$.

Many of our theorems hold literally when translated for abelian groups, and practically all have analogues in abelian group theory. In particular the results of the last section go over with the exception of the corollaries to Theorems 23 and 25. The former does not carry over since in a group the only multiplicatively closed coset of M is M itself; counterexamples to the latter are easily found.

We exhibit the basic analogies between the two systems in a table:

Join System $(G, \cdot)$	Abelian Group $(G', \cdot)$
convex set	multiplicatively closed set
linear set	subgroup
linear closure of S	subgroup generated by S
$x \equiv y (\text{mod } M)$ is $xM \approx yM$	$x \equiv y (\text{mod } M)$ is $x \in yM$
coset (half-space)	coset
coset $(a)_M = aM/M$	coset $(a)_M = aM$
M is a coset ($M = (m)_M$ for $m \subset M$)	M is a coset ($M = (i)_M$, where i is the identity of G)
opposite of $(a)_M$ is M/a	"inverse" of $(a)_M$ is $a^{-1}M$

Is it an accident that the two theories are so similar? The answer—as you probably realize by now—is no. Turning back to our postulates [(J1), $\cdots$, (J6) of Sec. 3] we observe that they involve no assumption concerning the number of elements in the join $a \cdot b$, except for the requirement in (J1) that it be nonvoid. Thus $a \cdot b$ may consist of a single element. Suppose then that $a \cdot b$ does consist of a single element for each pair a, b, that is $a \cdot b = (c)$. In accordance with our convention for identifying (c) and c we may write $a \cdot b = c$ and the operation $\cdot$ becomes a single-valued binary operation of the familiar type. In this case (J1), $\cdots$, (J4) reduce to the familiar postulates for an abelian group.

Now let $(G', \cdot)$ be any abelian group. Then $(G', \cdot)$ satisfies (J1), $\cdots$, (J4). Further, a/b must consist of a single element, so that (J5) reduces to the familiar proportion property, $a/b = c/d$ *implies* $ad = bc$, and is valid. (J6) will not in general be valid, since only the identity of G' will satisfy $aa = a = a/a$. In any case we have an unforseen but significant relation between classical geometry and modern algebra: Each theorem in the theory of join systems, whose proof is independent of (J6), is valid for abelian groups.* Our observation above (preceding the table), that the corollaries to Theorems 23 and 25 did not have valid analogues in abelian group theory, now falls into place: The proofs of these corollaries actually do depend on (J6).

* A chauvinistic geometer might be tempted to say that certain portions of modern algebra are merely special cases of a suitably formulated geometric theory.

Join systems are examples of a type of generalized group with many-valued composition called a multigroup. More precisely a multigroup is a system closed under an associative many-valued operation $\cdot$ (that is, it satisfies (J1) and (J3)), which contains elements x, y satisfying the relations, $a\cdot x \supset b$, $y\cdot a \supset b$ for a, b in the system (see [**3**], pp. 706, 707). Postulates (J1), $\cdots$, (J4) characterize commutative multigroups, which in view of our discussion include commutative or abelian groups.

Theories of multigroups, algebraically motivated, have been given by Ore and his students, among other mathematicians.* These do not seem to have much relevance to geometry. This paper is an outgrowth of my studies of types of multigroups suggested by classical geometries ([**9**], [**10**], [**12**]).

We have now completed the basic presentation of our approach to classical geometry. Join system, our central idea, has its roots in Euclid's operations of joining points and extending segments, and seems to provide an appropriate frame for the geometric study of the important contemporary ideas of convex sets, linear sets and half-spaces.

* For a list of references on multigroups see R. H. Bruck, A Survey of Binary Systems, Berlin, 1958.

II. FURTHER DEVELOPMENT OF THE THEORY OF JOIN SYSTEMS

In this part we probe more deeply into the theory of join systems and impose additional restrictions on them. The material is subtler and more difficult than that in Part I and some proofs are omitted.

12. The cosets form a geometry. Following a theme familiar in modern mathematics, especially so in algebra, we form a new mathematical system of the cosets $(a)_M$ by constructing a join operation applicable to these cosets. The general reference for this material is [**10**], Sec. 7.

Let M be a nonvoid linear set in G and let $G:M$ denote the set of cosets $(a)_M$ for $a \subset G$. ($G:M$ may be read G reduced modulo M, or G over M.) We define a join operation in $G:M$ rather naturally by taking $(a)_M \cdot (b)_M$ to be the set of cosets $(x)_M$ for which $x \subset ab$. That is, we form the join ab of representative elements a, b of the given cosets, and then take the set of all cosets of M determined by the elements of ab. It is not hard to see that the result is independent of the choice of the representative elements a, b. For if $(a)_M = (a')_M$ and $(b)_M = (b')_M$ then $a \equiv a'$ (mod M) and $b \equiv b'$ (mod M) so that $ab \equiv a'b'$ (mod M); thus the elements of ab determine the same set of cosets as those of $a'b'$, and $(a)_M \cdot (b)_M = (a')_M \cdot (b')_M$. This has important geometric content which is easily elicited (see the illustration of the theory of congruence at the end of Sec. 10 including Fig. 16).

$G:M$ with join or product as defined is called the *factor system* of G with respect to M * and bears many analogies to its parent system G. In studying $G:M$ we employ $\mathbf{a}, \mathbf{b}, \mathbf{c}, \cdots$ to represent its elements (that is, cosets of M) and adopt the same set-theoretic conventions concerning the use of $\subset$, $\supset$, $\approx$, etc., as in G. We define the inverse operation $\mathbf{a}/\mathbf{b}$ and extend $\cdot$ and / to sets just as in G. In view of this, certain of the results of Sections 4, 5 on join systems, which depend only on the definitions, automatically hold for $G:M$. These include Theorems 1 and 4 (the monotonic laws for join and extension), Theorem 5, and its corollary, *$a \approx bc$ if and only if $a/b \approx c$.*

The basic similarities of $G:M$ and G are given in

THEOREM 27. *$G:M$ satisfies* (J1), $\cdots$, (J5) *and the first idempotent law of* (J6).

Proof. (J1), (J2) are easily seen to hold in $G:M$.

To establish (J3), the associative law, suppose $\mathbf{x} \subset \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$. We reduce this to a corresponding relation in G in order to apply associativity in G. By definition of product of sets

* See [1], p. 154 for the analogous concept in group theory. We adopt the symbolism $G:M$ rather than G/M since the latter has already been employed to denote quotient of sets.

$$(12.1) \qquad \mathbf{x} \subset \mathbf{a} \cdot \mathbf{y}, \qquad \mathbf{y} \subset \mathbf{b} \cdot \mathbf{c}$$

holds for some $\mathbf{y}$. Let us examine the first relation in (12.1). Let $\mathbf{a} = (a)_M$, $\mathbf{y} = (y)_M$. Then by definition of $\cdot$, $\mathbf{a} \cdot \mathbf{y}$ or $(a)_M \cdot (y)_M$ is the set of cosets $(x)_M$ for which $x \subset ay$. Thus $\mathbf{x}$ is one of these cosets—that is, $\mathbf{x} = (x)_M$ for some $x \subset ay$. Similarly letting $\mathbf{b} = (b)_M$, $\mathbf{c} = (c)_M$, the second relation in (12.1) yields $\mathbf{y} = (y)_M$ for some $y \subset bc$.

We operate now in G. The relations $x \subset ay$, $y \subset bc$ imply $x \subset a(bc) = (ab)c$. This yields

$$(12.2) \qquad x \subset zc, \qquad z \subset ab$$

for some z.

What does (12.2) signify in $G:M$? It says, in view of the definition of $\cdot$,

$$(x)_M \subset (z)_M \cdot (c)_M, \qquad (z)_M \subset (a)_M \cdot (b)_M.$$

Letting $\mathbf{z} = (z)_M$, we have $\mathbf{x} \subset \mathbf{z} \cdot \mathbf{c}$, $\mathbf{z} \subset \mathbf{a} \cdot \mathbf{b}$, so that $\mathbf{x} \subset (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$. Thus

$$\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) \subset (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}.$$

The reverse inclusion may be proved similarly to complete the verification of (J3).

To verify (J4) show that if in G, $x \subset a/b$, then in $G:M$, $(x)_M \subset (a)_M/(b)_M$.

To verify (J5) suppose $\mathbf{a}/\mathbf{b} \approx \mathbf{c}/\mathbf{d}$. Then for some $\mathbf{x}$, $\mathbf{x} \approx \mathbf{a}/\mathbf{b}$, $\mathbf{x} \approx \mathbf{c}/\mathbf{d}$, so that $\mathbf{b} \cdot \mathbf{x} \approx \mathbf{a}$, $\mathbf{d} \cdot \mathbf{x} \approx \mathbf{c}$. Let $\mathbf{x} = (x)_M$, $\mathbf{b} = (b)_M$, $\mathbf{d} = (d)_M$. Then

$$(b)_M \cdot (x)_M \supset \mathbf{a}, \qquad (d)_M \cdot (x)_M \supset \mathbf{c}$$

so that $\mathbf{a} = (a)_M$, where $bx \supset a$, and $\mathbf{c} = (c)_M$, where $dx \supset c$. Thus $a/b \approx x \approx c/d$ and $ad \approx bc$. This yields $\mathbf{a} \cdot \mathbf{d} \approx \mathbf{b} \cdot \mathbf{c}$ and (J5) holds in $G:M$.

The idempotent law $aa = a$ in G, implies in $G:M$, $(a)_M \cdot (a)_M = (a)_M$ and our proof is complete.

We consider now the differences between $G:M$ and G. The second idempotent law of (J6) does not in general hold in $G:M$. This is a consequence of the existence of an identity element in $G:M$.

DEFINITION. *If* $\mathbf{a} \cdot \mathbf{i} = \mathbf{a}$ *for each* $\mathbf{a}$ *in* $G:M$ *we call* $\mathbf{i}$ *an identity element of* $G:M$.

THEOREM 28. *$G:M$ has a unique identity element, namely M.*

Proof. By Theorem 21 we have for arbitrary a in G, $am \equiv a (\mathrm{mod}\ M)$, where m is chosen to satisfy $m \subset M$. This is seen to yield in $G:M$

$$(a)_M \cdot (m)_M = (a)_M$$

and $(m)_M$ is an identity in $G:M$. By Theorem 24, $(m)_M = M$. Uniqueness is easily proved.

COROLLARY. *$G:M$ satisfies the idempotent law* $\mathbf{a}/\mathbf{a} = \mathbf{a}$ *only in the trivial case where it consists of the single element M.*

Proof. $\mathbf{a}\cdot M=\mathbf{a}$ implies $\mathbf{a}/\mathbf{a}\supset M$.

Since $G:M$ has an identity we naturally wonder whether elements have inverses. We adopt the

DEFINITION. *In $G:M$ if $\mathbf{a}\cdot\mathbf{x}\supset M$ we say $\mathbf{x}$ is an* ***inverse*** *of $\mathbf{a}$, or $\mathbf{a}$ and $\mathbf{x}$ are* ***inverse elements.***

This definition is reasonable in the present context, since the multiplication is many-valued—the requirement $\mathbf{a}\cdot\mathbf{x}=M$, suggested by conventional algebra, would be unduly restrictive.

In order to analyze the notion, suppose $(a')_M$ is an inverse of $(a)_M$. We have

$$(a)_M\cdot(a')_M\supset M. \tag{12.3}$$

Thus $M=(m)_M$ for some $m\subset aa'$. Clearly $m\subset M$ so that $aa'\approx M$. Consequently $(a)_M$ and $(a')_M$ are *opposite cosets* in the sense of Section 10 (Definition following Th. 24). Conversely if $(a)_M$ and $(a')_M$ are opposite then $aa'\approx M$ which implies (12.3). Thus the geometrically motivated notion of opposite cosets is equivalent to the algebraic notion of inverse elements in $G:M$.

This yields immediate information about $G:M$. Restating Theorem 26 on opposite cosets we have

THEOREM 29. *Each element of $G:M$ has a unique inverse.*

A deeper result ([**10**], p. 362, Th. 6) which we state without proof is

THEOREM 30. *In $G:M$, $\mathbf{a}/\mathbf{b}=\mathbf{a}\cdot\mathbf{b}'$, where $\mathbf{b}'$ is the inverse of $\mathbf{b}$.*

It seems remarkable that in $G:M$ "division" is reducible to multiplication as in school algebra. Thus in several ways $G:M$ is a simpler and algebraically more regular system than G.*

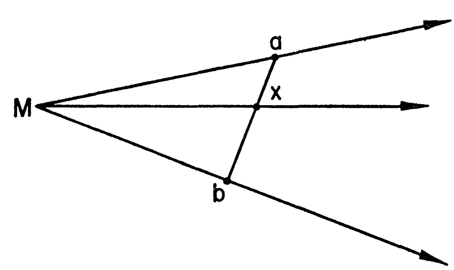


FIG. 17

To illustrate the geometric content of our discussion consider the example at the end of Section 10 in which G is a Euclidean 3-space and M a point. Then $G:M$, the family of cosets of M, consists of the rays of G from M, together with the "null ray" M. The join of the rays $(a)_M$, $(b)_M$ in $G:M$ (Fig. 17) consists of all

* See [**12**] for a study of multigroups of the type of $G:M$ and a relation to spherical geometries.

rays $(x)_M$ for which $x \subset ab$—that is, it is the set of rays from M which intersect the join ab. So in the typical case where $(a)_M$ and $(b)_M$ form a nonstraight angle (that is, a, b, M are noncollinear), their join $(a)_M \cdot (b)_M$ is actually the set of rays "between" $(a)_M$ and $(b)_M$ or equivalently the set of rays from M which are inside $\measuredangle aMb$.* The identity property asserts that if M is joined to point a, all points of their join determine the same ray (from M) as a. The inverse of any element of $G\colon M$, other than M, is merely its opposite ray. In fact if $(a)_M$, $(a')_M$ are opposite rays, $(a)_M \cdot (a')_M = ((a)_M, (a')_M, M)$. To summarize: In the given instance, $G\colon M$ is essentially the natural or intrinsic geometry of the family of rays from M, in which the notion of join (or of betweenness of rays) is that naturally induced by the join concept in G.

We can picture the given system $G\colon M$ more graphically by passing a sphere S centered at M, and projecting each ray from M into the point where it pierces S (Fig. 18). Then if a, b are points of S in general position (that is are distinct and not opposite to each other), the rays of the join $(a)_M \cdot (b)_M$ will pierce S in

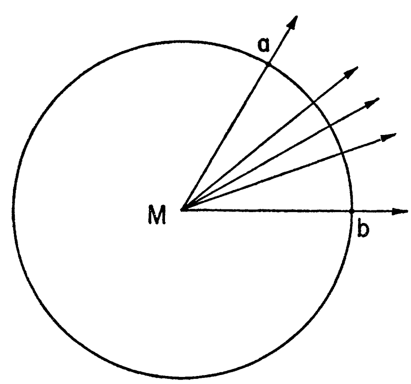


FIG. 18

the points of $\overset{\frown}{ab}$, the (open) minor arc of a great circle joining a and b. Thus our geometry $G\colon M$ is representable essentially as the natural geometry of a sphere in which the join of two points (in general position) is the minor arc of a great circle joining them.

To characterize more precisely this "spherical" representation of $G\colon M$, observe first that there is no point of S that represents the "null ray" M of $G\colon M$. To remedy this we adjoin point M, as a sort of ideal point, to the points of S forming a set S'. Then we define the join $a \cdot b$ for a, b in S' as follows:

(1) If a, b are points of S in general position, $a \cdot b$ is the minor arc of a great circle joining a and b; (2) if a, b are opposite points of S, $a \cdot b = (a, b, M)$; (3) $a \cdot a = a$; (4) $a \cdot M = M \cdot a = a$. This converts S' into what might be termed a "spherical join system" which is a representation of (or is isomorphic to) $G\colon M$.

* See [2], p. 163.

In this system S', point M is the identity element, and the inverse of any point of S is the opposite point of S. To illustrate the significance of the "quotient" a/b in system S', let a, b be points of S in general position. By definition a/b is the set of all elements x of S' such that the join bx contains a. All such x constitute the minor arc of a great circle $\widehat{ab'}$, where b' is the point of S opposite to b (Fig. 19). Thus $a/b=\widehat{ab'}$. Further we have $a/b=\widehat{ab'}=a\cdot b'$, giving a simple geometric interpretation for Theorem 30.

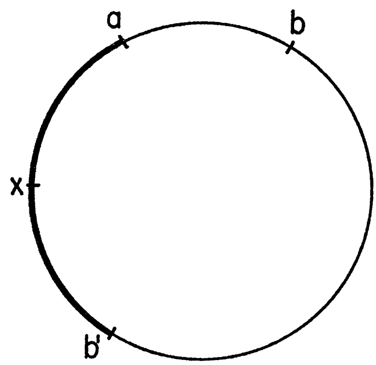


FIG. 19

These ideas are applicable if M is a nonvoid linear set in a Euclidean geometry G of arbitrary dimension—in this case $G:M$ is essentially the natural geometry of the family of half-spaces bounded by M, and also is representable as a "spherical system" ([**10**], p. 369, Theorem 1).*

We conclude this section with a simple property of factor systems which—as we shall see later—helps to illumine the concept of separation. As in group theory we define the *order* of $G:M$ to be its cardinal number. We call $G:M$ *trivial* if $G=M$, that is if it has order 1.

THEOREM 31. *A nontrivial factor system $G:M$ has order $\geqq 3$.*

Proof. Let $a\subset G$, $a\not\subset M$. Then $G:M$ contains the cosets M, $(a)_M$, and $(a')_M$ the opposite of $(a)_M$. Suppose two of these coincide. Then, since M is the only element of $G:M$ which is its own opposite (Theorem 25 Corollary 2), all three must coincide. Since $a\not\subset M$ this is impossible, and the theorem must hold.

13. Factor systems and homomorphisms. In this section we introduce the idea of homomorphism of a join system, relate it to the notion factor system, and indicate deeper analogies with group theory.

Let M be a given nonvoid linear set of join system G. Naturally we are interested in the mapping T, $x\rightarrow(x)_M$, which assigns to each element x of G the coset of M it determines. T maps G onto $G:M$. We have $a\rightarrow(a)_M$, $b\rightarrow(b)_M$. What can

* See [**12**] for a "join-theoretic" treatment of spherical geometries.

we say of the image of x, if $x \subset ab$? By definition of join for cosets (Sec. 12), we have $(x)_M \subset (a)_M \cdot (b)_M$. Conversely if $\mathbf{x} \subset (a)_M \cdot (b)_M$, we see that $\mathbf{x} = (x)_M$ for some $x \subset ab$. Thus T maps ab *onto* $(a)_M \cdot (b)_M$ or $T(ab) = T(a) \cdot T(b)$, and we say T maps joins onto joins or *preserves* joins. Note that T is not in general 1-1. Consider our perennial example: M is a point of Euclidean 3-space G. Then essentially T maps G in a "natural" manner onto the geometry of rays from M, or in effect T maps G onto an associated "spherical system."

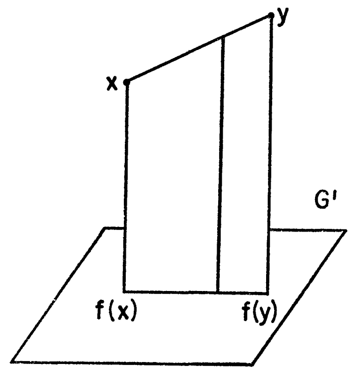


FIG. 20

These considerations call to mind the notion homomorphism of modern algebra.* For convenience we formulate the homomorphism concept for a general type of combinatory system not necessarily a join system. Consider a "combinatory system" $(S, *)$ consisting of a set S and a 2-term operation $*$ which associates to each pair a, b of elements of S a subset of S denoted $a * b$. Let $(S, *)$, $(S', *)$ be two such systems, where for convenience the same symbol is used to denote their operations. Suppose that there exists a mapping f of S on S' satisfying $f(x * y) = f(x) * f(y)$. Then we call f a *homomorphism* of $(S, *)$ on $(S', *)$ and say $(S, *)$ is *homomorphic* to $(S', *)$ or simply S is *homomorphic* to S'. If f is 1-1 we say S is *isomorphic* to S' and write $S \cong S'$.

We observe immediately that $x \rightarrow (x)_M$ is a homomorphism of join system G on $G:M$. You might expect, as in group theory ([**1**], p. 154, Th. 26; [**7**], Ch. I, Sec. 16), that every homomorphism of a join system G is associated with a factor system $G:M$. This is not so. The correct result is: Let f be a homomorphism of join system G on a combinatory system G', which has an identity i in the sense that $x * i = i * x = x$ for x in G'. Then G' is isomorphic to $G:M$, where M is the *kernel* of f, that is the set of elements mapped on i by f ([**10**], p. 366, Corol. 2). Thus the study of homomorphisms here is essentially more complex than in group theory, since there exist homomorphisms of a join system on a second join system which has no identity. Actually there is a very familiar example—an

* See, for example, [**1**], pp. 150–151; [**7**], pp. 41–43.

orthogonal projection f of Euclidean 3-space on plane G' (Fig. 20) clearly is a homomorphism of G on G' considered as join systems.

A sizable portion of the theory of groups involving factor groups and homomorphism is valid in our theory with suitable modification, including the Jordan-Holder theorem and various isomorphism theorems. An important example, rich in geometrical content, is

THEOREM 32 (ISOMORPHISM THEOREM). *Let A, B be linear sets in G; $A \cap B \neq \varnothing$. Then $\{A, B\}: A \cong B: (A \cap B)$.*

Here we are considering $\{A, B\}$ as a join system relative to the join operation in G (see last paragraph, Sec. 8) and A as a linear set in $\{A, B\}$; similarly for B and $A \cap B$. See [**10**], page 366, Theorem 3 for the proof; [**7**], page 136 for the corresponding result in group theory in generalized form.

Geometric content: In Euclidean 3-space G let A be a line, B a plane such that $A \cap B$ is a point (Fig. 21). Then $\{A, B\} = G$ and $\{A, B\}: A$ is essentially the geometry of the half-planes in G with edge A. Similarly $B: (A \cap B)$ is essentially the geometry of the rays in B from $A \cap B$. Clearly the two systems are isomorphic—essentially each is the natural geometry of a circle. Moreover we can establish the isomorphism by a simple geometric procedure, namely: Associate with each element of $\{A, B\}: A$ its intersection with plane B.

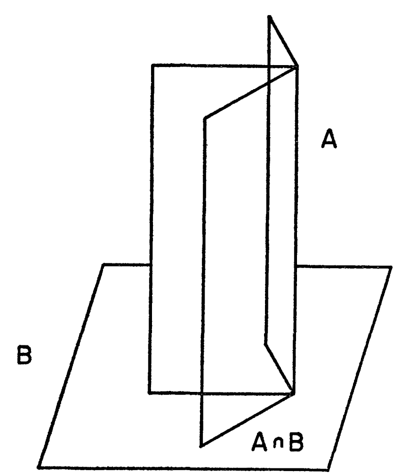


FIG. 21

14. Incidence relations and dimensionality. Our treatment of linear sets in an arbitrary join system (Secs. 8, 9) was quite general. It did not cover familiar incidence relations for example, "three noncollinear points belong to a unique plane," or intersection properties like "in a 3-space if two distinct planes intersect, their intersection is a line." In fact we have not defined "3-space" or even "plane." Nor have we introduced a notion of dimension to classify linear sets and organize them into a hierarchy. These related matters we consider now.

We can not, on the basis of (J1), $\cdots$, (J6), derive the familiar incidence properties of Euclidean geometry—in fact the property "two distinct points belong to a unique line" is independent of (J1), $\cdots$, (J6). This is justified by an example in the Appendix (Sec. E) of a join system which does not satisfy the stated property. Consequently to obtain the Euclidean theory of incidence it is necessary to postulate this property or an equivalent. It is interesting that the property is sufficient. Specifically we can assume it in the following weakened form:

(E) (EXCHANGE POSTULATE). *If* $b \subset \{a_1, a_2\}$, $b \neq a_2$ *then* $\{a_1, a_2\} = \{b, a_2\}$.

This says in effect that if b belongs to line a_1a_2, the pairs a_1, a_2 and b, a_2 determine the same line. Although (E) has not the simple algebraic character of (J1), $\cdots$, (J6), it is algebraic in nature and may be described as an "exchange" principle, since it permits us to exchange a_1 for b in $\{a_1, a_2\}$ without affecting the result.

We develop an incidence theory for join systems satisfying (E). First (E) is generalized to n terms:

THEOREM 33 (EXCHANGE LEMMA). *Suppose that* $b \subset \{a_1, \cdots, a_n\}$, $b \not\subset \{a_2, \cdots, a_n\}$. *Then* $\{a_1, \cdots, a_n\} = \{b, a_2, \cdots, a_n\}$.

Proof. The relation $b \subset \{a_1, \cdots, a_n\}$ implies, by definition of linear closure of a set, $\{b, a_2, \cdots, a_n\} \subset \{a_1, \cdots, a_n\}$. The reverse inclusion is immediate by the same argument, once we have established

$$a_1 \subset \{b, a_2, \cdots, a_n\}. \tag{14.1}$$

The proof of (14.1) is longer. We have

$$\begin{aligned} b &\subset \{a_1, \cdots, a_n\} \\ &\subset \{\{a_1, a_2\}, \{a_2, \cdots, a_n\}\} \\ &= \{a_1, a_2\}/\{a_2, \cdots, a_n\} \qquad \text{(Theorem 18).} \end{aligned}$$

This yields

$$b \subset p/q \tag{14.2}$$

where p, q satisfy

$$p \subset \{a_1, a_2\}, \tag{14.3}$$

$$q \subset \{a_2, \cdots, a_n\}. \tag{14.4}$$

Relation (14.2) implies

$$p \subset bq. \tag{14.5}$$

In order to apply the Exchange Postulate to (14.3) we need $p \neq a_2$. This is valid, since $p = a_2$ implies by (14.2), (14.4), $b \subset \{a_2, \cdots, a_n\}$, contrary to hypothesis.

Hence

$$\begin{aligned} a_1 &\subset \{a_1, a_2\} \\ &= \{p, a_2\} && \text{(Exchange Postulate)} \\ &\subset \{b, q, a_2\} && (14.5) \\ &\subset \{b, a_2, \cdots, a_n\} && (14.4). \end{aligned}$$

Thus (14.1) holds and the proof is complete.

In order to deal uniformly with n points in "general position," for example two distinct points or three noncollinear points, we introduce the idea of linear independence.

DEFINITION. *We say $a_1, \cdots, a_n$ are **linearly independent** or simply **independent** if they are never contained in a linear set generated by $n-1$ elements—that is, if the statement $\{x_1, \cdots, x_{n-1}\} \supset a_1, \cdots, a_n$ is false for every choice of $x_1, \cdots, x_{n-1}$.*

If $n=1$, $\{x_1, \cdots, x_{n-1}\}$ is taken to be $\{\varnothing\}$ which is $\varnothing$. Thus a single element a_1 is independent. Note that a_1, a_2 are independent if and only if $a_1 \neq a_2$.

We can show without difficulty: (1) if $a_1, \cdots, a_n$ are independent they are distinct; (2) that any nonvoid subset of an independent set is also independent.

The following theorem and corollary prepare for the notion of dimension.

THEOREM 34. *If $\{x_1, \cdots, x_m\} \supset a_1, \cdots, a_n$ and $a_1, \cdots, a_n$ are independent then $m \geqq n$.*

Proof. By definition of independence $m \neq n-1$. Suppose $m < n-1$. Then we can insert redundantly a particular one of the x's, say x_1, in the expression $\{x_1, \cdots, x_m\}$ a sufficient number of times, so that it appears as a linear set generated by $n-1$ elements. This is impossible by definition of independence and we infer $m \geqq n-1$.

COROLLARY. *Let $\{a_1, \cdots, a_m\} = \{b_1, \cdots, b_n\}$, where $a_1, \cdots, a_m$ are independent and $b_1, \cdots, b_n$ are independent. Then $m = n$.*

Proof. $\{a_1, \cdots, a_m\} \supset b_1, \cdots, b_n$ so that $m \geqq n$ by the theorem. Symmetrically $n \geqq m$.

When determining a linear set it is desirable, if possible, to choose a set of generators which is independent. This suggests the

DEFINITION. *If $a_1, \cdots, a_n$ is an independent set of generators of linear set A, that is, if $A = \{a_1, \cdots, a_n\}$ and $a_1, \cdots, a_n$ are independent, we call $a_1, \cdots, a_n$ a **basis** of A.*

Restating the last corollary in terms of the idea of basis, we have an important uniqueness result:

THEOREM 35. *All bases of linear set A have the same cardinal number.*

The unique cardinal number associated with linear set A indicates its relative complexity of structure and suggests the idea dimension:

DEFINITION. *If linear set A has a basis, the unique cardinal number of each of its bases is called its* **rank** *or* **dimension** *and is denoted by $d(A)$. We define $d(\varnothing)=0$.*

Dimension as defined exceeds by unity the conventional definition—for example, the dimension of a Euclidean line is 2, of a Euclidean plane is 3, etc. In the present context this is quite natural and very convenient. In our join theory the dimension of an element a is 1, that of a line ($\{a, b\}$, $a \neq b$) is 2, and we *define* a plane to be a linear set A such that $d(A)=3$.

Familiar determination properties of Euclidean lines and planes generalize to

THEOREM 36 (EXCHANGE THEOREM). *Suppose $A=\{a_1, \cdots, a_n\}$. Then any n independent elements of A form a basis of A.**

Proof. Suppose $\{a_1, \cdots, a_n\} \supset (b_1, \cdots, b_n)$ and the b's are independent. We employ the Exchange Lemma (Th. 33) to exchange a's for b's, one by one, in $\{a_1, \cdots, a_n\}$. Since $b_1, \cdots, b_n$ are independent, $\{a_2, \cdots, a_n\} \not\supset (b_1, \cdots, b_n)$ and one of the b's, for simplicity say b_1, is not in $\{a_2, \cdots, a_n\}$. By the Exchange Lemma we have

$$\{a_1, \cdots, a_n\} = \{b_1, a_2, \cdots, a_n\} = \{a_2, \cdots, a_n, b_1\}. \tag{14.6}$$

We apply a similar argument to $\{a_2, \cdots, a_n, b_1\}$. Clearly $\{a_2, \cdots, a_n, b_1\} \supset b_2, \cdots, b_n$. Further $\{a_3, \cdots, a_n, b_1\} \not\supset (b_2, \cdots, b_n)$. Hence one of the indicated b's, say b_2, is not in $\{a_3, \cdots, a_n, b_1\}$. By the Exchange Lemma

$$\{a_2, \cdots, a_n, b_1\} = \{b_2, a_3, \cdots, a_n, b_1\} = \{a_3, \cdots, a_n, b_1, b_2\},$$

and (14.6) implies $\{a_1, \cdots, a_n\} = \{a_3, \cdots, a_n, b_1, b_2\}$. Continuing to exchange a's for b's in this way we get eventually $\{a_1, \cdots, a_n\} = \{b_1, \cdots, b_n\}$ and the theorem holds.

The exchange theorem has many important consequences.

COROLLARY 1. *There is a unique linear set of dimension n containing a given set of n independent elements.*

Proof. Suppose $b_1, \cdots, b_n$ are independent. Then $\{b_1, \cdots, b_n\}$ satisfies the existence condition. Uniqueness follows, since if $d(A)=n$ and $A \supset b_1, \cdots, b_n$ then $A=\{b_1, \cdots, b_n\}$ by the theorem.

Taking $n=2$ we get the property: *Two distinct points belong to a unique line.*

Further we get a procedure for enlarging independent sets:

COROLLARY 2. *Let $a_1, \cdots, a_n$ be independent and $a_{n+1} \not\subset \{a_1, \cdots, a_n\}$. Then $a_1, \cdots, a_n, a_{n+1}$ are independent.*

* Similar properties occur in modern algebra, for example see [1] p. 169, Theorem 5 Corollary.

Proof. Suppose the conclusion false. Then

$$\{x_1, \cdots, x_n\} \supset a_1, \cdots, a_n, a_{n+1}$$

for some $x_1, \cdots, x_n$. Using the theorem,

$$a_{n+1} \subset \{x_1, \cdots, x_n\} = \{a_1, \cdots, a_n\},$$

contrary to hypothesis.

This easily yields an inductive criterion for independence:

COROLLARY 3. *$a_1, \cdots, a_n$ are independent if and only if*

$$a_i \not\subset \{a_1, \cdots, a_{i-1}\}, \qquad 1 \leqq i \leqq n. \tag{14.7}$$

Proof. Suppose (14.7). The case $n=1$ is trivial since a_1 is independent. Assume $n>1$. Let $i=2$; since a_1 is independent, Corollary 2 implies a_1, a_2 are independent. Similarly the latter implies a_1, a_2, a_3 are independent. Continuing in this way we get eventually the independence of $a_1, \cdots, a_n$.

To show the converse, suppose (14.7) fails for a given i. Then

$$\{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n\} \supset a_1, \cdots, a_n$$

and $a_1, \cdots, a_n$ are not independent.

Now the following familiar criterion for independence can be derived.

COROLLARY 4. *$a_1, \cdots, a_n$ are independent if and only if*

$$a_i \not\subset \{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n\}, \qquad 1 \leqq i \leqq n.$$

Corollary 3 also yields an important criterion for a basis:

COROLLARY 5. *A maximal independent subset of linear set A is a basis of A.*

Proof. Suppose $a_1, \cdots, a_n$ is an independent subset of A and no "larger" set has this property. Clearly $A \supset \{a_1, \cdots, a_n\}$. Let $x \subset A$. If $x \not\subset \{a_1, \cdots, a_n\}$ then Corollary 3 implies $a_1, \cdots, a_n, x$ are independent, contrary to hypothesis. Thus $x \subset \{a_1, \cdots, a_n\}$ and $A = \{a_1, \cdots, a_n\}$.

Now we have a criterion that a linear set have a dimension:

COROLLARY 6. *If a linear set has a finite set of generators it has a dimension.*

Proof. $\{a_1, \cdots, a_n\}$ has a maximal independent subset containing at most n elements in view of Theorem 34. Hence by Corollary 5 it has a basis and so a dimension.

Another consequence of Corollary 5 is the following monotonic property of dimension:

COROLLARY 7. *If A, B are linear, $A \subset B$, and B has a dimension, then A has a dimension and $d(A) \leqq d(B)$. Further, $d(A) = d(B)$ only if $A = B$.*

Now we consider intersection of linear sets. Familiar intersection properties

of Euclidean lines and planes are generalized in the following deep-lying theorem which we present without proof.

THEOREM 37 (DIMENSION PRINCIPLE). *Let A, B be linear sets which have dimensions and let $A \cap B \neq \varnothing$. Then*

$$d(\{A, B\}) + d(A \cap B) = d(A) + d(B).^*$$

As an application we prove: In a Euclidean 4-space if two intersecting planes are not contained in a 3-space, their intersection is a point.

Let A, B be the planes and C the 4-space. Clearly $\{A, B\} \subset C$ which implies

$$d(\{A, B\}) \leqq d(C) = 5$$

by Corollary 7 above, the monotonic property of dimension. Further $d(\{A, B\}) > 4$; since otherwise there would be a 3-space containing A and B. Thus $d(\{A, B\}) = 5$ and the Dimension Principle asserts

$$5 + d(A \cap B) = 3 + 3,$$

so that $d(A \cap B) = 1$ and the intersection of A and B is a point.

If $A \cap B = \varnothing$ the Dimension Principle is replaced by the inequality

$$d(\{A, B\}) \leqq d(A) + d(B). \tag{14.8}$$

This suggests in n-dimensional Euclidean geometry a general definition of parallelism and skewness of linear sets: If $A \cap B = \varnothing$ and (14.8) is strictly an inequality we say A and B are *parallel*, otherwise they are *skew*.

Finally we note that a linear set of a join system need not have a dimension as we have defined it—for it need not have a finite set of generators. In other words a join system may have linear sets to which our theory does not apply. However our theory of dimension can be extended to cover the linear sets of any join system. To do this a definition of (linear) independence is framed that is applicable equally to finite and infinite sets and is equivalent to our definition in the finite case (see [10], p. 355, Def. 1). Then it is proved that any linear set A has an independent set of generators; and that all independent sets of generators of A have the same cardinal number. This cardinal number is defined to be the dimension of A. If the dimension of A is a finite cardinal, A is said to be *finite* dimensional, otherwise it is *infinite* dimensional. Thus linear sets which have a dimension in our theory are finite dimensional in the generalized theory and the others are infinite dimensional in it. The dimension of a join system G is defined to be that of G as a linear set. In the Appendix (Sec. C) we give an example of an infinite dimensional join system. See [10], Section 9 for more information on the theory of dimension outlined above, and [8], Sections 3, 4 for its justification.†

* This result is proved in [8] for a class of lattices which covers the present situation.

† For such an unrestricted theory of dimension, in projective geometry see [9]; in the theory of vector spaces see R. Baer, Linear Algebra and Projective Geometry, New York, 1952, Chapter II, or N. Jacobson, Linear Algebra, vol. II, New York, 1953, Chapter IX.

15. The concept of order. We have been dwelling in the realm of higher dimensional linear sets, now we descend to consider the notion of order, which is concerned basically with the lowly linear question of how collinear points are related. Order, one of the most pervasive mathematical concepts, is particularly important in geometry. It is intimately involved not merely in purely linear matters like the relative position of points of a line, but is the basis for the study of separation (for example, a plane in a 3-space *separates* it into two half-spaces), and the idea of interiority (for example, a point is *inside* triangle abc if it is between a and a point which is itself between b and c).

Hilbert in his pioneer work [**5**] begins with incidence postulates relating the primitive terms point, line, plane. Then he introduces order in the form of a 3-term relation, "betweenness," which satisfies postulates ensuring that each line is a "linearly ordered" system.* Veblen in his classic foundations for Euclidean geometry [**13, 14**] takes point and a 3-term order (or betweenness) relation as primitive terms and defines segment, ray, line and plane in terms of them.

Order is implicit in our theory of join systems. For in Euclidean geometry we have defined (Sec. 2) the join ab of the distinct points a, b as the segment with endpoints a and b; so that any point of ab is between a and b. We convert the latter property into a definition of betweenness in the abstract theory of join systems:

DEFINITION. *If $x \subset ab$ and $a \neq b$ we say x is* **between** *a and b and we write (axb).*

Clearly $x \subset ab$ if and only if (axb) or $x = a = b$. In view of this (J1), $\cdots$, (J6) can be rephrased in terms of betweenness. For example, (J2) is equivalent to the familiar order property, (abc) implies (cba). Many other familiar properties of linear order are deducible from (J1), $\cdots$, (J6). The following theorems, which hold in an arbitrary join system (we are not assuming Postulate (E)), are examples of such properties.

THEOREM 38. *(abc) implies that a, b, c are distinct.*

Proof. By hypothesis $b \subset ac$ and $a \neq c$. Suppose $a = b$. Then $a \approx ac$ so that $a/a \approx c$ and (J6) implies $a = c$. This contradiction implies $a \neq b$. Similarly $b \neq c$.

THEOREM 39. *(abc), (acd) imply (abd), (bcd).*

Proof. By definition we have

$$b \subset ac, \qquad c \subset ad \tag{15.1}$$

where $a \neq c$, d. We eliminate c from (15.1) so: the first relation implies $b/a \supset c$, which yields with the second relation $b/a \approx ad$ and $b \approx aad = ad$. By definition (abd) holds.

Next we eliminate a from (15.1). We have $b/c \approx a$, $c/d \approx a$ so that $b/c \approx c/d$

* See [**6**] for characterizations of betweenness.

and $bd \approx c$. By Theorem 38, (abd) implies $b \neq d$ and (bcd) holds by definition.
Similarly we can prove that (abc), (bcd) imply (abd), (acd).

However all familiar properties of linear order do not hold in an arbitrary join system or even in one that satisfies Postulate (E). The most important example is the following

COMPARABILITY PROPERTY. *If p, q, r are distinct and collinear then at least one of (pqr), (qpr), (qrp) holds.*

An independence example for this property which satisfies (J1), $\cdots$, (J6) and (E), called a *partially ordered* join system, is described in the Appendix (Sec. G). Thus to obtain, in a join system, the familiar theory of linear order we shall have to postulate the Comparability Property or an equivalent.

Naturally we prefer to formulate the property in join terminology. Suppose then $x \subset \{a, b\}$ and assume $a \neq b$. Since x, a, b colline, the Comparability Property implies (xab), (axb), (abx) or $x = a$ or $x = b$, and so

$$x \subset a/b \cup ab \cup b/a \cup a \cup b,$$

which implies

$$\{a, b\} \subset a/b \cup ab \cup b/a \cup a \cup b. \tag{15.2}$$

Clearly (15.2) holds if $a = b$. The reverse inclusion to (15.2) has been established in Section 9 as relation (9.1). This suggests the following postulate which is a consequence of the Comparability Property:

(C) (COMPARABILITY POSTULATE). $\{a, b\} = a/b \cup ab \cup b/a \cup a \cup b$.

Since, when $a \neq b$, the right member of (C) is the conventional definition of line ab (Sec. 8, (8.1)) and the left member is the one we have adopted (Sec. 9), it may be good to discuss their connection and the significance of (C). Suppose we had defined line ab in the conventional way in Section 9. Immediately there would arise the awkward possibility that a line need not be a linear set. This is realized in the independence example for Postulate (E) given in the Appendix (Sec. E), where $a/b \cup ab \cup b/a \cup a \cup b$ is not linear and in fact is not even convex.

Adopting the suggested definition would consequently force us to postulate that line ab is a linear set, which implies line $ab = \{a, b\}$, $(a \neq b)$. Thus in effect we would have assumed Postulate (C) in Section 9. This would have restricted prematurely the scope of our theory in Sections 9–13, which is applicable to any join system at all. It would have excluded join systems in which Postulate (E) fails (see Appendix Sec. E), since (E) is a consequence of Postulate (C) (see below, Th. 40 Corol. 2), as well as "partially ordered" join systems in which (E) holds but (C) fails (see Appendix Sec. G).

Finally observe that although we have adopted (C) as a kind of comparability principle, motivated by the obvious importance of comparability in the theory of linear order, it is intrinsically worth studying as a property of join systems.

For in view of relation (9.1), a join system satisfying (C) is of the simplest possible type, since in it the expression for $\{a, b\}$ is as simple as possible.

This concludes our discussion of the significance of (C)—its consequences are developed in the next section.

16. Separation and factor systems. In this section we study separation of linear sets by linear sets, relating the idea to properties of factor systems. We derive a general separation theorem for arbitrary linear sets in a join system that satisfies Postulate (C). As a by-product we find that (C) implies the Exchange Postulate (E) of Section 14.

The classical treatment of linear separation proceeds in piecemeal fashion. First it is shown ([4], p. 51, Th. 8) that a line is separated by any of its points. Then it is proved ([4], p. 68, Th. 2) that a plane is separated by any contained line, on the basis of the famous

POSTULATE OF PASCH. *Suppose line* $L \subset$ *plane* $\{a, b, c\}$ *and* $L \not\supset a, b, c$. *Then* $L \approx ab$ *implies* $L \approx bc$ *or* $L \approx ac$, *but not both* (see Fig. 22).*

Finally Pasch's Postulate is generalized to 3-space ([4], p. 65, Th. 30) and it is shown ([4], p. 85, Th. 25) that any plane separates 3-space.

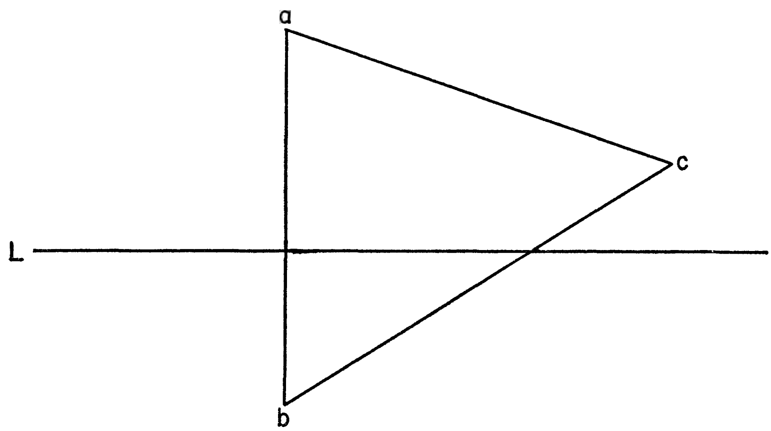


FIG. 22

We give a uniform, dimension-free treatment of the problem without using Pasch's postulate. First we formulate the separation idea in join-theoretic terms. Suppose in a Euclidean geometry, A, B are linear and B separates A; for example, a point separates a containing line or a 3-space separates a containing 4-space. The conventional definition of this may be expressed as follows: There exist nonvoid sets α, β such that (a) $A = \alpha \cup \beta \cup B$; (b) α, β, B are disjoint in pairs; (c) if $x \subset \alpha$, $y \subset \beta$ then $xy \approx B$; (d) if $x, y \subset \alpha$ or $x, y \subset \beta$ then $xy \approx B$ is false. (Fig. 23 illustrates the case where A is a plane and B a line.)

* Compare [4], p. 59, Theorems 16, 16.1.

Let $a \subset \alpha$ (see Fig. 23). It is not difficult to show that $\beta = B/a$, so that β is a coset of B by Theorem 26, Corollary. Similarly α is a coset of B (in fact α and β are opposite cosets by definition (Sec. 10) in view of (c)). Further (a), (b) imply that α, β, B constitute the coset decomposition of linear set A determined by its linear subset B. The essential point is that A decomposes into *three* cosets, which are distinct since the order of $A:B$ is at least 3 by Theorem 31. So our analysis yields this result: $A:B$ has order 3. This suggests the identification of the classical geometric notion "B separates A" with the join-theoretic idea "$A:B$ has order 3."

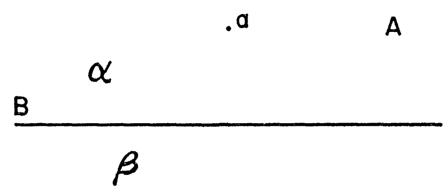


FIG. 23

To test the reasonableness of this suggestion, suppose in a Euclidean geometry A, B are linear sets such that $A:B$ has order 3. Let α, β be the elements of $A:B$ distinct from B. Then A is decomposed into α, β, B and (a), (b) above hold. Further since $A:B$ contains the opposite of each of its elements, and B alone is its own opposite (Th. 25, Corol. 2), the opposite of α must be β. Thus (c) holds by Theorem 25. Finally (d) holds since a coset is convex (Th. 23, Corol.), and B separates A in the conventional sense.

Thus we adopt the

DEFINITION. *If A, B are linear and the order of $A:B$ is 3, we say B* ***separates*** *A.*

In the remainder of this section we consider join systems that satisfy Postulate (C) (Sec. 15). The basic theorem on separation of linear sets is preceded by three lemmas.

LEMMA 1. *$\{a, b\}:b$ has order 3, provided $a \neq b$.*

Proof. Since $\{a, b\} \neq b$, $\{a, b\}:b$ is nontrivial and its order $r \geqq 3$ by Theorem 31. Let $(x)_b$ be any element of $\{a, b\}:b$. We show $(x)_b$ is one of the following cosets, $(a)_b$, its opposite b/a, or b itself. This is trivial if $x = a$ or $x = b$. Suppose $x \neq a, b$. By Postulate (C) one of the following holds:

$$(1)\ x \subset a/b, \qquad (2)\ x \subset ab, \qquad (3)\ x \subset b/a.$$

Suppose (1). Then $xb \approx a$ so that $xb \approx ab$ and $x \approx ab/b = (a)_b$ by Theorem 23. Thus $(x)_b = (a)_b$. Similarly (2) implies $xb \approx ab$, again yielding $(x)_b = (a)_b$. Finally (3) implies $(x)_b = b/a$. Thus $r = 3$ and the lemma holds.

LEMMA 2. *Suppose $A \supset B \supset C$, where A, B, C are linear and $C \neq \varnothing$. Then the order of $A:B$ does not exceed that of $A:C$.*

Proof. By Theorem 23, for $a \subset A$, $(a)_B = aB/B \supset aC/C = (a)_C$. The lemma follows.

LEMMA 3. *If* $A:C = B:C$, *then* $A = B$.

Proof. A and B are the unions of the elements of $A:C$ and $B:C$ respectively.

Now we are prepared to substantiate the general separation theorem for linear spaces. Roughly stated it says if linear space A contains linear space B and is of "next higher rank" then B separates A. How can we clarify the somewhat vague phrase, A is of next higher rank than B, as for example when A is a plane and B a line? We might be tempted to define this by $d(A) = d(B) + 1$, but this imposes the unnecessary restriction that A and B have dimensions (see discussion at end of Sec. 14). The answer is given in the following

DEFINITION. *Let* A, B *be linear sets such that* $A \supset B$, $A \neq B$, *and there exists no linear set* X *"between"* A *and* B, *that is,* $A \supset X \supset B$ *for linear* X *implies* $X = A$ *or* $X = B$. *Then we say* A ***covers*** B.*

The separation theorem is now expressible in the following simple form:

THEOREM 40 (SEPARATION THEOREM FOR LINEAR SETS). B *separates* A *if and only if* A *covers* B *and* $B \neq \emptyset$.

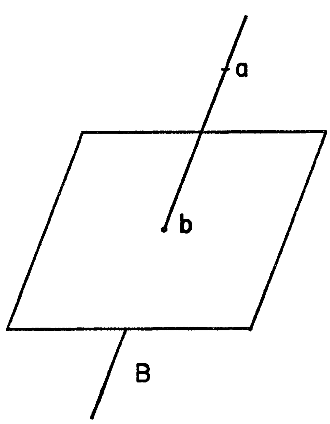


FIG. 24

Proof. Suppose A covers B, $B \neq \emptyset$. Let $a \subset A$, $a \not\subset B$ and $b \subset B$. (Fig. 24 illustrates the case where B is a plane.) We show $A:B$ isomorphic to a factor system of the form $\{a, b\}:B'$. We have

$$A \supset \{B, a\} \supset B, \qquad \{B, a\} \neq B.$$

These imply, by definition of A covers B, $A = \{B, a\} = \{B, \{a, b\}\}$. By the

* This is related to the notions of maximal ideal or maximal subgroup of modern algebra and has been abstracted to lattice theory; see [1], p. 349.

Isomorphism Theorem (Th. 32)

$$A:B = \{B, \{a, b\}\}:B \cong \{a, b\}:(B \cap \{a, b\}) = \{a, b\}:B',$$

where $B' = B \cap \{a, b\}$. We have

$$\{a, b\} \supset B' \supset b, \qquad \{a, b\} \neq B'.$$

Hence, by Lemma 2, the order r of $\{a, b\}:B'$ does not exceed that of $\{a, b\}:b$, which, by Lemma 1, is 3. However $r \geqq 3$ by Theorem 31, so that $r = 3$. Thus the order of $A:B$ must be 3, and B separates A by definition.

Conversely suppose B separates A. Clearly $A \supset B$, $A \neq B$. Suppose $A \supset X \supset B$, $X \neq B$, where X is linear. Then in view of Theorem 23 we have

$$A:B \supset X:B. \tag{16.1}$$

In (16.1) $A:B$ has order 3 by definition of separation, and $X:B$ has order at least 3 by Theorem 31. Hence $A:B$ and $X:B$ both have order 3 and (16.1) implies $A:B = X:B$. By Lemma 3 this yields $X = A$, and A covers B by definition.

Note in the theorem that there is no dimensionality restriction on the join system considered, or on its linear sets A and B; for example, A and B may be infinite dimensional (see discussion at end of Sec. 14). Further, the theorem includes the necessity of the covering condition for separation, which is not considered in the conventional treatment.

COROLLARY 1. *Suppose A covers B, $B \neq \emptyset$, $a \subset A$, $a \not\subset B$, $aa' \approx B$. Then A has a partition of the form $A = B \cup B/a \cup B/a'$.*

Proof. By the theorem, B separates A, so that A has a partition composed of three cosets of B. It is easily shown that B, B/a, B/a' are distinct.

COROLLARY 2. *In a join system Postulate* (C) *implies Postulate* (E).

Proof. Suppose $b \subset \{a_1, a_2\}$, $b \neq a_2$. Then $a_1 \neq a_2$ and by Lemma 1, $\{a_1, a_2\}:a_2$ has order 3. Thus a_2 separates $\{a_1, a_2\}$ and $\{a_1, a_2\}$ covers a_2 by the theorem. But

$$\{a_1, a_2\} \supset \{b, a_2\} \supset a_2, \qquad \{b, a_2\} \neq a_2.$$

Therefore $\{a_1, a_2\} = \{b, a_2\}$ and Postulate (E) is verified.

When we introduced Postulate (C) in Section 15 we motivated it as a *consequence* of the Comparability Property, emphasizing the importance of the latter as an essential property of linear order. Now we are able to complete the discussion of the relation between these properties.

COROLLARY 3. *In a join system Postulate* (C) *implies the Comparabilty Property of Section* 15.

Proof. Suppose p, q, r distinct and collinear. By Corollary 2, Postulate (E) holds and so the results of Section 14 are valid. Thus $\{q, r\}$ is the only line containing q, r (Th. 36, Corol. 1) and $p \subset \{q, r\}$. By Postulate (C)

$$p \subset q/r \cup qr \cup r/q$$

so that (pqr) or (qpr) or (prq) and the Comparability Property follows.

The separation theorem, despite its broad scope, does not immediately justify familiar Euclidean properties such as, a point separates a containing line, or a line separates a containing plane. For the required covering conditions —that a line cover a contained point, or a plane a contained line—have not yet been established. Using Corollary 2 above we take care of this now, in general terms, by relating covering conditions and dimensionality properties.

THEOREM 41. *If A covers B and $d(A)$ exists then $d(A)=d(B)+1$. Conversely if $A\supset B$ and $d(A)=d(B)+1$ then A covers B.*

Proof. Observe that by Corollary 2 above, we may apply the dimension theory of Section 14. Suppose A covers B and $d(A)$ exists. If $B=\varnothing$ then $d(A)=1=d(B)+1$. Suppose $B\neq\varnothing$. By Theorem 36, Corollary 7, $d(B)$ exists. Let $B=\{b_1, \cdots, b_n\}$, where $b_1, \cdots, b_n$ are independent. Let $a\subset A$, $a\not\subset B$. Then

$$A \supset \{a, b_1, \cdots, b_n\} \supset B,$$

so that $A=\{a, b_1, \cdots, b_n\}$. The indicated set of generators of A is independent by Theorem 36, Corollary 2. Therefore $d(A)=n+1=d(B)+1$.

Conversely suppose $A\supset B$ and $d(A)=d(B)+1$. Note $A\neq B$. Suppose $A\supset X\supset B$, where X is linear. By Theorem 36, Corollary 7,

$$d(A) \geqq d(X) \geqq d(B),$$

so that $d(X)=d(A)$ or $d(X)=d(B)$. By Theorem 36, Corollary 7, $X=A$ or $X=B$, and A covers B by definition.

COROLLARY 1. *Suppose $A\supset B\neq\varnothing$, and $d(A)=d(B)+1$. Then B separates A.*

Proof. A covers B by the theorem and B separates A by Theorem 40.

COROLLARY 2. *In a join system Postulate* (C) *implies the Postulate of Pasch.*

Proof. Suppose line $L\subset$plane $\{a, b, c\}$, $L\not\supset a, b, c$ and $L\approx ab$. Since

$$d(\{a, b, c\}) = 3 = d(L) + 1,$$

the theorem implies $\{a, b, c\}$ covers L. By Theorem 40, Corollary 1 we have the partition $\{a, b, c\}=L\cup L/a\cup L/b$. Hence $c\subset L/a$ or $c\subset L/b$ but not both, which implies $ac\approx L$ or $bc\approx L$ but not both.

For an extension of the methods developed here to the more complicated problem of the decomposition of a linear set of dimension n effected by an n-simplex, see [10], Section 11. This also contains ([10], Sec. 10) a more sophisticated treatment of the theory of separation of linear sets, based on a deeper analysis of the factor systems involved.

17. Conclusion. This concludes our approach to classical geometry. We have given a join-theoretic basis for the classical theories of incidence and order of Euclidean geometry. Since we have assumed no parallel postulate our treatment is equally valid in hyperbolic (Lobachevskian) geometry. In fact our study of join systems that satisfy Postulate (C) is a dimensionally (and existentially) unrestricted theory of the ordered linear geometries which have been studied in the foundations of geometry under the name descriptive geometries.*

We have approached but not reached classical geometry—much remains to be done. It is not wholly clear which is the best path to follow in introducing parallel and congruence properties into a join system. Although there is no existence theorem for a best path, we might try to search for a treatment that will be as satisfactory to us as Euclid's must have appeared to his contemporaries.

* See [2], Chapter VIII for a discussion of descriptive geometry and [10] for the relation between descriptive geometries and join systems.

APPENDIX. EXAMPLES OF JOIN SYSTEMS

In this appendix examples and modes of construction of join systems are presented, in order to give body to our theory and to serve as independence examples for the exchange, comparability and separation properties of a join system.

A. The arithmetic affine plane. Let R_2 be the set of all ordered pairs (x_1, x_2) of real numbers. The elements of R_2 may be called points or vectors. Using vectorial shorthand, we write x for (x_1, x_2) and define addition and subtraction in R_2, and multiplication of elements of R_2 by real numbers, as follows:

$$x \pm y = (x_1, x_2) \pm (y_1, y_2) = (x_1 \pm y_1, x_2 \pm y_2),$$
$$\lambda x = \lambda(x_1, x_2) = (\lambda x_1, \lambda x_2).$$

In R_2 we define join in this way: ab is the set of all elements x of R_2 expressible in the form $x=\lambda a+\mu b$, where λ, μ are real numbers which satisfy $0<\lambda, \mu; \lambda+\mu=1$. This definition is suggested by the familiar point of division formula of elementary analytic geometry. We are saying in effect that the join of a and b is to be the set of points "between" a and b, which is the set of points that divide "segment ab" in a positive ratio.

R_2, with the given definition of join, is a join system.

To verify (J1) observe first that $ab \subset R_2$ by definition of join, and second that $ab \neq \emptyset$ since it contains, for example, $\frac{1}{2}a+\frac{1}{2}b$. (J2) and the first part of (J6) are easily seen to be valid in R_2.

Verification of (J3). First we show that

$$x \subset (ab)c \tag{A.1}$$

holds if and only if x is expressible in the form

$$x = \lambda a + \mu b + \nu c, \qquad 0 < \lambda, \mu, \nu, \qquad \lambda + \mu + \nu = 1. \tag{A.2}$$

Suppose (A.1). Then

$$x \subset yc, \qquad y \subset ab \tag{A.3}$$

holds for some y. By definition of join, (A.3) yields

$$x = \alpha y + \beta c, \qquad 0 < \alpha, \beta, \qquad \alpha + \beta = 1; \tag{A.4}$$

$$y = \gamma a + \delta b, \qquad 0 < \gamma, \delta, \qquad \gamma + \delta = 1. \tag{A.5}$$

Eliminating y between (A.4), (A.5) and determining λ, μ, ν by

$$\lambda = \alpha\gamma, \qquad \mu = \alpha\delta, \qquad \nu = \beta, \tag{A.6}$$

we obtain (A.2).

Conversely suppose that x satisfies (A.2). We retrace our steps to (A.1). First we determine α, β, γ, δ to satisfy (A.6) and the relation $\alpha+\beta=1$. We get

$$\alpha = 1 - \nu = \lambda + \mu, \qquad \beta = \nu, \qquad \gamma = \lambda/(\lambda + \mu), \qquad \delta = \mu/(\lambda + \mu).$$

Then $0 < \alpha, \beta, \gamma, \delta$ and $\gamma + \delta = 1$. Further $x = \alpha(\gamma a + \delta b) + \beta c$. Determine y by (A.5), so that (A.4) follows. Relations (A.4), (A.5) imply (A.3) and (A.1) follows.

Similarly we show that $x \subset a(bc)$ is equivalent to (A.2). Thus $x \subset (ab)c$ is equivalent to $x \subset a(bc)$ and (J3) is verified.

(J4) is easily verified in R_2. By definition $a/b \subset R_2$, and a/b is nonvoid since it contains, for example, $2a-b$. To see this, note $a = \frac{1}{2}(2a-b) + \frac{1}{2}b$, $0 < \frac{1}{2}$.

Verification of (J5). Suppose in R_2, $a/b \approx c/d$. Then for some element x of R_2 $x \approx a/b$, $x \approx c/d$. Hence $a \approx bx$, $c \approx dx$, which imply

$$a = \lambda b + \mu x, \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1, \tag{A.7}$$

$$c = \lambda' d + \mu' x, \qquad 0 < \lambda', \mu', \qquad \lambda' + \mu' = 1. \tag{A.8}$$

Eliminating x between (A.7), (A.8) we have

$$\mu' a + \mu\lambda' d = \mu'\lambda b + \mu c. \tag{A.9}$$

Further

$$\mu' + \mu\lambda' = \mu' + \mu(1 - \mu') = \mu'(1 - \mu) + \mu = \mu'\lambda + \mu.$$

Thus (A.9) implies

$$\frac{\mu' a + \mu\lambda' d}{\mu' + \mu\lambda'} = \frac{\mu'\lambda b + \mu c}{\mu'\lambda + \mu}. \tag{A.10}$$

Let y denote either member of (A.10). By the definition of join, $y \subset ad$ and $y \subset bc$, so that $ad \approx bc$ and (J5) is verified.

It remains only to verify the idempotent law, $a/a = a$. Suppose $x \subset a/a$. Then $ax \supset a$ and

$$a = \lambda a + \mu x, \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1.$$

Solving for x we have $x = \mu^{-1}(1-\lambda)a = a$, and we infer $a/a = a$.

Thus we have proved that R_2 satisfies (J1), $\cdots$, (J6) and is a join system. R_2, as a system of points and lines, is an ordered affine plane—we have in a natural manner converted R_2 into a join system, which might be called an affine planar join system.

Furthermore R_2 satisfies (E), the Exchange Postulate of Section 14 and (C), the Comparability Postulate of Section 15. In order to show this we introduce two lemmas which are of some interest in themselves.

LEMMA 1. *a/b is the set of all elements x of R_2 expressible in the form*

$$x = \lambda a + \mu b, \qquad \mu < 0, \qquad \lambda + \mu = 1. \tag{A.11}$$

Proof. Let $x \subset a/b$. Then $a \subset bx$ so that

$$a = \rho b + \sigma x, \qquad 0 < \rho, \sigma, \qquad \rho + \sigma = 1.$$

Solving for x we get $x=\sigma^{-1}a-\sigma^{-1}\rho b$. Let $\lambda=\sigma^{-1}$, $\mu=-\sigma^{-1}\rho$. Then $\mu<0$ and $\lambda+\mu=1$ so that (A.11) is valid. Conversely, it is easily shown that (A.11) implies $x\subset a/b$, and the lemma holds.

LEMMA 2. *$\{a, b\}$ is the set of all elements x of R_2 expressible in the form*

$$x = \lambda a + \mu b, \qquad \lambda + \mu = 1. \tag{A.12}$$

Proof. Let $x\subset\{a, b\}$. Since R_2 is a join system we have $\{a, b\}=ab/ab$ by Theorem 16, Corollary. Thus $x\subset ab/ab$ so that $x\subset p/q$ where $p, q\subset ab$. Lemma 1 implies

$$x = \rho p + \sigma q, \qquad \rho + \sigma = 1; \tag{A.13}$$

and the definition of join in R_2 yields

$$p = \alpha a + \beta b, \qquad \alpha + \beta = 1, \tag{A.14}$$

$$q = \gamma a + \delta b, \qquad \gamma + \delta = 1. \tag{A.15}$$

Eliminating p, q between (A.13), (A.14), (A.15) and letting $\lambda=\rho\alpha+\sigma\gamma$, $\mu=\rho\beta+\sigma\delta$ we obtain (A.12).

Conversely suppose (A.12). If $\lambda=0$ or $\mu=0$ then $x=b$ or $x=a$ and $x\subset\{a, b\}$. Suppose λ, $\mu\neq 0$. Then we have

$$\mu < 0, \qquad \lambda < 0, \quad \text{or} \quad 0 < \lambda, \mu. \tag{A.16}$$

This implies, by Lemma 1 and the definition of join in R_2,

$$x \subset a/b \cup b/a \cup ab \subset \{a, b\},$$

and the lemma is established.

Verification of Postulate (E). Suppose that

$$b \subset \{a_1, a_2\} \tag{A.17}$$

holds, where $b\neq a_2$. We show that $\{a_1, a_2\}=\{b, a_2\}$. By definition of linear closure of a set (Sec. 9), relation (A.17) implies $\{b, a_2\}\subset\{a_1, a_2\}$. To establish the reverse inclusion we apply Lemma 2 to (A.17), obtaining

$$b = \lambda a_1 + \mu a_2, \qquad \lambda + \mu = 1.$$

Here $\lambda\neq 0$ since $b\neq a_2$. Thus we can solve for a_1, getting $a_1=\lambda^{-1}b-\lambda^{-1}\mu a_2$. Lemma 2 applies and yields $a_1\subset\{b, a_2\}$. Thus $\{a_1, a_2\}\subset\{b, a_2\}$ and we conclude that $\{a_1, a_2\}=\{b, a_2\}$.

Verification of Postulate (C). Suppose that $x\subset\{a, b\}$. By Lemma 2, $x=\lambda a+\mu b$, where $\lambda+\mu=1$, that is, (A.12) holds. Then by the argument in the second paragraph of the proof of Lemma 2

$$x \subset a/b \cup ab \cup b/a \cup a \cup b.$$

Thus

$$\{a, b\} \subset a/b \cup ab \cup b/a \cup a \cup b.$$

The reverse inclusion is always valid ((9.1) of Sec. 9) so that Postulate (C) holds in R_2.

Having verified (C) in the join system R_2, we could immediately infer that (E) holds by Theorem 40, Corollary 2. We have verified (E) independently, in order to make the appendix less dependent on the body of the paper and because our proof is valid for certain join systems in which (C) fails (see Sec. G below).

Observe that all the results of Sections 14, 15, 16 on join systems that satisfy (E) or (C) are valid for R_2.

In order to find the significance of independence in R_2 and to determine its dimension, it is most convenient to have a formula for $\{a_1, \cdots, a_r\}$ in R_2. We generalize Lemma 2 in the following theorem.

THEOREM A. *$\{a_1, \cdots, a_r\}$ is the set of all elements x of R_2 expressible in the form*

$$x = \lambda_1 a_1 + \cdots + \lambda_r a_r, \qquad \lambda_1 + \cdots + \lambda_r = 1. \tag{A.18}$$

Proof. Let $S(a_1, \cdots, a_r)$ be the set of x which satisfy (A.18). First we establish, using induction,

$$\{a_1, \cdots, a_r\} \supset S(a_1, \cdots, a_r). \tag{A.19}$$

Relation (A.19) is valid for $r=1$ (and for $r=2$ by Lemma 2). Suppose (A.19) is valid for $r=k$. Let $x \subset S(a_1, \cdots, a_{k+1})$, which implies

$$x = \lambda_1 a_1 + \cdots + \lambda_k a_k + \lambda_{k+1} a_{k+1}, \qquad \lambda_1 + \cdots + \lambda_{k+1} = 1. \tag{A.20}$$

At least one of the λ's is not 1. It is not restrictive to suppose $\lambda_{k+1} \neq 1$. Then $\lambda = \lambda_1 + \cdots + \lambda_k \neq 0$ and (A.20) implies

$$x = \lambda x' + \lambda_{k+1} a_{k+1}, \tag{A.21}$$

where $x' = \lambda^{-1}(\lambda_1 a_1 + \cdots + \lambda_k a_k)$. By definition of $S(a_1, \cdots, a_k)$ and the induction supposition,

$$x' \subset S(a_1, \cdots, a_k) \subset \{a_1, \cdots, a_k\}.$$

By Lemma 2, (A.21) implies

$$x \subset \{x', a_{k+1}\} \subset \{a_1, \cdots, a_k, a_{k+1}\},$$

so that $\{a_1, \cdots, a_{k+1}\} \supset S(a_1, \cdots, a_{k+1})$, and (A.19) holds for $r = k+1$. Thus (A.19) is valid by induction.

Now we show that $S(a_1, \cdots, a_r)$ is linear. Suppose that $p, q \subset S(a_1, \cdots, a_r)$, and $x \subset \{p, q\}$. A straightforward algebraic argument using Lemma 2 shows that $x \subset S(a_1, \cdots, a_r)$. Thus

$$S(a_1, \cdots, a_r) \supset \{p, q\} \supset pq, p/q$$

and $S(a_1, \cdots, a_r)$ is linear by definition.

Finally note that $S(a_1, \cdots, a_r) \supset a_1, \cdots, a_r$. Thus

$$S(a_1, \cdots, a_r) \supset \{a_1, \cdots, a_r\}$$

by definition of $\{a_1, \cdots, a_r\}$, and the theorem is established.

Now we obtain an algebraic criterion for independence of elements of R_2. The criterion involves the vector "zero," that is the ordered pair $(0, 0)$, which we denote by 0. The context will always indicate whether 0 denotes the real number zero, or the vector zero of R_2.

THEOREM B. *$a_1, \cdots, a_r$ are independent, if and only if, the relations*

$$\text{(A.22)} \qquad \lambda_1 a_1 + \cdots + \lambda_r a_r = 0, \qquad \lambda_1 + \cdots + \lambda_r = 0$$

imply for all choices of $\lambda_1, \cdots, \lambda_r$

$$\text{(A.23)} \qquad \lambda_1 = \cdots = \lambda_r = 0.$$

Proof. Suppose $a_1, \cdots, a_r$ are not independent. We show that (A.22) does not imply (A.23) for certain $\lambda_1, \cdots, \lambda_r$. By Theorem 36, Corollary 4

$$a_i \subset \{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r\}$$

for some a_i. By the last theorem we have

$$a_i = \lambda_1 a_1 + \cdots + \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \cdots + \lambda_r a_r,$$

where $\lambda_1 + \cdots + \lambda_{i-1} + \lambda_{i+1} + \cdots + \lambda_r = 1$. Letting $\lambda_i = -1$, these yield

$$\lambda_1 a_1 + \cdots + \lambda_r a_r = 0, \qquad \lambda_1 + \cdots + \lambda_r = 0.$$

Thus (A.22) holds but (A.23) fails.

Conversely suppose (A.22) does not always imply (A.23). Then there exist numbers $\lambda_1, \cdots, \lambda_r$ not all zero which satisfy (A.22) but not (A.23). Suppose $\lambda_i \neq 0$. Then we can solve (A.22) for a_i in terms of the other a's, and apply the last theorem to yield

$$a_i \subset \{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r\}.$$

Thus by Theorem 36, Corollary 4, $a_1, \cdots, a_r$ are not independent, and our proof is complete.

Now we can show that R_2, as a join system, has dimension 3. Let $e^0 = (0, 0)$, $e^1 = (1, 0)$, $e^2 = (0, 1)$. We show that e^0, e^1, e^2 form a basis (Sec. 14) of R_2. Suppose that

$$\lambda_0 e^0 + \lambda_1 e^1 + \lambda_2 e^2 = 0, \qquad \lambda_0 + \lambda_1 + \lambda_2 = 0.$$

Then $(\lambda_1, \lambda_2) = 0 = (0, 0)$ so that $\lambda_1 = \lambda_2 = 0$. Clearly $\lambda_0 = 0$, and e^0, e^1, e^2 are independent by the last theorem. Further by Theorem A, $\{e^0, e^1, e^2\}$ is the set of elements of the form $\lambda_0 e^0 + \lambda_1 e^1 + \lambda_2 e^2 = (\lambda_1, \lambda_2)$ where $\lambda_0 + \lambda_1 + \lambda_2 = 1$. Thus

$\{e^0, e^1, e^2\} = R_2$ so that e^0, e^1, e^2 form a basis of R_2, and the dimension of R_2 is 3.

B. Generalization to affine n-space. The discussion above can be generalized in several ways. First we can extend it, with little trouble, from ordered pairs to ordered n-tuples of real numbers.

Let R_n be the set of ordered n-tuples $(x_1, \cdots, x_n)$ of real numbers. As in R_2 we write x for $(x_1, \cdots, x_n)$ and define

$$x \pm y = (x_1, \cdots, x_n) \pm (y_1, \cdots, y_n) = (x_1 \pm y_1, \cdots, x_n \pm y_n),$$
$$\lambda x = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n).$$

The element $(0, \cdots, 0)$ of R_n is called the *zero* of R_n and is denoted by 0.

We define join in R_n exactly as in R_2. Then the theory presented in Section A holds just as well for R_n as for R_2 (except for the discussion in the last paragraph of Sec. A on the dimension of R_2). This is not very remarkable, since the algebraic properties employed are independent of the number of components and so are equally valid in R_n and R_2. In fact, if you are familiar with the notion of vector space, you may observe that the discussion of Section A (except for the part on dimension) is applicable to any vector space over the real field (or an ordered field). Thus the definition of join of Section A serves to convert any such vector space into a join system.

To determine the dimension of R_n, define $e^0 = (0, \cdots, 0)$, $e^1 = (1, 0, \cdots, 0)$, $\cdots$, $e^n = (0, \cdots, 0, 1)$ and show, using Theorems A, B that $e^0, e^1, \cdots, e^n$ form a basis of R_n.

Summary. R_n is a join system that satisfies Postulates (E) and (C) and has dimension $n+1$.

C. An infinite-dimensional join system. The restriction to n components in the above discussion is not essential. We can easily extend it to "vectors" with infinitely many components.

Let R^* be the set of all infinite sequences $(x_1, \cdots, x_n, \cdots)$ of real numbers. We write x for $(x_1, \cdots, x_n, \cdots)$ and define

$$x \pm y = (x_1, \cdots, x_n, \cdots) \pm (y_1, \cdots, y_n, \cdots) = (x_1 \pm y_1, \cdots, x_n \pm y_n, \cdots),$$
$$\lambda x = \lambda(x_1, \cdots, x_n, \cdots) = (\lambda x_1, \cdots, \lambda x_n, \cdots).$$

The infinite sequence $(0, \cdots, 0, \cdots)$ is called the *zero* of R^* and denoted by 0.

We define join in R^* as in R_2. The discussion of Section A, being vector algebraic in character, is still independent of the number of components and is applicable to R^*. Thus R^*, with join as defined, is a join system that satisfies Postulates (E) and (C). However unlike R_n, R^* is infinite dimensional (in the sense of Sec. 14, last paragraph).

To show this, let $e^0 = (0, \cdots, 0, \cdots)$ and e^r $(r = 1, 2, \cdots)$ be the element of R^* in which the rth term is 1 and the other terms are 0. By Theorem B, $e^0, e^1, \cdots, e^{n-1}$ are independent for each n. Thus R^* contains n independent

elements for each natural number n—consequently it has no finite basis and is infinite dimensional.

D. The direct product of two join systems. We present in this section a formal procedure for constructing a join system from two given ones.

Let $(G_1, \cdot)$, $(G_2, \cdot)$ be any two join systems—for the sake of convenience we use the same symbol to denote their operations. If $A_1 \subset G_1$, $A_2 \subset G_2$ we use the symbol (A_1, A_2) to denote the cartesian product of A_1 and A_2, that is the set of ordered pairs (a_1, a_2) where $a_1 \subset A_1$, $a_2 \subset A_2$. Let $G = (G_1, G_2)$, the cartesian product of G_1 and G_2. We define a join operation $\cdot$ in G as follows:

$$(a_1, a_2)\cdot(b_1, b_2) = (a_1\cdot b_1, a_2\cdot b_2).$$

We call G with join as defined the *direct product** of the join systems G_1, G_2. We show that G is a join system.

In G the operation $\cdot$ is extended to sets and the inverse operation / is defined in the usual way. Then we can prove

(D.1) $$(A_1, A_2)\cdot(B_1, B_2) = (A_1\cdot B_1, A_2\cdot B_2),$$

(D.2) $$(a_1, a_2)/(b_1, b_2) = (a_1/b_1, a_2/b_2).$$

We justify (D.2). Suppose

(D.3) $$(x_1, x_2) \subset (a_1, a_2)/(b_1, b_2).$$

By definition of /, $(b_1, b_2)\cdot(x_1, x_2) \supset (a_1, a_2)$, so that by definition of join in G $(b_1\cdot x_1, b_2\cdot x_2) \supset (a_1, a_2)$. This implies, by definition of cartesian product,

$$b_1\cdot x_1 \supset a_1, \qquad b_2\cdot x_2 \supset a_2.$$

Thus $x_1 \subset a_1/b_1$, $x_2 \subset a_2/b_2$ and we infer, again using the definition of cartesian product,

(D.4) $$(x_1, x_2) \subset (a_1/b_1, a_2/b_2).$$

Conversely, supposing (D.4), we can retrace the steps to (D.3) so that (D.2) is proved.

Using (D.1), (D.2) it is not difficult to show that (J1), $\cdots$, (J6) hold in G, since the operations which occur in (J1), $\cdots$, (J6) may be performed component by component. As an illustration we show that (J5) holds in G. Suppose that

$$(a_1, a_2)/(b_1, b_2) \approx (c_1, c_2)/(d_1, d_2).$$

By (D.2) $(a_1/b_1, a_2/b_2) \approx (c_1/d_1, c_2/d_2)$ so that $a_1/b_1 \approx c_1/d_1$, $a_2/b_2 \approx c_2/d_2$. Since (J5) is valid in G_1 and G_2, $a_1d_1 \approx b_1c_1$, $a_2d_2 \approx b_2c_2$. Thus

* Compare the idea of direct product in group theory; see, for example, K. S. Miller, Elements of Modern Abstract Algebra, New York, 1958.

$$(a_1d_1, a_2d_2) \approx (b_1c_1, b_2c_2), \qquad (a_1, a_2)\cdot(d_1, d_2) \approx (b_1, b_2)\cdot(c_1, c_2)$$

and (J5) is valid in G.

E. The direct product of two lines. We consider now an interesting case of the direct product of two join systems which serves as an important counterexample.

Let G_1 be the real number system and let a join operation be defined in G_1 as follows: $a\cdot b$, if $a\neq b$, is the set of real numbers between a and b; and $a\cdot a=a$. It is not hard to show that G_1, with $\cdot$ as defined, is a join system—in fact it is equivalent to the join system R_n of Section B when n is taken to be 1.

Now let G be the direct product of G_1 and G_1. Note that G may be considered to be the cartesian plane with an unusual kind of join operation. G is a join system which does not satisfy the Exchange Postulate (E) of Section 14—its existence attests that (E) and the familiar Euclidean theory of incidence (Sec. 14) are independent of (J1), $\cdots$, (J6).

We show that Postulate (E) fails in G. Let x denote the arbitrary element (x_1, x_2) of G. Choose $a=(a_1, a_2)$ and $b=(b_1, b_2)$ such that $a_1<b_1$, $a_2<b_2$ (see Fig. 25). Then $a\cdot b$, the join of a and b in G, consists of all x such that

$$a_1 < x_1 < b_1, \qquad a_2 < x_2 < b_2.$$

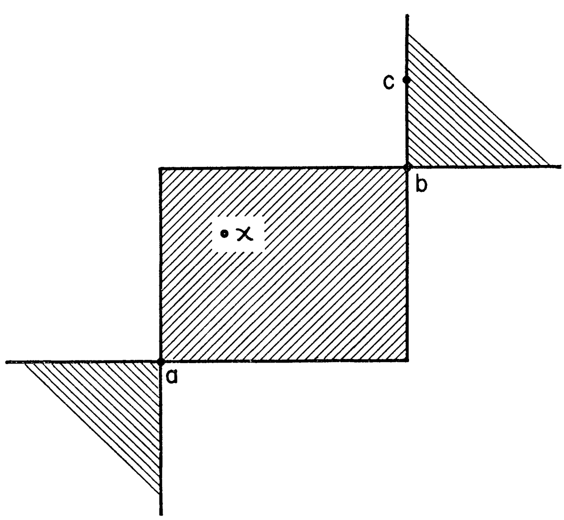


FIG. 25

Thus $a\cdot b$ is represented by the interior of the rectangle which has horizontal and vertical sides and a, b as a pair of opposite vertices (Fig. 25). Furthermore $\{a, b\}$ is seen to contain all the elements of G, and is represented by the cartesian plane (Fig. 25).

Now choose c so that $c_1=b_1$, $c_2\neq b_2$ (Fig. 25). We have

(E.1) $$c \subset \{a, b\}, \qquad c \neq b.$$

Observe that $\{c, b\}$ consists of all y ($y=(y_1, y_2)$) such that $y_1=b_1$, and is represented by the vertical cartesian line determined by b and c. Thus $\{a, b\} \neq \{c, b\}$. This with (E.1) shows that Postulate (E) is false in G.

It immediately follows that the Comparability Postulate (C) of Section 15 fails in G, since Postulate (C) implies Postulate (E) in any join system (Th. 40, Corol. 2).

Furthermore the join system G serves to indicate the restrictiveness in the conventional definition of line, since in G, $a/b \cup ab \cup b/a \cup a \cup b$ is neither a linear nor a convex set (see shaded area in Fig. 25). Note also that the elements a, b, c of G do not satisfy the Comparability Property of Section 15—for they are distinct and collinear but neither of the three is between the other two, since neither is in the join of the other two. This suggests the possibility of a theory of "partially ordered" join systems (or geometries) in which the Comparability Property fails (see Sec. G).

F. Subsystems of a join system. We show now how to construct a join system from a given one by using a suitable type of convex subset. The procedure is similar to that employed at the end of Section 8 to convert a linear subset of a join system into a join system.

Let K be a convex set of a join system G. Since K is closed under join it is natural to ask about its properties relative to the join operation restricted in its application to the elements of K. In particular is K (relative to this operation) a join system? To answer this observe that (J1) holds in K since K is a subset of G which is closed under join. (J2), (J3) and the idempotent law $aa=a$ hold in K since they are formal laws valid for any elements of G. The transposition law (J5), that $a/b \approx c/d$ implies $ad \approx bc$, requires a bit more thought for two reasons: first, it involves the defined operation extension, and the extension operation in K need not coincide with that in G; second, it is not simply a formula since its hypothesis and conclusion are existential statements. However we can express (J5) equivalently as: If $a \approx bx$ and $c \approx dx$ for some x then $ad \approx bc$. This is valid when the elements are restricted to K, since it is valid in G and K is a convex set in G. Similarly the idempotent law $a/a=a$ can be rewritten, $ax \approx a$ if and only if $x=a$, and holds in K.

Summary. (J1), (J2), (J3), (J5), and (J6) are valid in K.

(J4) need not hold in K. Counterexamples are easily found. For example, in a Euclidean geometry (considered as a join system), take K as a closed interval with endpoints p, q or a closed triangular region with vertices p, q, r. In each case (J4) fails, since the extension operation in K satisfies $p/q=\varnothing$. Note however that a segment (open interval) or a triangle interior does satisfy (J4). Thus we have an answer to our question concerning the system K; K is a join system if and only if it satisfies (J4).

Let us examine this more closely. Actually all we require of (J4) is the condition $a/b \neq \emptyset$, since $a/b \subset K$ by definition. Stated a bit more precisely the condition is: If $a, b \subset K$ then $a/b \approx K$ or, equivalently, $a \approx bK$. That is, $a \subset K$ implies $a \subset bK$ for $b \subset K$. This is equivalent to: $K \subset xK$ for $x \subset K$. This is an important property of a convex set which we signalize in the

DEFINITION. *Let K be a convex set in join system G such that $K \subset xK$ for $x \subset K$. Then we say K is an* **open** *convex set or simply K is* **open.**

We summarize our discussion more formally and perspicuously by introducing the term subsystem.

DEFINITION. *Let A be a subset of join system G which is a join system relative to the join operation of G restricted to the elements of A. Then A is a* **subsystem** *(or a sub-join-system) of G.*

Then we may assert the

THEOREM. *Let K be a convex subset of join system G. Then K is a subsystem of G if and only if K is an open convex set.*

Thus every open convex subset of G gives rise to an "induced" join system or subsystem in a natural manner. Can nonconvex sets behave similarly? The answer is no—since if A is a subsystem of G, it must, by definition, satisfy Postulate (J1) and be convex. It is interesting to compare our results with conditions in group theory that a subset be a subgroup.

The theorem yields a new mode of construction of join systems provided we can "find" the open convex sets of a given join system. We show now how to construct the open convex sets of a join system from its convex sets. The Euclidean examples mentioned above suggest the conjecture that if we delete the "boundary" of a convex set, we get an open convex set. We proceed to clarify and justify this for any join system.

Suppose a convex set K is not open—that is, $K \subset xK$ for all $x \subset K$ is false. Then there exist $a, b \subset K$ such that $a \subset bK$ is false, or equivalently $a/b \approx K$ is false. In Euclidean geometry, this means that ray a/b is disjoint to K and consequently that there are points not in K which are as close to a as we please. Thus in the Euclidean case, a would be a boundary point of K. This suggests the

DEFINITION. *Let K be a convex set in a join system and $a \subset K$. Suppose $a \not\subset bK$ (or, equivalently, $a/b \approx K$ is false) for some $b \subset K$. Then we call a, a* **boundary** *element of K. In the contrary case a is an* **interior** *element of K. That is, a is an interior element of K if $a \subset xK$ for $x \subset K$. The* **boundary (interior)** *of K is the set of its boundary (interior) elements.*

Comparing this definition with that of open convex set we note that a convex set is open if and only if it is identical with its interior, or equivalently its boundary is void.

By the definition the interior of convex set K is merely K "minus" its boundary, so we may state our conjecture as follows:

THEOREM. *The interior of a convex set is an open convex set.*

To prove this we introduce the

LEMMA. *Let I be the interior of convex set K. Then $a \subset I$ implies $aK = I$.*

Proof. Let $p \subset aK$. Suppose that $x \subset K$; then by definition of I, $a \subset xK$. Thus

$$p \subset aK \subset xKK = xK$$

and $p \subset I$. Conversely, $p \subset I$ implies $p \subset aK$ by definition of I.

Proof of Theorem. Let K be a convex set and I its interior. The convexity of I follows directly from the lemma, since $a, b \subset I$ implies $ab \subset aK = I$. To show that I is open, let $x \subset I$. Then we have, applying the lemma twice, $I = xK = xxK = xI$ and I is open by definition.

A final remark. Experience with convex sets in Euclidean geometries suggests that a nonvoid convex set has a nonvoid interior. This is not true for all join systems. As an example consider R^*, the infinite dimensional join system discussed in Section C, and let K be the set of infinite sequences of real numbers with no negative terms and only a finite number of positive ones. K is easily shown to be convex. But each element a of K is a boundary element. For if $a = (a_1, \cdots, a_m, 0, 0, \cdots)$ and we let $b = (b_1, \cdots, b_m, 1, 0, 0, \cdots)$ we can show that $a \not\subset bK$.

G. Join systems over a partially ordered field. In Section A we constructed a join system R_2 using algebraic and order properties of the real number system. This was generalized in Section B to a join system of arbitrary finite dimension and in Section C to one of infinite dimension. Now we develop a very different type of generalization: We alter the base field rather than the number of components. The proof in Section A that R_2 is a join system did not employ all the basic order properties of the real numbers. For example the continuity property was not needed. Our treatment in Section A would have worked just as well if we had taken the rational number system as our base field. Evidently we need merely a field satisfying certain familiar order properties in order to obtain the results of Sections A, B, C. There is a well-known concept of *ordered field* (see [1], pp. 48, 8) which is a generalization of the rational and real number systems and our treatment in Sections A, B, C holds without change if we replace R, the real number field, by any ordered field. However the notion of ordered field has been generalized to a concept of *partially ordered field*, over which join systems can be constructed by the method of Section A. Thus we shall introduce the notion of a partially ordered field and show that the construction of Section A yields a join system when applied to any partially ordered field—the discussion automatically covers ordered fields. However a join system over a partially

ordered field need not satisfy all the properties of R_2. We shall give an example of one which satisfies Postulate (E) but not Postulate (C). This is an interesting join system—its points and lines form an affine geometry but neither (C) nor Pasch's Postulate (Section 16) holds. It is thus an independence example for these principles relative to the postulate set composed of (J1), $\cdots$, (J6) and (E).

We introduced the notion of a partially ordered field in [**11**] with the object of constructing a "partially ordered geometry"—this may be described essentially as a join system which satisfies (E) but not (C). A partially ordered field is defined there to be a field in which a 3-term relation (betweenness) is assigned, which satisfies postulates suggested by certain betweenness properties of the real numbers. Recently Dubois has introduced a definition of partially ordered field* based on the notion of positive element. His formulation is quite simple and lends itself to our present purpose. We state it as follows:

DEFINITION. *A partially ordered field is a field with a prescribed subset P, called the set of* **positive** *elements, which satisfies the following postulates:*

(O1). *P is nonvoid.*

(O2). *P is closed under addition, multiplication and the operation of forming reciprocals.*

A partially ordered field F is called **fully ordered** *or simply an* **ordered field** *if it satisfies the trichotomy law:*

(O3). *If a is an element of F just one of the following alternatives holds: a is an element of P, $a=0$, or $-a$ is an element of P.*

We quickly sketch the elementary properties of partially ordered fields. We begin as in the theory of ordered fields (or ordered integral domains) (see [**1**], pp. 8, 9), by defining "less than" and deriving its elementary properties which are independent of trichotomy. We define $a<b$ (or $b>a$) to mean $b-a$ is positive, that is $b-a$ is in P. If $a<0$ we say a is *negative*. Note that a is positive if and only if $0<a$, or equivalently $-a$ is negative. The relation $<$ has the following properties: transitiveness, $a<b$ and $b<c$ imply $a<c$; asymmetry, $a<b$ implies $b<a$ false; irreflexiveness, $a<a$ is false. The monotonic laws hold: $a<b$ implies (1) $a+x<b+x$ for arbitrary x; (2) $ax<bx$ for positive x; and (3) $bx<ax$ for negative x. The familiar multiplicative laws for positive and negative numbers are valid: The product of two positive or two negative numbers is positive, and the product of a positive and a negative is negative.

Similar properties hold for quotients. First observe that Postulate (O2) implies P is closed under division. Then if b is negative, so is its reciprocal. For $b<0$ implies $-b$ is positive, so that its reciprocal $1/(-b)=-1/b$ is positive, and $1/b$ is negative. Then we can assert that the quotient of two positive or two nega-

* See D. W. Dubois, On partly ordered fields, Proc. Amer. Math. Soc., vol. 7, 1956, pp. 918–930. I suspect that the two notions of partially ordered field are equivalent but I have not yet proved this.

tive numbers is positive, and the quotient of a positive by a negative or a negative by a positive is negative.

For an ordered field the trichotomy law can be extended ([1], p. 8) to the form: Exactly one of the statements $a<b$, $a=b$, $b<a$ holds.

Now we indicate that any partially ordered field F has a subfield isomorphic to the rational number field. We begin with the order relations of the elements 0, 1 of F. By Postulate (O1), there is an element a in P. By Postulate (O2), P contains $a \div a = 1$. Thus $0<1$. Adding 1 to both sides we get $1<2$. Similarly $2<3$ and so on. That is we have

$$0 < 1 < 2 < 3 < \cdots .$$

Thus the positive integral elements of F are ordered in the usual way. It follows that the integral elements of F form an infinite set (F has zero characteristic) and that they behave (as regards $+$, $\times$, $<$) exactly like the ordinary signed integers. Furthermore the *rational elements* of F (quotients of integral elements) behave like the rational numbers—that is, they form a subsystem of F isomorphic to the ordered field of rational numbers. Note in particular that the rational elements of F are ordered in the familiar way, for example, in F, $0<1/2<1$.

Now let F be any partially ordered field. We form F_2 the set of ordered pairs (x_1, x_2) of elements of F. We define join in F_2 exactly as in R_2 (Sec. A). We assert that F_2 is a join system. To justify this, merely reexamine the proof that R_2 is a join system in Section A and observe that the order properties of real numbers employed are valid in any partially ordered field. Nowhere in the verification of (J1), $\cdots$, (J6) did we assume trichotomy. For example in verifying (J3) and (J5) we employed the property that the sum, product and quotient of positive numbers is positive; in the verification of (J4) we used $0<1/2$.

Furthermore the proof of Lemma 1 of Section A holds for F_2; that of Lemma 2 fails however since the trichotomy principle is required to establish relation (A.16). Indeed Lemma 2 does not hold for all partially ordered fields F. As a counterexample, we can take the complex number field and convert it into a partially ordered field by specifying P to be the set of positive real numbers.

Thus Sections A, B, C are not completely valid when we replace the real number system by an arbitrary partially ordered field. However it is easily verified that the theory is valid for any *ordered* field. In particular if F is an ordered field, the join system F_2 satisfies Postulates (E) and (C).

Now we come to our main problem: To construct a join system which satisfies (E) and not (C). Our procedure is to construct a partially ordered field G which is not fully ordered, but such that Lemma 2 holds for the join system G_2. Then we employ Lemmas 1 and 2 to establish Postulate (E) for G_2 and show that Postulate (C) fails because G is not fully ordered.

Let G be the field of rational functions in a real variable y with real coefficients, and let the set P of positive elements be composed of the rational functions other than zero which assume no negative value. G, with P as assigned, is a partially ordered field. In G, $a<b$ means that $a\neq b$ and $a(y)\leqq b(y)$ for all y

for which a and b are defined. G is not fully ordered since the identity function θ defined by $\theta(y) = y$ satisfies none of the conditions, $\theta < 0$, $\theta = 0$, $0 < \theta$. The following property of G is crucial in our discussion: For each a in G there exists b in G such that $a < b$ and $0 < b$. That is each element of G is "dominated" by a positive element. To show this observe that $a(y) < 1 + a^2(y)$.

We recall that Lemma 1 of Section A holds for G_2, since it is valid for any partially ordered field. Next we prove Lemma 2 valid for G_2. We must show that given elements a, b of G_2, $\{a, b\}$ is the set S of all x of G_2 which satisfy

$$x = \lambda a + \mu b, \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1, \tag{G.1}$$

where λ, μ are in the base field G. Reviewing the proof of Lemma 2, Section A as given for the real field R, we see that the first part is valid for any partially ordered field. Thus $\{a, b\} \subset S$, and we have merely to show the reverse inclusion.

Suppose then x is in S, that is, x satisfies (G.1). We introduce auxiliary elements c, d to relate x to a and b. Let λ' satisfy: $\lambda < \lambda'$, $1 < \lambda'$. We determine c, d by

$$c = \lambda' a + \mu' b, \qquad \mu' = 1 - \lambda'; \qquad d = (\lambda' + 1)a + (\mu' - 1)b.$$

A formal calculation verifies that $x = (1 + \lambda' - \lambda)c + (\lambda - \lambda')d$. Here $\lambda - \lambda' < 0$ and Lemma 1 implies $x \subset c/d$. Further $\mu' < 0$ so that $c \subset a/b$ by Lemma 1. Similarly $d \subset a/b$. Thus $x \subset \{a, b\}$ and $S \subset \{a, b\}$. This completes the verification of Lemma 2 for G_2.

It is immediate that G_2 satisfies Postulate (E), for its verification in Section A depends merely on Lemma 2.

G_2 does not satisfy Postulate (C). To see this suppose the contrary and let a, b be chosen as the elements $(1, 0)$, $(0, 0)$ of G_2. Let $c = (\theta, 0)$ where θ is the identity function of G. Note that θ bears no order relation to 0 or 1. We have

$$c = \theta a + (1 - \theta)b. \tag{G.2}$$

Lemma 2 implies that c is in $\{a, b\}$. Thus by Postulate (C) $c \subset ab \cup a/b \cup b/a \cup a \cup b$. Since $c \neq a, b$ we have

$$(1)\ c \subset ab, \qquad (2)\ c \subset a/b, \quad \text{or} \quad (3)\ c \subset b/a.$$

Note that the coefficients of a and b in (G.2) are uniquely determined. In view of (G.2), (1) implies by definition of join that $0 < \theta$, (2) implies by Lemma 1 that $1 - \theta < 0$ or $1 < \theta$, and similarly (3) yields $\theta < 0$. Thus G_2 cannot satisfy Postulate (C).

We conclude with some remarks concerning the join system G_2. In some respects it is a familiar kind of system. Its points and lines form an affine geometry. Actually its lines are representable as linear equations over field G just as the lines of a Euclidean plane are representable by linear equations over the real field. In fact $x = \lambda a + \mu b$, $\lambda + \mu = 1$, $(a \neq b)$ is a sort of parametric equation for the line $\{a, b\}$ and we can show that

$$(a_2 - b_2)x_1 + (b_1 - a_1)x_2 + a_1 b_2 - a_2 b_1 = 0$$

is an equivalent expression for it in ordinary linear (nonparametric) form.

Moreover G_2 admits a notion of betweenness for collinear points (see the first definition of Sec. 15) which enjoys many of the familiar properties of betweenness in Euclidean geometry, but does not satisfy the Comparability Property of Section 15, that is there exist three distinct collinear points no one of which is between the other two. Thus the points of a line in G_2 need not be "fully ordered" by the betweenness relation and form, relative to it, a *partially ordered* set in the sense of [11].

A comparable situation exists in the plane, for Pasch's Postulate (Sec. 16) fails in G_2—that is, there exists a line and three distinct points not on it such that no two of the points are on the same side of the line. To show this, let line $L=\{p, q\}$, where $p=(0, 0)$, $q=(1, 0)$ and let $a=(0, 1)$, $b=(0, -1)$, $c=(1, \theta)$, where θ is the identity function of G. In view of Lemma 2, L is the set of points x $(x=(x_1, x_2))$ which satisfy $x_2=0$. Thus $L \not\supset a, b, c$. Similarly we show $c \not\subset \{a, b\}$ so that a, b, c are independent. Observe that the discussion at the end of Section A on the dimension of R_2 applies to G_2, since it is based on Lemmas 1 and 2 and Postulate (E). Thus G_2 has dimension 3 and $L \subset G_2=\{a, b, c\}$. Finally L meets ab, since L contains (0, 0). Thus if Pasch's Postulate is valid in G_2, L meets ac or bc. Suppose the former. Then

$$(x_1, 0) = \lambda(0, 1) + \mu(1, \theta), \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1$$

holds for some x_1, λ, μ of G. Thus $0=\lambda+\mu\theta$ so that $\theta=-\lambda\cdot\mu^{-1}<0$, which is impossible. Similarly, L meets bc implies $\theta>0$ which is equally impossible, and G_2 does not satisfy Pasch's Postulate. As is well known this implies that (plane) G_2 is not separated by all of its lines.

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BERNOULLI NUMBERS MODULO 27000

J. S. FRAME, Michigan State University

1. Introduction. Bernoulli numbers play an important role in many interesting series expansions related to trigonometric and hyperbolic functions. They may be defined either as the coefficients B_k in the expansion

$$(1.1)\quad 1 - \tfrac{1}{2}x \cot \tfrac{1}{2}x = B(x) = \sum_1^\infty B_k \frac{x^{2k}}{(2k)!} = \frac{1}{6}\frac{x^2}{2!} + \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} + \cdots$$

or as the related coefficients b_k in the expansion

$$(1.2)\quad \frac{x}{e^x - 1} = b(x) = \sum_0^\infty b_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} - \cdots.$$

The function $b(x)+\frac{1}{2}x$ is equal to $\frac{1}{2}x \coth \frac{1}{2}x$ so the b_k are given in terms of B_k by the formulas

$$(1.3)\quad b_0 = 1,\ b_1 = -\tfrac{1}{2}; \qquad b_{2k} = (-1)^{k-1}B_k,\ b_{2k+1} = 0 \qquad \text{for } k > 0.$$

Our purpose in this paper is to derive a number of congruence relations* for the B_k, using powers of 2, 3, and 5 and their products as moduli. Perhaps the most interesting of these relations is the congruence

$$(1.4)\qquad 30B_{2m} \equiv 1 + 600\binom{m-1}{2} \pmod{27000}.$$

2. Some important properties of Bernoulli numbers. From the identities

$$(2.1)\qquad x \tan \tfrac{1}{2}x = x \cot \tfrac{1}{2}x - 2x \cot x = 2[B(2x) - B(x)],$$

$$(2.2)\qquad x \csc x = x \cot \tfrac{1}{2}x - x \cot x = 1 + B(2x) - B(x),$$

it follows that the coefficients T_k in $x \tan \frac{1}{2}x$ and D_k in $x \csc x$ defined by

$$(2.3)\qquad x \tan \tfrac{1}{2} x = \sum_{k=1}^\infty T_k \frac{x^{2k}}{(2k)!}, \qquad x \csc x = \sum_{k=0}^\infty D_k \frac{x^{2k}}{(2k)!},$$

are the following simple multiples of B_k:

$$(2.4)\qquad T_k = 2(4^k - 1)B_k, \qquad D_k = (4^k - 2)B_k = \tfrac{1}{2}T_k - B_k.$$

Since the coefficients T_k satisfy the recurrence relation [2]

$$(2.5)\quad T_1 = 1, \quad T_k = \binom{k}{2}T_{k-1} - \binom{k}{4}T_{k-2} + \binom{k}{6}T_{k-3} - \cdots, \ k > 1,$$

they are integers. Also since the coefficient of $x^{2k-1}/(2k-1)!$ in the expansion of $\tan x$ is the integer $4^{k-1}T_k/k$, it follows that T_k is an odd integer divisible by

* The author wishes to acknowledge the assistance of Mr. Ted Petrie in searching the literature on this subject.

the highest odd factor of k. On computing the first eight T_k from (2.5), and the corresponding B_k from (2.4), we find the following values:

(2.6)

k	1	2	3	4	5	6	7	8
T_k	1	1	3	17	155	2073	38227	929569
B_k	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$

It is clear from (2.4) that the denominator of B_k is twice an odd factor of 4^k-1. More precisely, this denominator is known by the Staudt-Clausen theorem [6] to be the product of all those distinct primes $p_{k,r}$ such that $p_{k,r}-1$ divides $2k$. In fact, Clausen and von Staudt [6] proved the congruence

$$(-1)^k B_k \equiv \sum_r p_{k,r}^{-1} \pmod 1, \tag{2.7}$$

where $(p_{k,r}-1)\,|\,2k$. For example

$$-B_1 = -\frac{1}{6} = \frac{1}{2} + \frac{1}{3} - 1, \quad B_6 = \frac{691}{2730} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} - 1. \tag{2.8}$$

Such equations imply congruences for the numerators N_k of B_k modulo each of the prime divisors of the denominator.

Besides these congruences for B_k with prime moduli, there are exact formulas for expressing B_k in terms of the sum of the series $\sum n^{-2k}$, and thus obtaining simple approximations to its magnitude. The function $(\sin \pi x)/(\pi x)$ vanishes at all the positive and negative integers and has the factorization

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right). \tag{2.9}$$

Differentiation of the logarithm of both members gives

$$\pi \cot \pi x - 1/x = -2x \sum_{n=1}^{\infty} (n^2 - x^2)^{-1}. \tag{2.10}$$

Comparing (2.10) and (1.1) we then see that

$$B(2\pi x) = 2 \sum_{n=1}^{\infty} \frac{x^2/n^2}{1 - x^2/n^2} = 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x^{2k}/n^{2k}. \tag{2.11}$$

Then comparing coefficients of $2x^{2k}$ we find

$$\frac{B_k}{2} \frac{(2\pi)^{2k}}{(2k)!} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \zeta(2k) = \prod (1 - p^{-2k})^{-1}, \tag{2.12}$$

where the last member of (2.12) is the factored form of the zeta function and the product is over all primes p. Use of Stirling's formula for $(2k)!$ yields the

approximation

$$(2.13)\qquad B_k \sim 2\sqrt{(4\pi k)}(k/\pi e)^{2k}\zeta(2k) \sim \sqrt{(50k)}(0.117k)^{2k}\cdot 1 \quad \text{for } k > 4.$$

Also from (2.12) we may derive a close inequality for the ratio B_k/B_{k-1}:

$$(2.14)\qquad 1 - \frac{1}{4^{k-1}-2} < \frac{\pi^2}{k(k-\frac{1}{2})}\,\frac{B_k}{B_{k-1}} < 1 - \frac{3}{4^k - 1}, \qquad k > 2.$$

The inequalities (2.14), together with the exact Staudt-Clausen formula for the denominator of B_k, are sufficient to determine the integer N_k if it is less than 4^k. For example, given $B_5 = 5/2\cdot 3\cdot 11$ and $B_6 = N_6/2\cdot 3\cdot 5\cdot 7\cdot 13$, (2.14) becomes for $k=6$

$$(2.15)\qquad 1 - \frac{1}{1022} < \frac{4\pi^2}{12\cdot 11}\,\frac{2\cdot 3\cdot 11}{2\cdot 3\cdot 5\cdot 7\cdot 13}\,\frac{N_6}{5} < 1 - \frac{1}{1365}$$

and we find

$$(2.16)\qquad \frac{6825}{\pi^2}\cdot\frac{1021}{1022} = 690.840 < N_6 < \frac{6825}{\pi^2}\cdot\frac{1364}{1365} = 691.010.$$

Hence N_6 is the integer 691.

The exact computations in (2.16) can be avoided, however, if we use (1.4) together with a rough estimate of B_k. Since π^2 is nearly 10, we obtain from (2.13) or (2.14) the rough estimates $\frac{1}{4}$, 1, 7, 50, and 500 for $B_6, B_7, B_8, B_9, B_{10}$, and the corresponding rough estimates 680+, 6+, 3570+, 40000+, and 165,000 for their numerators. But by (1.4) we have

$$(2.17)\qquad \begin{aligned} N_6 &= 91(30B_6) \equiv 91[1+600] \equiv 54691 \equiv 691 \pmod{27000},\\ N_8 &= 17(30B_8) \equiv 17[1+1800] \equiv 30617 \equiv 3617 \pmod{27000},\\ N_{10} &= 11(30B_{10}) \equiv 11[1+3600] \equiv 39611 \equiv 174611 \pmod{27000}. \end{aligned}$$

Thus the numerators N_6, N_8, and N_{10} are exactly 691, 3617, and 174611.

To derive the congruence (1.4) and a more complicated one with modulus 81000000 we first study congruences for B_k with moduli 4^9, 3^5, and 5^6.

3. Congruences modulo 4^9.

We write the identity

$$(3.1)\qquad \frac{2x}{e^x - 1} - \frac{2x}{e^{2x}-1} = \frac{2xe^x}{e^{4x}-1} - \frac{2xe^{-x}}{e^{-4x}-1}$$

in the form

$$(3.2)\qquad 2b(x) - b(2x) = \tfrac{1}{2}[e^x b(4x) + e^{-x} b(-4x)].$$

Then we equate coefficients of $x^{2n}/(2n)!$ using (2.1) and obtain

$$(3.3)\qquad (2-4^n)b_n = 1 + 4b_1\binom{2n}{1} + 4^2 b_2\binom{2n}{2} + \cdots + 4^r b_r\binom{2n}{r} + \cdots.$$

Making use of the binomial identity

$$\binom{2n}{r} = \sum_k 2^{2k-r}\binom{k}{2k-r}\binom{n}{k} \tag{3.4}$$

and then replacing r by $2k-2s$ for even r, we obtain

$$\begin{aligned}(2-4^n)b_{2n} &= 1 - 4n + \sum_{k>s} 4^{2k-s}b_{2k-2s}\binom{k}{2s}\binom{n}{k} \\ &= 1 - 4n/3 + \sum_{k=2}^{\infty} 4^k\binom{n}{k}\sum_{s<k}\binom{k}{2s}4^{k-s}b_{2k-2s} \\ &= 1 - 4n/3 + \sum_{k=2}^{\infty} 4^k\binom{n}{k}\sum_{s<k}\binom{k}{2s}b_{2k-2s},\end{aligned} \tag{3.5}$$

where the last simplification is obtained by substituting T_k from (2.4) into (2.5). Thus B_n satisfies the congruence

$$\begin{aligned}(-1)^{n-1}(2-4^n)B_n &\equiv 1 - \frac{4n}{3} + \frac{4^3}{30}\binom{n}{2} - \frac{4^5}{210}\binom{n}{3} + \frac{4^6}{210}\binom{n}{4} \\ &\quad - \frac{4^8}{462}\binom{n}{5} + \frac{191.4^9}{10010}\binom{n}{6} \pmod{2^{21}}.\end{aligned} \tag{3.6}$$

This was obtained by Frobenius by another method ([3], p. 821) with the comment that these results are hardly accessible by a derivation with the help of the exponential function.

In a recent paper [1], Carlitz stated that

$$2b_{2n} \equiv 1 \pmod{2^{r+1}}, \tag{3.7}$$

where 2^r is the highest power of 2 dividing $2n$. Clearly his congruence is a special case of (3.6).

For even n we set $n=2m$ in (3.6) and multiply by -15. Thus

$$\begin{aligned}(1-2^{4m-1})30B_{2m} &\equiv -15 + 40m - 2^5\binom{2m}{2} + \frac{2^9}{7}\binom{2m}{3} \\ &\quad - \frac{2^{11}}{7}\binom{2m}{4} + \frac{5\cdot 2^{15}}{77}\binom{2m}{5} - 5\cdot 2^{17}\binom{2m}{6} \pmod{2^{21}}.\end{aligned} \tag{3.8}$$

Changing binomial coefficients to the exponent m we then have

$$\begin{aligned}&(1-2^{4m-1})30B_{2m} \\ &\quad \equiv -15 + 8m - 2^7\binom{m}{2} - 3\cdot 2^{12}\binom{m}{3} + 9\cdot 2^{15}\binom{m}{4} \pmod{2^{19}}.\end{aligned} \tag{3.9}$$

Correcting for the term 2^{4m-1} for $m = 1, 2, 3, 4$ we have an expression for

$30B_{2m} (\text{mod } 2^{18})$.

$$(3.10)\quad \begin{aligned} 30B_{2m} &\equiv 1 + 2^3(m - 2 + \delta_{m1}) - 2^7\left[\binom{m}{2} - \delta_{m2}\right] - 2^{11}\left[6\binom{m}{3} - 9\delta_{m3}\right] \\ &\quad + 2^{15}\left[9\binom{m}{4} + \delta_{m4}\right] \pmod{2^{18}}. \end{aligned}$$

The corresponding formula for odd Bernoulli numbers is found to be

$$(3.11)\quad 6B_{2m+1} \equiv (2\delta_{m0} - 1) + 8(4\delta_{m1} - 3)/7 + \frac{11\cdot 2^7}{7}\binom{m}{2} - 2^9\delta_{m2} - 2^{12}\binom{m}{3} \pmod{2^{15}}.$$

4. Congruences with the modulus 3^5. We next write the identity

$$(4.1)\quad 3b(x) - b(3x) = e^x b(3x) + e^{-x} b(-3x)$$

and equate coefficients of $x^{2n}/(2n)!$ on both sides. Thus

$$(4.2)\quad \begin{aligned} (3 - 3^{2n})b_{2n} &= 2\sum_{r=0}^{2n}\binom{2n}{r}3^r b_r \\ &\equiv 2 - 6n + 3\binom{2n}{2} - \frac{3^3}{5}\binom{2n}{4} \pmod{3^5} \\ &\equiv 2 - 3n - 42\binom{n}{2} + 81\binom{n}{3} + 108\binom{n}{4} \pmod{3^5}, \end{aligned}$$

$$(4.3)\quad \begin{aligned} &(-1)^n(1 - 3^{2n-1})30B_n \\ &\quad \equiv -20 + 30n - 66\binom{n}{2} - 81\binom{n}{3} + 108\binom{n}{4} \pmod{3^5}. \end{aligned}$$

Using (3.4) again to change the binomial coefficients when $n = 2m$ is even we have

$$(4.4)\quad (1 - 3^{4m-1})30B_{2m} \equiv -20 - 6m + 33\binom{m}{2} - 27(64)\binom{m}{4} \pmod{3^5}.$$

Only when $m = 1$ does the term involving 3^{4m-1} appear, so

$$(4.5)\quad 30B_{2m} \equiv 1 + 6\binom{m-1}{2} + 27\left[\delta_{m1} - 2 + m + \binom{m-1}{2} - 64\binom{m}{4}\right] \pmod{3^5},$$

$$(4.6)\quad 30B_{2m} \equiv 1 + 600\binom{m-1}{2} + 27\left[\delta_{m1} - 2 + m - 3\binom{m-1}{2} - 64\binom{m}{4}\right] \pmod{3^5}.$$

A corresponding congruence for the odd Bernoulli numbers is

$$6B_{2m+1} \equiv 3\delta_{m0} - 2 - 102m + 96\binom{m}{2} + 27\binom{m}{3} + 54\binom{m}{4} \pmod{3^5}. \tag{4.7}$$

5. Congruences with the modulus 5^6. Here we use the identity

$$5b(x) - b(5x) = [\cosh \tfrac{1}{2}x + \cosh \tfrac{3}{2}x] 5x \operatorname{csch} \tfrac{5}{2}x. \tag{5.1}$$

Comparing coefficients of $x^{2n}/(2n)!$ and recalling (2.2), we have

$$\begin{aligned}(5 - 5^{2n})b_{2n} \equiv{}& 2\left[\left(\frac{5}{4} - 1\right)^n + \left(\frac{5}{4} + 1\right)^n\right]\binom{2n}{0} \\ & - \frac{2}{3}\left[\left(\frac{5}{4} - 1\right)^{n-1} + \left(\frac{5}{4} + 1\right)^{n-1}\right]\binom{2n}{2}\left(\frac{5}{2}\right)^2 \\ & + \frac{14}{15}\left[\left(\frac{5}{4} - 1\right)^{n-2} + \left(\frac{5}{4} + 1\right)^{n-2}\right]\binom{2n}{4}\left(\frac{5}{2}\right)^4 \pmod{5^6}.\end{aligned} \tag{5.2}$$

Separate expansions must be used for odd and even n, and we consider only the case $n = 2m$, in expanding the binomials in 5/4.

$$\begin{aligned}(5 - 5^{2n})b_{2n} \equiv{}& 4 + 4\binom{n}{2}\left(\frac{5}{4}\right)^2 + 4\binom{n}{4}\left(\frac{5}{4}\right)^4 - \binom{2n}{3}\left(\frac{5}{2}\right)^3 \\ & - \frac{1}{6}\binom{n-1}{3}\binom{2n}{2}\left(\frac{5}{2}\right)^5 + \frac{14}{3}\binom{2n}{4}\left(\frac{5}{2}\right)^3 \pmod{5^6},\end{aligned} \tag{5.3}$$

$$\begin{aligned}(5 - 5^{2n})B_n \equiv{}& -4 - \binom{n}{2}\left(\frac{5}{2}\right)^2 + \binom{2n}{3}\left(\frac{5}{2}\right)^3 - \frac{5^4}{4^3}\binom{n}{4} \\ & + \frac{2}{3}(2n - 1)\binom{n}{4}\left(\frac{5}{2}\right)^5 - \frac{14}{3}\binom{2n}{4}\left(\frac{5}{2}\right)^3 \pmod{5^6}.\end{aligned} \tag{5.4}$$

Since $(2n+2)\binom{n}{4}$ is divisible by 5, the right member is a polynomial in n of degree 4 (mod 5^6). It has the simplified expansion

$$(5 - 5^{4m})B_{2m} \equiv -4 - \frac{2}{3}\binom{2m}{2}5^2 - 6\binom{2m}{3}5^3 + 4\binom{2m}{4}5^3 \pmod{5^6}. \tag{5.5}$$

We multiply by 6 and correct for the term in 5^{4m-1} when $m = 1$.

$$\begin{aligned}&(1 - 5^{4m-1})30B_{2m} \\ &\equiv 1 - 25 - 4\binom{2m}{2}5^2 - 36\binom{2m}{3}5^3 + 24\binom{2m}{4}5^3 \pmod{5^6} \\ &\equiv 1 - 25 - 100m + 225\binom{m}{2} + 384\binom{m}{4}5^3 \pmod{5^6},\end{aligned} \tag{5.6}$$

$$(5.7)\quad 30B_{2m} \equiv 1 + 600\binom{m-1}{2} + 5^3\left[+ m - 2 + \delta_{m1} - 3\binom{m-1}{2} + 384\binom{m}{4}\right] \pmod{5^6}.$$

We shall merely state without proof the corresponding congruence for the odd-numbered Bernoulli numbers, namely

$$(5.8)\quad 6B_{2m+1} \equiv 1 + 5(\delta_{m0} - 1) + 897m - 85\binom{m}{2} + 700\binom{m}{3} - 375\binom{m}{4} \pmod{5^5}.$$

6. Congruences modulo 27000 **and** 7776000000. We now combine the congruences for $30B_{2m}$ that we have found with modulus 2^{18} in (3.11), with modulus 3^5 in (4.6), and with modulus 5^6 in (5.7), noting that in all three cases the right member is a fourth-degree polynomial in m for $m>4$. Our main congruence (1.4) follows immediately, and we define the integers α, β, γ, ϵ and the rational number R with denominator prime to 30 by the abbreviations

$$(6.1)\qquad \alpha = m - 2 + \delta_{m1},\quad \beta = \binom{m-1}{2},\quad \gamma = \binom{m}{4},\quad \epsilon = \binom{m}{2} - \delta_{m2},$$

$$(6.2)\qquad 30B_{2m} = 1 + 600\beta + 27000R.$$

THEOREM. *The rational number R defined by* (6.2), *whose denominator is the same as that of $30B_{2m}$, satisfies the congruence*

$$(6.3)\quad R \equiv 11(\alpha - 3\beta) + 2500(5\alpha + 3\beta) + 224\gamma + 54000(\alpha - 3\beta - 2\gamma - \epsilon) \pmod{288000}.$$

(By substituting this value of R in (6.2), a congruence for $30B_{2m}$ is obtained modulo $2^{11}\cdot 3^5\cdot 5^6 = 7776000000$.)

Proof. Using the abbreviation (6.2) in (3.11), (4.6) and (5.7), and dividing by 2^3, 3^3 and 5^3, respectively, we have

$$(6.4)\qquad (15)^3R \equiv \alpha - 75\beta - 16\epsilon \pmod{2^8},$$

$$(6.5)\qquad (10)^3R \equiv \alpha - 3\beta - 64\gamma \pmod{3^2},$$

$$(6.6)\qquad 6^3R \equiv \alpha - 3\beta + 9\gamma \pmod{5^3}.$$

Since $(1+5)^{-3} \equiv 11 \pmod{5^3}$, we satisfy (6.6) by introducing a fraction S with denominator prime to 5 such that

$$(6.7)\qquad R = 11(\alpha - 3\beta + 9\gamma) + 125(\gamma + S).$$

The congruence (6.5) is now satisfied by introducing T so that

$$(6.8)\qquad S = 100\alpha + 60\beta + 9T.$$

To simplify (6.4) we multiply both sides by -49, which is congruent to

$(-1+16)^{-3} \pmod{2^8}$. Substitution of the value of R from (6.7) in (6.4) then gives

$$(6.9)\qquad 11(\alpha - 3\beta) + 224\gamma + 125S \equiv -49\alpha + 49(75)\beta + 16\epsilon \pmod{2^8}.$$

Solving for $125S$ and using (6.8) we obtain

$$(6.10)\qquad \begin{aligned} 125(100\alpha + 60\beta + 9T) &\equiv -60\alpha + 124\beta + 32\gamma + 16\epsilon \pmod{2^8}, \\ -3(25\alpha + 15\beta + 9T/4) &\equiv -15\alpha + 31\beta + 8\gamma + 4\epsilon \pmod{2^6}. \end{aligned}$$

Solving for $T/4$ we find

$$(6.11)\qquad T/4 \equiv 12(\alpha - 3\beta - 2\gamma - \epsilon) \pmod{2^6}.$$

The congruence (6.3) now follows from (6.7), (6.8) and (6.11).

Although it is less elegant than the congruence for B_{2m}, we include an analogous congruence for B_{2m+1} derived from (3.13), (4.7) and (5.8):

$$(6.12)\qquad \begin{aligned} 6B_{2m+1} \equiv 1 &- 6m/7 + 2040\binom{m}{2} - 60\binom{m}{3} \\ &+ 8130\left[\delta_{m0} + \binom{m-1}{3} + \binom{m+1}{3}\right] \pmod{27000}. \end{aligned}$$

In conclusion we compute $B_{12} = N_{12}/2730$ by (6.2) and (6.3), using (2.13) to obtain the approximate magnitude

$$(6.13)\qquad N_{12} \sim \sqrt{(600)}(1.404)^{24}.2730 \sim 2.3 \times 10^8.$$

For $m=6$, (6.1) gives the parameter values

$$(6.14)\qquad \alpha = 4, \quad \beta = 10, \quad \gamma = 15, \quad \epsilon = 15,$$

and (6.2) gives

$$(6.15)\qquad N_{12} = 91(30B_{12}) = 546091 + 27000(91R).$$

From (6.13) and (6.15) the magnitude of $91R$ is about 9000. From the congruence in our main theorem we have

$$(6.16)\qquad \begin{aligned} 91R &\equiv 1001(-26) + 91(2500)(50) + 91(3360) \pmod{18000} \\ &\equiv 11654734 \equiv 8734 \pmod{18000}. \end{aligned}$$

Hence $91R$ is 8734 and

$$(6.17)\qquad N_{12} = 546091 + 27000(8734) = 236364091.$$

The twelfth Bernoulli number is $B_{12} = 236364091/2730$.

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RATIONAL TRIANGULATIONS

FRED SUPNICK, City College, New York

1. Introduction. A triangle with rational sides and rational area is referred to as a *rational triangle.* Let S_1 denote the configuration consisting of three mutually tangent circles which are not tangent at one point (some but not all three circles may be straight lines; if two are straight lines then they must be parallel). Let S_i denote the configuration consisting of S_{i-1} and all circles each of which is tangent to three mutually tangent circles of S_{i-1}. Let S denote the smallest configuration containing each S_i $(i=1, 2, \cdots)$. Let w be a triangle whose vertices are the centers of three mutually tangent circles of finite radius of S. Let W denote the set of all triangles w.

We prove the following theorem:

THEOREM 1. *Either all or none of the triangles of W are rational.*

The theorem can be applied to obtain a class of solutions of the functional equation

$$f(x_1^*)f(x_2^*)f(x_3^*)(f(x_1^*) + f(x_2^*) + f(x_3^*)) = R^2, \tag{1.1}$$

R rational (*i.e.*, functions $f(x^*)$ and suitable triplets (x_1^*, x_2^*, x_3^*) can be determined satisfying (1.1)). Suppose that the triangles of W were rational. If it were possible to obtain a function $f(x^*)$ defined over a domain D, and triplets $\{x_{1,i}^*, x_{2,i}^*, x_{3,i}^*\} \subset D$ $(i=1, 2, \cdots)$, so that $f(x_{1,i}^*)$, $f(x_{2,i}^*)$, $f(x_{3,i}^*)$ were the lengths of the radii of mutually tangent circles $C(x_{1,i}^*)$, $C(x_{2,i}^*)$, $C(x_{3,i}^*)$ of S with no interior point in common, then the area

$$(f(x_{1,i}^*)f(x_{2,i}^*)f(x_{3,i}^*)(f(x_{1,i}^*) + f(x_{2,i}^*) + f(x_{3,i}^*)))^{1/2}$$

(by Heron's Formula) of the triangle having the centers of $C(x_{1,i}^*)$, $C(x_{2,i}^*)$, $C(x_{3,i}^*)$ as vertices would be rational. In Section 3 we show how such functions $f(x^*)$ and triplets (x_1^*, x_2^*, x_3^*) may indeed be obtained. Namely, by executing a geometric inversion on the circle configuration attributed to A. Speiser (cf. [1]; also [2]) into a configuration S whose circles have no interior points in common, and whose triangles W are all rational. The formula for the radius of the inverse of a given circle gives a suitable $f(x^*)$, and those consecutive triplets (α, β, γ) of any Farey series F_j (cf. Sec. 3 below) for which $\beta \notin F_{j-1}$ gives a triplet (x_1^*, x_2^*, x_3^*).

By this method, we may obtain classes of (integral or rational) solutions of the Diophantine equation

$$(1.2) \qquad xy + yz + xz = R^2$$

(cf. Remark 1 below). We shall make use of the following theorem of H. E. Blichfeldt [3]:

If the sides of a triangle are a, b, c, then by letting

$$a = \left| m + \frac{1}{m} \right|, \qquad b = \left| n + \frac{1}{n} \right|, \qquad c = \left| m - \frac{1}{m} + n - \frac{1}{n} \right|,$$

and giving m and n all rational values integral or fractional, positive or negative, we form all the triangles with rational sides and having rational areas.

We illustrate the method outlined above by deriving those solutions of the Diophantine equation (1.2) which are associated with some one, say the $(+, +, +)$ case, of the eight possibilities occurring in Blichfeldt's theorem:

$$(1.3) \qquad \begin{gathered} 0 < a = \pm\left(m + \frac{1}{m}\right), \qquad 0 < b = \pm\left(n + \frac{1}{n}\right), \\ 0 < c = \pm\left(m - \frac{1}{m} + n - \frac{1}{n}\right), \end{gathered}$$

namely, the case $m, n > 0$, $mn > 1$ (cf. Remark 2 below) which is expressed by the following theorem:

THEOREM 2. *Let p_1/q_1, p_2/q_2, p_3/q_3 be consecutive elements of a Farey series, F_j, such that $p_2/q_2 \notin F_{j-1}$. Let*

$$(1.4) \qquad \begin{aligned} G(m, n, p, q) \equiv{} & q^2n^2(m^2 + 1) + 2mn(m + n)q(q - p) \\ & + m^2(n^2 + 1)(q - p)^2 - mn(mn - 1), \end{aligned}$$

$G_1 \equiv G(m, n, p_1, q_1)$, $G_2 \equiv G(m, n, p_2, q_2)$, $G_3 \equiv G(m, n, p_3, q_3)$. If $m, n \geqq 0$, $mn \geqq 1$, then

$$(1.5) \qquad G_1G_2 + G_1G_3 + G_2G_3 = R^2,$$

where R is integral or rational accordingly as both m and n are integral or rational.

Remark 1. We note that (1.2) is a generalization of the "Pell" equation

$$(1.6) \qquad 3t^2 - R^2 = 1,$$

since, if we let

$$(1.7) \qquad x = t, \qquad y = t - 1, \qquad z = t + 1$$

in (1.2), we obtain (1.6). Thus any solution of (1.6) provides through (1.7) a solution of (1.2).

Remark 2. By the $(+, +, +)$ case we mean, of course, the choice of the "+" signs in the right expressions of (1.3). Noticing that

$$m - \frac{1}{m} + n - \frac{1}{n} = (m + n)\left(\frac{mn - 1}{mn}\right),$$

and that a, b, c must be positive, we see that the $(+, +, +)$ case is equivalent to $m, n > 0, mn > 1$. For the $(+, +, -)$ case we have

$$\left(m + \frac{1}{m}\right) > 0, \quad \left(n + \frac{1}{n}\right) > 0, \quad -\left(m - \frac{1}{m} + n - \frac{1}{n}\right) > 0,$$

which is equivalent to $m, n > 0, mn < 1$; etc.

The method of obtaining solutions of (1.2) in the $(+, +, +)$ case, illustrated in the proof of Theorem 2, may be applied to the other cases $(+, +, -)$, $(+, -, +)$, $(+, -, -)$, $(-, +, +)$, $(-, +, -)$, $(-, -, +)$, $(-, -, -)$ of (1.3), each leading to a class of solutions of (1.2).

2. Some lemmas and proof of the theorem. We prove

LEMMA 1. *Let C_x, C_y, C_z, C_t be mutually tangent circles (i.e. each tangent to the other three) with radii of finite lengths x, y, z, t and centers X, Y, Z, T respectively. If ΔXYZ is rational, then*

$$\Delta YZT,\ \Delta XYT,\ \Delta XZT \tag{2.1}$$

are also rational.

Proof. I. Suppose C_x, C_y, C_z, C_t are mutually external (*i.e.*, have no interior point in common). Since the sides of ΔXYZ are rational, x, y, z must be rational. By a geometric inversion (which leaves C_x invariant, with center at the point of tangency between C_y and C_z), it may be shown that

$$t = \frac{xyz}{x(y + z) \pm 2A + yz}, \tag{2.2}$$

where $A \equiv (xyz(x+y+z))^{\frac{1}{2}}$ is the area of ΔXYZ, and upper or lower signs are to be used accordingly as T is inside or outside ΔXYZ (cf. note 1 below). Since A is rational, t is rational. Thus the sides of triangles (2.1) are rational. The areas

$$(tyz(t + y + z))^{1/2}, \quad (txy(t + x + y))^{1/2}, \quad (txz(t + x + z))^{1/2}$$

of triangles (2.1) are

$$t(y + z \pm (A/x)), \quad t(x + y \pm (A/z)), \quad t(x + z \pm (A/y))$$

respectively (cf. Note 2 below), and are therefore rational (upper or lower signs taken as before).

II. Suppose C_x, C_y, C_z are mutually external, but are all inside C_t. Then

$$t = \frac{xyz}{-(x(y+z) - 2A + yz)} \cdot \tag{2.3}$$

Thus the sides of triangles (2.1) are rational. The areas are also rational; thus, *e.g.*, the area of ΔTYZ is $(tyz(t-y-z))^{\frac{1}{2}}$, which is equal to $t(y+z-(A/x))$.

III. Suppose C_t and two elements C_g, C_h of $\{C_x, C_y, C_z\}$ are mutually external, and C_t, C_g, C_h are all inside $C_k(C_k \in \{C_x, C_y, C_z\}, C_k \neq C_g, C_k \neq C_h)$. Thus, *e.g.*, suppose C_y, C_z, C_t are mutually external and are all inside C_x. The area A' of ΔXYZ is $(xyz(x-y-z))^{\frac{1}{2}}$ (which is rational by hypothesis). Now,

$$t = \left| \frac{xyz}{x(y+z) \pm 2A' - yz} \right| \cdot \tag{2.4}$$

If $y+z=x$, then $A'=0$. If $y+z\neq x$, then there are two possible circles C_t (of unequal radii); the $+$ or $-$ is taken accordingly as C_t is the smaller or larger of these possibilities. The sides of triangles (2.1) are of course rational. The areas

$$(tyz(t+y+z))^{1/2}, \qquad (txy(x-y-t))^{1/2}, \qquad (txz(x-z-t))^{1/2}$$

of triangles (2.1) are

$$t(y+z\pm(A'/x)), \qquad t(x-y\pm(A'/z)), \qquad t(x-z\pm(A'/y))$$

respectively, and are therefore rational.

Note 1. Proof of the "+" case of (2.2) (the "−" case, (2.3), (2.4), (2.5), (2.6) and (2.7) are done in an analogous manner): Suppose C_x, C_y, C_z, C_t are mutually and *externally* tangent (*i.e.*, they have no interior point in common) and T is inside ΔXYZ. Let I be the inversion which leaves C_x invariant and whose center is at the point P of tangency between C_y and C_z. Then $I(C_y)$ (the inverse of C_y) and $I(C_z)$ are lines which we denote by l_y and l_z respectively. Let l_y and l_z intersect the line $l(Y, Z)$ (determined by Y and Z) in the points L_y and L_z, respectively. Let k denote the length of the radius of the circle of inversion. Then $k^2=\overline{PL_y}\cdot 2y$, $k^2=\overline{PL_z}\cdot 2z$ and $\overline{PL_y}+\overline{PL_z}=2x$. Eliminating $\overline{PL_y}$ and $\overline{PL_z}$ we obtain $k^2=4xyz/(y+z)$. Let D_t denote $I(C_t)$, and d_t be the center of D_t; the radius of D_t is x. Let h be the distance from X to $l(Y, Z)$. Now, P is an external homothetic center between C_t and D_t. Let one of the common tangents to C_t and D_t which pass through P touch them in the points A and B, respectively. Suppose (with no loss of generality) that $z\geqq y$. Then, since

$$h = 2\{xyz(x+y+z)\}^{1/2}/(y+z), \qquad \overline{PL_z} = 2xy/(y+z),$$

we obtain

$$\begin{aligned} t &= x\cdot\frac{\overline{PA}}{\overline{PB}} = x\cdot\frac{\overline{PA}\cdot\overline{PB}}{\overline{PB}^2} = \frac{x\cdot k^2}{\overline{Pd_t}^2 - x^2} \\ &= \frac{x\cdot k^2}{(x-\overline{PL_z})^2+(h+2x)^2-x^2} = \frac{xyz}{x(y+z)+2\{xyz(x+y+z)\}^{1/2}+yz} \cdot \end{aligned}$$

Note 2. Thus, let K denote the denoninator of (2.2). Then

$$[tyz(t+y+z)]^{1/2} = \left[\frac{xyz}{K}\cdot yz\left(\frac{xyz}{K}+y+z\right)\right]^{1/2}$$

$$= \frac{yz}{K}[x\{xyz+(y+z)K\}]^{1/2} = \frac{yz}{K}[\{x(y+z) \pm [xyz(x+y+z)]^{1/2}\}^2]^{1/2}$$

$$= \frac{yz}{K}(x(y+z) \pm A) = t\left(y+z\pm\frac{A}{x}\right).$$

LEMMA 2. *Let mutually and externally tangent circles C_x, C_y, C_z with centers at X, Y, Z, and radii of rational lengths x, y, z respectively ($x<\infty$, $y<\infty$, $z<\min(x,y)$), be tangent to a line L. Let C_t be a circle tangent to L and any two elements C_g, C_h of $\{C_x, C_y, C_z\}$; let G, H, T be the centers, and g, h, $t(t<\infty)$ be the lengths of the radii of C_g, C_h, C_t respectively. Then ΔGHT is rational.*

Proof. Suppose $C_t\in\{C_x, C_y, C_z\}$. Then $\Delta XYZ\equiv\Delta GHT$. We note that ΔXYZ is rational. Its sides are of course rational. Since

$$z = \frac{xy}{x+y+2(xy)^{1/2}}, \tag{2.5}$$

then $(xy)^{\frac{1}{2}}$ is rational. But the area $(xyz(x+y+z))^{\frac{1}{2}}$ of ΔXYZ is also rational since it is equal to $z(x+y+(xy)^{\frac{1}{2}})$. Let us therefore assume that $C_t\notin\{C_x, C_y, C_z\}$.

Suppose $C_g\equiv C_y$, $C_h\equiv C_z$. Let x_{-1} be the radius of the circle $C_{x_{-1}}(\neq C_z)$ which is tangent to C_x, C_y and L. Let x_2 be the radius of the circle $C_{x_2}(\neq C_x)$ which is tangent to C_z, C_y and L. Then

$$x_{-1} = \frac{xy}{x+y-2(xy)^{1/2}}, \tag{2.6}$$

$$x_2 = \frac{xy}{4x+y+4(xy)^{1/2}}. \tag{2.7}$$

Since $(xy)^{\frac{1}{2}}$ is rational (cf. (2.5)) x_{-1} and x_2 are rational.

Suppose $C_t\equiv X_{x_2}$. The sides of ΔTYZ are rational. The area $(zx_2y(z+x_2+y))^{\frac{1}{2}}$ of ΔTYZ (using (2.5) and (2.7)) is $zx_2(3x+3(xy)^{\frac{1}{2}}+y)/x$, and is therefore rational.

Similarly, if $C_g\equiv C_x$, $C_h\equiv C_z$ we can show that ΔTZX is rational.

Suppose $C_g\equiv C_y$, $C_h\equiv C_x$. There is no loss of generality if we assume $x<y$ (if $x=y$, then $t=\infty$ which is contrary to hypothesis). Let $C_t=C_{x_{-1}}$. The sides of ΔXYT are rational. The area $(x_{-1}xy(x_{-1}+x+y))^{\frac{1}{2}}$ of ΔXYT is $x_{-1}(x+y-(xy)^{\frac{1}{2}})$ and is therefore rational. This completes the proof of Lemma 2.

Proof of the theorem. There is no loss of generality if we assume that S_1 consists of mutually tangent circles (not all tangent at one point) with centers at X, Y, Z ($x<\infty$, $y<\infty$, $z<\infty$). Let us assume that ΔXYZ is rational. Let w_i

denote the set of all (nondegenerate) triangles whose vertices are three mutually tangent circles of S_i.

If no S_i has a straight line as one of its "circles," then by Lemma 1 each element of w_i $(i=1, 2, \cdots)$ is rational.

Suppose S_{i_0} does not have a straight line as one of its "circles," and S_{i_0+1} does. Then by Lemma 1 each element of w_{i_0} is rational, and by Lemmas 1 and 2 each element of w_i $(i>i_0)$ is rational.

COROLLARY. *If any four mutually tangent circles of S have radii of (finite) rational length, then all elements of W are rational.*

Proof. This is an immediate consequence of (2.2), (2.3), (2.4) and the theorem.

3. A class of solutions of (1.1) and proof of Theorem 2. Let F_j denote the Farey series of order j, *i.e.*, the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed j, the numbers 0 and 1 included in the form 0/1 and 1/1 (cf. [4]).

Consider the circle $C_{p/q}$ with center at $(p/q, 1/2q^2)$ (cartesian coordinates) and radius $1/2q^2$, where p and q are relatively prime integers, and $0 \leqq p/q \leqq 1$. Let S' denote the configuration consisting of all such circles $C_{p/q}$ and no others (cf. [1], [2]). If p_1/q_1, p_2/q_2, p_3/q_3 are consecutive terms of F_j for which $p_2/q_2 \notin F_{j-1}$, then C_{p_i/q_i} $(i=1, 2, 3)$ are mutually and externally tangent.

The triangle with vertices $(0, \frac{1}{2})$, $(1, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{8})$ is rational. Therefore any triangle whose vertices are the centers of three mutually tangent circles of S' are *rational* triangles. The area of the triangle with vertices $(p_i/q_i, 1/2q_i^2)$ $(i=1, 2, 3)$ is therefore rational, *i.e.*

$$\left\{\frac{1}{2q_1^2}\cdot\frac{1}{2q_2^2}\cdot\frac{1}{2q_3^2}\left(\frac{1}{2q_1^2}+\frac{1}{2q_2^2}+\frac{1}{2q_3^2}\right)\right\}^{1/2}$$

is rational. Thus the function $f(x^*)=1/2q^2(x^*=p/q, (p, q)=1)$ satisfies (1.1) for all triplets $x_1^* \equiv p_1/q_1$, $x_2^* \equiv p_2/q_2$, $x_3^* \equiv p_3/q_3$.

Now, let I be an inversion which takes any three mutually tangent circles of S' into mutually external circles whose centers are vertices of a rational triangle. Let $f(p/q)$ be the length of the radius of $I(C_{p/q})$. Then

$$f(p_1/q_1)f(p_2/q_2)f(p_3/q_3)[f(p_1/q_1)+f(p_2/q_2)+f(p_3/q_3)] = R^2, \tag{3.1}$$

R rational. We now show how to construct such an inversion I, and obtain the corresponding function $f(p/q)$ explicitly.

Let m and n be positive rational numbers for which $mn>1$. Let a triangle τ have sides a, b, c with

$$a = m + m^{-1}, \qquad b = n + n^{-1}, \qquad c = m - m^{-1} + n - n^{-1}.$$

Let $a=x+y$, $b=y+z$, $c=x+z$. Then

$$x = m - n^{-1}, \qquad y = m^{-1} + n^{-1}, \qquad z = n - m^{-1}. \tag{3.2}$$

Consider mutually and externally tangent circles C_x, C_y, C_z with radii of lengths x, y, z (cf. (3.2)) and centers at X, Y, Z the common endpoints of sides a and c, a and b, b and c, respectively. Let D be a circle, with diameter of length one and center at J, which is externally tangent to C_y and C_z, where J and X are in the same half-plane determined by the line $l(Y, Z)$.

Let us now consider the inversion I which leaves D invariant and whose center is at the point of tangency between C_y and C_z. $I(C_y)$ and $I(C_z)$ are lines which we denote by l_y and l_z respectively. Let $l(Y, Z)$ intersect l_y and l_z in the points L_y and L_z respectively. Let C_t be the circle tangent to but not overlapping C_x, C_y, C_z. Let E and F be the points of tangency between l_y and the inverses $I(C_t)$ and $I(C_x)$ of C_t and C_x respectively. Let l_E be the line perpendicular to l_y at E. Let l_y and l_E be the axes of a cartesian coordinate system with E: (0, 0) and F: (1, 0). Let us consider the circle configuration S' on this coordinate system, and let $C_{p/q}$ be defined as above, its center being denoted by G.

We note that P is an external homothetic center between $C_{p/q}$ and $I(C_{p/q})$. Let one of the common tangents to $C_{p/q}$ and $I(C_{p/q})$ which pass through P, touch $C_{p/q}$ and $I(C_{p/q})$ in the points B and A, respectively.

There is no loss in generality if we assume $z \geqq y$. Since I leaves D invariant, the power k of I satisfies

$$k^2 = \overline{PL}_y \cdot 2y, \qquad k^2 = \overline{PL}_z \cdot 2z, \qquad \overline{PL}_y + \overline{PL}_z = 1. \tag{3.3}$$

Eliminating $\overline{PL}_y$ and $\overline{PL}_z$ we obtain

$$k^2 = 2yz/(y + z). \tag{3.4}$$

Furthermore, using (3.4) in the first expression of (3.3) we obtain

$$\overline{PL}_y = z/(y + z). \tag{3.5}$$

Let h denote the distance from X to $l(Y, Z)$. Then

$$h = 2(xyz(x + y + z))^{1/2}/(y + z). \tag{3.6}$$

$I(C_x)$ has center at X': $(1, \frac{1}{2})$. Let H be the distance from X' to $l(Y, Z)$. Then $H/h = \overline{X'P}/\overline{XP} = \frac{1}{2}/x$, *i.e.*,

$$H = (xyz(x + y + z))^{1/2}/x(y + z). \tag{3.7}$$

The radius $f(p/q)$ of $I(C_{p/q})$ may now be computed as follows:

$$f(p/q) = \frac{1}{2q^2}\frac{\overline{PA}}{\overline{PB}} = \frac{1}{2q^2}\frac{\overline{PA}\cdot\overline{PB}}{\overline{PB}^2} = \frac{1}{2q^2}\frac{k^2}{\overline{PG}^2 - \{1/(2q^2)\}^2}.$$

Thus

$$f(p/q) = \frac{\{1/(2\cdot q^2)\}k^2}{(\overline{PL}_y - \{(1/(2\cdot q^2)\})^2 + (H + 1 - (p/q))^2 - \{1/(2\cdot q^2)\}^2}. \tag{3.8}$$

Applying (3.4), (3.5), (3.7) to (3.8), and then (3.2), we obtain finally

$$(3.9)\quad f(p/q) = \frac{(m+n)(mn-1)}{q^2n^2(m^2+1)+2mn(m+n)q(q-p)+m^2(n^2+1)(q-p)^2-mn(mn-1)}.$$

We recall that

$$(3.10)\qquad C_{p_1/q_1},\ C_{p_2/q_2},\ C_{p_3/q_3}$$

are mutually and externally tangent since p_1/q_1, p_2/q_2, p_3/q_3 are consecutive terms of F_j such that $p_2/q_2 \notin F_{j-1}$. Since the center of inversion P is not inside any circle of (3.10), the inverses of (3.10),

$$(3.11)\qquad I(C_{p_1/q_1}),\ I(C_{p_2/q_2}),\ I(C_{p_3/q_3}),$$

must be mutually and *externally* tangent. As the triangle τ with sides a, b, c is rational (by Blichfieldt's theorem), the triangle joining the centers of (3.11) is rational (by Th. 1). Thus $f(p/q)$ satisfies (1.1) for all consecutive triplets $(p_1/q_1, p_2/q_2, p_3/q_3)$ of any Farey series F_j for which $p_2/q_2 \notin F_{j-1}$.

We now complete the proof of Theorem 2. Equation (3.1) is valid for $f(p/q)$ defined by (3.9) over the set Γ of all pairs of positive rational numbers (m, n) for which $mn > 1$. Substituting (3.9) into (3.1) we see that (1.5) is valid over Γ. We note now, that if m and n are integers, the left side of (1.5) is an integer, and this implies that R is integral.

Finally, we note that (1.5) holds for $G(0, n, p, q)$, $G(m, 0, p, q)$, or where $mn = 1$. This is a consequence of the identity

$$(M^2 + MN + N^2)^2 = (MN)^2 + \{M(M+N)\}^2 + \{N(M+N)\}^2$$

(cf. [5]), and the fact that $p_2 = p_1 + p_3$ and $q_2 = q_1 + q_3$ (since $p_2/q_2 \notin F_{j-1}$).

Remark 3. In the above argument, the center of inversion P was the point of contact between C_y and C_z. The cases where the inversion center P is at the contact point between C_z and C_x, or between C_x and C_y, is treated analogously. There are, in general, three positions for P for each of the eight cases of (1.3).

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NEW LOOPS FROM OLD GEOMETRIES

J. MARSHALL OSBORN, University of Wisconsin

In this paper we shall exhibit a simple but unusual way of defining an algebraic system on the points of a Euclidean plane. Although our ultimate purpose is to produce examples of certain algebraic systems the theory of which has been treated elsewhere,* we shall try to make this paper as self-contained as possible, since we expect the method to be of wider interest than the resulting examples.

We begin by defining a new binary operation on the complex numbers. Let a be a fixed complex number not zero or one, and define

$$x \circ y = x + a(y - x) = (1 - a)x + ay \tag{1}$$

for any two complex numbers x and y. From this definition it is easy to check that

(i) If any two of the three complex numbers x, y, z are chosen, there exists a unique choice for the third one so that the relation $x \circ y = z$ is satisfied,

(ii) For any four complex numbers w, x, y, and z, $(w \circ x) \circ (y \circ z) = (w \circ y) \circ (x \circ z)$,

(iii) For any complex number x, $x \circ x = x$.

A set with a binary operation satisfying these three postulates is called an *idempotent Abelian quasigroup*. Thus, for each choice of the complex number a, we have made the set of complex numbers into an idempotent Abelian quasigroup, which we shall denote by Q.

Let us next think of the complex numbers as the points of a plane E, and consider what happens geometrically under the operation "o." If $a = re^{-i\theta}$ for real numbers $r > 0$ and $0 \leqq \theta < 2\pi$, then (1) may be expressed geometrically by saying that $z = x \circ y$ is that point of the plane such that the directed angle between the segments $\overline{xz}$ and $\overline{xy}$ is θ, and such that the length of the segment $\overline{xz}$ is r times the length of the segment $\overline{xy}$. But from this geometric description of Q it is clear that any translation of the plane induces an automorphism on Q. This can be expressed algebraically by the identity

$$x + (y \circ z) = (x + y) \circ (x + z), \tag{2}$$

where "+" is the usual operation of addition of complex numbers. We thus have two operations ("+" and "o") defined on the set of complex numbers (or on the points of the plane E), one of which distributes over the other—a situation which is reminiscent of a field. It will prove worth while here to alter our system for the purpose of strengthening this resemblance.

Let L be the set consisting of the symbol u and the points of E under a new operation "o" which is exactly the same as the old operation "o" for the product

* Specifically, we desire examples of cross inverse property loops, which have been discussed in [1], [2] and [7].

of any two distinct points of E, and which is defined otherwise by $x \circ u = u \circ x = x$ and $x \circ x = u$. Now L still satisfies (i), and in addition has a two-sided identity element u. Such a system is called a *loop*. Making the obvious definitions $x+u=u+x=u+u=u$, the distributive law (2) still holds, giving an algebraic system which now only fails in satisfying the axioms of a field because the operation "o" is not associative. A set with two binary operations satisfying all the axioms of a field except for the associative law for one or both of these operations is called a *neofield.** We may then summarize the above remarks by saying that we have exhibited a class of neofields.

Let us next restrict our attention to the special case of the loop L mentioned above in which r and θ have the values 1 and $\frac{1}{3}\pi$, respectively. Geometrically, this is the case where the points x, y and $x \circ y$ form the vertices of an equilateral triangle. Rotating this triangle, we may then conclude that

$$(3) \qquad (x \circ y) \circ x = y,$$

a relation which still holds if either x or y is the identity u. It is examples of loops satisfying this relation (known as the cross inverse property†) that we wish to derive. If we ask for the subloop K of L generated by two distinct points of E, we will find that we get exactly the vertices of a regular triangular tessellation of the plane‡ (illustrated in Fig. 1) in addition to u. And if we ask for the subloop K_1 of K generated by two vertices of this tessellation which are not adjacent, we get a proper subset of the vertices which themselves are the vertices of a regular triangular tessellation (for example, the set of vertices which are labelled "1" in Fig. 1). Considering the translates of the tessellation K_1^* corresponding to K_1 within the tessellation K^* corresponding to K, it is clear that K^* may be broken up into a finite number of equivalence classes, each of which is a tessellation. But since the subloops of L are just the subquasigroups of Q with the identity adjoined, and since it is known that the cosets of an Abelian quasigroup with respect to any subquasigroup also form a quasigroup,§ we may conclude that the cosets of K modulo K_1 form a finite homomorph H of K. For example, identifying together the vertices in Figure 1 which have the same label, we get a quasigroup of order 7, and adding an identity element gives a loop of order 8. As in the case of the whole plane, we may identify each translation of the tessellation K^* with an element of H to make H into a neofield.

We shall determine next the orders of the loops arising from the homomorphs of K. Given two vertices of K_1^* which are a minimal distance apart, we observe that there is a unique parallelogram P in K^* with these two points as the

* A more precise definition of a neofield may be found in [8].

† In the usual definition of this property, one of the x's in (3) is replaced by an inverse of x; but since we have defined $x \circ x = u$, x and its inverses are the same here.

‡ A tessellation is an infinite collection of polygons which cover the whole plane without overlapping, and which have the property that every edge of each polygon belongs also to one other polygon.

§ See [5].

acute-angled vertices of P. Let the minimal distance between vertices in K^* be one unit, and let the lengths of two nonparallel sides of this parallelogram be m and n (in Fig. 1, m and n have the values 2 and 1, respectively, for example). Then the minimal distance between vertices in K_1^* is just the length of the longer diagonal of P, which is easily computed to be $t=\sqrt{(m^2+mn+n^2)}$. But the number of disjoint translates of K_1^* in K^* is the same as the ratio of the density of the vertices of K^* in the plane to the density of the vertices of K_1^* in the plane. And this is the same as the square of the ratio of the minimal distances occurring between vertices in K_1^* and K^* respectively, or $t^2=m^2+mn+n^2$. Since we can find a subtessellation of K^* using any nonnegative integral values of m and n we wish (m, n not both zero), we have shown that the orders of the loops arising by adjoining an identity element to each of the proper homomorphs of K, are exactly the integers of the form m^2+mn+n^2+1.

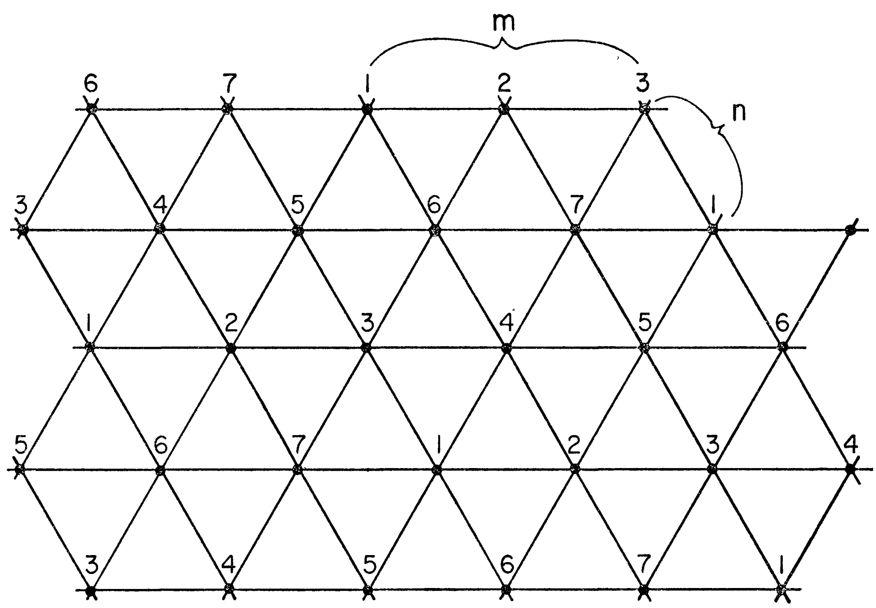

FIG. 1

The orders occurring here may be characterized a little more neatly using the following result which we quote from number theory.*

LEMMA. *A positive integer may be expressed in the form m^2+mn+n^2, where m and n are nonnegative integers, if and only if its square-free part is not divisible by a prime of the form $6t+5$.*

We are now ready to state formally what we have proved.

* For example, see Chapter VI and page 265 of [6].

THEOREM. *Let n be a positive integer whose square-free part is not divisible by a prime of the form $6t+5$. Then there exists a simple loop of order $n+1$ with the cross inverse property. This loop may be made into a neofield by identifying each element except the identity with an appropriate automorphism of the loop.*

The simplicity of the loop mentioned in this theorem is due to a theorem of Bruck, which states, in part, that a finite loop with a transitive automorphism group is either simple or associative.*

There are several variations of our construction which should also be pointed out. First of all, if G is an affine plane with n points on a line where $n \equiv 2 \pmod 3$, or any affine plane with infinitely many points on a line, then we may divide the collection of parallel classes of lines into equivalence classes such that each equivalence class contains exactly three parallel classes, and such that the parallel classes in each equivalence class are given a cyclical order. Then if x, y, z are distinct points of G, we let z be the product of x and y whenever the three lines $\overline{xy}$, $\overline{yz}$, $\overline{xz}$ belong respectively to distinct parallel classes C_1, C_2, C_3 which constitute an equivalence class, and which are arranged here in the correct cyclical order. Developing this situation like the previous one, we would find that an affine plane with n points on a line leads to a cross inverse property loop of order n^2+1. This loop can be made into a neofield if and only if the affine plane is Desarguesian.

A second variation arises by using a sphere instead of a Euclidean plane in our original construction. If x and y are two points on a sphere S which are not antipodal, we define multiplication between them using r and θ just as in the case of the Euclidean plane. To take care of antipodal points, we adjoin a symbol e with the relations $e \circ e = e$, $e \circ x = x \circ e = x'$, and $x \circ x' = e$ for any pair of antipodal points x and x'. This gives an idempotent quasigroup (it will not be Abelian any more) which may be made into a loop as before, and which will again have the cross inverse property for $r=1$ and $\theta = \frac{1}{3}\pi$. This time, instead of tessellations, one is led to the three regular polyhedra with triangles as faces, giving three cross inverse property loops of order 5, 8, and 14. The first of these loops has appeared in the literature several times as an example of a loop which is not a group, and the second one turns out to be the quaternion group with minor alterations in its multiplication table. The automorphism groups of these three loops are exactly the same as the symmetry groups of the regular polyhedra from which they arise (although in the case of the icosohedron, one must remember to include the "outer automorphism" of Coxeter†).

Finally, our construction may be generalized by replacing the complex plane by an n-dimensional vector space V. Instead of a fixed complex number α, we choose a fixed $n \times n$ matrix A, and define

$$X \circ Y = X + A(Y - X) = (I - A)X + AY$$

* See [3].

† See page 106 of [4].

for any two vectors X, Y of V. If both A and $(I-A)$ are nonsingular matrices (to satisfy (i)), then the construction of Abelian quasigroups and neofields proceeds as before, except that we cannot use tessellations for $n>2$.

Since the construction of our class of neofields may suggest the construction used by Paige in [8] to some readers, it might be remarked that if the matrix A is chosen to be the negative of the identity matrix (this means $r=1$, $\theta=\pi$ in the case of the plane), we get neofields of the type mentioned by Paige, but that otherwise ours are new. Like Paige's neofields, the class given here cannot be used to define a projective geometry.

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ON CHROMATIC GRAPHS

LÉOPOLD SAUVÉ, St. Patrick's College, Ottawa, Canada

1. Introduction. Let N points be given in general position in space (no three in a line, and no four in a plane). All of the segments joining them in pairs are drawn and then colored, some of the segments red and some of the segments blue. What is the smallest possible number of monochromatic triangles, *i.e.*, triangles whose three sides are of the same color?

This problem was first solved by A. W. Goodman [1], and the answer is given in Theorem 1 below. However, his method is a little complicated. We give here, in Section 2, a new and much simpler proof of his result. Although so far the new method has not been helpful in solving the related open problems mentioned by Goodman [1] and by Greenwood and Gleason [2], there is still the hope that eventually the methods used here will be effective in obtaining new information on chromatic graphs.

Finally, an extract from a letter of P. Erdös is included in Section 3 (with his kind permission) in which he settles one of the questions left open by Goodman [1].

2. The minimum number of monochromatic triangles.

THEOREM 1. *Let B and R be the number of solid blue triangles and solid red triangles respectively. Then in any complete coloring of a configuration of N points, using only the colors red and blue, we have*

$$B + R \geqq \begin{cases} \frac{1}{3}u(u-1)(u-2), & \text{if } N = 2u; \\ \frac{2}{3}u(u-1)(4u+1), & \text{if } N = 4u+1; \\ \frac{2}{3}u(u+1)(4u-1), & \text{if } N = 4u+3; \end{cases} \tag{1}$$

where u is a nonnegative integer. Further, this lower bound is sharp for each positive integer N.

Proof. To each colored configuration (chromatic graph) we will attach a weight W determined as the sum of the weights w_P at each point P. At each point P there are $N-1$ line segments emanating, and hence C_2^{N-1} distinct pairs of segments. To each pair of segments at P we assign the weight 2 if the segments are of the same color, and -1 if they are of different colors. Then w_P, the weight at P, is the sum of the weights for all pairs of distinct line segments at P. The weight of a triangle is the sum of the weights of the pairs of segments intersecting at each vertex. Since the points are in general position, each pair of intersecting segments belongs to precisely one triangle; hence the sum of the weights of all the triangles must be W. Clearly each monochromatic triangle has weight 6, and all of the others have weight 0. Hence $B+R$, the total number of monochromatic triangles, is $\frac{1}{6}W$. We must find the minimum value of $\frac{1}{6}W$ for all possible coloring schemes.

Case 1: $N=2u$. At each point w_P will be a minimum when the maximal number of pairs of segments at P are of different colors. This will occur when k segments are of one color and $k-1$ segments are of the other color. When this occurs

$$w_P = 2C_2^u + 2C_2^{u-1} - u(u-1) = (u-1)(u-2).$$

Hence for any coloring scheme $W \geqq 2u(u-1)(u-2)$ and therefore

$$R + B = \tfrac{1}{6}W \geqq \tfrac{1}{3}u(u-1)(u-2). \tag{2}$$

Case 2: $N=4u+1$. In this case w_P will be a minimum when there are $2u$ segments of each color at P. When this occurs

$$w_p = 4C_2^{2u} - 4u^2 = 4u(u-1).$$

Hence for any coloring scheme $W \geqq (4u+1)4u(u-1)$ and therefore

$$R + B = \tfrac{1}{6}W \geqq \tfrac{2}{3}u(u-1)(4u+1). \tag{3}$$

Case 3: $N=4u+3$. It is not possible to have $2u+1$ segments of each color at each point, for then the number of segments of a given color in the whole

configuration would be $\frac{1}{2}(4u+3)(2u+1)$ and this is not an integer. Accordingly, a lower bound is given by the best possible alternative, namely, $2u+1$ segments of each color at $4u+2$ of the points (all but one), and at the remaining point $2u$ segments of one color and $2u+2$ segments of the other color. At each of the first $4u+2$ points we have

$$w_p = 4C_2^{2u+1} - (2u+1)^2 = 4u^2 - 1.$$

At the remaining point

$$w_p = 2C_2^{2u+2} + 2C_2^{2u} - (2u+2)2u = 4u^2 + 2.$$

Hence for any coloring scheme $W \geqq 4u(u+1)(4u-1)$ and therefore

$$R + B = \tfrac{1}{6}W \geqq \tfrac{2}{3}u(u+1)(4u-1). \tag{4}$$

We will now show that the lower bounds found above are sharp by exhibiting coloring schemes whereby equality is attained in (2), (3) and (4).

Equality holds in (2) if and only if at each point there are u segments of one color and $u-1$ segments of the other. Let the points be numbered $P_1, P_2, \cdots, P_{2u}$. Color the segment P_iP_j red if $i+j$ is odd, and blue if $i+j$ is even. This yields equality in (2).

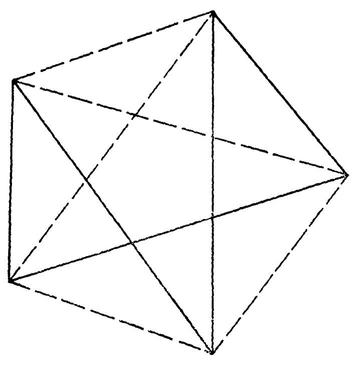

FIG. 1

Equality holds in (3) if and only if at each point there are $2u$ segments of each color, and in (4) if and only if there are $2u+1$ segments of each color at $4u+2$ of the points, and $2u$ segments of one color and $2u+2$ segments of the other at the remaining point. We proceed by induction. Figure 1 shows that a coloring scheme exists which yields equality in (3) for $N=5$. We will assume that a coloring scheme has been found that yields equality in (3) for $N=4u+1$ and show how to obtain one which does the same thing for $N=4u+5$. Furthermore, in doing so we will have found a coloring scheme which yields equality in (4). By assumption, from each point P_i, $i=1, 2, \cdots, 4u+1$, there emanate $2u$ segments of each color. Add the points P_{4u+2} and P_{4u+3} to the configuration. Color the segment $P_{4u+2}P_i$, $i=1, 2, \cdots, 4u+1$ blue if i is odd, red if i is even. Color

the segment $P_{4u+3}P_i$, $i=1, 2, \cdots, 4u+1$ red if i is odd, blue if i is even. Finally color the segment $P_{4u+2}P_{4u+3}$ red. The resulting coloring scheme yields equality in (4) for $N=4u+3$.

Now add the points P_{4u+4} and P_{4u+5} to the configuration. Color the segments $P_{4u+4}P_i$, $i=1, 2, \cdots, 4u+2$ blue if i is odd and red if i is even. Color the segments $P_{4u+5}P_i$, $i=1, 2, \cdots, 4u+2$ red if i is odd and blue if i is even. Finally color the segments $P_{4u+3}P_{4u+4}$, $P_{4u+3}P_{4u+5}$, and $P_{4u+4}P_{4u+5}$ blue, blue, and red respectively. The resulting coloring scheme yields equality in (3) for $N=4u+5$. This completes the proof of Theorem 1.

3. A contribution by P. Erdös. We will call a graph extremal if the equality sign holds in Theorem 1. Goodman has raised the question of whether it is possible to have an extremal graph in which either R or B is zero. In other words, can there be an extremal graph in which all of the monochromatic triangles have the same color? In case N is even the answer is in the affirmative, as was shown in [1], but for odd N the question was left open. We now settle this question by proving that if $N>7$ *is odd, then in any extremal graph both R and B are positive.* Because of the symmetry it will suffice to prove that $R>0$.

Let us consider first the case that $N=4u+1$ with $u>1$. In an extremal graph, every vertex has $2u$ red lines emanating from it. Let O be one of the vertices, and denote by $P_1, P_2, \cdots, P_{2u}$ the vertices connected with O by a red line. Similarly denote by $Q_1, Q_2, \cdots, Q_{2u}$ the vertices connected with O by a blue line. If $R=0$, no P_i and P_j can be connected with a red line, hence each point P_i is connected to exactly $2u-1$ points Q_j with red lines. Now there are $2u$ red lines at each point Q_j, so u of these red lines must join pairs of points Q_jQ_k. Let us suppose that P_1 is joined by red lines to $Q_1, Q_2, \cdots, Q_{2u-1}$. Then no two of these Q's can be connected with a red line, for otherwise we would have a red triangle. But then Q_{2u} must be joined by u red lines to Q-points, say $Q_1, Q_2, \cdots, Q_u$. But $2u$ red lines emanate from Q_{2u}, so u of these red lines must connect Q_{2u} with points P_i. Let P_2 be one of these points. But P_2 must be connected by red lines with $2u-1$ of the points Q, and, since $u>1$, it must be connected by a red line to one of the points $Q_1, Q_2, \cdots, Q_u$, say Q_i. Then $P_2Q_iQ_{2u}$ is a red triangle.

Assume next that $N=4u+3$. Let O be the exceptional point for the extremal graph, that is at O there are $2u+2$ lines of one color and $2u$ lines of the other color. Hence we must consider two subcases.

Case 1. Suppose that at O there are $2u+2$ red lines and $2u$ blue lines, and at every other point of the graph there are $2u+1$ lines of each color. Let $P_1, P_2, \cdots, P_{2u+2}$ be the points connected to O with red lines and let $Q_1, Q_2, \cdots, Q_{2u}$ be the points connected to O with blue lines. If $R=0$, no P_i and P_j can be connected with a red line. Therefore, since $2u+1$ red lines issue from each P_i, each P_i is connected with each Q_j by a red line. But then each Q_j has $2u+2$ red lines, and this is impossible.

Case 2. Suppose that at O there are $2u$ red lines and $2u+2$ blue lines. Let $P_1, P_2, \cdots, P_{2u}$ be the points connected to O by red lines, and let $Q_1, Q_2, \cdots,$

Q_{2u+2} be the points connected to O by blue lines. If $R=0$, each point P_i is connected by red lines with $2u$ of the points Q_j. At the points Q_j there is a total of $(2u+2)(2u+1)$ red lines, of which none connect with O, and $(2u)^2$ lines connect with points P_i. Hence there are $\frac{1}{2}[(2u+2)(2u+1)-4u^2]=3u+1$ red lines interconnecting points Q_i. Since $N>7$, we have $u>1$. Hence among the $2u+2$ points Q_j there is at least one that is connected to three others by red lines. Suppose these red connections are Q_1Q_2, Q_1Q_3, and Q_1Q_4. If Q_1 is connected by a red line to any P, say P_1, then P_1 cannot be connected with a red line to either Q_2, Q_3, or Q_4, for otherwise we would have a red triangle. But P_1 is connected by red lines to $2u$ of the Q's, and this is impossible. Therefore if $R=0$, Q_1 cannot be connected by a red line to any point P_j. Since $2u+1$ red lines issue from Q_1, it must be connected by red lines with all of the other Q's. So if $R=0$, no other pair Q_i, Q_j $(i>1, j>1)$ can be connected with a red line. But this means there are only $2u+1$ red lines among the Q's, and this contradicts the fact that we have $3u+1$ such lines. Hence $R>0$.

Finally we must consider $N=3$, 5, and 7. The first two cases are trivial since we can have both $R=0$ and $B=0$ (see Fig. 1). In case $N=7$, equation (1) gives $R+B=4$ for the extremal configuration. Now Figure 2 shows that such a chromatic graph is possible with $B=4$ and $R=0$. In that figure the dotted lines denote the blue lines and the solid ones denote the red lines.

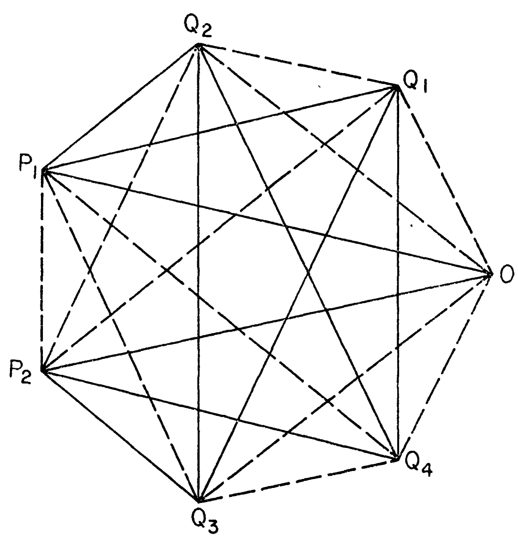

FIG. 2

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THE WEDGE PRODUCT

GERALD BERMAN, University of Waterloo, Ontario, Canada

1. Introduction. If a suitable product is defined for vectors in two dimensions, the resulting system is isomorphic to the complex numbers, and the usual vector operations can be interpreted in terms of operations on complex numbers. This leads to an elegant alternative approach to topics in which two-dimensional vectors are used. In a similar manner, quaternions may be used in place of three-dimensional vectors. The main disadvantage of complex numbers and quaternions is that they may be used only in two and three dimensions respectively.

The purpose of this note is to discuss another type of algebraic system, the Grassman algebra, which may be used to give a unified approach to several different topics, including vector algebra and its application. Only one new operation, called the wedge product, is needed to express dot products, cross products, etc., in a simple way. The system is easily extended to handle vector analysis. A differential operator is defined in terms of which gradient, curl, etc., may easily be expressed. This operator may also be used, together with the wedge product, to give elegant formulations of other topics such as the Jacobian of a transformation and the integral theorems.

2. The Grassman algebra Λ. Let $B(\epsilon_1, \cdots, \epsilon_n)$, $\epsilon_i = 0, 1$; $i = 1, \cdots, n$, denote 2^n basis vectors of a vector space of 2^n dimensions over the real numbers. Any vector may be written in the form

$$\alpha = \sum_{\epsilon} a(\epsilon_1, \cdots, \epsilon_n) B(\epsilon_1, \cdots, \epsilon_n), \tag{2.1}$$

where the summation is taken over all choices of the ϵ's. The symbols β, γ, etc., will represent similar sums in which b's, c's, etc., replace the a's. The wedge product of two elements $\alpha \wedge \beta = \gamma$ is defined by the identities

$$c(\epsilon_1, \cdots, \epsilon_n) = \sum_{\mu,\nu} \delta\begin{pmatrix} \mu_1, & \cdots, & \mu_n \\ \nu_1, & \cdots, & \nu_n \end{pmatrix} a(\mu_1, \cdots, u_n) b(\nu_1, \cdots, \nu_n), \tag{2.2}$$

where the summation is taken over all values of the μ's and ν's, and

$$\begin{aligned} \delta\begin{pmatrix} \mu_1, & \cdots, & \mu_n \\ \nu_1, & \cdots, & \nu_n \end{pmatrix} &= 0 \text{ if } \mu_i = \nu_i \text{ for any } i, \\ &= \prod_{i=1}^{n} (-1)^{\nu_i}(\mu_i + \mu_{i+1} + \cdots + \mu_n), \text{ otherwise.} \end{aligned} \tag{2.3}$$

The vector space, together with this wedge product, defines the Grassman algebra Λ [**3**].

It can be shown that the wedge product is associative, and both distributive laws hold, that is,

(2.4) $(\alpha\wedge\beta)\wedge\gamma=\alpha\wedge(\beta\wedge\gamma),\ \alpha\wedge(\beta+\gamma)=\alpha\wedge\beta+\alpha\wedge\gamma,\ (\alpha+\beta)\wedge\gamma=\alpha\wedge\gamma+\beta\wedge\gamma,$

so that Λ is a linear associative algebra. The basis elements satisfy the identities

$$B(\mu_1, \cdots, \mu_n) \wedge B(\nu_1, \cdots, \nu_n) = \delta\begin{pmatrix}\mu_1, & \cdots, & \mu_n\\ \nu_1, & \cdots, & \nu_n\end{pmatrix}B(\mu_1 + \nu_1, \cdots, \mu_n + \nu_n). \tag{2.5}$$

The wedge product is thus defined by stipulating the rules (2.4) and assuming the identities (2.5) for products of basis elements.

An examination of (2.5) indicates a simpler formulation. Since $B(0, 0, \cdots, 0) \wedge\alpha=\alpha$ for every α, it is convenient to set

$$B(0, 0, \cdots, 0) = 1 = E_0, \tag{2.6}$$

and identify it with the real number 1. Further set

$$B(0, \cdots, 0, 1, 0, \cdots, 0) = E_i, \tag{2.7}$$

where the 1 occurs in the ith position. The basis vectors may be expressed in terms of these vectors. In fact,

$$B(\epsilon_1, \cdots, \epsilon_n) = E_{1\epsilon_1} \wedge \cdots \wedge E_{n\epsilon_n}. \tag{2.8}$$

The terms in which $\epsilon_i=0$ can be omitted.

It follows that $E_0=1, E_1, \cdots, E_n$, form a basis for the linear associative algebra Λ, and that the identities (2.5) may be replaced by the simpler identities.

$$\begin{aligned} E_i \wedge E_j &= -E_j \wedge E_i, & i, j &= 1, \cdots, n. \\ E_0 \wedge E_i &= E_i \wedge E_0 = E_i, & i &= 1, \cdots, n. \end{aligned} \tag{2.9}$$

For example, in the case $n=2$, Λ is a 4-dimensional vector space with basis $E_0=1$, E_1, E_2, $E_1\wedge E_2$, and the wedge product of two elements is given by

$$\begin{aligned}&(a_0 + a_1E_1 + a_2E_2 + a_3(E_1 \wedge E_2)) \wedge (b_0 + b_1E_1 + b_2E_2 + b_3(E_1 \wedge E_2))\\ &= a_0b_0 + (a_1b_0 + a_0b_1)E_1 + (a_0b_2 + a_2b_0)E_2 + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)(E_1 \wedge E_2).\end{aligned}$$

It is convenient to choose the 2^n basis vectors $B(\epsilon_1, \cdots, \epsilon_n)$ of Λ (considered as a vector space) to be unit vectors orthogonal in pairs. Length of vectors will be defined in the usual way (and denoted as usual by $|\alpha|$).

A change of basis for **V** (see below) induces a change of basis for Λ in a natural way.

3. An automorphism in Λ. Let $*$ denote the 1-1 mapping which carries α, given by (2.1), into

$$\alpha^* = \sum_{\epsilon} a(\epsilon_1, \cdots, \epsilon_n)B((1 - \epsilon_1, \cdots, 1 - \epsilon_n). \tag{3.1}$$

It can easily be verified that $*$ is an automorphism of the Grassman algebra.

Let **V** denote the n-dimensional subspace spanned by the vectors $E_1, \cdots, E_n$. Every element of **V** has the form

$$A = \sum_{i=1}^{n} a_i E_i, \qquad a_i \text{ real.} \tag{3.2}$$

(The letters B, C, etc., will denote vectors in which the a's have been replaced by b's, c's, etc.)

The automorphism $*$ maps the vector E_i on the vector E_i^* given by

$$E_i^* = E_1 \wedge \cdots \wedge E_{i-1} \wedge E_{i+1} \wedge \cdots \wedge E_n. \tag{3.3}$$

These vectors determine a vector space $\mathbf{V}^*$ isomorphic to $\mathbf{V}$ under $*$.

It is clear that $\mathbf{V}$ represents the general n-dimensional real vector space. The algebra Λ in which $\mathbf{V}$ is embedded and the isomorphism $*$ are helpful in studying the properties of $\mathbf{V}$. For example, the scalar product of vectors A and B of $\mathbf{V}$ is given by

$$A \cdot B = (A \wedge B^*)^*. \tag{3.4}$$

4. Geometrical interpretations and applications of the wedge product.

(a) *Volume.* The product $A \wedge B$ of two n-dimensional vectors of $\mathbf{V}$ is a vector in an $\frac{1}{2}n(n-1)$-dimensional orthogonal subspace of Λ. The length of $A \wedge B$, $|A \wedge B|$, represents the area of the parallelogram having edges parallel to A and B.

More generally, let $v(A_1, \cdots, A_2)$ represent the volume of a parallelopiped with edges parallel to $A_1, \cdots, A_r$. The function v is characterised by the properties

(1) $v(A_1, \cdots, A_r) \geqq 0$;
(2) $v(A_1, \cdots, A_{i-1}, A_i + A_j, A_{i+1}, \cdots, A_r) = v(A_1, \cdots, A_r)$ $i \neq j$;
(3) $v(A_1, \cdots, A_{i-1}, aA_i, A_{i+1}, \cdots, A_r) = |a| v(A_1, \cdots, A_r)$, a real;
(4) $v(U_1, \cdots, U_r) = 1$, $U_1, \cdots, U_r$ the edges of an r-dimensional cube; [5].

It is immediate that $|A_1 \wedge \cdots \wedge A_r|$, considered as a function of $A_1, \cdots, A_r$, has these properties and

$$v(A_1, \cdots, A_r) = |A_1 \wedge \cdots \wedge A_r|. \tag{4.1}$$

Let $d(1), \cdots, d\binom{n}{r}$ denote the $r \times r$ determinants that can be formed from the array of components of the vectors $A_1, \cdots, A_r$. Then

$$|A_1 \wedge \cdots \wedge A_r|^2 = d^2(1) + \cdots + d^2\binom{n}{r},$$

so that the volume is easily expressible in terms of the components.

(b) *A generalization of Lagrange's identity.* The volume of the parallelopiped with edges parallel to $A_1, \cdots, A_r$ may be expressed in another simple form. It is easily verified that

$$v(A_1, \cdots, A_r) = \sqrt{(\det (A_i \wedge A_j^*)^*)} \tag{4.2}$$

also satisfies the axioms characterizing volume. Thus, comparing with (4.1),

$$\det(A_i \wedge A_j^*)^* = |A_1 \wedge \cdots \wedge A_r|^2. \tag{4.3}$$

Formula 4.3 is a generalization of the well-known Lagrange identity [2]. In fact, setting $n=3$ and $r=2$ in (4.3) yields

$$\begin{aligned}(a_{11}^2 + a_{12}^2 + a_{13}^2)(a_{21}^2 + a_{22}^2 + a_{32}^2) - (a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23})^2 \\ = (a_{12}a_{23} - a_{13}a_{22})^2 + (a_{13}a_{21} - a_{11}a_{13})^2 + (a_{11}a_{22} - a_{12}a_{21})^2,\end{aligned} \tag{4.4}$$

where

$$A_i = (a_{i1}, a_{i2}, a_{i3}), \qquad i = 1, 2.$$

(c) *Linear independence.* The vectors $A_1, \cdots, A_r$ are linearly independent if and only if the corresponding parallelepiped has nonzero volume. Thus: *The vectors $A_1, \cdots, A_r$ are linearly independent if and only if $A_1 \wedge \cdots \wedge A_r \neq 0$.*

This theorem may be used to deduce the equation of an r-space in compact form. Let P, P_0, P_1, etc., denote position vectors of points of E_n, Euclidean space of n dimensions. If the points $P_0, P_1, \cdots, P_r$ determine an r-space S, the vectors $P-P_0, P-P_1, \cdots, P-P_r$ will determine a parallelepiped of zero volume if and only if P lies in S. Thus the equation of S is

$$(P - P_0) \wedge (P - P_1) \wedge \cdots \wedge (P - P_r) = 0. \tag{4.5}$$

(d) *Vector algebra* [2]. The scalar product $A \cdot B$ of elements of $\mathbf{V}$ may be expressed in terms of the wedge product:

$$A \cdot B = (A \wedge B^*)^*. \tag{4.6}$$

In line with the discussion in (c), the hyperplane through P_1 having normal A is given by $(P-P_1) \wedge A^* = 0$.

In case $\mathbf{V}$ is 3-dimensional, vector products are usually defined. In terms of the wedge product

$$A \times B = (A \wedge B)^*. \tag{4.7}$$

Thus: *Nonzero vectors A, B are perpendicular (parallel) if and only if $A \wedge B^* = 0$ ($A \wedge B = 0$).*

The other vector products and identities can be translated into forms involving the wedge product. For example

$$A \times B \cdot C = (A \wedge B \wedge C)^*, \qquad (A \times B) \times C = ((A \wedge B)^* \wedge C)^*;$$

and the vector identity $(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$ is equivalent to the wedge identity

$$(A \wedge B)^* \wedge C = (C^* \wedge B)^* A^* + (A \wedge C^*)^* B^*.$$

(e) *Determinants* [5]. Let $A_i = a_{i1}E_1 + \cdots + a_{in}E_n$ $(i = 1, \cdots, n)$ denote

the vectors corresponding to the rows of the $n\times n$ determinant $d=|a_{ij}|$. Then $d=(A_1\wedge \cdots \wedge A_n)^*$. The theory of determinants is easily developed from this point of view. For example, to deduce the usual expansions in terms of cofactors, set $\overline{A}_i=\bar{a}_{i1}E_1+\cdots+\bar{a}_{in}E_n$, where $\overline{A}_i^*=(-1)^{i+1}A_1\wedge\cdots\wedge A_{i-1}\wedge A_{i+1}\wedge\cdots\wedge A_n$. The numbers $\bar{a}_{ij}$ are the cofactors of a_{ij} $(i,j=1,\cdots,n)$, and

$$\begin{aligned} d^* &= A_1\wedge\overline{A}_1=\cdots=A_n\wedge\overline{A}_n \\ &= a_{i1}\bar{a}_{i1}+a_{i2}\bar{a}_{i2}+\cdots+a_{in}\bar{a}_{in} \qquad (i=1,\cdots,n). \end{aligned}$$

The general Laplace expansion may be deduced in the same way by considering the products $A_{i1}\wedge\cdots\wedge A_{ir}$.

The vector properties lead to the usual rules for determinants and methods of simplification. As an example, consider the Vandermonde determinant

$$\Delta=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Let $A=E_1+aE_2+a^2E_3$, $B=E_1+bE_2+b^2E_3$, $C=E_1+cE_2+c^2E_3$,

$$B_1=\frac{B-A}{b-a}=E_2+(b+a)E_3, C_1=\frac{C-A}{c-a}=E_2+(c+a)E_3, C_2=\frac{C_1-B_1}{c-b}E_3.$$

Then

$$\begin{aligned}\Delta^*&=A\wedge B\wedge C=A\wedge(b-a)B_1\wedge(c-a)C_1=A\wedge(b-a)B_1\wedge(c-a)(c-b)C_2\\ &=(b-a)(c-a)(c-b)E_1\wedge E_2\wedge E_3.\end{aligned}$$

5. Augmented vector fields. With every point $(x_1,\cdots,x_n)$ of E_n associate a vector $A(x_1,\cdots,x_n)$ of $\mathbf{V}$. This defines a *vector field* A in E_n. Continuity, partial derivatives, etc., are easily defined. More generally, an element $\alpha(x_1,\cdots,x_n)$ of Λ can be associated with every point of E_n. This defines an *augmented vector field*. Derivatives are defined in the natural way [2]. Let D denote the set of augmented vector fields which have continuous partial derivatives in some region.

A transformation ∇ may be defined in D, which generalizes the gradient operator. Let ∇f denote the gradient of f when f denotes a real-valued function of n variables, that is,

$$\nabla f=\frac{\partial f}{\partial x_1}E_1+\cdots+\frac{\partial f}{\partial x_n}E_n. \tag{5.1}$$

Notice that $\nabla x_i=E_i$, $i=1,\cdots,n$. Let

$$\nabla E_i=0, \qquad i=1,\cdots,n, \tag{5.2}$$

and assume

(5.3) $\nabla(\alpha+\beta) = \nabla\alpha + \nabla\beta, \qquad \nabla(\alpha\wedge\beta) = \nabla\alpha\wedge\beta + \alpha\wedge\nabla\beta.$

In particular, if f is defined as above, $f\wedge\alpha = f\alpha$ so that

(5.4) $$\nabla(f\alpha) = \nabla f\wedge\alpha + f\nabla\alpha.$$

The relations (5.1), (5.2), (5.3), and (5.4) define $\nabla\phi$ for every element ϕ of D. We now give some applications of ∇.

6. Applications of augmented vector fields.

(a) *Jacobian of a transformation* [1]. Consider the transformation of coordinates defined by the equations

(6.1) $$x_i = f_i(u_1, \cdots, u_n), \qquad i = 1, \cdots, n.$$

If the functions f_i, $i=1, \cdots, n$, have continuous derivatives, and if the Jacobian J is not zero, the variables u_i, $i=1, \cdots, n$, are continuous differentiable functions of $x_1, \cdots, x_n$, and u_i, $i=1, \cdots, n$, are well defined. Applying ∇ to the equations (6.1) yields $E_1, \cdots, E_n$ in terms of $\nabla u_1, \cdots, \nabla u_n$. A straightforward calculation yields

(6.2) $$\nabla x_1\wedge\cdots\wedge\nabla x_n = E_1\wedge\cdots\wedge E_n = J(\nabla u_1\wedge\cdots\wedge\nabla u_n).$$

Equation (6.2) may be used to evaluate J. To illustrate the calculations involved, consider the transformation to spherical polar coordinates ρ, θ, ϕ in E_3, given by

(6.3.) $$x = \rho\cos\theta\cos\phi, \qquad y = \rho\cos\theta\sin\phi, \qquad z = \rho\sin\theta.$$

Let $\rho_1 = \rho\cos\theta$, so that $x=\rho_1\cos\phi$, $y=\rho_1\sin\phi$, and $\nabla x\wedge\nabla y = (\nabla\rho_1 - \rho_1\sin\phi\nabla\phi)\wedge(\nabla\rho_1\sin\phi + \rho_1\cos\phi\nabla\phi) = \rho_1\nabla\rho_1\wedge\nabla\phi$. Similarly, $\nabla\rho_1\wedge\nabla z = \rho\nabla\rho\wedge\nabla\theta$. Thus, $\nabla x\wedge\nabla y\wedge\nabla z = (\rho_1\nabla\rho_1\wedge\nabla\phi)\wedge\nabla z = -\rho_1(\nabla\rho_1\wedge\nabla z)\wedge\nabla\phi = -\rho_1\rho\nabla\rho\wedge\nabla\theta\wedge\nabla\phi$. The Jacobian is thus $-\rho_1\rho = -\rho^2\cos\theta$.

To illustrate the type of calculations involved in the generalization to n dimensions, consider the transformation to spherical polar coordinates in n dimensions. This transformation, which is of importance in statistics, is given by

(6.4) $$x_i = \rho\cos\theta_1\cdots\cos\theta_{n-i}\sin\theta_{n-i+1}, \quad \theta_0 = 0, \ \theta_n = \tfrac{1}{2}\pi; \qquad i = 1, \cdots, n.$$

Let $\rho_i = \rho\cos\theta_1\cdots\cos\theta_i$, $i=1, \cdots, n-2$, and set $\rho_0=\rho$. Then $x_1 = \rho_{n-2}\cos\theta_{n-1}$, $x_i = \rho_{n-i}\sin\theta_{n-i+1}$, and $\rho_{n-i+1} = \rho_{n-i}\cos\theta_{n-i+1}$, $i=2, \cdots, n$. As before,

(6.5) $$\nabla x_1\wedge\nabla x_2 = \rho_{n-2}\nabla\rho_{n-2}\wedge\nabla\theta_{n-1},$$

(6.6) $$\nabla x_i\wedge\nabla\rho_{n-i+1} = \rho_{n-i}\nabla\rho_{n-i}\wedge\nabla\theta_{n-i+1}, \qquad i = 2, \cdots, n.$$

Using (6.5) and (6.6) in an induction on r, it easily follows that

(6.7) $$\begin{aligned}&\nabla x_1\wedge\cdots\wedge\nabla x_r \\ &\quad = (-1)^{(r-1)(r-2)/2}\rho_{n-2}\cdots\rho_{n-r}\,\nabla\rho_{n-r}\wedge\nabla\theta_{n-(r-1)}\wedge\cdots\wedge\nabla\theta_{n-1}.\end{aligned}$$

In particular, taking $r=n$, the Jacobian of the transformation is given by

$$J = (-1)^{(r-1)(r-2)/2}\rho_{n-2}\rho_{n-3} \cdots \rho_0$$
$$= (-1)^{(r-1)(r-2)/2}\rho^{n-1} \cos^{n-2} \theta_1 \cos^{n-3} \theta_2 \cdots \cos \theta_{n-2}.$$

This Jacobian is used in developing the probability density function of χ^2 [4].

(b) *Vector analysis.* Applying ∇ to the elements of D corresponding to $\mathbf{V}$ and $\mathbf{V}^*$ of Λ in three dimensions is equivalent to the ordinary operations in vector analysis [2]. In fact,

$$(6.8) \quad \nabla A = (\text{curl } A)^*, \quad \nabla A^* = (\text{div } A)^*, \quad \nabla(\nabla f) = 0, \quad \nabla(\nabla f)^* = (\text{Lap} f)^*$$

Any calculations involving curl, div, etc., can thus be carried out in terms of ∇ in any coordinate system.

For example, consider the transformation to spherical polar coordinates given by (6.3). Applying ∇ to the equations yields equations for $\nabla\rho$, $\nabla\theta$, $\nabla\phi$. It is easily verified that

$$(6.9) \quad \begin{aligned} \nabla\rho &= \cos\theta\cos\phi E_1 + \cos\theta\sin\phi E_2 + \sin\theta E_3, \\ \rho\nabla\theta &= -\sin\theta\cos\phi E_1 - \sin\theta\sin\phi E_2 + \cos\theta E_3, \\ \rho\cos\theta\nabla\phi &= -\sin\phi E_1 + \cos\phi E_2; \end{aligned}$$

$$(6.10) \quad \begin{aligned} \nabla\rho \wedge (\nabla\rho)^* &= E_1 \wedge E_2 \wedge E_3, \quad \nabla\rho \wedge (\nabla\theta)^* = \nabla\rho \wedge (\nabla\phi)^* = 0, \\ \nabla\theta \wedge (\nabla\theta)^* &= \rho^{-2}(E_1 \wedge E_2 \wedge E_3), \quad \nabla\theta \wedge (\nabla\phi)^* = 0, \\ \nabla\phi \wedge (\nabla\phi)^* &= \rho^{-2}\sec^2\theta\,(E_1 \wedge E_2 \wedge E_3). \end{aligned}$$

The Laplacians of ρ, θ, and ϕ can now be evaluated. For example,

$$\begin{aligned} \nabla(\nabla\rho)^* = {} & \nabla(\cos\theta\cos\phi)(E_2 \wedge E_3) + \nabla(\cos\theta\sin\phi)(E_3 \wedge E_1) \\ & + \nabla\sin\theta(E_1 \wedge E_2) = (2/\rho)(E_1 \wedge E_2 \wedge E_3), \end{aligned}$$

so that Lap $\rho = 2/\rho$ by (6.8). In a similar manner, Lap θ and Lap ϕ may be calculated.

$$(6.11) \quad \text{Lap}\,\rho = 2/\rho, \qquad \text{Lap}\,\theta = -(\tan\theta)/\rho^2, \qquad \text{Lap}\,\phi = 0.$$

Now, consider any function V of ρ, θ, ϕ which is twice differentiable. Using subscripts to indicate partial derivatives

$$(\nabla V)^* = V_\rho(\nabla\rho)^* + V_\theta(\nabla\theta)^* + V_\phi(\nabla\phi)^*;$$

so that, using (6.10)

$$\begin{aligned} \nabla(\nabla V)^* &= \nabla V_\rho \wedge (\nabla\rho)^* + \nabla V_\theta \wedge (\nabla\theta)^* + (\nabla V_\phi) \wedge (\nabla\phi)^* \\ &\quad + V_\rho\nabla(\nabla\rho)^* + V_\theta\nabla(\nabla\theta)^* + V_\phi\nabla(\nabla\phi)^* \\ &= V_{\rho\rho}\nabla\rho \wedge (\nabla\rho)^* + V_{\theta\theta}\nabla\theta \wedge (\nabla\theta)^* + V_{\phi\phi}\nabla\phi \wedge (\nabla\phi)^* \\ &\quad + V_\rho\nabla(\nabla\rho)^* + V_\theta\nabla(\nabla\theta)^* + V_\phi\nabla(\nabla\phi)^*. \end{aligned}$$

Finally, using (6.8), (6.10), (6.11) yields

$$\text{(6.12)} \qquad \operatorname{Lap} V = V_{\rho\rho} + \frac{1}{\rho^2} V_{\theta\theta} + \frac{\sec^2\theta}{\rho^2} V_{\phi\phi} + \frac{2}{\rho} V_\rho - \frac{\tan\theta}{\rho^2} V_\theta.$$

(c) *Substitution in triple integrals* [1]. If the transformation (6.1) is applied to the n-tuple integral of $F(x_1, \cdots, x_n)$ over the region R, it is well known that, under suitable assumptions,

$$\int \underset{R}{\cdots} \int F(x_1, \cdots, x_n) dx_1 \cdots dx_n = \int \underset{\bar{R}}{\cdots} \int \bar{F}(u_1, \cdots, u_n) J du_1 \cdots du_n,$$

where

$$\text{(6.13)} \qquad \bar{F}(u_1, \cdots, u_n) = F[f_1(u_1, \cdots, u_n), \cdots, f_n(u_1, \cdots, u_n)],$$

and $\bar{R}$ is the region corresponding to R under the transformation. A comparison with (6.2) suggests the use of $\nabla x_1 \wedge \cdots \wedge \nabla x_n$ instead of $dx_1 \cdots dx_n$ in the symbol for the integral. Changes in coordinates could then be made by expressing each of the symbols in the new coordinates, as in the case of ordinary Riemann integrals, that is,

$$\text{(6.14)} \qquad \begin{aligned} &\int \underset{R}{\cdots} \int F(x_1, \cdots, x_n) \nabla x_1 \wedge \cdots \wedge \nabla x_n \\ &\qquad = \int \underset{\bar{R}}{\cdots} \int \bar{F}(u_1, \cdots, u_n) J \nabla u_1 \wedge \cdots \wedge \Delta u_n, \end{aligned}$$

using (6.2) and (6.13).

(d) *Integral theorems* [2]. The previous remarks indicate the advantage of using ∇x, ∇y, ∇z instead of dx, dy, and dz in integration in E_3. Define line and surface integrals of other elements of D by

$$\int_C A = \int_C a_1 dx + a_2 dy + a_3 dz, \qquad \iint A^* = \int_S a_1 dydz + a_2 dzdx + a_3 dxdy.$$

For a suitably well-behaved region R whose boundary is a surface S, the divergence theorem states that $\iint_S A^* = \iiint_R \nabla A^*$. Similarly, for a suitably well-behaved orientable surface R bounded by a curve C, Stokes' theorem states that $\int_C A = \iint_R \nabla A$.

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THE COMPLEX SUM OF DIVISORS

ROBERT SPIRA, University of California, Berkeley

0. Introduction. The gaussian integers have been the subject of numerous investigations. In fact, for every property of the natural numbers one can ask if the same is true for the gaussian integers, or for other unique factorization domains. One of the first examples of this is Dirichlet's extension of his theorem on primes in arithmetic progressions to the complex case. Many other topics have been considered, such as quadratic reciprocity, quadratic forms, the Pell equation, sums of squares, the Fermat problem and asymptotic laws.

Natural questions certainly arise about number theory functions such as the totient function and the number-of-divisors function; and these have been defined for the gaussian integers and shown to have analogous properties to their real counterparts. These last functions are "counting" functions, from the complex numbers to the reals. Gegenbauer [1] has defined a function in which the norms of the divisors are summed. One can ask if there is a complex sum-of-divisors function, from the complexes to the complexes. This paper takes up the definition and simplest properties of such a function. Also treated are complex Mersenne and complex perfect numbers. The methods are elementary.

The method used to define the complex σ-function is to first define it for powers of primes and then to extend the definition multiplicatively. For powers of primes we try the guess $\sigma(\pi^n)=1+\pi^*+\cdots+\pi^{*n}$, where π^* is an associate of π, and ask: How does the would-be definition of $\sigma(\pi^n)$ depend on the choice of the associate π^*. The remarkable answer to this question is that for the reasonable property $|\sigma(\pi^n)|\geqq|\pi^n|$, the associate chosen must lie in the right half-plane. An opposite inequality results from choices in the left half-plane.

1. Definition of $\sigma(a+bi)$. There are, *a priori*, infinitely many possible definitions of the function $\sigma(a+bi)$. For one can choose for each divisor of $a+bi$ any one of the four associates and add them up. For instance, one could define $\sigma(1+i)$ as $\alpha+\beta$, where $\alpha\in\{1,-1,i,-i\}$ and $\beta\in\{1+i,1-i,-1+i,-1-i\}$, giving 16 choices for the definition, and similarly for other gaussian integers. From among this chaos of possibilities it is possible to select a well-behaved function, having the natural property that $|\sigma(a+bi)|\geqq|a+bi|$. In the following, I will use Greek letters for gaussian integers, π denoting a gaussian prime.

We wish, first of all, to have a multiplicative function, *i.e.*, $\sigma(\alpha\cdot\beta)=\sigma(\alpha)\cdot\sigma(\beta)$ for $(\alpha,\beta)=1$. To achieve this, we set $\sigma(\epsilon)=1$, for ϵ a unit. Then we set $\sigma(\pi^k)=\sigma((i\pi)^k)=\sigma((-\pi)^k)=\sigma((-i\pi)^k)=$ a certain number defined below, independent of the choice of associate of π. Then if $\eta=\prod\pi_i^{k_i}$, we set $\sigma(\eta)=\prod\sigma(\pi_i^{k_i})$. By these means, we clearly obtain a multiplicative sum-of-divisors function.

It now remains to define $\sigma(\pi^k)$. This definition is based on the following lemma.

LEMMA. *For complex integers α, such that $|\alpha| > 1$,*

$$(1)\qquad |1+\alpha+\cdots+\alpha^n| \geqq |\alpha^n|, \textit{for } \mathfrak{R}(\alpha) > 0;$$

$$(2)\qquad |1+\alpha+\cdots+\alpha^n| \leqq |\alpha^n|, \textit{for } \mathfrak{R}(\alpha) < 0.$$

Proof. Let $\alpha = a+bi$, then for $\mathfrak{R}(\alpha) > 0$, setting the norm of $\eta = \|\eta\|$,

$$\left\|\frac{\alpha^{n+1}-1}{\alpha-1}\right\| = \left\|\frac{\alpha^{n+1}-1}{a+bi-1}\right\| > \left\|\frac{\alpha^{n+1}-1}{a+bi}\right\| = \left\|\frac{\alpha^{n+1}-1}{\alpha}\right\| \geqq \|\alpha^n\| - 1/\|\alpha\|.$$

Thus $\|1+\alpha+\cdots+\alpha^n\| > \|\alpha^n\| - 1/\|\alpha\|$; but since the left-hand side is an integer and $\|\alpha^n\|$ is an integer, we can neglect the term $1/\|\alpha\|$ (being <1) if we replace $>$ by $\geqq$. Thus (1) is proved for norms, and hence is true for absolute values. The proof of (2) is similar.

This lemma was discovered geometrically by plotting values of $1+\alpha+\cdots+\alpha^n$ and values of α^n on the same sheet of paper for α's in various parts of the plane.

According to the lemma, if we fix on an associate π^* of π with $\mathfrak{R}(\pi^*) > 0$, and set $\sigma(\pi^k) = 1+\pi^*+\cdots+\pi^{*k}$, we will have $|\sigma(\pi^k)| \geqq |\pi^k|$. To fix our ideas, the first quadrant is thought of as containing the positive real semiaxis, but not the imaginary semiaxis. Also, from the two associates lying in the right half plane, we arbitrarily select the π^* lying in the first quadrant. Other possibilities are discussed later in Section 4. The foregoing paragraphs are summed up in the following definition.

DEFINITION. *Let η be a gaussian integer. To calculate $\sigma(\eta)$, factor η into a product of powers of distinct primes, $\eta = \epsilon\prod \pi_i^{k_i}$, ϵ a unit, each π_i lying in the first quadrant, then $\sigma(\eta) = \prod(\pi_i^{k_i+1}-1)/(\pi_i-1)$.*

Clearly σ is multiplicative and $|\sigma(\eta)| \geqq |\eta|$. Where the domains of the rational powers of primes and gaussian powers of primes overlap, the sum-of-divisors functions coincide. Thus in a natural sense, the complex σ-function is an extension of the real σ-function. However, for instance, for the real σ-function, $\sigma(2) = 3$, but for the complex σ-function, $\sigma(2) = 2+3i$. A small table of the complex σ-function is given in Table 1 below.

For the natural numbers we have the well-known laws of parity:

$$\text{odd} + \text{odd} = \text{even} + \text{even} = \text{even},$$

$$\text{odd} + \text{even} = \text{even} + \text{odd} = \text{odd}.$$

It is an unusual fact that these laws of parity hold also for the gaussian integers, where one defines: α is even iff $1+i$ divides α. The proof of the parity laws consists in checking the 16 cases arising from the fact that $a+bi$ is odd iff a and b have different real parity.

Using these laws of parity one can prove, in a manner similar to the real case, the following theorem: $\sigma(\alpha)$ is odd iff α is a square or $1+i$ times a square or an associate of such.

TABLE 1

$\sigma(a+bi)$

$a \backslash b$	0	1	2	3	4	5	6
1	1	$2+i$	$2+2i$	$5+5i$	$2+4i$	$6+8i$	$2+6i$
2	$2+3i$	$3+i$	$5i$	$3+3i$	$-2+10i$	$3+5i$	$-5+15i$
3	4	$2+6i$	$4+2i$	$8+4i$	$6+5i$	$9+7i$	$8+8i$
4	$-4+5i$	$5+i$	$3+11i$	$-1+6i$	$-8+i$	$5+5i$	$-3+15i$
5	$4+8i$	$3+9i$	$6+2i$	$10i$	$6+4i$	$20i$	$6+6i$
6	$8+12i$	$7+i$	$-10+10i$	$12+4i$	$2+16i$	$7+5i$	$20i$

2. Complex Mersenne numbers. The class of numbers $(q^p-1)/(q-1)$, with p and q rational primes, I call general Mersenne numbers. If $q=2$, these are called Mersenne numbers. Similarly, the numbers $(\pi^p-1)/(\pi-1)$, π a gaussian prime and p a rational prime, I call the general complex Mersenne numbers, and if $\pi=1+i$, these are called complex Mersenne numbers, and are denoted as M_p. These classes of numbers are useful in the study of σ-functions.

Table 2 gives the real and imaginary parts of M_p, where $h=\frac{1}{2}(p-1)$.

TABLE 2

	$p\equiv 1(8)$	$p\equiv 3(8)$	$p\equiv 5(8)$	$p\equiv 7(8)$
$\mathcal{R}(M_p)$	2^h	2^h	-2^h	-2^h
$\mathcal{I}(M_p)$	$-(2^h-1)$	2^h+1	2^h+1	$-(2^h-1)$

If $p\equiv \pm 1 \pmod 8$, then $\|M_p\|=2^{2h+1}-2^{h+1}+1$, and if $p\equiv \pm 3 \pmod 8$, then $\|M_p\|=2^{2h+1}+2^{h+1}+1$. On the other hand,

$$2^{2p}+1=(2^{2h+1}-2^{h+1}+1)(2^{2h+1}+2^{h+1}+1),$$

so that $\|M_p\|$ divides $2^{2p}+1$ in all cases (p an odd prime). The numbers $2^{2p}+1$ have been studied by Cunningham [2], and others. Mr. John Brillhart is at present searching for factors of these numbers, and will present a complete report shortly.

3. Complex perfect numbers. One possible definition of complex perfect numbers would be to say that α is perfect iff $\sigma(\alpha)=2\alpha$. These numbers, however, can be regarded as multiply perfect. As $1+i$ is the "prime of least norm," we take the definition: α is perfect iff $\sigma(\alpha)=(1+i)\alpha$. We have, using this definition, the following analog to Euclid's theorem on perfect numbers:

THEOREM. *If M_p is a complex Mersenne prime and if $p \equiv 1 \pmod 8$, then $(1+i)^{p-1} \cdot M_p$ is perfect.*

Proof. $M_p = -i[(1+i)^p - 1]$ and

$$\sigma[(1+i)^{p-1} \cdot M_p] = -i[(1+i)^p - 1]\sigma[(1+i)^p - 1].$$

To find $\sigma[(1+i)^p - 1]$, we pick the associate π, with $\mathcal{R}(\pi) > 0$ and $\mathcal{I}(\pi) > 0$. It is easy to see that this associate is $[(1+i)^p - 1]$ itself iff $p \equiv 1 \pmod 8$. In this case $\sigma[(1+i)^p - 1] = (1+i)^p$, and the theorem is proved.

For $p \equiv 1 \pmod 8$, the M_p mentioned in the previous section are all composite for $p < 250$ except for the cases $p = 73$, 241, which have recently been shown to be prime by D. H. Lehmer and John Brillhart.

In the real case, we have also the converse to the analog of the above theorem. The proofs of this converse do not appear to generalize to the complex case. However, we do have the following theorem.

THEOREM. *If $a+bi$ is odd then $(1+i)^{k-1} \cdot (a+bi)$ can be perfect only if $k \equiv 0$ or $k \equiv \pm 1 \pmod 8$.*

Proof. Table 3 gives the real and imaginary parts of $(1+i)^k$, where $h = [\frac{1}{2}k]$.

TABLE 3

$k \equiv$	0	1	2	3	4	5	6	7	(mod 8)
$\mathcal{R}$	2^h	2^h	0	-2^h	-2^h	-2^h	0	2^h	
$\mathcal{I}$	0	2^h	2^h	2^h	0	-2^h	-2^h	-2^h	

If $(1+i)^{k-1} \cdot (a+bi)$ is perfect, then we have $\sigma[(1+i)^{k-1} \cdot (a+bi)] = -i[(1+i)^k - 1]\sigma(a+bi) = (1+i)^k \cdot (a+bi)$. However, $|(1+i)^k - 1| > |(1+i)^k|$ iff $k \equiv 2, 3, 4, 5, 6 \pmod 8$, and always $|\sigma(a+bi)| \geqq |a+bi|$; hence $|-i[(1+i)^k - 1]\sigma(a+bi)| > |(1+i)^k \cdot (a+bi)|$ and the theorem is proved.

The complex perfect numbers are included in the class of numbers for which $||\sigma(\alpha)|| = 2||\alpha||$. These numbers I call norm-perfect. A curiosity of this class is the number $2+i$, an odd norm-perfect number.

Abundant and deficient numbers can be defined as numbers α for which $||\sigma(\alpha)|| > 2||\alpha||$, and $||\sigma(\alpha)|| < 2||\alpha||$, respectively. Thus $1+i$ is abundant, $1+2i$ is deficient. One could also, in an obvious manner, define multiply-perfect and norm-multiply-perfect numbers.

4. Other possible definitions. In the definition of Section 1, we chose, of the two associates of π with positive real part, the one lying in the first quadrant. One could define other σ-functions by taking associates in the fourth quadrant, or even mixing the choice of associates in the first and fourth quadrants. This would certainly be justified, if by so doing one could prove, say, an analog of

the Euler pentagonal-number recurrence formula. So far, I have found no such relation. The choice made in this paper was made on the basis of simplicity. It makes no difference in the theory of complex Mersenne numbers, since, if we change the choice, we would be dealing merely with the conjugate numbers.

However, this could make a difference in the theory of perfect numbers. For instance, if one defined: α is perfect iff $\sigma(\alpha)=(1-i)\alpha$, and let $\sigma((1+i)^k) = -i[(1-i)^{k+1}-1]$, then $\sigma[(1-i)^6(-i)(7+8i)]=(1-i)^7(-i)(7+8i)$ and we would have a perfect number corresponding to an associate of M_7. One could possibly force a number to be perfect by mixing the choice of associates in the right and left half-planes, but this has little significance.

Another natural possibility for defining a sum-of-divisors function would be to use the second part of the lemma in Section 1, and construct a σ-function satisfying $|\sigma(\alpha)| \leqq |\alpha|$. One can even define in the real case a function of this type, setting $\sigma^*(p^k)=1-p+p^2- \cdots +(-1)^k p^k$.

5. Conclusions. It has been shown in this paper how to define a sum-of-divisors function for the gaussian integers. An important property of this function is that $|\sigma(\alpha)| \geqq |\alpha|$. Much of the real σ-function theory, such as Mersenne and perfect numbers, can be carried over to the complex case.

The strictness of the above inequality was checked for powers of primes up to the ninth power. A proof of this strictness for all powers would certainly be desirable. One would also like to know if all even complex perfect numbers are of "Euclid's" type, and if there are any odd complex perfect numbers.

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DESIGN OF MIXED DOUBLES TOURNAMENTS

E. N. GILBERT, Bell Telephone Laboratories, Murray Hill, New Jersey

1. Introduction. This paper will discuss combinatorial problems which arise in designing mixed doubles tournaments. In each match of a mixed doubles tournament two teams, each containing one man and one woman, compete with one another (as in tennis, bridge, etc.). In the literature on the design of doubles tournaments (see F. Scheid [5] and G. W. Beynon [1]) the added requirement that each team be a man-woman pair seems not to have been considered.

For scoring purposes, ordinary doubles tournaments are designed to have a high degree of symmetry; *e.g.*, each player appears once as partner to every other player and twice as opponent to every other player. In a mixed tournament only much weaker symmetries are possible. In the designs which follow each player is to meet many other players without undue repetitions. One constraint will be

I. *No two players shall appear together in more than one match of the tournament (regardless of whether they play on the same or on opposite teams in that match).*

If I limits the number of matches in the tournament too severely, an alternative constraint is

II. *No two players of the same sex shall appear together in more than one match; two players of opposite sex can play in* 0, 1, *or* 2 *matches but in the last case they must appear once as teammates and once as opponents.*

In designing a tournament subject to I or II we will try to maximize the number T of matches in the tournament. It will be assumed that there is the same number n of men as of women in the tournament. If I is adopted, then

$$T \leqq \left[\tfrac{1}{2}n\left[\tfrac{1}{2}n\right]\right]. \tag{1}$$

To prove (1), note that each man can appear in at most $[\frac{1}{2}n]$ matches and that

$$2T = \sum_{\text{man } i} \{\text{number of matches containing man } i\}.$$

Similarly, if II is adopted, then

$$T \leqq \tfrac{1}{2}n(n-1) \tag{2}$$

for now each man can appear in at most $n-1$ matches.

The bounds (1), (2) are generally found to be very close to the maximum values of T and are achieved for infinitely many values of n. Whenever (1) is attained for even n, each player meets $\frac{1}{2}n$ players of the same sex and all n players of opposite sex. Whenever (2) is attained, each player meets all other players of the same sex and at least $n-1$ players of opposite sex.

2. Design for I with $n=12k\pm 2$. The design of this section is based on a suggestion of Conyers Herring. Let $N=\frac{1}{2}n=6k\pm 1$. Number the $2N$ men using the labels $1, \cdots, N, 1^*, \cdots, N^*$. Also number the women with the same set of labels. Each match of the tournament will be described by a quadruple (m_1, m_2, w_1, w_2) where m_1, m_2, w_1, w_2 are the labels given to the man on team #1, the man on team #2, $\cdots$, the woman on team #2.

Pick three integers a, b, c such that each of $a, b, c, a-b, b-c, c-a$ is relatively prime to N. For example, with $N=6k\pm 1$, $a=1$, $b=2$, $c=3$ suffice. The tournament is a two-parameter family of matches $M(i, j)$, $i=1, \cdots, N$,

$j=1, \cdots, N$. Match $M(i, j)$ is the quadruple

$$M(i,j) = (i, (i + aj)^*, i + bj, (i + cj)^*),$$

where the additions are modulo N. Note, in this design, that two players of the same sex appear together only if one has a starred number and the other has an unstarred number.

To verify that no pair of players meets twice (Condition I) one may consider pairs of players of six possible kinds: a pair of men (m and m^*), a pair of women (w and w^*), and four kinds of man-woman pairs (m and w, m and w^*, m^* and w, m^* and w^*). Six necessary and sufficient conditions, each to prevent one of the six pairs from meeting twice, are that a, $c-a$, b, c, $b-a$, and $c-a$ be relatively prime to N. Proofs in the six cases are similar. For example, if (contrary to I) the pair m^*, w appears both in $M(i, j)$ and in $M(I, J)$, then

$$m^* \equiv i + aj \equiv I + aJ \pmod{N}, \qquad w \equiv i + bj \equiv I + bJ \pmod{N};$$

whence $(a-b)(j-J)\equiv 0 \pmod{N}$. Since $a-b$ is relatively prime to N, $j \equiv J \pmod{N}$ and then $i\equiv I \pmod{N}$, a contradiction. Conversely, if D divides both N and $a-b$, then the pair m^*, w which appears in $M(i, j)$ reappears in $M(i-a(N/D), j+(N/D))$.

Of the six numbers, a, b, c, $a-b$, $b-c$, $c-a$, at least one is even and at least one is divisible by 3. Then the only N's which can be relatively prime to all six numbers are those of the form $6k\pm 1$. Thus this design does not generalize to other values of N.

In this design $[\frac{1}{2}n[\frac{1}{2}n]]=N^2$, the number of matches; thus the bound (1) is attained. Moreover, as mentioned in the introduction, each player will meet all players of the opposite sex since n is even.

In some tournaments it is required that the matches be arranged into rounds of some given number s of simultaneous matches. This is an added restriction inasmuch as a player can now appear at most once in any of the matches within a round. The tournaments of this section can be arranged in rounds of N, the jth round containing the matches $M(1, j), \cdots, M(N, j)$.

3. Generators. Most of the designs which follow use a method of generators suggested by the technique of R. C. Bose [2] for designing incomplete balanced block experiments. For the moment, the men will be assumed numbered $0, 1, \cdots, n-1$ and likewise the women. A *generator* will be a particular match from which $n-1$ other matches may be derived. If the generator has the quadruple $G=(m_1, m_2, w_1, w_2)$ (interpreted as in Sec. 2), then for $t=1, \cdots, n-1$, other matches are

$$G(t) = (m_1 + t, m_2 + t, w_1 + t, w_2 + t),$$

where the addition is modulo n.

Now a list of g generators $G_1, \cdots, G_g$ will specify a tournament of ng matches $G_i(t)$ $i=1, \cdots, g$, $t=0, \cdots, n-1$. These ng matches will not satisfy

I or II, or even be distinct unless the generators G_i are chosen with care.

Table I below lists sets of generators for tournaments satisfying I for various values of n. For example when there are 8 men and 8 women the generators are (0102) and (0256). These generate a 16 match tournanent:

(0102), (1213), (2324), (3435), (4546), (5657), (6760), (7071),

(0256), (1367), (2470), (3501), (4612), (5723), (6034), (7145).

It remains to characterize the sets of generators which will produce tournaments satisfying I or II. To each generator $G=(m_1, m_2, w_1, w_2)$, associate four sets of numbers called MM differences, WW differences, WM_0 differences, WM_t differences as follows.

$$\begin{aligned} &MM \text{ differences are } m_2 - m_1 \text{ and } m_1 - m_2,\\ &WW \text{ differences are } w_2 - w_1 \text{ and } w_1 - w_2,\\ &WM_0 \text{ differences are } w_2 - m_1 \text{ and } w_1 - m_2,\\ &WM_t \text{ differences are } w_2 - m_2 \text{ and } w_1 - m_1.\end{aligned}$$

These differences are to be computed modulo n. For example in the case of the two generators for $n=8$ the differences are listed below.

Generator	Differences			
	MM	WW	WM_0	WM_t
0102	± 1	± 2	2, 7	0, 1
0256	± 2	± 1	3, 6	4, 5

This set of differences satisfies the hypotheses of the following theorem.

Theorem I. *The tournament generated by g generators $G_1, \cdots, G_g$ satisfies* I *if and only if*

(i) *no two of the* $2gMM$ *differences are equal* modulo n,
(ii) *no two of the* $2gWW$ *differences are equal* modulo n,
(iii) *no two of the* $4g$ WM *differences are equal* modulo n.

Theorem II. *The tournament generated by g generators $G_1, \cdots, G_g$ satisfies* II *if and only if*

(i) *no two of the* $2g$ MM *differences are equal* modulo n,
(ii) *no two of the* $2g$ WW *differences are equal* modulo n,
(iii) *no two of the* $2g$ WM_0 *differences are equal* modulo n,
(iv) *no two of the* $2g$ WM_t *differences are equal* modulo n.

The proofs of Theorems I and II are similar; only Theorem I will be proved in detail.

Man a and woman b play together twice in the tournament if and only if there are two generators (possibly the same) G and G^* such that

1. G contains a man m and a woman w and $a\equiv m+t$, $b\equiv w+t$ for some t.

2. G^* contains a man m^* and a woman w^* and $a \equiv m^* + t^*$, $b \equiv w^* + t^*$ for some t^*. Rewriting 1 and 2 as $w - m \equiv b - a \equiv w^* - m^*$, man a and woman b play together twice if and only if $b - a$ appears twice among the WM differences. Similarly the possibility of two men or two women meeting twice leads to the remaining conditions (i) and (ii) of Theorem I.

To achieve the upper bound (1) using Theorem I we must have $T = ng = [\frac{1}{2}n[\frac{1}{2}n]]$, which is possible only if $n \equiv 0$ or 1 (mod 4) and $g = [\frac{1}{4}n]$. For every such value of n in $5 \leqq n \leqq 17$ it has been found possible to find $[\frac{1}{4}n]$ generators and it seems reasonable to conjecture that even for larger $n \equiv 0$ or 1 (mod 4) the upper bound (1) can be achieved. Although Table I was easy to compute by cut and try methods a systematic way of producing $[\frac{1}{4}n]$ generators is known only when n is a power of an odd prime (next section). When $n \leqq 4$ it is impossible to find even a single generator satisfying the hypotheses of the theorem. When $n = 4$ there can be only two matches, such as (0101), (2323). When $n = 2$ or 3 only one match is possible.

When $n \equiv 2$ or 3 (mod 4) the method of generators can never produce a tournament of $[\frac{1}{2}n[\frac{1}{2}n]]$ matches. There is some evidence to show that this number of matches might still be had by other means. For example when $n = 6$ the upper bound is 9; a 9-match tournament is (0101), (0223), (0345), (1424), (1535), (2405), (2514), (3413), (3502). When $n = 7$ the upper bound is 10; a 10 match tournament is (0101), (0223), (0345), (1424), (1535), (2306), (2415), (3612), (4603), (5646). Unlike the tournaments of Section 2 or those constructed from generators, each player does not appear the same number of times in the 10 match tournament for $n = 7$; man 5 and woman 6 play only twice while the others play three times.

In any tournament of $[\frac{1}{2}n[\frac{1}{2}n]]$ matches satisfying I, and in which n is even, each man meets every woman. Similarly whenever the upper bound is achieved for a value of $n \equiv 1$ (mod 4) each man meets every woman but one. If the players consist of n married couples the tournament can then be arranged so that every man meets every woman except his own wife. For example, if the design is obtained from $[\frac{1}{4}n]$ generators there will be just one residue R (mod n) which does not appear among the WM differences. Then the men and women are to be numbered so that, for every $x = 0, \cdots, n-1$, man x is married to woman $x + R$ (mod n).

Table II lists sets of generators for tournaments satisfying II. For example when $n = 5$ the generators (0103), (0240) generate the tournament (0103), (1214), (2320), (3431), (4042), (0240), (1301), (2412), (3023), (4134). In this tournament man x plays against every other man and both with and against every woman except woman $x + 1$.

In a tennis match it may be a slight advantage to play on a particular side of the court (because of the position of the sun or the direction of wind). Suppose we always interpret the quadruple $(m_1 m_2 w_1 w_2)$ as a match in which m_1 and w_1 play on the better side of the court and m_2 and w_2 play on the other. Then each player appears g times on the better side and g times on the worse side in a tour-

nament constructed from g generators.

To generate a tournament by rounds of s, one may insist further that the generators $G_1, \cdots, G_g$ each have no MM or WW differences $0, \pm 1, \cdots$, or $\pm(s-1) \pmod{n}$. For suppose $G=(m_1, m_2, w_1, w_2)$ is a generator with such MM and WW differences. Consider the two matches $G(t)$ and $G(T)$, generated by G and containing a given player, say woman b. Then $b \equiv w_1+t \equiv w_2+T \pmod{n}$ and $T-t \equiv w_1-w_2 \pmod{n}$. Thus the player appears at most once in any collection of s consecutive matches of the sequence $G, G(1), \cdots, G(n-1)$. For $j=1, \cdots, [n/s]$, the set of s matches

$$R(j) = G(sj), \qquad G(sj+1), \cdots, G(sj+s-1)$$

is a round which can be played simultaneously. Thus the tournament provides $g[n/s]$ rounds of s. If s does not divide n, there will be $g(n-[n/s])$ other matches which can be played in g smaller rounds. If constraint I is adopted, the additional restriction on the MM and WW differences is not always severe. For example for $n=15$ and $s=5$ there are still three generators (0, 5, 0, 6)(0, 7, 9, 14), (0, 6, 4, 11).

Table I

Sets of Generators for Tournaments Satisfying I

n	Generators
5, 6, or 7	(0 1 0 2)
8, 9, 10, or 11	(0 1 0 2), (0 2 5 6)
12	(0 1 0 4), (0 2 7 10), (0 4 1 6)
13	(0 1 0 2), (0 2 7 8), (0 5 3 9)
14 or 15	(0 1 0 4), (0 2 7 10), (0 4 1 6)
16, 19 and over	(0 1 0 6), (0 4 12 13), (0 6 1 10), (0 5 3 7)
17 and over	(0 1 0 2), (0 2 9 10), (0 8 6 11), (0 7 4 12)

Table II

Sets of Generators for Tournaments Satisfying II

n	Generators
5	(0 1 0 3), (0 2 4 0)
6	(0 1 0 2), (0 2 5 4)
7	(0 1 0 2), (0 2 3 0), (0 3 6 5)
8 and over	(0 1 0 2), (0 2 3 6), (0 3 6 5)

4. Designs for n a power of an odd prime. This section will give a systematic procedure for designing tournaments satisfying I or II when $n=p^B$, p an odd prime, B some integer $\geqq 1$. The method of generators will be used. Since the cases $n=p^B$ with $B \geqq 2$ require a slight generalization of the results of Section 3, the simpler case $n=p$ will be given in detail first.

Every prime n has a primitive root (see Nagell [4]), *i.e.*, an integer a such that $a, a^2, \cdots, a^{n-1}\equiv 1$ are all different modulo n. Set $k=[\frac{1}{4}n]$ and construct a generator

$$G = (0,\ a^k - 1,\ a^k,\ -1). \tag{3}$$

Its differences are

$$\begin{aligned} MM \text{ differences} &= \pm\ (a^k - 1),\\ WW \text{ differences} &= \pm\ (a^k + 1),\\ WM_0 \text{ differences} &= \pm\ 1,\\ WM_t \text{ differences} &= \pm\ a^k. \end{aligned}$$

The other generators will be $G_1, G_2, \cdots$, where

$$G_i = a^iG = (0,\ a^i(a^k - 1),\ a^{i+k},\ -a^i). \tag{4}$$

The differences of G_i are those of G multiplied by a^i.

If constraint II is adopted, the $\frac{1}{2}(n-1)$ generators $G, G_1, \cdots, G_{\frac{1}{2}(n-3)}$ will be used. In Theorem II, each set of differences contains constant multiples of $\pm 1, \pm a, \cdots, \pm a^{\frac{1}{2}(n-3)}$. These differences are all distinct. For, if $a^i\equiv \pm a^j$, then $a^{i-j}\equiv\pm 1$, which implies that $\frac{1}{2}(n-1)$ divides $i-j$. Now Theorem II applies and $G, G_1, \cdots, G_{\frac{1}{2}(n-3)}$ generate a tournament which satisfies II and achieves the bound (2).

If I is adopted, we use only $[\frac{1}{4}n]$ generators $G, G_1, \cdots, G_{k-1}$ in order to prevent equalities between WM_0 and WM_t differences. Theorem I applies and the tournament of $n[\frac{1}{4}n]$ matches satisfies I. If $n=4k+1$, the bound (1) is achieved. The only man-woman pairs which do not meet are those with $m\equiv w \pmod n$.

Similar designs are obtained for $n=p^B$ (where $B\geqq 2$) by using a field of p^B elements. The names of the players are no longer integers modulo n but are elements of the field. Generators are defined as in Section 3 but with $+$ the addition operation of the field and t ranging over all p^B field elements. Theorems I and II still hold; indeed their proofs require only that $+$ be a commutative group operation.

Again a primitive root a exists such that $a, a^2, \cdots, a^{n-1}=1$ are the nonzero field elements [3]. Set $k=[\frac{1}{4}n]$ and construct generators G, G_i as quadruples of field elements using (3) and (4). Again $G, G_1, \cdots, G_{\frac{1}{2}(n-3)}$ and $G, G_1, \cdots, G_{k-1}$ satisfy Theorems II and I.

For illustration take $n=3^2$. A field of 9 elements consists of 9 polynomials $0, \pm 1, \pm x, \pm(x+1), \pm(x-1)$ with coefficients integers modulo 3. The product of two elements is found by multiplying the two polynomials in the natural way, dividing the result by a mod 3 irreducible polynomial x^2-x-1, and keeping only the remainder. In this field the polynomial x is a primitive root. In the design for constraint I, the two generators and their differences are listed below:

GENERATOR	MM	WW	WM_0	WM_t
$G=(0, x, x+1, -1)$	$\pm x$	$\pm(x-1)$	± 1	$\pm(1-x)$
$xG=(0, x+1, 1-x, -x)$	$\pm(x+1)$	± 1	$\pm x$	$\pm(1-x)$

The generalization fails if $n=2^B$. In a field of 2^B elements, $-d=d$ for all elements d; then the MM differences cannot be distinct.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

THE SEPARATION OF PARTIAL DIFFERENTIAL EQUATIONS WITH MIXED DERIVATIVES

H. L. HUNZEKER, The University of Nebraska

The classical argument for the separation of variables in partial differential equations is often too weak to separate equations with mixed derivatives. We present a stronger separability argument which may also be used for partial differential equations with mixed derivatives.

To define separability, let $L[U]$ be a partial differential operator defined on a domain D of an n-dimensional Euclidean space. The partial differential equation $L[U]=0$ is said to be separable in the $x=(x_1, \cdots, x_n)$ variables if there exists a solution to $L[U]=0$ in the form

$$U(x) = \prod_{i=1}^{n} X^i(x_i), \tag{1}$$

where each X^i, $i=1, 2, \cdots, n$, is a function of x_i only.

A sufficient condition for separability is:

LEMMA 1. *Let $M^i[V_i]$, $i=1, \cdots, n$, denote partial differential operators on the functions $V^i(\bar{x}_i)$ defined on D with $\bar{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$,*

$i=1, \cdots, n$. Let $N^i[X^i(x_i)]$, $i=1, \cdots, n$, denote ordinary differential operators defined on D. Then the partial differential equation $L[U]=0$ is separable if there exist operators M^i and N^i such that solutions to the n ordinary differential equations $N^i[X^i]=0$ exist in D and such that

$$L[U] = \sum_{i=1}^{n} M^i[V^i]N^i[X^i]$$

for x in D.

The proof is obvious since the solution in the form of equation (1) can be constructed at once from solutions to the n ordinary differential equations $N^i[X^i]=0$.

In general, the above lemma yields the same solutions as the classical argument when the classical argument is applicable. Furthermore, it is more straightforward. For example, consider the biharmonic equation $\nabla^4 U(x, y)=0$. Here, we assume that $U(x, y)\equiv X(x)Y(y)$ and write

$$X^{(\text{iv})}Y + 2X''Y'' + XY^{(\text{iv})} \equiv [Y'' + Y(d^2/dx^2)](X'' + kX) + [X'' + X(d^2/dy^2)](Y'' - kY) = 0$$

where k is arbitrary. We may then use solutions to the ordinary differential equations $X''+kX=0$ and $Y''-kY=0$ to construct solutions to the biharmonic equations.

It is not difficult to find a non-orthogonal coordinate system in which Laplace's equation is separable [1]. For this, let

$$x = u, \qquad y = \tfrac{1}{2}(u - v), \qquad z = w - \tfrac{1}{4}\sqrt{2}(3u - v).$$

In the nonorthogonal u, v, w-coordinates,

$$\nabla^2 F \equiv F_{,uu} + 5F_{,vv} + 2F_{,ww} + 2F_{,uv} + \sqrt{2}F_{,uw} - \sqrt{2}F_{,vw}.$$

Now let $F(u, v, w) = U(u)V(v)W(w)$, and Laplace's equation may be written in the form

$$L(U' - kU) + M(V' + kV) + 2UV(W'' + \sqrt{2}kW' + 2k^2W) = 0$$

where k is arbitrary and

$$L = V'W + \sqrt{2}VW' + VW\frac{d}{du}, \qquad M = U'W - 4kUW - \sqrt{2}UW' + 5UW\frac{d}{dv}\cdot$$

A solution to the partial differential equation may now be constructed from solutions to the ordinary differential equations

$$U' - kU = 0, \qquad V' + kV = 0, \quad \text{and} \quad W'' + \sqrt{2}kW' + 2k^2W = 0.$$

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ON AN OPTIMAL SEARCH PROCEDURE

MARTIN SANDELIUS, Statistical Institute, University of Gothenburg, Sweden

One of the n objects of a set S will obtain a certain property by a random choice such that the ith object has the probability P_i of being chosen $(\sum_{i=1}^{n} P_i=1)$. Afterwards the chosen object, called A, will be identified by the following test sequence. Assuming that after h tests $(h=0, 1, \cdots)$ A is found to belong to a subset of m_h objects $(m_0=n)$, the procedure is terminated if $m_h=1$, otherwise the $(h+1)$th test consists in splitting this subset into two subsets and determining which of these contains A. Zimmerman [1] proved that the reversion of a combination procedure, such that in each stage the objects with the two smallest probabilities are combined into a new object and the sum of the two probabilities is assigned as the probability of this object, gives a test procedure with minimum expected number of tests.

In this note we shall consider the particular case when all P_i are equal. Writing $m_h=2^{p_h}+k_h$, where p_h and k_h are nonnegative integers and $k_h<2^{p_h}$ we shall prove the following

THEOREM. *When all P_i are equal the minimum expected number of tests is $p_0+(2k_0/n)$, and this will be attained if and only if the number of objects of each of the two groups which are tested in the $(h+1)$th test $(h=0, 1, \cdots)$ belongs to the closed interval $(2^{p_h-1}, 2^{p_h})$.*

Instead of using the combination device we base our proof directly on the splitting procedure.

Proof. The theorem holds for $n=1$, since no testing is needed in this case. Assuming that the theorem holds for $1, \cdots, n-1$ objects, we shall show that it holds for n objects. We divide the set S into two subsets of n_1 and n_2 objects, with $n_1 \leqq n_2$. Dropping the subscripts of p_0 and k_0 we shall prove that for n objects the minimum number of tests equals $p+(2k/n)$, and that this minimum is attained if and only if n_1 and n_2 both belong to the closed interval $(2^{p-1}, 2^p)$ and subsequent testing follows the condition of the theorem.

We shall write $n_1=2^{p-1-r}+i$ and $n_2=2^{p-1+s}+j$, where $p-1-r$, $p-1+s$, r, s, i, j are nonnegative integers, $i<2^{p-1-r}$ and $j<2^{p-1+s}$. Further $p-1+s\leqq p$, so that s equals 0 or 1. Let $m(n_1, n_2)$ denote the minimum expected number of tests (n_1, n_2 fixed); $p(A, n_i)$, the probability that A belongs to the subset with n_i objects; and $\mu(A, n_i)$, the minimum expected number of tests given that A belongs to the subset with n_i objects. Then holding n_1, n_2 fixed, we have, since by assumption the theorem applies to $1, \cdots, n-1$ objects,

$$\begin{aligned} m(n_1, n_2) &= 1 + \sum_{i=1}^{2} p(A, n_i)\mu(A, n_i) \\ &= 1 + (n_1/n)[p - 1 - r + 2(i/n_1)] + (n_2/n)[p - 1 + s + 2(j/n_2)] \\ &= p + (2k/n) + R, \end{aligned}$$

where R, by means of the above expressions for n, n_1, n_2, can be written

$$R = (1/n)(n_2 s - n_1 r + 2^{p+1} - 2^{p-r} - 2^{p+s}).$$

Since s can take only the values 0 and 1 we can write

$$R = (1/n)(2^p - n_1 r - 2^{p-r} + a), \qquad a = \begin{cases} 0, & s = 0, \\ j, & s = 1. \end{cases}$$

Hence $R=0$ for (i) $r=0$, $s=0$, and for (ii) $r=0$, $s=1$, $j=0$, while $R>0$ for (iii) $r=0$, $s=1$, $j>0$, and, since $n_1<2^{p-r}$, also for the remaining case, (iv) $r>0$. Since n_1 and n_2 both belong to the closed interval $(2^{p-1}, 2^p)$ if and only if (i) or (ii) holds, and since, by assumption, any departure, in subsequent testing, from the condition of the theorem will add a positive quantity to the above expression for the minimum expected number of tests, the theorem follows.

The theorem can be applied to testing a system consisting of a set of n components and having the following properties. (i) While the system functions all of its components have the same probability of failing within a unit of time. (ii) Only one component fails at a time. (iii) The system breaks down as soon as one of its components fails. (iv) No further component will fail while the system is tested and repaired. (v) Any subset of i components $(1 \leqq i \leqq n)$ can be tested in a single test. (vi) The testing is error free.

REMARK. In [2], which deals with the different problem of searching for all A-objects in a set where initially each object independently of the others has the probability P of becoming an A-object, some limiting results are obtained, to which the above theorem is closely related (cf. formulas (20a), (21), and (43b) of [2]).

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ON SOLUTIONS OF HOMOGENEOUS, LINEAR, DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

DAVID ZEITLIN, Remington Rand Univac, St. Paul, Minnesota

THEOREM. (a) *Let*

$$(1) \qquad y_k = \sum_{j=1}^{t}\left(\sum_{i=1}^{n_j} \beta_{ij} k^{i-1}\right) r_j^k + \sum_{i=(n_1+\cdots+n_t)+1}^{n} \alpha_i r_i^k, \qquad k = 0, 1, \cdots,$$

be the general solution of a homogeneous, linear, difference equation of order n with real, constant coefficients, where $n_j \geqq 1$, $j=1, \cdots, t$, and $(n_1+\cdots+n_t) \leqq n$. β_{ij} and α_i are arbitrary constants, and r_j, $j=1, \cdots, t$, and r_i, $i=n, n-1, \cdots, (n_1+\cdots+n_t)+1$, are distinct, real or complex numbers. Let m be a positive integer, and let $p\neq 0$, $q\neq 0$ be arbitrary real constants such that $p+qr_j^m\neq 0$,

$j=1, 2, \cdots, t$, and $p+qr_i^m \neq 0$, $i=n, n-1, \cdots, (n_1+\cdots+n_t)+1$. If we define

$$Y_{k,m} = \sum_{s=0}^{k} \binom{k}{s} p^{k-s} q^s y_{sm}, \tag{2}$$

then

$$Y_{k,m} = \sum_{j=1}^{t} \left\{ \beta_{1j} + \sum_{\nu=1}^{n_j-1} \gamma_{\nu j} k^\nu \right\} (p + qr_j^m)^k + \sum_{i=(n_1+\cdots+n_t)+1}^{n} \alpha_i (p + qr_i^m)^k, \tag{3}$$

where the $\gamma_{\nu j}$ are arbitary constants. Thus, $Y_{k,m}$ is the general solution of some homogeneous, linear, difference equation of order N with real, constant coefficients, where $N \leqq n$. If the values of r_j^m, $j=1, \cdots, t$, and r_i^m, $i=n, n-1, \cdots, (n_1+\cdots+n_t)+1$, are distinct, then $N=n$ (for $m=1$, this is always so).

(b) *In* (1), *let $n_j=1$ and $\beta_{1j}=\alpha_j$, $j=1, \cdots, t$, noting that r_i, $i=1, \cdots, n$, are distinct, real or complex numbers. If $r_i^n \equiv C \neq 0$, $i=1, \cdots, n$, where C is a real constant, and if $m=\sigma n$, $\sigma=1, 2, \cdots$, then $N=1$, and $Y_{k,m=\sigma n} \equiv y_0(p+qC^\sigma)^k$ is the general solution. If* (1) *(with $n_j=1, j=1, \cdots, t$) is the general solution of*

$$y_{k+n} = qy_{k+m} + py_k, \qquad pq \neq 0, \ 1 \leqq m < n, \tag{4}$$

then $Y_{k,m} \equiv y_{kn}$, $k=0, 1, \cdots$. If (1) *(with $n_j=1, j=1, 2, \cdots, t$) is the general solution of* (5),

$$py_k + qy_{k+n} = y_{k+1}, \qquad pq \neq 0, \tag{5}$$

then $Y_{k,n} \equiv y_{kn}$, $k=0, 1, \cdots$.

Remarks. A Fibonacci sequence $\{f_k\}$ is defined by $f_{k+2}=f_{k+1}+f_k$, $k \geqq 0$, where f_0 and f_1 are two arbitrary positive integers. If we set $p=1$, $q=1$, $m=1$, and $n=2$ in (4) and (2), we obtain

$$f_{2k} = \sum_{s=0}^{k} \binom{k}{s} f_s. \tag{6}$$

(6) solves the elementary problem E1347 [1959, 61] of this MONTHLY, vol. 66, p. 592. If we set $p=-1$, $q=1$, and $n=2$ in (5) and (2), we obtain

$$f_k = (-1)^k \sum_{s=0}^{k} (-1)^s \binom{k}{s} f_{2s}. \tag{7}$$

(7) is of interest when k is odd. In Jordan ([1], p. 132), we find the summation formula,

$$\sum_{s=0}^{k} (-1)^s \binom{k}{s} g(s) = (-1)^k \Delta^k g(0). \tag{8}$$

Thus, (8) also yields (7) if $g(s) \equiv f_{2s}$, since $\Delta^k f_{2s} = f_{2s+k}$. If $g(s) \equiv f_{2s+1}$, then (8)

yields

$$f_{k+1} = (-1)^k \sum_{s=0}^{k} (-1)^s \binom{k}{s} f_{2s+1}. \tag{9}$$

(9) is of interest when k is odd. The paper by Rao [2] has an extensive list of identities satisfied by the Fibonacci numbers, but fails to list either (7) or (9) which, apparently, are not too well known.

Proof of the theorem. (a) Substitution of y_{sm}, given by (1), into (2), yields, upon interchanging summations,

$$Y_{k,m} = \sum_{j=1}^{t} \sum_{i=1}^{n_j} \beta_{ij} m^{i-1} \sum_{s=0}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} + \sum_{i=(n_1+\cdots+n_t)+1}^{n} \alpha_i (p + qr_i^m)^k.$$

Now,

$$\begin{aligned} A_j &\equiv \sum_{i=1}^{n_j} \beta_{ij} m^{i-1} \sum_{s=0}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} \\ &= \beta_{1j}(p + qr_j^m)^k + \sum_{i=2}^{n_j} \beta_{ij} m^{i-1} \sum_{s=1}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} \\ &= (p + qr_j^m)^k \left[\beta_{1j} + \sum_{i=2}^{n_j} \beta_{ij} m^{i-1} \sum_{s=1}^{i-1} \mathfrak{S}_{i-1}^{s} (qr_j^m)^s (p + qr_j^m)^{-s} k^{(s)} \right], \end{aligned}$$

since

$$\sum_{s=1}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} = (p + qr_j^m)^k \sum_{s=1}^{i-1} \mathfrak{S}_{i-1}^{s} (qr_j^m)^s (p + qr_j^m)^{-s} k^{(s)}. \tag{10}$$

(10) follows from ([1], pp. 196–197, formulae (2) and (7), where $u(t) = (p+qr_j^m t)^k$ in (7)). In (10), $\mathfrak{S}_{i-1}^{s}$ is a Stirling number of the second kind, *i.e.*, $k^t = \sum_{s=1}^{t} \mathfrak{S}_t^s k^{(s)}$, where $k^{(s)} = k(k-1) \cdots (k-s+1) = \sum_{\nu=1}^{s} S_\nu^s k^\nu$ and S_ν^s is a Stirling number of the first kind. Let

$$h(s, i) = \mathfrak{S}_{i-1}^{s} (qr_j^m)^s (p + qr_j^m)^{-s}.$$

Then

$$\sum_{s=1}^{i-1} h(s, i) \sum_{\nu=1}^{s} S_\nu^s k^\nu = \sum_{\nu=1}^{i-1} \left(\sum_{s=\nu}^{i-1} h(s, i) S_\nu^s \right) k^\nu.$$

Let $H(i, \nu) = \sum_{s=\nu}^{i-1} h(s, i) S_\nu^s$. Then

$$\begin{aligned} (p + qr_j^m)^{-k} A_j &= \beta_{1j} + \sum_{i=2}^{n_j} \beta_{ij} m^{i-1} \sum_{\nu=1}^{i-1} H(i, \nu) k^\nu \\ &= \beta_{ij} + \sum_{\nu=1}^{n_j-1} \left(\sum_{i=\nu}^{n_j-1} m^i H(i + 1, \nu) \beta_{(i+1)j} \right) k^\nu = \beta_{1j} + \sum_{\nu=i}^{n_j-1} \gamma_{\nu j} k^\nu, \end{aligned}$$

where

$$(11)\qquad \gamma_{\nu j} = \sum_{i=\nu}^{n_j-1} m^{i} H(i+1,\nu)\beta_{(i+1)j} = \sum_{i=\nu}^{n_j-1}\sum_{s=\nu}^{i} m^{i} S_{\nu}^{s}\mathfrak{S}_{i}^{s}(qr_j^{m})^{s}(p+qr_j^{m})^{-s}\beta_{(i+1)j}.$$

(b) If $n_j=1$ and $\beta_{1j}=\alpha_j$, $j=1,\cdots,t$, then $y_k=\sum_{i=1}^n \alpha_i r_i^k$ and $Y_{k,m} = \sum_{i=1}^n \alpha_i(p+qr_i^m)^k$. If y_k is the general solution of (4), then $p+qr_i^m=r_i^n$, $i=1,\cdots,n$. If y_k is the general solution of (5), then $p+qr_i^n=r_i$, $i=1,\cdots,n$.

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ON THE ORDER OF A BIAS

W. H. WILLIAMS,* McMaster University

A quantity which often arises in statistics is u/v where $u=(\bar{x}-\mu_x)^2$, $v=\sum_{i=1}^n (x_i-\bar{x})^2$, and $\bar{x}$ is the mean of a sample $(x_1,\cdots,x_n)$ drawn from a population with mean μ_x. The form arises, for example, in evaluating the variance of the linear regression estimator of a population mean, see Cochran [1]. It has been shown ([3], [4]) that u and v are independently distributed if and only if the x distribution is Gaussian. Much literature has been devoted to the effects of non-normality on u/v, most of which is directed at the distribution properties and analysis of Type I and Type II errors. In general $E(u/v) = E(1/v)Eu+\mathrm{Cov}(u, 1/v)$, where E denotes mathematical expectation and Cov denotes the covariance of two random variables. If the higher moments are finite, it can be shown by an interesting straightforward method that the covariance term is of lower order in n than $E(1/v)Eu$.

If u/v is written as $U(1+\delta_u)(1+\delta_v)^{-1}/V$ and $1/v$ as $(1+\delta_v)^{-1}/V$ where $Eu=U, Ev=V\neq 0$, $\delta_u=(u-U)/U$ and $\delta_v=(v-V)/V$ then $\mathrm{Cov}(u,1/v) = (U/V)E\{\delta_u(1+\delta_v)^{-1}\} \doteq -\mathrm{Cov}(u,v)/V^2$. The approximation involves dropping the terms $UE\{\delta_u\delta_v^{2+i}\}/V$, $i=0,1,\cdots$. The terms inside the expectation cannot be greater than $O(n^{-1})$ by the methods of Fisher [2]. Further, $U=\sigma_x^2/n$ and $V=(n-1)\sigma_x^2$ so that the neglected terms are no more than $O(n^{-3})$.

Next, let $d_i=x_i-\mu_x$ so that $Ed_i=0$ and $Ed_i^2=\sigma_x^2$, then

$$\mathrm{Cov}(u,v) = \frac{1}{n^3}E\left\{\left[(n-1)\sum_{i=1}^n d_i^2 - \sum_{i\neq j}^n d_id_j\right]\left[\sum_{i=1}^n d_i^2+\sum_{i\neq j}^n d_id_j\right]\right\} - \frac{n-1}{n}\sigma_x^4.$$

which will reduce to

$$(1)\qquad \frac{n-1}{n^2}\{Ed_i^4 - 3\sigma_x^4\}$$

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* Now at Bell Telephone Laboratories, Inc.

or alternatively, since by definition the variance of d_i^2 is $\mathrm{Var}(d_i^2)=Ed_i^4-\sigma_x^4$,

$$\frac{n-1}{n^2}\left\{\mathrm{Var}(d_i^2)-2\sigma_x^4\right\}. \tag{2}$$

Hence, $\mathrm{Cov}(u, v)$ is $O(n^{-1})$ and $\mathrm{Cov}(u, 1/v)$ is $O(n^{-3})$. Furthermore, Eu is $O(n^{-1})$ and $E(1/v)$ is $O(n^{-1})$ so that $EuE(1/v)$ is $O(n^{-2})$. Notice the interesting agreement of (1) and (2) with the known fact that $\mathrm{Cov}(u, v)=0$ for normal populations. The straightforward approach used to derive these expressions can also be used to verify the stated orders of the terms neglected in the approximation.

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AN ELEMENTARY ANALYSIS OF AN INTEGRAL QUADRATIC FORM

WILLIAM EDWARD CHRISTILLES, San Antonio, Texas

The quadratic form,

$$x^2 + hxy + ky^2 \tag{1}$$

in integers x and y, with integral coefficients, can be written as the determinant of the matrix,

$$(xI + yA), \tag{2}$$

where I is the two by two identity matrix and A is a two by two matrix with a trace and determinant of h and k respectively. The corresponding matrix form (2) defines a commutative ring with identity over the integers and an integral domain if A has no rational characteristic root. We shall obtain some general and particular results concerning binary integral quadratic forms with a leading coefficient of one by analysis of the ring defined by (2).

For a two by two matrix, H, let $|H|$, $\overline{H}$, and $\mathrm{tr}(H)$ denote, respectively, the determinant, the adjoint, and the trace of H. For a positive integer, n, and square matrices of integers U and V of like order, let the relation, $U\equiv V(\mathrm{mod}\ n)$, mean that n divides each element of $U-V$. Let "diagonal matrix" mean "square matrix whose off-diagonal elements are zero." Let "scalar matrix" mean "diagonal matrix with each diagonal element equal to the same number."

The following relations are readily verified for two by two matrices, $A, B, K, \cdots$.

1.1 $\overline{uA+vI}=u\overline{A}+vI$, where u and v are numbers.
1.2 $A+\overline{A}=(\text{tr } A)I$
1.3 $(A-\overline{A})^2=I[\text{tr}(A^2)-2|A|]=-|A-\overline{A}|I$
1.4 $|A|=|\overline{A}|$ and $A\overline{A}=|A|I$
1.5 $|tI-A|=t^2-t(\text{tr } A)+|A|=\frac{1}{4}|(2t-\text{tr } A)I-(A-\overline{A})|$ $=\frac{1}{4}[(2t-\text{tr } A)^2+|A-\overline{A}|]$, for each number t.
1.6 $\overline{AB}=\overline{B}\overline{A}$
1.7 $(A-\overline{A})(K-\overline{K})$ is a scalar matrix if K is of the form $(xI+yA)$, for numbers x and y.
1.8 If $AB=BA$ and B is a diagonal matrix, then A is a diagonal matrix or B is a scalar matrix.

It should be emphasized that these statements, with the exception of (1.6), are valid only for two by two matrices.

We now introduce the following two lemmas:

LEMMA 1. *If U and V are two by two nondiagonal matrices of integers which commute, then there exist integers a, b, and c, where $ab\neq 0$, and $(a, b, c)=1$, such that $aU=bV+cI$.*

LEMMA 2. *If U and V are two by two nondiagonal matrices of integers which commute, and p is a prime integer which divides $|U|$ and $|V|$, then $UV\equiv 0 \pmod p$ or $\overline{U}V\equiv 0 \pmod p$.*

Proof of Lemma 1. It will suffice to show that there exists a rational linear combination of U and V which is a scalar matrix. Denote the elements of U and V by U_{ij} and V_{ij}, respectively, where the first index denotes the row position and the second, the column position.

The equation

$$\begin{pmatrix} U_{12} & V_{12} \\ U_{21} & V_{21} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a rational nontrivial solution since, owing to the commutativity of U and V, $U_{12}V_{21}-V_{12}U_{21}=0$. Hence, $xU+yV$ is a diagonal matrix and commutes with U and V. Further, by (1.8) above, $xU+yV$ is a scalar, since U and V are not diagonal matrices. Neither xU nor yV is a scalar; hence, neither x nor y is zero.

Proof of Lemma 2. By Lemma 1, and relations (1.1), (1.2), and (1.4) above, there exist integers, a, b, and c, where $ab\neq 0$, and $(a, b, c)=1$, such that:
2.1 $aU=bV+cI=-b\overline{V}+(c+b \text{ tr } V)I$ and
2.2 $a\overline{U}=b\overline{V}+cI$.
Hence, $aUV=-b|V|I+(c+b\text{ tr } V)V$ and $a\overline{U}V=b|V|I+cV$. Since, by (1.4), (2.1), and (2.2), $a^2|U|I=a^2U\overline{U}=[b^2|V|+c(c+b \text{ tr } V)]I$, it follows that p divides c or p divides $(c+b \text{ tr } V)$. Therefore $aUV\equiv 0 \pmod p$ or $a\overline{U}V\equiv 0 \pmod p$. Suppose p divides a and c; then it cannot divide b since $(a, b, c)=1$ and hence divides $\overline{V}$. This establishes the lemma for, if p divides a and c (or a and $c+b \text{ tr } V$), then $V\equiv\overline{V}\equiv 0$ (mod p.).

Suppose now that A is a two by two matrix of integers, with $h=\operatorname{tr} A$, $k=|A|$, and $D=|A-\bar{A}|$. Impose the restrictions that $-|A-\bar{A}|$ is not the square of an integer. By (1.5) above, this implies that A has no rational characteristic root. Let S denote the set of matrices $\{xI+yA\}$, M the set of integers $\{|uI+vA|\}$, where x, y, u, and v are integers, with $(u, v)=1$. Let P denote the set of positive prime factors of integers in M, and let P' denote the set of odd primes in P. A matrix U in S will be called an *S-unit* if $|U|^2=1$, an *S-composite* if it is the product of two nonunit matrices of S, and an *S-prime* if it is neither an S-unit nor an S-composite. It follows that S is an integral domain and is, in fact, isomorphic to the quadratic integral domain defined by the characteristic root field of A.

We now state and prove the following theorems:

THEOREM 1. *If p belongs to P and to M, and u is an integer such that pu belongs to M, then u belongs to M.*

THEOREM 2. *If to each p in P' there corresponds an odd integer m in M, less than p^2, such that p divides m, then P' is a subset of M. If P' is a subset of M, and either M contains no even integer or M contains an even integer not divisible by* 4, *then P is a subset of M.*

THEOREM 3. *If U is an S-prime and P is a subset of M, then $|U|$ is a prime integer.*

THEOREM 4. *If P is a subset of M, then S has unique factorability, up to an S-unit.*

Proof of Theorem 1. By Lemma 2, $(UV)/p$ or $(\bar{U}V)/p$ is in S. This proves the theorem since $|(UV)/p| = |(\bar{U}V)/p| = u$.

Proof of Theorem 2. Suppose P' is not a subset of M. Then let p be the least prime in P' which is not in M. But, by hypothesis, p divides an odd integer in M with a resulting odd quotient less than p. Thus each prime factor of this quotient is less than p, and hence, is in M. But by Theorem 1, it would follow that p is in M. Further, if P is a subset of M, and M contains no even integer, then P' is P; if M contains an even integer not divisible by 4, then M contains 2, since all odd prime factors of this integer are in P'.

Proof of Theorem 3. Suppose $|U|$ is not a prime and not a unit. Then, since P is a subset of M, there exist matrices B and C in S such that $|U|=|B||C|$, where $|B|$ is a prime p, and $|C|^2>1$. Then

$$U = \frac{UB\bar{B}}{p} = \frac{U\bar{B}}{p}B = \frac{UB}{p}\bar{B}.$$

One of these two forms gives U as the product of two matrices in S, neither of which has determinant ± 1.

Proof of Theorem 4. Let $\{U_i\}$, $1 \leqq i \leqq n_1$, and $\{V_i\}$, $1 \leqq i \leqq n_2$, be sets of S-primes such that $\prod_{i=1}^{n_1} U_i = \prod_{i=1}^{n_2} V_i$ and hence such that $\left|\prod_{i=1}^{n_1} U_i\right| = \left|\prod_{i=1}^{n_2} V_i\right|$. Thus, by Theorem 3 and the division algorithm for integers, for each i from 1 to n_1 and some j from 1 to n_2, it follows that $|U_i| = |V_j|$. Hence $n_1 = n_2$.

Suppose one of the U's, say U_1, is not the product of an S-unit and one of the V_i. But, $U_1\overline{U}_1 = |U|I = |V_j|I = V_j\overline{V}_j$ for some integer j from 1 to n_1. Then

$$U_1 = \left(\frac{\overline{V}_j U_1}{|U_1|}\right) V_j = \left(\frac{V_j U_1}{|U_1|}\right) \overline{V}_j$$

and $(\overline{V}_j U_1)/|U_1|$ or $(V_j U_1)/|U_1|$ is an S-unit. But from the assumption on U_1, $(\overline{V}_j U_1)/|U_1|$ cannot be an S-unit. Thus for some S-unit T, $U_1 = T\overline{V}_j$. If $\overline{V}_j$ is not one of the V_i, then $T|V_j|I\prod U_i = V_j^2 \prod V_i$, where in the first product, $2 \leqq i \leqq n_1$, and in the second, $1 \leqq i \leqq n_1$, $i \neq j$. Since $|V_j|$ is a prime, then for one of the V's, say V_k, and some S-unit Z, $V_j V_k = V_j \overline{V}_j Z$. Hence $V_k = \overline{V}_j Z$, which contradicts the assumption that none of the V_i's is $\overline{V}_j$. This completes the proof.

We now proceed to state criteria which will aid in determining the presence of unique factorability.

It is evident that by a reversible linear transformation over the integers the form (1) can be reduced to one of the forms:

$$u^2 + uv + nv^2, \tag{3}$$

$$u^2 + nv^2, \tag{4}$$

where (3) is obtainable if h is odd and (4) if h is even.

Clearly, if $n > 0$, unity is the only number in M which is less than n, and n is in M. If, in addition, P' is a subset of M and n has an odd prime divisor, p, then $p = n$ since p belongs to P'; therefore to M; hence is not less than n. Therefore, with $n > 0$ and P' a subset of M, n is an odd prime or has the form 2^j for some positive integer j. Now, M, defined by (3), contains $n+2$ and $4n-1$, and M, defined by (4), contains $n+1$, $n+4$, and $n+9$. Then if n is positive and even, M contains $2(2^{j-1}+1)$ if defined by (3), and $4(2^{j-2}+1)$ if defined by (4). Since $2^{j-1}+1$ is odd unless $j=1$ and $2^{j-2}+1$ is an odd integer unless $j=1$ or 2, it follows that if n is even, P' is not a subset of M, if M is defined by (3) and $n > 2$, or if M is defined by (4) and $n > 4$.

Again, if n is odd and positive and M is defined by (4), P' is not a subset of M if $n+1$ has an odd prime divisor, for such a divisor would be less than n, hence not in M. If $n+1$ has the form 2^j, for some positive integer j, then 2^j+8 is in M, defined by (4), so that if $j > 3$, the integer $2^{j-3}+1 < n$ and has an odd prime divisor which is in P'. Therefore P' is not a subset of M defined by (4) if n is odd, positive, and distinct from each of the values: 1, 3, 7.

Now if p is in P' (that is, if p is an odd prime divisor of a member of M) and if $(n, p) = 1$, then there exist positive integers r, s, and t, where s is odd, t even, $2r \leqq p-1$, $s \leqq p-2$, and $t \leqq p-1$, such that if M is defined by (3), p

divides $\frac{1}{4}[(2r+1)^2+D]$ and $(s+1)^2+D$, each of which is in M; and if M is defined by (4), then p divides t^2+n and s^2+n, each of which is in M.

Now from the restrictions given on r, s, and t, we obtain:

3.1 $0<\frac{1}{4}[(2r+1)^2+D]<\frac{1}{4}[p^2+D]<p^2$, if $0<D<3p^2$

3.2 $0<(s+1)^2+D\leqq p^2-[2p-(D+1)]<p^2$, if $0<D<2p-1$

3.3 $0<t^2+n\leqq p^2-[2p-(n+1)]<p^2$, if $0<n<2p-1$

3.4 $0<s^2+n\leqq p^2-[4p-(n+4)]<p^2$, if $0<n<4(p-1)$.

From these inequalities, the above limitations on n, and Theorem 2, together with the observation that p belongs to P' only if $-D$ is a quadratic residue of p, we have the following conclusions:

4.1 2 is the only even positive value of n for which P' is a subset of M, defined by (3), in which case P is a subset of M.

4.2 With M defined by (4) the only positive values of n for which P' is a subset of M are 1, 2, 3, 4, and 7, with P a subset of M only if $n=1$ or 2.

4.3 If n is positive and odd and M is defined by (3), $P'=P$ and P is a subset of M only if there exists an odd prime q, with $D<3q^2$, such that $-D$ is not a quadratic residue of any odd prime less than q, in which case n is the least member of P and M contains no composite integer below n^2.

As an illustration, let $n=41$ for form (3). Then $D=163$, and -163 is seen to be a quadratic nonresidue of 3, 5, and 7, while $163<3(11)^2$. Hence, x^2+x+41 is a prime for $x=0, 1, 2, \cdots, 39$. Note that the elimination of merely 3, 5, and 7 here automatically eliminates from P all primes below 41.

Some positive values of D for which similar results are seen to occur are 3, 7, 11, 19, 43, 67, and 163; and the corresponding n values are 1, 2, 3, 5, 11, 17, and 41. Hardy and Wright list these special values of D and indicate that there is at most one more positive integer with this remarkable property.* In view of the exacting conditions required of D, it seems unlikely that there exists another such positive value.

Consider now form (3), where $n<0$. Then $D<0$. If p is in P' there exists an even integer s and an odd integer t, $s^2<p^2$, $t^2<p^2$, such that p divides s^2+D and t^2+D, and either s^2+D or $\frac{1}{4}(t^2+D)$ is an odd integer in M, according as n is even or odd, respectively. If this odd integer is positive, it clearly is less than p^2. Therefore, P' is a subset of M only if each prime factor of x^2+D is in M, for every integer x with $x^2+D<0$ and $x+n$ even.

As a conclusion, we note that if P' is a subset of M and P is not a subset of M, then S has unique factorability up to an S-composite whose determinant is 2^j, for some integer $j\geqq 2$, and that with M defined by 4, and $n\equiv -1 \pmod 4$, the corresponding S set is a subset of the S set for which M is defined by (3) and $\frac{1}{4}n+\frac{1}{4}$ the coefficient of y^2. Should the former lack unique factorability,

* G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1954, p. 213.

certain difficulties are surmounted if the latter possesses this desired property. As an instance of this, the general integral solution of the equation, $x^2+11y^2=z^n$, n a positive integer, can be obtained by regarding the form $u^2+uv+3v^2$, of which the form x^2+11y^2 is a special case.

ON A THEOREM OF M. EIDELHEIT CONCERNING RINGS OF CONTINUOUS FUNCTIONS

JU-KWEI WANG, Academia Sinica, Taiwan

It is the intention of this paper to give a simplified proof of the following theorem of M. Eidelheit [1]:

THEOREM. *Let X be a compact metrizable space and let $C(X)$ be the ring of all real-valued continuous functions on X endowed with the supremum norm. Let P be a closed subring of $C(X)$ containing the constant functions. Then P is isometrically isomorphic to the ring $C(Y)$ of all real-valued continuous functions on a compact metrizable space Y.*

Our proof will be a simple application of the Stone-Weierstrass theorem [3] and the following result [2]:

PROPOSITION. *Let X be a compact Hausdorff space. In order that X is metrizable, it is necessary and sufficient that $C(X)$ is separable.*

Proof of the theorem. In X we define a binary relation R as follows: for x and x' in X, we define xRx' to mean that $f(x)=f(x')$ for all functions f in P. The relation R is evidently an equivalence relation. Let Y be the quotient space X/R. Since the set $\{(x, x'): xRx'\}$ is closed in the product space $X\times X$, Y is Hausdorff. Also since X is compact and since the partition map is continuous, Y is compact.

Let f be a function in P and let y be a point in Y. If x and x' are representatives of the equivalence class y, then $f(x)=f(x')$. Hence there is no ambiguity if we define $(Tf)(y)=f(x)$. The mapping T is then a linear isometry of P into $C(Y)$. By our construction of Y, $T(P)$ separates points. As P is uniformly closed, we see that $T(P)$ is the whole of $C(Y)$.

We still have to show that the space Y is metrizable. By assumption, X is metrizable. Hence by the Proposition, $C(X)$ is separable. As $C(X)$ is both metrizable and separable, its subspace P is separable. Thus $T(P)=C(Y)$ is also separable. Therefore, by our Proposition again, Y is metrizable. This finishes the proof.

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A SIMPLE PROOF OF A KNOWN RESULT IN PARTITIONS

E. M. WRIGHT, University of Aberdeen, Scotland

Let $a_1, \cdots, a_k$ be a set of different positive integers, let a be their least common multiple and let $p(n)$ be the number of partitions of n into parts chosen from this set, *i.e.*, $p(n)$ is the number of solutions of the Diophantine equation

$$n = m_1a_1 + m_2a_2 + \cdots + m_ka_k,$$

where the m_i are nonnegative integers. We give here a very simple proof of the

THEOREM. $p(n) = \sum_{t=1}^{k} c_{kt}(n)n^{k-t}$, *where* $c_{kt}(n)$ *depends on the residue of* n (mod a) *but not otherwise on* n.

The theorem is not new, for it is implicit in the results of Sylvester and Glaisher, who investigated the properties of $p(n)$ at great length [**1**, **3**].

Recently Rieger [**2**] gave an entirely different proof of a similar result for the particular case in which $a_j = j$. His result was slightly weaker, as the modulus a was replaced by k!

If we put $p(0) = 1$ and take $|x| < 1$, we have

$$F(x) = \prod_{j=1}^{k} (1 - x^{a_j})^{-1} = \sum_{n=0}^{\infty} p(n)x^n.$$

But $(1-x^a)(1-x^{a_j})^{-1}$ is a polynomial in x of degree $a-a_j$ and so

$$\sum_{n=0}^{\infty} p(n)x^n = P(x)(1 - x^a)^{-k} = P(x)\sum_{h=0}^{\infty} Q(h)x^{ah},$$

where $P(x) = \sum_{r=0}^{R} b_rx^r$ is a polynomial in x of degree $R = \sum_{j=1}^{k} (a-a_j) < ka$ and $Q(h) = (h+1)(h+2) \cdots (h+k-1)/(k-1)!$ is a polynomial in h of degree $k-1$. Since $Q(h) = 0$ for $h = -1, -2, \cdots, -(k-1)$, we have

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{r=0}^{R} b_rx^r \sum_{h=-(k-1)}^{\infty} Q(h)x^{ah}.$$

Hence, for $n > 0 \geqq R - ka$, we have

$$p(n) = \sum_{r\equiv n \,(\mathrm{mod}\ a)} b_rQ(\{n - r\}/a) = \sum_{t=1}^{k} c_{kt}(n)n^{k-t},$$

where

$$c_{kt}(n) = a^{t-k}\{(k - t)!\}^{-1} \sum_{r\equiv n\,(\mathrm{mod}\ a)} b_rQ^{(k-t)}(-r/a).$$

Thus $c_{kt}(n)$ depends on the residue of n (mod a) but not otherwise on n.

Much more information about $p(n)$ can, of course, be found by expressing $F(x)$ in partial fractions. This is essentially the method used by Sylvester and Glaisher.

References

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THE CONGRUENCE $(p-1/2)! \equiv \pm 1 \pmod p$

L. J. MORDELL, University of Colorado and St. John's College, Cambridge, England

Let p be an odd prime. Then Wilson's classical result states that $(p-1)!+1 \equiv 0 \pmod p$. On noting that $p-r\equiv -r \pmod p$, this gives, when $p\equiv 1 \pmod 4$, as is well known,

$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 + 1 \equiv 0 \pmod p.$$

However, when $p\equiv 3 \pmod 4$, we have

$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 - 1 \equiv 0 \pmod p.$$

Hence

$$\left(\frac{p-1}{2}\right)! \equiv (-1)^a \pmod p, \tag{1}$$

where $a=0$ or 1. In view of the history of the question, it may perhaps be worth while to state and prove the

THEOREM.* *If p is a prime $\equiv 3 \pmod 4$ and $p>3$, then in* (1)

$$a \equiv \tfrac{1}{2}[1 + h(-p)] \pmod 2 \tag{2}$$

where $h(-p)$ is the class number of the quadratic field $k\{\sqrt{(-p)}\}$.

This result does not appear to have been explicitly stated or at any rate does not seem well known. It is, however, implicit in the literature, and it is now a trivial deduction from results long known, *e.g.*, an old one of Dirichlet's (1828) given here as (3). In fact, Jacobi (1832) conjectured a result equivalent to (2) at a time when the class-number formula was not known. For the history of the subject, see Dickson's *History of the Theory of Numbers*, Vol. 1, page 275.

Write $E=[\frac{1}{2}(p-1)]!$ Denote by $r_1, r_2, \cdots$ the R quadratic residues of p less than $\frac{1}{2}p$, and by $n_1, n_2, \cdots$ the N quadratic nonresidues less than $\frac{1}{2}p$. Then the quadratic residues $r_1', r_2', \cdots$ greater than $\frac{1}{2}p$ are given by $p-n_1, p-n_2, \cdots$ since $p\equiv 3 \pmod 4$. Then

$$E = r_1r_2\cdots n_1n_2\cdots \equiv (-1)^N r_1r_2\cdots r_1'r_2'\cdots \equiv (-1)^N \bmod p \text{ if } p > 3, \tag{3}$$

* Professor Chowla informs me that he found the result about the same time that I did.

since $(-1)^N E \equiv g^{2+4+\cdots+(p-1)} = g^{\frac{1}{4}(p^2-1)} = (g^{\frac{1}{2}(p+1)})^{\frac{1}{2}(p+1)} \equiv 1 \pmod{p}$, where g is a primitive root of p.

Now $R+N=\frac{1}{2}(p-1)$, and it is known* from the class-number formula that

$$R - N = \delta h(-p), \qquad \begin{cases} \delta = 1 \text{ if } p \equiv 7 \pmod{8}, \\ \delta = 3 \text{ if } p \equiv 3 \pmod{8}, p > 3. \end{cases}$$

Hence $2N=\frac{1}{2}(p-1)-\delta h(-p)$. Then if $p\equiv 7 \pmod 8$, $2N\equiv 3-h(-p) \pmod 4$, and if $p\equiv 3 \pmod 8$, $2N\equiv 1-3h(-p) \pmod 4$. The first is $N\equiv\frac{1}{2}[-1-h(-p)] \pmod 2$ and the second is $N\equiv\frac{1}{2}[1+h(-p)] \pmod 2$. These are both included in $N\equiv\frac{1}{2}[1+h(-p)] \pmod 2$.

* L. Holzer, Zahlentheorie II, 1959, Leipzig, pp. 91–93, and H. Hasse, Vorlesungen über Zahlentheorie, 1950, Berlin, pp. 386–390.

A GENERALIZED TURÁN EXPRESSION FOR THE BESSEL FUNCTIONS

WALFED A. AL-SALAM, University of Baghdad, Iraq

1. In a recent paper Toscano [3] has proved the formula

$$\tag{1.1} \begin{aligned} &\sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} H_{n+r}(x) H_{n-r}(x) \\ &\qquad = \frac{(2m)!(n-m)!}{m!} \sum_{j=m}^{n} \binom{j-1}{m-1} \frac{H^2_{n-j}(x)}{(n-j)!} \qquad (m \leqq n) \end{aligned}$$

where $H_n(x)$ is the Hermite polynomial of order n. The expression in the left hand side may be regarded as a generalization of the Turán expression $H_n^2(x) - H_{n+1}(x)H_{n-1}(x)$. Indeed (1.1) reduces to the Demir-Hsü formula [2] when $m=1$. Other proofs of (1.1) as well as extensions to the Laguerre and ultraspherical polynomials and other hypergeometric functions are given in [1].

In the present note we obtain a similar formula involving the Bessel functions. We prove

$$\tag{1.2} \begin{aligned} \Omega_n^{(m)}(x) &= \sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} J_{n-r}(x) J_{n+r}(x) \\ &= \frac{4^m(2m)!}{x^{2m} m!(m-1)!} \sum_{k=0}^{\infty} (n+m+2k)(k+1)_{m-1}(n+k-1)_{m-1} J^2_{n+m+2k}(x) \end{aligned}$$

where $(a)_m = a(a+1)(a+2)\cdots(a+m-1)$, $(a)_0=1$. For definition of the Bessel function $J_n(x)$ see [4]. This formula reduces, for $m=1$, to Lommel's formula [4, p. 152]

$$\tag{1.3} \tfrac{1}{4}x^2\{J_n^2(x) - J_{n+1}(x)J_{n-1}(x)\} = \tfrac{1}{4}x^2\Delta_n(x) = \sum_{k=0}^{\infty} (n+1+2k) J^2_{n+1+2k}(x).$$

It is also a positive representation as sum of squares.

We shall also discuss the cases $m=2$; $m=2$, $n=0$; and $n=0$ of (1.2).

2. *Proof of formula* (1.2). We first note that the formula

$$\sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} \cos 2r\theta = 2^{2m} \sin^{2m} \theta \tag{2.1}$$

can be proved very easily by substituting $\frac{1}{2}(e^{2r\theta i}+e^{-2r\theta i})$ for $\cos 2r\theta$ in the left hand side of (2.1) and summing the resulting two series separately.

It now follows from the formula [4, p. 150]

$$J_\mu(x)J_\nu(x) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2x\cos\theta)\cos(\mu-\nu)\theta d\theta \tag{2.2}$$

and (2.1) that

$$\Omega_n^{(m)}(x) = \frac{2^{2m+1}}{\pi}\int_0^{\frac{1}{2}\pi} J_{2n}(2x\cos\theta)\sin^{2m}\theta d\theta \tag{2.3}$$

This formula can also be written in the form

$$x^{2m}\Omega_n^{(m)}(x) = \frac{2}{\pi}\int_0^{2x} J_{2n}(y)\{4x^2-y^2\}^{m-\frac{1}{2}} dy. \tag{2.4}$$

Put $\phi_m(x)=x^{2m}\Omega_n^{(m)}(x)$. Then $\phi_m(0)=0$ $(m\neq 0)$, and $\phi_m(-x)=\phi_m(x)$. It is also easy to see from (2.4) that

$$\frac{d}{dx}\phi_{m+1}(x) = 4(2m+1)x\phi_m(x). \tag{2.5}$$

We shall prove by induction that $\phi_m(x)$, and hence $\Omega_n^m(x)$ is nonnegative for $m=1, 2, \cdots$. We know from (1.3) that $\phi_1(x)\geqq 0$. Suppose $\phi_m(x)\geqq 0$. Then it follows from (2.5) and the fact that $\phi_{m+1}(0)=0$ that $\phi_{m+1}(x)$ is nonnegative if $x\geqq 0$. But $\phi_{m+1}(x)$ is even. Therefore $\phi_{m+1}(x)$ is everywhere nonnegative.

If we put $m=0$ in (2.5) we get the relation

$$\phi_1(x) = x^2\Delta_n(x) = 4\int_0^x xJ_n^2(x)dx;$$

the right-hand side is evaluated by means of [4, p. 151]

$$\int_0^x xJ_n^2(x)dx = 2\sum_{k=0}^{\infty}(n+1+2k)J_{n+1+2k}^2(x). \tag{2.6}$$

More generally, (2.5) yields the integral representation

$$\phi_m(x) = 2^m\frac{(2m)!}{m!}\int_0^x x\int_0^x x\int_0^x \cdots \int_0^x xJ_n^2(x)(dx)^m. \tag{2.7}$$

Now (2.6) and (2.7) yield

$$x^{2m}\Omega_n^{(m)}(x) = \frac{4^m(2m)!}{m!}\sum_{k=0}^{\infty}(n+m+2k)C_k(m)J^2_{n+m+2k}(x),$$

where $C_k(1)=1$ and

$$C_k(m) = \sum_{\substack{s_1+s_2+\cdots+s_m=k\\ s_1,s_2,\cdots s_m\geqq 0}}\prod_{r=1}^{m-1}(n+r+2s_1+2s_2+\cdots+2s_r)\qquad (m\geqq 1).$$

The author is grateful to Professor L. Carlitz for pointing out that the coefficients $C_k(m)$ can be evaluated as

$$C_k(m) = \frac{1}{(m-1)!}(k+1)_{m-1}(n+k+1)_{m-1}. \tag{2.8}$$

The proof is as follows.

$$\begin{aligned} C_k(m+1) &= \sum_{s_1+s_2+\cdots+s_{m+1}=k}\prod_{r=1}^{m}(n+r+2s_1+2s_2+\cdots+2s_r) \\ &= \sum_{s=0}^{k}\sum_{s_1+s_2+\cdots+s_m=k-s}(n+m+2s_1+2s_2+\cdots+2s_m) \\ &\quad\cdot\prod^{m-1}(n+r+2s_1+\cdots+2s_r) \\ &= \sum_{s=0}^{k}(n+m+2s)C_s(m). \end{aligned}$$

We can now prove (2.8) by induction.

The justification for term by term integration to get (2.7) is by means of the inequality

$$|J_n(x)| \leqq (1/n!)\,|x/2|^n e^{|x|^2},$$

from which it follows that the series is uniformly convergent in every closed interval before and after each integration.

If we put $m=2$ in (1.2) we get

$$\begin{aligned} &x^4\{6J_n^2(x) - 8J_{n+1}(x)J_{n-1}(x) + 2J_{n+2}(x)J_{n-2}(x)\} \\ &\qquad = 2^6\cdot 3\sum_{k=0}^{\infty}(k+1)(n+k+1)(n+2+2k)J^2_{n+2+2k}(x). \end{aligned}\tag{2.10}$$

If in addition we put $n=0$ in (2.10) we obtain the result

$$x^4\{3J_0^2(x) + 4J_1^2(x) + 2J_2^2(x)\} = 2^6\cdot 3\sum_{k=1}^{\infty}k^3J^2_{2k}(x). \tag{2.11}$$

If in (1.2) we put $n=0$, we get

$$\binom{2m}{m}J_0^2(x)+2\sum_{r=1}^{m}\binom{2m}{m-r}J_r^2(x) = \frac{4^m(2m)!}{x^{2m}m!(m-1)!}\sum_{k=0}^{\infty}(m+2k)\{(k+1)_{m-1}\}^2J_{m+2k}^2(x). \tag{2.12}$$

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

ON THE SUM OF POWERS OF NATURAL NUMBERS

JOHN G. CHRISTIANO, Northern Illinois University

The purpose of this paper is to find an expression for $\sum_{x=1}^{n} x^k$. This problem and its result are not new. Usually they are to be found in works on the calculus of finite differences.* However, it is the author's objective to present the solution from the point of view of elementary concepts of algebra in the hope of interesting those without training in the finite calculus technique.

Let us write the identity

$$(x-1)^{k+1}=x^{k+1}-\binom{k+1}{1}x^k+\binom{k+1}{2}x^{k-1}-\binom{k+1}{3}x^{k-2}+\cdots+(-1)^{k+1}, \tag{1}$$

and form the sum

$$\sum_{x=1}^{n}(x-1)^{k+1}=\sum_{x=1}^{n}x^{k+1}-\binom{k+1}{1}\sum_{x=1}^{n}x^k+\binom{k+1}{2}\sum_{x=1}^{n}x^{k-1} -\binom{k+1}{3}\sum_{x=1}^{n}x^{k-2}+\cdots+(-1)^{k+1}\sum_{x=1}^{n}1. \tag{2}$$

* See, for example, Charles Jordan, Calculus of Finite Differences, New York, 1957.

Rewrite the left member of (2) as

$$\sum_{x=1}^{n}(x-1)^{k+1}=\sum_{x=1}^{n}x^{k+1}-n^{k+1}. \tag{3}$$

Making use of (3) in (2) and solving for $\sum x^k$, one obtains

$$\begin{aligned}\sum x^k &= \frac{n^{k+1}}{k+1}+\frac{1}{2}\binom{k}{1}\sum x^{k-1}-\frac{1}{3}\binom{k}{2}\sum x^{k-2}\\ &\quad+\frac{1}{4}\binom{k}{3}\sum x^{k-3}+\cdots(-1)^m\frac{1}{m}\binom{k}{m-1}\sum x^{k-m+1}+\cdots.\end{aligned} \tag{4}$$

Equation (4) is a recursion relation and is taken as defining $\sum_{x=1}^{n} x^k$ so that $\sum x^{k-1}$ is obtained from (4) by replacing k by $k-1$. Similarly, $\sum x^{k-2}$ is obtained by replacing k by $k-2$, etc. Applying (4) to the second member of (4) one gets

$$\begin{aligned}\sum_{x=1}^{n} x^k &= \frac{n^{k+1}}{k+1}+\frac{1}{2}\binom{k}{1}\left[\frac{n^k}{k}+\frac{1}{2}\binom{k-1}{1}\sum x^{k-2}-\frac{1}{3}\binom{k-1}{2}\sum x^{k-3}+\cdots\right]\\ &\quad-\frac{1}{3}\binom{k}{2}\sum x^{k-2}+\frac{1}{4}\binom{k}{3}\sum x^{k-3}-\frac{1}{5}\binom{k}{4}\sum x^{k-4}+\cdots.\end{aligned} \tag{5}$$

Simplify (5) and obtain

$$\begin{aligned}\sum_{x=1}^{n} x^k &= \frac{n^{k+1}}{k+1}+\frac{n^k}{2}+\frac{k(k-1)}{12}\sum x^{k-2}-\frac{k(k-1)(k-2)}{24}\sum x^{k-3}\\ &\quad+\frac{k(k-1)(k-2)(k-3)}{80}\sum x^{k-4}+\cdots.\end{aligned} \tag{6}$$

Again apply (4) to third member of (6) and get

$$\sum_{x=1}^{n} x^k = \frac{n^{k+1}}{k+1}+\frac{n^k}{2}+\frac{kn^{k-1}}{12}-\frac{k(k-1)(k-2)(k-3)}{720}\sum x^{k-4}+\cdots.$$

By repeated application of (4) one can conclude

$$\begin{aligned}\sum_{x=1}^{n} x^k &= \frac{n^{k+1}}{k+1}+\frac{n^k}{2}+\frac{kn^{k-1}}{12}+0\cdot n^{k-2}-\frac{1}{720}k(k-1)(k-2)n^{k-3}\\ &\quad+0\cdot n^{k-4}+\frac{k(k-1)(k-2)(k-3)(k-4)}{30{,}240}n^{k-5}+\cdots.\end{aligned}$$

This result may conveniently and symmetrically be written as

$$\sum_{x=1}^{n} x^k = \frac{n^{k+1}}{k+1}+\frac{n^k}{2}+\sum_{j=2}^{k}\frac{1}{j}B_j\binom{k}{j-1}n^{k-j+1},$$

where the B_j are the well-known Bernoulli numbers ($B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$,

$B_4=-\frac{1}{30}$, $B_6=\frac{1}{42}$, $\cdots$, and $B_{2k+1}=0$ for $k>0$).

Example. Find $\sum_{x=1}^{n} x^3$.

Solution.

$$\sum_{x=1}^{n} x^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{2}B_2\binom{3}{2}n^2 + \tfrac{1}{3}B_3\binom{3}{3}n$$
$$= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 = [\tfrac{1}{2}n(n+1)]^2.$$

NOTE ON EVALUATING CERTAIN REAL INTEGRALS BY CAUCHY'S RESIDUE THEOREM

ORIN J. FARRELL, Union College, and BERTRAM ROSS, Korell Company

The evaluation of a real integral of the form $\int_0^\infty x^m dx/(x^n+a)$ where $a>0$, n is an even integer $\geqq 2$, and m is a nonnegative even integer $\leqq n-2$, by Cauchy's residue theorem is often presented in textbooks as follows. After observation that the value of the integral in question is half that of $\int_{-\infty}^{\infty} x^m dx/(x^n+a)$, the value of the latter is obtained by the usual limiting process, starting with an integral $\int_C z^m dz/(z^n+a)$, where z is complex and C is the closed contour running from $-R$ to R on the real axis, thence around the upper half of the circle $|z|=R$.

For all sufficiently large R the value of the integral around C is of course equal to $2\pi i$ times the sum of the residues at the zeros of z^n+a which lie in the upper half plane. An alternative procedure which avoids the addition of residues is given by Franklin.* This alternative, moreover, applies even when m and n are not integral. It does, however, require an extra change of variable.

The writers wish to present here a modification of the usual procedure which, while not as general in result as that given by Franklin in that m and n are still restricted to being nonnegative integers, does not require addition of residues and does not require an extra change of variable. Moreover, the integers m and n may now be odd or even and the integral from 0 to ∞ can be evaluated directly by the residue theorem without having recourse to the integral from $-\infty$ to ∞. The essential feature of the modification lies in using an appropriately chosen contour C which encloses *just one* of the zeros of z^n+a. This is done, for instance, in evaluating an integral of the form $\int_{-\infty}^{\infty} e^{px}dx/(1+e^x)$.

Let C denote the contour made up of the segment of the real axis from $x=0$ to $x=R$, thence along $|z|=R$ to the ray arg $z=\exp(2\pi i/n)$, thence along this ray to the origin. Thus for all sufficiently large R the contour C encloses just one of the zeros of z^n+a, where n is a positive integer, namely $z_1=a^{1/n}\exp(\pi i/n)$. Then we have

$$\int_C \frac{z^m dz}{z^n+a} = \frac{2\pi i z_1^m}{n z_1^{n-1}} = \frac{2\pi i}{n a^{(n-m-1)/n}\exp[(n-m-1)\pi i/n]}.$$

* Methods of Advanced Calculus, New York, 1944, Ex. 83, p. 248.

When the integral around C is written, as usual, as the sum of three integrals, the integral along the ray $z=r\exp(2\pi i/n)$ may be written as

$$\int_R^0 \frac{r^m \exp[(m+1)2\pi i/n]dr}{r^n+a} = -\int_0^R \frac{x^m \exp[(m+1)2\pi i/n]dx}{x^n+a}.$$

Combining this last integral with the integral along the real axis from $x=0$ to $x=R$ and letting $R\to\infty$, we have, for $0<m+1<n$,

$$\int_0^\infty \frac{x^m dx}{x^n+a} = \frac{2\pi i}{\{1-\exp[(m+1)2\pi i/n]\}na^{(n-m-1)/n}\exp[(n-m-1)\pi i/n]}$$
$$= \frac{\pi}{na^{(n-m-1)/n}\sin[(m+1)\pi/n]}.$$

INTEGRATION

D. G. Mead, Pratt Institute

Although we frequently think of indefinite integration as being more complex than the determination of roots of algebraic equations, we shall see that in some respects this is not the case. It is well known that solutions of some algebraic equations cannot be written in terms of radicals, and also that certain functions do not possess elementary integrals. However, due to the length and depth of the proofs of these facts, few texts go beyond merely exhibiting some examples. Confusing a sufficient condition with one which is both necessary and sufficient, many a student finds security in knowing that all equations of degree $\leqq 4$ have been solved. With respect to integration, however, he is well aware of his rather unhappy position, that of knowing that some functions cannot be integrated (in finite terms) but of possessing no way of determining whether a particular function is in this class or not. This latter situation need not be accepted, for Liouville obtained a test [1] which is both necessary and sufficient for the integrability of any one of a rather large class of functions. We shall recall this test and show both how it can be applied by the college freshman and its utility as an integration technique.

The foundation for the test is the following theorem which is sufficiently "natural" to be readily accepted and easily remembered.

Liouville's Theorem. *If $\int fe^g dx$ is an elementary function, f and g are rational functions of x, and the degree of $g>0$, then*

$$\int fe^g dx = Re^g \tag{1}$$

where R is a rational function of x.

This is actually a special case of Liouville's original theorem (see [1] page 114 or [2] page 47), but it is sufficiently general for our purposes.

In this note, by rational function we mean the quotient of two polynomials with coefficients in any field of characteristic zero, for example, the complex numbers. The term elementary function is more difficult to define (indeed, Ritt takes 12 pages to do so). However, the student is willing to accept as meaningful such statements as "the general algebraic equation of degree 5 cannot be solved in terms of radicals," although we know that "in terms of radicals" requires some preliminary discussion. Therefore, he finds no difficulty with the following definition: An elementary function is one which can be constructed by means of any finite combination of the operations addition, subtraction, multiplication, division, raising to powers, taking roots, forming trigonometric functions and their inverses, taking exponentials and logarithms. In short, no matter how complicated the function, if we can write down all of its terms, the function is elementary. (Actually, the construction of elementary functions includes the forming of algebraic functions, but it seems advisable to omit this generality for the beginning calculus student.)

We return to the test. If we wish to determine whether fe^g can be integrated (*i.e.* has an integral which is an elementary function, or, as we shall also say, $\int fe^g dx$ is elementary), we know, by Liouville's Theorem the form of the integral. Differentiating equation (1) and cancelling the nonzero e^g we find $f=R'+Rg'$ or, letting $R=P/Q$, where P and Q are relatively prime polynomials in x,

$$fQ^2 = P'Q - PQ' + PQg'. \tag{2}$$

Thus $\int fe^g dx$ is elementary if and only if there exist polynomials P and Q satisfying the differential equation (2).

Besides Liouville's theorem, the test requires only one further fact; namely, the following

LEMMA. *If the polynomial $f(x)$ has an r-fold zero at $x=\alpha$ and $r>0$, then $f'(x)$ has an $(r-1)$-fold zero at $x=\alpha$; in other words, if $f(x)=(x-\alpha)^r h(x)$ where $r>0$, $h(x)$ is a polynomial and $h(\alpha)\neq 0$, then $f'(x)=(x-\alpha)^{r-1}k(x)$ where $k(\alpha)\neq 0$.*

The proof of the lemma is a simple differentiation exercise which the student can supply.

It will make things easier if we define the term multiplicity. The number α is called a zero of the polynomial $f(x)$ of multiplicity r (or a root of $f(x)=0$ of multiplicity r) if $f(x)=(x-\alpha)^r h(x)$, where the polynomial $h(\alpha)\neq 0$. In terms of multiplicity the lemma reads: If α is a zero of the polynomial $f(x)$ of multiplicity $r>0$, then α is a zero of $f'(x)$ of multiplicity $r-1$.

By examining some of the examples most frequently quoted in texts, we shall show how easily the analysis of equation (2) can be carried out, even by the student who has not previously encountered differential equations.

Example 1: e^{-x^2}. If $\int e^{-x^2}dx$ is elementary, then $\int e^{-x^2}dx=Re^{-x^2}$ or $1=R'-2xR$. Letting $R=P/Q$, where P and Q are relatively prime polynomials and $Q\neq 0$, we find:

$$Q^2 = QP' - PQ' - 2xPQ \tag{1.2}$$

which is equation (2). Rearranging, we obtain

$$Q(Q - P' + 2xP) = - PQ'. \tag{1.3}$$

Let us assume that the degree of Q is positive. Then $Q=0$ has a root; let α be such a root and call its multiplicity $r (r>0)$. Since P and Q are relatively prime, $P(\alpha)\neq 0$. Now, α is a zero of the left side of (1.3) of multiplicity $\geqq r$ but α is a zero of the right side of multiplicity $r-1$. This is a contradiction, hence our assumption that the degree of Q is positive must be false. Q is a constant $(\neq 0)$ which we can assume is unity.

From (1.3) we obtain

$$P' - 2xP = 1. \tag{1.4}$$

Since P is a polynomial in x, it is clear that the degree of $-2xP>$degree of P', and the degree of $-2xP>0$. The degree of the left side of (1.4) is always greater than the degree of the right side, which is a contradiction. We have proved that there is no polynomial P satisfying (1.4), hence no rational function satisfying (1.2). Consequently $\int e^{-x^2}dx$ is not elementary.

Example 2: e^{bx}/x, with b a nonzero constant. If $\int(e^{bx}/x)dx$ is elementary, then $\int(e^{bx}/x)dx = Re^{bx}$ or $(1/x) = R' + bR$. Letting $R=P/Q$, where P and Q are relatively prime polynomials, $Q\neq 0$, we find:

$$Q^2 = xQP' - xPQ' + xbPQ \tag{2.2}$$

$$Q(Q - xP' - bxP) = - xPQ'. \tag{2.3}$$

If we assume that Q has positive degree, then $Q=0$ has a root. Let α be such a root and call its multiplicity r. If $\alpha\neq 0$, we encounter the same contradiction met in the first example, that α is a zero of the left side of (2.3) of multiplicity $\geqq r$, while α is a zero of the right side of multiplicity $r-1$. Thus α must be zero, and $Q=cx^r$, for some $c\neq 0$. Putting this expression for Q in (2.2) we have $cx^{r+1}(cx^{r-1}-P'-bP) = -crx^rP$. Again there is a contradiction, for the number 0 is a zero of the left side of multiplicity $\geqq r+1$, while it is a zero of the right side of multiplicity r. Our assumption that Q has positive degree is no longer tenable; hence Q is a constant, which we can assume is unity.

From (2.3) we obtain

$$xP' + bxP = 1. \tag{2.4}$$

As before, since P is a polynomial in x, the degree of the left side = degree of $(bxP)>0$ = degree of the right side. We have proved that there is no polynomial P satisfying (2.4), hence no rational function satisfying (2.2). Consequently $\int(e^{bx}/x)dx$ with $b\neq 0$ is not elementary.

Example 3: $(\sin x)/x$. It is clear that if $f(x) = u(x) + iv(x)$, where $u(x)$ and $v(x)$ are real valued functions, then

$$\mathcal{R}\int f(x)dx = \int \mathcal{R}f(x)dx = \int u(x)dx,$$

$$\mathcal{I}\int f(x)dx = \int \mathcal{I}f(x)dx = \int v(x)dx,$$

and if $\int f(x)dx$ is elementary, both $\int u(x)dx$ and $\int v(x)dx$ are elementary. ($\mathcal{R}$ and $\mathcal{I}$ stand for "the real and imaginary parts of," respectively.)

Although $(\sin x)/x$ is not in the form of Liouville's Theorem, by Euler's relation ($e^{ix}=\cos x+i \sin x$), we have $(\sin x)/x=\mathcal{I}(e^{ix}/x)$. Since e^{ix}/x does not possess an elementary integral, by example 2, neither does $\mathcal{I}(e^{ix}/x)=(\sin x)/x$.

Example 4: $1/\log x$. Again Liouville's theorem is not immediately applicable. If $y=\log x$, then $\int(1/\log x)dx=\int(e^y/y)dy$. By Example 2, the latter integral is not elementary, hence the same is true of the former.

Example 5: $(x^2+ax+b)e^x/(x-1)^2$, with a and b constants. Not only do the usual integration techniques require a considerable amount of skill, but there is no a priori assurance that we could find the integral, if it exists. Let us apply the test.

Assume $\int[(x^2+ax+b)e^x/(x-1)^2]dx=Re^x=Pe^x/Q$. Then

$$(x^2 + ax + b)Q^2 = (P'Q - Q'P + PQ)(x - 1)^2 \tag{5.2}$$

$$Q(Q(x^2 + ax + b) - (x - 1)^2P' - (x - 1)^2P) = - Q'P(x - 1)^2. \tag{5.3}$$

Assume Q has positive degree, and let α be a zero of Q of multiplicity r. If $\alpha\neq 1$, α is a zero of the left side of multiplicity $\geqq r$, but a zero of the right side of multiplicity $r-1$. This is a contradiction, hence $\alpha=1$ and $Q=(x-1)^r$. Substituting this into (5.3) we find

$$(x - 1)^r[(x - 1)^r(x^2 + ax + b) - (x - 1)^2P' - (x - 1)^2P] = - r(x - 1)^{r+1}P.$$

Using the fact that the multiplicities of 1 as a zero of the left and right sides must be the same, we see that $r=1$; *i.e.* $Q=(x-1)$. In the last equation we can cancel a common factor of $(x-1)^2$ from both sides, giving

$$(x^2 + ax + b) - (x - 1)P' - (x - 1)P = - P,$$

$$(x - 1)P' + (x - 2)P = x^2 + ax + b.$$

P is clearly linear, $P=cx+d$, which means

$$cx - c + cx^2 + dx - 2cx - 2d = x^2 + ax + b.$$

Since these two polynomials are identical, the coefficients of like powers are the same, $c=1$, $d-c=a$, $-c-2d=b$, whence $b=-2a-3$.

Consequently $\int[(x^2+ax+b)e^x/(x-1)^2]dx$ is elementary if and only if $b=-2a-3$, in which case the integral is $e^x(x+a+1)/(x-1)+C$.

By using one unproved theorem, we have seen how it is possible for even the beginning calculus student to test the integrability of certain transcendental

functions, and the logical structure of the test, though not trivial, is sufficiently similar to others he has seen for it to be easily grasped (*e.g.*, the test for the rationality of say $\sqrt{2}$). Although it is obvious that a table of functions which cannot be integrated could be constructed by a careful analysis of examples, to do so for the student would be no better than is done at present. Rather we feel that the student, by applying the test to a few functions, will have a fruitful introduction to differential equations and will gain well-founded confidence in his ability to follow and reconstruct proofs requiring more than one or two steps. Also, the last example illustrates the usefulness of the test as an integration technique.

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THE INTEREST RATE IN INSTALLMENT CONTRACTS

HUGH E. STELSON, Michigan State University

1. Introduction. In an installment contract* in which the time price differential is an add-on, the yield return to the lender increases as the contract gets longer up to a point. Thereafter the yield declines as the length of the contract increases.

Example. For a 6% add-on rate (*i.e.* $\frac{1}{2}$% of the original loan is added to the loan cost for each month of the duration of the loan) in an installment contract the effective rate is 8.98% per year for a 3-month contract and 10.21% for a 6-month contract. The yield builds up to a maximum of 11.13% around a 26-month contract. Thereafter it declines. The yield is 10.21% for a 120-month contract and it continues to decline to 6.2% in a 500-year contract.

The formula for the present value of a loan on a monthly basis is

$$B = Ra_{\overline{n}|} \text{ at } (i) \tag{1}$$

where B = cash price or present value of loan, R = monthly payment, n = number of months, and i = interest rate per month. Let c = per cent of add-on per month (*i.e.* at a 6% yearly add-on, $c = .005$); then $R = B(1+cn)/n$ and (1) becomes $B = B(1+cn)[1-(1+i)^{-n}]/ni$, or it can be put in the form

$$(1 + cn - in)(1 + i)^n - (1 + cn) = 0. \tag{2}$$

2. Maximum value of i for a constant value of c. The value of i as given by (2) is expressed approximately [1] [2] by

$$i_b = \frac{6cn}{3(n + 1) + cn(n - 1)}, \tag{3}$$

* This problem was presented to me by M. R. Neifeld, Beneficial Management Corporation, Morristown, New Jersey.

where

$$i - i_b < \frac{2(n-1)(n+2)}{9(n+1)^3} c^3 n^3.$$

From (3), we now obtain $\partial i_b/\partial n$ which is equated to zero. Solving for n gives

$$n_b = \sqrt{(3/c)}. \tag{4}$$

Since i_b is always smaller than i, and since the error, $i-i_b$, is an increasing function of n, it is apparent geometrically that n_b is too small. Although the value given by (4) is quite simple, it gives results which are about 1 too small. We present (4) only because it is useful in a more accurate determination of n (maximum).

A more exact value [2] of i is given by

$$i_c = \frac{2cn}{n+1}\left[\frac{3(n+1)+(n+2)cn}{3(n+1)+(2n+1)cn}\right] \tag{5}$$

where

$$i_c - i < \frac{2(n-1)(2n+1)(n+2)}{135(n+1)^4} c^4 n^4.$$

From (5), $\partial i_c/\partial n$ equated to zero gives

$$(c^2+3c)n^4 - 2c^2n^3 - (9+15c+2c^2)n^2 - 6(3+2c)n - 9 = 0. \tag{6}$$

Now (6) has only one positive root, n_c, for $c>0$. Since i_c is always greater than i, and since the error i_c-i is an increasing function of n, it is apparent geometrically that n_c is always slightly large.

To find n_c, we take $n_a=\sqrt{(3/c)}+1$ as a starting point and apply Newton's formula, which gives

$$n = (\sqrt{(3/c)}+1) - \frac{18+24c\sqrt{(3/c)}+30c+6c^2\sqrt{(3/c)}+3c^2}{18\sqrt{(3/c)}+72+18c\sqrt{(3/c)}-12c}$$

or

$$n_c = \sqrt{(3/c)} + 1 + \sqrt{(c/3)} + 2\sqrt{(c/3)^3} - \tfrac{1}{3}c^2 + \cdots.$$

Since in practical cases $0<i<.1$ and since $c<i$, we retain only the first three terms of n_c which gives

$$n_c \doteq \sqrt{(3/c)} + 1 + \sqrt{(c/3)} \text{ (maximum value)}. \tag{7}$$

3. Minimum value of c for a constant value of i. Solving (2) for c, gives

$$c = \frac{i}{1-(1+i)^{-n}} - \frac{1}{n},$$

$$\frac{\partial c}{\partial n} = \frac{1}{n^2} - \frac{\ln(1+i)}{s_{\overline{n}|} - a_{\overline{n}|}}. \tag{8}$$

Equating $\partial c/\partial n$ to zero, and expanding $\ln(1+i)$, $(1+i)^n$, and $(1+i)^{-n}$ to the seventh power of i, we solve the result for n^2 which gives

$$n^2 = \frac{1 - \frac{7}{6}i + \frac{7}{6}i^2 - \frac{137}{45}i^3 + \frac{7}{5}i^4}{\frac{1}{6}i - \frac{1}{3}i^2 + \frac{17}{36}i^3 - \frac{7}{12}i^4 + \frac{1}{180}n^2i^3 - \frac{1}{180}n^2i^4}. \tag{9}$$

Dividing the numerator of the right member of (9) by the denominator gives $n^2 = 6/i + \cdots$. Neglecting all but the first term gives $n^2 = 6/i$. Substituting this value for n^2 in the denominator of (9), we then divide again obtaining $n^2 = 6/i + 3.8 + \cdots$. If the first two terms are retained, and this value of n^2 is substituted in the denominator, there results by division

$$n \doteq \sqrt{(6/i + 3.8 - .32i)} \text{ (minimum value).} \tag{10}$$

Formula (10) will be accurate to two decimal places as checked by central difference interpolation formula.

4. Some examples*

CRITICAL VALUE OF n	VALUE OF i	CORRESPONDING VALUE OF c
7.98	.10	.062444
17.43	.02	.011146
34.62	.005	.002644
84.87	1/1200	.0004264845

The critical value of n is a maximum for i with c constant or a minimum for c with i constant.

5. Discount contract. In cases of a discount loan, the sum of the payments is B and the present value of the loan is $B - Bcn$. For example, if $\frac{1}{2}\%$ a month is charged, for a loan of 12 months \$94 would be received by the borrower for the total sum of 12 payments of \$100. In general, the formula for this type of loan is

$$B(1 - cn) = (B/n)a_{\overline{n}|} \tag{11}$$

or $a_{\overline{n}|} = n - cn^2$ for $0 < cn < 1$, $0 < a_{\overline{n}|} < n$. As n increases, with c constant, $a_{\overline{n}|}$ decreases monotonically relative to n, and i increases correspondingly, hence there is no relative maximum in this case.

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* By use of the recursion formula

$$c_1 = 1,\ c_{n+1} = \frac{[c_n + (1/n)](1+i)}{c_n + (1/n) + 1} - \frac{1}{n+1},$$

which is obtained by combining the formulas $c_n = (1/a_{\overline{n}|}) - (1/n)$, $c_{n+1} = (1/a_{\overline{n+1}|}) - (1/(n+1))$ and making use of the relation $a_{\overline{n}|} + 1 = (1+i)a_{\overline{n+1}|}$. Professor Gerald Weeg, Michigan State University, used the MYSTIC computer to make a table of add-on equivalent of effective periodic interest rates to 8 decimal places for i from $\frac{1}{4}\%$ to 2% by $\frac{1}{4}\%$ and n from 1 to 102.

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A VERY INDEPENDENT AXIOM SYSTEM

FRANK HARARY, University of Michigan

1. Introduction. One of the most popular mathematics courses at the University of Michigan is Mathematics 195, *Foundations of Mathematics*. This course has been given for over two decades by Professor R. L. Wilder and his excellent book [7], is used as the text. I wish to thank the students in my Mathematics 195 course this semester (Spring 1960) for their assistance in our stumbling onto an apparently new and stronger concept of the independence of an axiom system.

Let $S=\{A_1, \cdots, A_n\}$ be a system of axioms over a given collection of primitives. We shall assume throughout that this axiom system is *satisfiable*, that is, that it has a model. Following Wilder [7], an axiom A in S is *independent* if the modified system $(S-A)+\sim A$ is also satisfiable; and S itself is *independent* if each of its axioms is independent. Continuing, the axiom system S is *completely independent* if for any subset S_1 of axioms, the modified axiom system $(S-S_1)+\sim S_1$ has a model.

There was considerable interest in constructing completely independent axiom systems for groups, boolean algebras, and other algebraic systems in the beginning decades of this century; see for example the works of Bernstein [1] and Huntington [4].

In the above definition of an independent axiom A it is only necessary to have a model in which A does *not always* hold. However, it is also sometimes possible to construct a model in which all the other axioms hold and A *never* holds. In this case, we shall say that A is a *very independent axiom.* An axiom *system* S is *very independent* if each of its axioms is very independent. We restrict consideration to axioms of the form "p implies q": we admit the possibility of taking p as the universally true statement. The hypothesis p of this statement may be a condition of the form: x and y are elements of some set U. We shall say that an axiom A of the form "p implies q" never holds in a model M if the hypothesis p occurs at least once in M and furthermore whenever p is true, q is false, *i.e.*, p implies not q.

In an entirely analogous manner, we say that the axiom system S is *absolutely independent* if for every subset S_1 of axioms, there is a model in which $S-S_1$ holds and each axiom in S_1 never holds.

A *relation* is a set of ordered couples. For any relation R, we write aRb to indicate that $(a, b)\in R$. The elements $a, b, c, \cdots$ which may be in the relation

R to each other constitute the *field* F of the relation. With this notation, one usually has the following definitions which hold for all $x, y, z \in F$:

R is *reflexive* if xRx;
R is *symmetric* if xRy implies yRx;
R is *transitive* if xRy and yRz imply xRz.

The above definition of transitivity is given in the books [2, 3, 7]. However, there is an alternative definition of a transitive relation in König [5], and Lewis and Langford [6]:

R is *distinctly transitive* if for any three distinct elements $x, y, z \in F$, xRy and yRz imply xRz.

We now include the corresponding properties of relations which are opposite to conditions r, s, t (reflexive, symmetric, distinctly transitive).

R is *irreflexive* if xRx does not hold for any $x \in F$.
R is *asymmetric* if whenever $x \neq y$ and xRy holds, yRx does not;
R is *intransitive* if for distinct $x, y, z \in F$, xRy and yRz imply that xRz is false.

We abbreviate the last three properties by the symbols $\bar{r}$, $\bar{s}$, $\bar{t}$ respectively. An *anti-equivalence relation* is irreflexive, asymmetric, and intransitive ($\bar{r}$, $\bar{s}$, $\bar{t}$). We do not know of any general theorems for such relations.

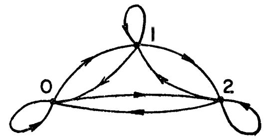
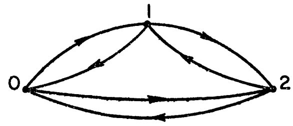
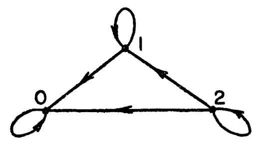
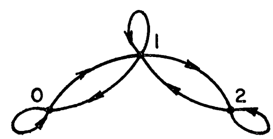
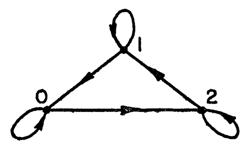
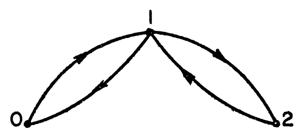
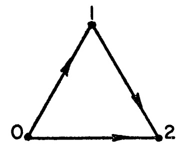
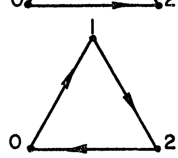
2. A very independent axiom system. An *equivalence relation* is usually defined as one which is reflexive, symmetric, and transitive. In this form, the statement is often made that symmetry and transitivity imply reflexivity. This assertion is usually accompanied by considerable discussion since it needs to be qualified by appropriate conditions in order to be incontrovertibly true; but this does not concern us here. We now state an axiom system for equivalence relations which is not only independent, but is very independent; and indeed, even absolutely independent!

The primitives or undefined terms in this system are a nonempty set F of elements $x, y, z, \cdots$ and a relation R whose field is F. The axioms are:

r. R is reflexive, s. R is symmetric, t. R is distinctly transitive.

In order to show that R is very independent, it is necessary to construct four models, one for each of the sets of conditions rst, $\bar{r}st$, $r\bar{s}t$, $rs\bar{t}$. Then in order to verify the further statement that this axiom system is absolutely independent, it is necessary to produce four additional models which satisfy: $\bar{r}\bar{s}t$, $\bar{r}s\bar{t}$, $r\bar{s}\bar{t}$, $\bar{r}\bar{s}\bar{t}$.

We shall now exhibit eight relations whose field is $\{0, 1, 2\}$ which satisfy the conditions listed above. It is convenient to include a graphical representation of these relations in which there is a directed line from point i to point j if and only if iRj and there is a directed line from point i to itself if and only if iRi.

Conditions	xRy where $F=\{0, 1, 2\}$	Diagram of the relation on $\{0, 1, 2\}$
rst	$x=y$ or $x\neq y$	
$\bar{r}st$	$x\neq y$	
$r\bar{s}t$	$x\geqq y$	
$rs\bar{t}$	$\|y-x\|\leqq 1$	
$r\bar{s}\bar{t}$	$y\neq(x+1)$ (mod 3)	
$\bar{r}s\bar{t}$	$\|y-x\|=1$	
$\bar{r}\bar{s}t$	$x<y$	
$\bar{r}\bar{s}\bar{t}$	$y=(x+1)$ (mod 3)	

It is interesting to write eight corresponding relations on the field of all people; some of these are given in Copi [**3**]. For example, "sibling" is $\bar{r}st$, "uncle" is $\bar{r}\bar{s}\bar{t}$, "taller than" is $\bar{r}\bar{s}t$, and "has the same first name" is rst. Continuing, "at least as tall" is $r\bar{s}t$, "within one year of age" is $rs\bar{t}$, "exactly one year difference" is $\bar{r}s\bar{t}$, and "same age or exactly one year older" is $r\bar{s}\bar{t}$.

3. Problem. What other very independent axiom systems are there? It may be regarded as fortuitous that a very independent axiom system was found at all. We note that the usual axiom system for groups is independent but not very. For taking the primitives as a set with a binary operation, and the postulates as closure, associativity, the existence of an identity element, and the existence of inverse elements, we see that the associative law certainly holds whenever at least one of the three elements involved is the identity. Hence, whenever the identity postulate does hold, associativity cannot always fail.

Recently, Evans [8] gave a postulational description of a number theory in which a collection of numbers generated by a set corresponding to the natural numbers is closed under addition. These have the property that the associative law for addition never holds, and neither does the commutative law.

Euclid's parallel postulate is very independent! This is shown by both kinds of non-Euclidean geometry. The postulate states that if L is any line and A is a point not on L, then there exists a unique line M containing A and disjoint from (parallel to) L. In the Bolyai-Lobachewsky geometry, there is always more than one such line M, in fact an infinite number. But in Riemannian geometry, there is never any such line M, for any two lines meet in exactly one point.

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LOGARITHMIC NUMBERS AND SOME THEOREMS ON PERMUTATIONS

J. M. GANDHI, Government College, Bhilwara, India

The author has defined the *related logarithmic polynomials** $A_r^{(n)}(t)$ by the generating function

$$\sum_{r=0}^{\infty} A_r^{(n)}(t)x^r/r! = (x^{n-1}e^{-xt})/(1 - x^n). \tag{1}$$

Whence expanding and equating the coefficients of x^r we get

* See [1], where the author has defined the logarithmic polynomials $G_r^{(n)}(t)$ by $e^{-xt}\log(1-x^n) = -\sum_{r=1}^{\infty} G_r^{(n)}(t)x^r/r!$. If this expression is differentiated with respect to x, (1) follows on putting $A^{(n)}(t)=(tG^{(n)}(t)+G_{r+1}^{(n)}(t))/n=A_r^{(n)}(t)$.

$$A_r^{(n)}(t)/r! = \sum_i (-t)^{r-in+1}/(r - in + 1)!. \tag{2}$$

The special cases $t=1$ and $t=-1$ of $A_r^{(n)}(t)$ are called *logarithmic numbers*.

We first show that canonical generators for $A_r^{(n)}(-1)$ and $A_r^{(n)}(1)$ are

$$A_r^{(n)}(-1) = r^{(n)} A_{r-n}^{(n)}(-1) + r^{(n-1)}, \tag{3}$$

$$A_r^{(n)}(1) = r^{(n)} A_{r-n}^{(n)}(1) + (-1)^{r-n+1} r^{(n-1)}, \tag{4}$$

where $r^{(n)}=r(r-1)\cdots(r-n+1)$. Putting $t=-1$ in (2) we get

$$\begin{aligned} A_r^{(n)}(-1) &= r! \sum_i 1/(r - in + 1)! = r^{(n)}(r - n)! \sum_i 1/(r - in + 1)! \\ &= r^{(n)}(r - n)! \sum_i 1/(r - n - in + 1)! + r^{(n)} \cdot (r - n)!/(r - n + 1)! \\ &= r^{(n)} A_{r-n}^{(n)}(-1) + r^{(n-1)}. \end{aligned}$$

This proves (3) and, similarly, (4) can be proved.

The case $n=1$ is of special interest in the present paper. Putting $n=1$, $t=-1$ and $t=1$ in (2), we get after some simplification

$$A_r(-1) = \sum_{t=0}^{r} {}_rP_t, \qquad A_r(1) = \sum_{t=0}^{r} (-1)^t {}_rP_t,$$

where ${}_rP_t \equiv r!/(r-t)!$ Putting $n=1$ in (3) and (4) we get

$$A_r(-1) = rA_{r-1}(-1) + 1, \qquad A_r(1) = rA_{r-1}(1) + (-1)^r.$$

Now referring to Mullin's paper [2], we note that his $\Phi(r)$ is the same as $A_r(-1)$ so that the logarithmic numbers include $\Phi(r)$ as a special case. It was proved in [2] that

$$\Phi(r) \sim r!e \quad \text{or} \quad A_r(-1) \sim r!/e. \tag{5}$$

In this paper we prove that

$$A_r(1) \sim r!/e. \tag{6}$$

We have

$$\begin{aligned} A_r(1) &= r!\left\{ \sum_{i=1}^{r+1} (-1)^{r-i+1}/(r - i + 1)! \right\} = r! \sum_{t=0}^{r} (-1)^t/t! \\ &= r!\left\{ \sum_{t=0}^{\infty} (-1)^t/t! - \sum_{t=r+1}^{\infty} (-1)^t/t! \right\} = r!\{1/e + o(1)\}, \end{aligned}$$

which proves (6).

We get a simple asymptotic formula for the A's by dividing (5) by (6):

(7) $$A_r(-1)/A_r(1) \sim e^2.$$

Using Stirling's formula for $r!$, (5) and (6) may be written

(8) $$A_r(-1) \sim r^{r+\frac{1}{2}}e^{-r+1}\sqrt{(2\pi)},$$

(9) $$A_r(1) \sim r^{r+\frac{1}{2}}e^{-r-1}\sqrt{(2\pi)}.$$

Multiplying (8) and (9) we get, using (7),

$$A_r(-1)A_r(1) \sim r^{2r+1}e^{-2r}(2\pi) \sim 2\pi r^{2r+1}\{A_r(1)/(A_r(-1)\}^r.$$

Hence $\frac{1}{2}r^{-2r-1}\{A_r(-1)\}^{r+1}/\{A_r(1)\}^{r-1} \sim \pi$.

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A NOTE ON HAUSDORFF SEPARATION

EDWIN HALFAR, University of Nebraska

The examples usually given as instances of topological spaces that have T_1-separation but not T_2-separation (Hausdorff) also have the property that some compact subset is not closed. This with the classic result concerning closedness of compact subsets of a Hausdorff space suggests the question of the equivalence of Hausdorff separation and the condition that the class of compact subsets be a subclass of the class of the closed subsets of a given space. The following is a simple result of this type and may be of some use in an introductory course in point set topology.

THEOREM. *If X is a space satisfying the first axiom of countability, then a necessary and sufficient condition that X be a Hausdorff space is that the class of compact subsets of X be a subclass of the class of closed subsets of X.*

Only the sufficiency need be considered here.

Since points are compact, it is immediate that X must be at least T_1. Also it can be assumed that the neighborhood base $\{V_n/n=1, 2, \cdots\}$ at each point is such that $V_n \subseteq V_m$, $n \geqq m$. Suppose there exist points x and y such that there are no disjoint pairs of neighborhoods of x and y respectively. Then a sequence $\{x_n/n=1, 2, \cdots\}$ may be selected by choosing each x_n in the intersection of the nth sets of the neighborhood bases of x and y. The set $\{x_n/n=1, 2, \cdots\} \cup [x]$ is compact but is not closed since y is an accumulation point.

That the assumption of a local countable base or some other restriction is necessary is seen from the following example.

Let X be an uncountable set with a topology such that a set is open if and only if it is X, the null set or the complement of a countable set. The space is not Hausdorff and does not satisfy the first axiom of countability. However the only compact sets are finite sets and hence closed.

THE BETA-GAMMA FUNCTION IDENTITY

JAMES W. BROWN, University of Michigan

The Beta-Gamma function identity

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

is traditionally demonstrated by use of a double integral in polar coordinates. There is a more compact proof which uses the Laplace transformation.

In general,

$$L\{t^{k-1}\} = \Gamma(k)s^{-k} \qquad (k > 0). \tag{1}$$

Using the convolution integral and the substitution $x = t^{-1}\tau$,

$$\begin{aligned} \Gamma(p)s^{-p}\Gamma(q)s^{-q} &= L\left\{\int_0^t \tau^{p-1}(t-\tau)^{q-1}d\tau\right\} \\ &= L\left\{t^{p+q-1}\int_0^1 x^{p-1}(1-x)^{q-1}dx\right\} = L\{t^{p+q-1}B(p, q)\}. \end{aligned} \tag{2}$$

But

$$\Gamma(p+q)s^{-p-q}B(p, q) = L\{t^{p+q-1}B(p, q)\}. \tag{3}$$

From equations (2) and (3),

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p, q); \qquad \text{Re}\,(p),\ \text{Re}\,(q) > 0. \tag{4}$$

This is the desired identity.

COMPLETE CONTINUITY FOR FUNCTIONS

HELEN F. CULLEN, University of Massachusetts

Introduction. Levine* has defined strong continuity for functions. A function f: $(X, S) \to (Y, T)$, where S and T are topologies for X and Y respectively, is strongly continuous if and only if the inverse image of every set in the range is open in the domain. It is to be recalled that continuity demands only that the inverse images of open sets in the range-space be open in the domain-space. It is the purpose of this note first to generalize slightly Levine's Theorem 2 and then to redescribe strong continuity in terms of different topologies for Y.

1. Quasi-constant functions.

DEFINITION 1. *A function f defined on a space (X, S) is called quasi-constant if and only if f is constant on each quasi-component of (X, S).*

THEOREM 1. *Let f: $(X, S) \to (Y, T)$ be strongly continuous, then f is quasi-constant.*

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* Norman Levine, Strong continuity in topological spaces, this MONTHLY, vol. 67, 1960, p. 269.

Proof. Let a and b be two points of X that lie in the same quasi-component of X. Assume that $f(a)=\alpha\neq\beta=f(b)$. Now $f^{-1}(\alpha)\cap f^{-1}(\sim\{\alpha\})=\emptyset$ where $\sim\{\alpha\}$ denotes the complement of α in Y. $X=f^{-1}(\alpha)\cup f^{-1}(\sim\{\alpha\})$ and $f^{-1}(\alpha)$ and $f^{-1}(\sim\{\alpha\})$ are both open. Thus $f^{-1}(\alpha)\cup f^{-1}(\sim\{\alpha\})$ is a separation of X. Further, a is in $f^{-1}(\alpha)$ and b is in $f^{-1}(\sim\{\alpha\})$; this contradicts the fact that a and b lie in the same quasi-component of X. Hence f is quasi-constant.

COROLLARY 1.1. *If f: $(X, S)\to(Y, T)$ is strongly continuous then f is constant on the components of X.*

Proof. Components are contained in quasi-components.

Corollary 1.1 is a rephrasing of Levine's Theorem 2.

COROLLARY 1.2. *If (X, S) is a connected space then the only strongly continuous functions are the constant functions.*

Proof. (X, S) consists of one component, X.

Example. Let (X, S) be the subspace of the space, E^2, (the euclidean plane) based on the set, $\{(x, y)\mid x=1/n \text{ and } 0\leqq y\leqq 1 \text{ for } n=1, 2, \cdots\}\cup\{(0, 0)\}\cup\{(0, 1)\}$ (Fig. 1). Now, $\{(0, 1)\}$ constitutes a component of (X, S) as also

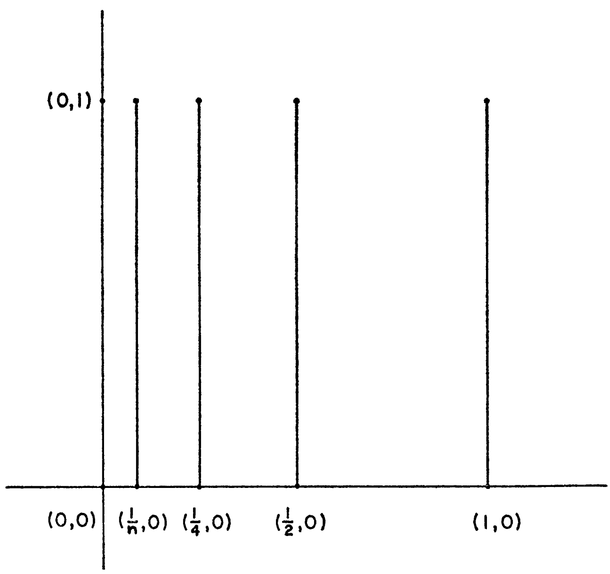

FIG. 1

does $\{(0, 0)\}$. However, $\{(0, 1)\}\cup\{(0, 0)\}$ constitutes one quasi-component, since in any separation $A\cup B$ of X, $(0, 1)$ and $(0, 0)$ would both have to be in the same part of this separation. Hence, if f is to be strongly continuous on (X, S) not only does each of the verticals, $\{(x, y)\mid x=1/n \text{ and } 0\leqq y\leqq 1\}$ have

to have a one point image but (0, 1) and (0, 0) must be assigned the same image. Of course, not every quasi-constant function is strongly continuous. Levine's example can be used to show this since in his example components also happen to be quasi-components or in the above space define $f(1/n, y)=(1/n, 0)$ and $f(0, 0)=f(0, 1)=(0, 1)$. Assign the relative topology to the range of f. f is quasi-constant but, since $f^{-1}(\{(0, 1)\})$ is not open in the domain (X, S), f is neither strongly continuous nor even continuous.

2. Completely continuous functions. In introductory analysis, continuity is usually considered as an intrinsic property of a function; this is because the topologies on the domains and ranges are fixed. However, any function $f: X \to Y$ can be made continuous by assigning "right" topologies S and T to X and Y, respectively. Now, continuity for f depends on the relative "size" of S to T and T to S, since if f is to be continuous the inverse image of every set in T must be in S. Thus so long as S is "big enough" relative to T or equivalently T is "small enough" relative to S, f is continuous; otherwise f, the same f, is not continuous. For example, if $X=Y=$ set of all real numbers, if S consists of X and $\varnothing$ only, *i.e.*, S is the trivial topology for X, and if T consists of all subsets of Y, *i.e.*, T is the discrete topology for Y, and if f is the "identity" map $i: (X, S) \to (Y, T)$, where $i(x)=x$, then i is not continuous [*e.g.*, $\{0\}$ is open in (Y, T) but $i^{-1}(\{0\})$ is not open in (X, S)]. Now if we assign the topology T to X and the topology S to Y, then the "same" function $i: X \to Y$, where $i(x)=x$, is continuous.

THEOREM 2. *If f is a mapping from a topological space (X, S) into a topological space (Y, T) where S is the discrete topology for X, then f is strongly continuous.*

Proof. Since every subset of X is open in (X, S), every inverse image set is open in (X, S) and f is strongly continuous.

THEOREM 3. *A one-to-one mapping from a space (X, S) into a space (Y, T) is strongly continuous if and only if S is the discrete topology for X.*

Proof. The "if" part follows from Theorem 2. Conversely if f is strongly continuous, every one-point subset $\{x\}$, the inverse image of $f(x)$, is open in (X, S). Hence, every subset of X is open in (X, S).

COROLLARY 3.1. *A homeomorphism, f, from a space (X, S) onto a space (Y, T) is strongly continuous if and only if S and T are the discrete topologies for X and Y, respectively.*

Proof. If f is strongly continuous, by Theorem 3, every subset in X is open. Since f is a homeomorphism, open sets have open images, hence every subset of Y is open.

DEFINITION 2. *A mapping f defined on a space (X, S) is called completely continuous on (X, S) if and only if f is continuous with each and every topology on the range.*

THEOREM 4. *A function f from a space (X, S) into a space (Y, T) is strongly continuous if and only if f is completely continuous on (X, S).*

Proof. If f is strongly continuous, then since the inverse image of every subset of Y is open in X, f is continuous, using the discrete topology on Y. So f is completely continuous. Conversely, if f is completely continuous on (X, S), then f is continuous with any topology on the range, including, then, the discrete topology. Hence, it follows that the inverse image of every subset of Y is open in (X, S) and so f is strongly continuous.

Thus, for strong continuity, the discrete topology on the domain is sufficient, quasi-constancy is necessary and complete continuity is necessary and sufficient.

RECURSION FORMULAS FOR DERIVATIVES OF TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

F. H. STEEN, Allegheny College

Recursion formulas for the values at the origin of nth-derivatives of trigonometric and hyperbolic functions are of interest in connection with Maclaurin Series expansions. In this paper we establish two such formulas, each applicable with only a single change to many functions, and more economical than those generally known.

We wish to evaluate $f^{(n)}(0)=A_n$ for certain functions $f(x)$. To accomplish this we consider the product $\phi(x) = M(x)f(x)$ where $M(x) = \cos x \cosh x = \Re[\cosh(1+i)x]$. Setting $M^{(k)}(0)=M_k$, we see that $M_{2r+1}=0$, $r=0, 1, 2, \cdots$, and $M_{2r}=\Re[(1+i)^{2r}\cosh(1+i)x]_0=\Re[(1+i)^{2r}]=(-4)^t$ when $r=2t$, 0 when $r=2t+1$, $t=0, 1, 2, \cdots$.

Applying Leibniz's formula for the nth derivative of a product to $\phi(x) = M(x)f(x)$, assuming all A_k to exist, and setting $\phi^{(n)}(0)=B_n$, we obtain

$$B_n = \sum_{t=0}^{[\frac{1}{4}n]} \binom{n}{4t} (-4)^t A_{n-4t},$$

which may be written in the form

$$\text{(I)} \qquad A_n = B_n + 4\binom{n}{4} A_{n-4} - 4^2\binom{n}{8} A_{n-8} + 4^3\binom{n}{12} A_{n-12} - \cdots$$

This is a generic recursion formula applicable to functions $f(x)$ for which the corresponding B_n is known. For example, if $f(x)=\sec x$, $\phi(x)=\cos x \cosh x \sec x = \cosh x$, and then $B_{2t}=1$, $B_{2t+1}=0$, $t=0, 1, 2, \cdots$. Then (I) gives $A_{2t+1}=0$ and

$$A_{2t} = 1 + 4\binom{2t}{4} A_{2t-4} - 4^2\binom{2t}{8} A_{2t-8} + 4^3\binom{2t}{12} A_{2t-12} - \cdots,$$

whence $A_0=1$, $A_2=1$, $A_4=5$, $A_6=61$, $\cdots$. These are the numerical values of the Euler numbers. The recursion formula is very effective. To obtain A_{10}, for example, it is merely necessary to know A_2 and A_6, since $A_{10}=1+840A_6-720\,A_2=50{,}521$.

If $f(x)=\operatorname{sech} x$, $\phi(x)=\cos x$, whereupon $B_n=0$ for $n=2t+1$, 1 for $n=4t$, and -1 for $n=4t+2$. Consequently the recursion formula for $A_n'=[D^n \operatorname{sech} x]_0$ is exactly that for $A_n=[D^n \sec x]_0$ when $n=2t+1$ or $4t$, wherefore $A'_{2t+1}=A_{2t+1}=0$, and $A'_{4t}=A_{4t}$. When $n=4t+2$, $B_n'=-1=-B_n$. Addition of (I) for sec x and sech x then shows that $A_{4t+2}+A'_{4t+2}=0$, or $A'_{4t+2}=-A_{4t+2}$. Hence the derivatives of sech x at $x=0$ are those of sec x, but with alternating signs.

To obtain formulas for the derivatives at $x=0$ of sec x sech x, tan x, tanh x, e^x sec x, and e^x sech x, it is merely necessary to substitute in (I) the corresponding values of B_n. These can be obtained by the method used above in finding M_k. They are given in Table I.

TABLE I

$f(x)$	$\phi(x)=\cos x \cosh x\, f(x)$	B_n
sec x sech x	1	$1,\ n=0$ $0,\ n>0$
tan x	sin x cosh x	$0,\ n=2t$ $(-4)^t,\ n=4t+1$ $2(-4)^t,\ n=4t+3$
tanh x	cos x sinh x	$0,\ n=2t$ $(-4)^t,\ n=4t+1$ $-2(-4)^t,\ n=4t+3$
e^x sec x	e^x cosh x	$1,\ n=0$ $2^{n-1},\ n>0$
e^x sech x	e^x cos x	$(-4)^t,\ n=4t,\ 4t+1$ $0,\ n=4t+2$ $-2(-4)^t,\ n=4t+3$

A companion formula to (I), obtained by the same method but using $M(x)=\sin x \sinh x$, is

$$\text{(II)}, \qquad \binom{n}{2}A_{n-2}=\tfrac{1}{2}B_n+4\binom{n}{6}A_{n-6}-4^2\binom{n}{10}A_{n-10}+\cdots.$$

Use has been made of the fact that $\sin x \sinh x=\Im[\cosh(1+i)x]$ which gives $M_{4t+2}=(-1)^t 2^{2t+1}$, $M_n=0$, $n\neq 4t+2$. This new recursion formula is useful in determining the coefficients in the expansion of such functions as $x\csc x$, $x \operatorname{csch} x$, $x^2 \csc x \operatorname{csch} x$, $x\cot x$, and $x\coth x$. The corresponding values of B_n are given in Table II.

TABLE II

$f(x)$	$\phi(x)=\sin x \sinh x f(x)$	B_n $(B_0=0)$
$x \csc x$	$x \sinh x$	$0, n=2t+1$ $n, n=2t+2$
$x \operatorname{csch} x$	$x \sin x$	$0, n=2t+1$ $n, n=4t+2$ $-n, n=4t+4$
$x^2 \csc x \operatorname{csch} x$	x^2	$2, n=2$ $0, n\neq 2$
$x \cot x$	$x \cos x \sinh x$	$0, n=2t+1$ $n(-4)^t, n=4t+2$ $-2n(-4)^t, n=4t+4$
$x \coth x$	$x \sin x \cosh x$	$0, n=2t+1$ $n(-4)^t, n=4t+2$ $2n(-4)^t, n=4t+4$

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

REPORT ON THE PROGRAM OF VISITING LECTURERS TO COLLEGES, 1960–61

R. A. ROSENBAUM, Wesleyan University

This note is a supplement to one by Rothwell Stephens (this MONTHLY, vol. 67, 1960, pp. 463–465).

During 1959–60, the six visiting lecturers spent a total of 257 days at 129 institutions, the average length of a visit thus being two days. In addition, there were 35 visiting mathematicians who lectured for a total of 83 days, in most instances making one-day visits.

The itineraries of the visiting lecturers in the current academic year are as follows (minor changes may still be made):

R. D. Anderson

Oct. 31, Nov. 1	Grinnell College
Nov. 2	Iowa State Teachers College
Nov. 3, 4	Wartburg College
Nov. 7, 8	Luther College
Nov. 9	College of St. Teresa, Minnesota
Nov. 10	St. Mary's College, Minnesota
Nov. 11	Hamline University
Nov. 15	St. Cloud College
Nov. 16	College of St. Benedict
Nov. 17	St. John's University, Minnesota
Nov. 28, 29	East Texas State College
Nov. 30, Dec. 1	North Texas State College
Dec. 2	East Central State College
Dec. 5, 6	Oklahoma City University
Dec. 7	Central State College, Oklahoma
Dec. 8, 9, 10	University of Oklahoma
Dec. 12	Kansas State University
Dec. 13, 14	Washburn University
Dec. 15	Kansas University
Dec. 16	Baker University

Charles W. Curtis

Feb. 2, 3	Earlham College
Feb. 6, 7, 8	Mississippi State University
Feb. 9, 10	Millsaps College
Feb. 13, 14	Mississippi Southern College
Feb. 15, 16	Loyola University, Louisiana
Feb. 17, 18	Louisiana-Mississippi Section of M.A.A.
Feb. 20, 21	Talladega College
Feb. 23, 24	University of the South
Mar. 4, 6	Mundelein College
Mar. 7, 8	Rosary College
Mar. 9, 10	Saint Xavier College
Mar. 13, 14	Bradley University
Mar. 16, 17	Southwestern at Memphis
Mar. 20, 21	Anderson College
Mar. 23, 24	Marquette University
Mar. 27, 28	Wisconsin State College, River Falls
Mar. 29, 30	Wisconsin State College, Eau Claire

Andrew M. Gleason

Apr. 3, 4	Vanderbilt University
Apr. 6, 7	Berea College
Apr. 10, 11	Indiana University
Apr. 13, 14	Eastern Illinois University
Apr. 17, 18	Purdue University
Apr. 19	Butler University
Apr. 20, 21	Indiana Technical College
Apr. 24, 25, 26	University of Wisconsin
Apr. 28	Western Illinois University

H. H. Goldstine

Oct. 31, Nov. 1, 2	University of Buffalo
Nov. 3, 4	St. Bonaventure University
Nov. 7, 8, 9	University of Ottawa, Carleton University
Nov. 28, 29, 30	Wesleyan University, Trinity College, Connecticut
Dec. 1, 2, 3, 4, 5	Smith College, University of Massachusetts, Mt. Holyoke College
Dec. 7, 8, 9	Dartmouth College
Apr. 3, 4, 5	Rutgers, The State University
Apr. 6, 7	Haverford College
Apr. 10, 11	Franklin and Marshall College
Apr. 12, 13	Bucknell University
Apr. 14	Geneva College
Apr. 24, 25	Merrimack College
Apr. 26, 27, 28	College of the Holy Cross, Clark University, Worcester Polytechnic Institute
May 1, 2	Southern Illinois University
May 4, 5	Auburn University, Alabama

Leon Henkin

Apr. 5, 6, 7	Miami University, Ohio
Apr. 10, 11	Denison University
Apr. 13, 14	Ohio Wesleyan University
Apr. 17, 18, 19, 20	American University, Catholic University, George Washington University, Howard University, District of Columbia Teachers College
Apr. 24, 25	Oberlin College
Apr. 27, 28	University of Dayton
May 1	University of Michigan
May 2, 3	Hope College
May 4, 5	Central Michigan College

Leo Moser

Mar. 16, 17	Eastern New Mexico University
Mar. 20, 21	Pomona College
Mar. 22, 23, 24	University of California, Santa Barbara
Mar. 27, 28	California State Polytechnic College
Apr. 6, 7	George Pepperdine College
Apr. 10, 11	University of Nevada
Apr. 12, 13, 14	University of Arizona
Apr. 17, 18	Colorado State University
Apr. 20, 21	Utah State University
Apr. 24, 25	Humboldt State College
May 1, 2	San Jose State College, University of Santa Clara
May 3	St. Mary's College of California
May 4, 5	University of California, Davis
May 8, 9	Sacramento State College
May 11, 12	Willamette University
May 15, 16	Montana State University
May 17	Northwest Nazarene College
May 18, 19	Montana State College
May 22, 23	Washington State University
May 25, 26	University of Idaho

Hartley Rogers, Jr.

Feb. 27, 28	University of Alberta
Mar. 1, 2	University of British Columbia
Mar. 3	Victoria (B.C.) College
Mar. 6, 7, 8, 9, 10	University of Washington and neighboring institutions
Mar. 13, 14	University of Alaska
Mar. 16, 17	Eastern Montana College

Frank M. Stewart

Feb. 2, 3	St. John's University, Jamaica, New York
Feb. 6, 7	Mansfield State College
Feb. 9, 10	Grove City College
Feb. 13, 14, 15	State Teachers College, Indiana, Pennsylvania
Feb. 16, 17	Washington and Jefferson College
Feb. 27, 28	Alfred University
Mar. 1, 2	State U. College of Education, Buffalo
Mar. 3, 4, 5, 6	University of Rochester
Mar. 7, 8	Hobart and William Smith Colleges
Mar. 9, 10	Hamilton College
Mar. 13, 14, 15	The St. Lawrence University, Clarkson College of Technology
Mar. 16, 17, 18	University of Montreal
Apr. 10, 11	Vassar College
Apr. 12, 13	Simmons College
Apr. 14, 15, 16, 17	Tufts University
Apr. 24, 25, 26, 27	Shippensburg State College, Wilson College, Dickinson College
Apr. 28, 29	Montclair State College
May 1, 2	Lafayette College

Robert M. Thrall

Jan. 30, 31	Antioch College
Feb. 2, 3	Hiram College
Feb. 6, 7	Bowling Green State University
Feb. 9, 10	Baldwin-Wallace College
Feb. 13, 14	East Carolina College
Feb. 15	University of North Carolina
Feb. 16, 17	North Carolina State College
Feb. 20, 21, 22	Sweet Briar College, Randolph-Macon Woman's College, Lynchburg College
Mar. 6, 7	University of Richmond
Mar. 9, 10	College of William and Mary
Mar. 13, 14, 15	Florida A. and M. University
Mar. 16, 17	Florida State University
Mar. 21, 22	University of Florida
Mar. 23, 24	Florida Presbyterian College
Apr. 4, 5	University of Miami
Apr. 8	Southeastern Sectional meeting of M.A.A.
Apr. 10, 11	University of Delaware
Apr. 13, 14	West Virginia University
Apr. 17, 18	University of South Carolina
Apr. 20, 21	Agnes Scott College
Apr. 24, 25, 26, 27, 28	University of Georgia, Emory University, Georgia Institute of Technology

May 1, 2	Knoxville College
May 4, 5	Xavier University
May 15, 16	Andrews University
May 18, 19	Western Michigan University
May 22, 23	General Motors Institute
May 25	Michigan State University Oakland

Schedules for W. T. Guy, Jr., and Paul C. Rosenbloom were not available at the time that this note was written.

Plans are being made for a continuance of the Visiting Lecturer Program into 1961–62; announcement will be made about May 1, 1961.

REPORT ON THE PROGRAM OF VISITING LECTURERS TO SECONDARY SCHOOLS, 1959–60

Plan of operation. The program was operated during the academic year, 1959–60, in nine regions: Alabama-Arkansas, California, Connecticut-Maine-Rhode Island-Vermont, Delaware-District of Columbia-Maryland-West Virginia, Idaho-Oregon-Washington, Illinois, Kansas-Missouri-Nebraska, Montana-North Dakota-South Dakota, North Carolina-South Carolina.

TABLE SHOWING NUMBER OF LECTURERS AND SCHOOLS VISITED 1959–60

	Northwest	California	Montana-North Dakota-South Dakota	Kansas-Nebraska-Missouri	Arkansas-Alabama	Illinois	Delaware-District of Columbia-Virginia-West Va.	North Carolina-South Carolina	New England	Total
Number of Mathematicians	26	60	5	6	3	9	9	1	1	120
Number of Schools	90	150	69	51	30	44	73	85	72	664
Number of Days	91	80	45.5	42	28	44	48	54	72	504.5
Number of States	3	2	3	3	3	1	4	2	4	25
Audience	16,800	15,000	9,975	9,900	4,051	9,423	6,500	10,880	10,500	93,029

In addition to members of the Committee on Visiting Lecturers to Secondary Schools, five other mathematicians served as regional representatives. Regional representatives assumed responsibility for regional publicity, obtaining applications from schools in their regions, selecting lecturers from an approved list prepared by the Committee, and scheduling lectures.

The lecturers were prepared not only to give lectures on mathematical topics but to confer with students on future opportunities in study and employment, and discussed teaching problems and curriculum with members of the staff. In short, the lecturers cooperated with the schools in all ways possible toward the furtherance of the aims of the program.

Schools were informed that a maximum daily load for a lecturer would be two lectures to students and one to teachers and that a lesser load would be preferable. The Committee recommended that the lecturer not address a general assembly of students. It was suggested that regional representatives encourage schools to make contributions in partial payment of lecturer's expenses.

Two regions were served by lecturers on leave of absence during the second semester. Professor Israel Rose of the University of Massachusetts lectured in New England states and Professor W. Norman Smith of the University of Wyoming was "full-time" lecturer in North and South Carolina. California and the states of the Northwest made use of a larger number of lecturers than other regions. In general, lecturers in the West Coast regions made trips from their colleges to schools in the neighboring cities and counties. The remaining regions were served by 3 to 9 lecturers, who in a number of instances made trips of a week's duration to several schools in a particular part of a state or region.

Evaluation. The reports from schools show that the lecturers were very well received. Schools in all of the regions asked that the program be continued another year. One regional representative reports that the visits by all but two of the lecturers were unanimously highly praised, and that one of the others apparently improved with time as each report was more favorable than the preceding one. A number of schools in this region reported some regrets for having used the lecturer for speaking to rather large groups of ninth and tenth grade students. Many felt that the lecturer was much more effective in stimulating the advanced students.

A few typical comments from teacher reporters are:

"The lecturer met the students on their level. They were most enthusiastic. His time with us was too short. Could we in the future have a lecturer for two days?"

"Some students came for all four lectures and wished for more."

"The principal thought the visit was very worthwhile."

"All enjoyed it very much. Math has taken on a note of greater importance."

"The math faculty is very happy about the program. The anticipation was good for us and the realization better."

"The lecturer captivated his audience."

Program participants. A partial list of the visiting lecturers includes:

Henry Alder
Bradford Arnold
Wilfred Barnes
Ross A. Beaumont
William E. Briggs
Z. W. Birnbaum
K. A. Bush
Donald Bushaw
J. R. Byrne
Theodore S. Chihara
Leon Cohen
H. W. Crowley
Mary P. Dolciani
Roy Dubisch
Wade Ellis
Robert E. Gaskell
George Hufford
V. James
George R. Johnson
Burton W. Jones
H. S. Kaltenborn
Antony E. Labarre, Jr.
Calvin Long
Arvid Lonseth
Richard P. Mayer
Rogert E. McKelvey
Albert Nijenhuis
Cletus Oakley
Thurman S. Peterson
Pasquale Porcelli
R. W. Rampfer
L. A. Ringenberg
S. T. Rio
J. B. Roberts
Israel Rose
Paul Rosenbloom
Hans Sagan
Norman Smith
S. Stein
Alfred Willcox
William B. Woolf
James H. Zant

The regional representatives for 1959–60 were:

Russell N. Bradt, University of Kansas, Lawrence; John A. Brown, University of Delaware, Newark; Roy Dubisch, Fresno State College; W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts; Harvey M. Gelder, Western Washington College of Education, Bellingham; Adrien L. Hess, Montana State College, Bozeman; Houston T. Karnes, Louisiana State University, Baton Rouge; Thomas D. Reynolds, Duke University, Durham, North Carolina; Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Indiana.

Plans for 1960–61. The program will be carried on during 1960–61 on much the same basis as last year. For this year there are ten regions. The regions, with the names of regional representatives given in parentheses are: Alabama-Florida-Georgia (Houston T. Karnes); California-Nevada (Roy Dubisch); Iowa-Minnesota-Wisconsin (H. Vernon Price); Kentucky-Michigan-Ohio (Marie S. Wilcox); Maryland-Virginia-West Virginia (Malcolm W. Oliphant); New Jersey-Pennsylvania (John A. Brown); New York (John F. Randolph); Idaho-Oregon-Washington (Harvey M. Gelder); Oklahoma-Texas (William T. Guy); Arizona-New Mexico-Utah (Charles Wexler).

The Committee has approved a list of lecturers, including about the same number as last year, and in which there are some duplications with last year and a number of new persons who are assisting with the program. There will be no lecturers during 1960–61 on a leave-of-absence basis. The new program is made possible by a third grant from the National Science Foundation to the Mathematical Association for this purpose. Brochures announcing the program may be obtained from any of the regional representatives. A new member of the committee on visiting lecturers is Professor F. A. Ficken, New York University.

This year, while the committee in general believes that lecturers to general assemblies of students are not to be recommended, it does recognize that in some instances these can be quite successful; hence, a special list of lecturers is being prepared so that if schools make requests for a general assembly speaker, the regular lecturer or a different one may be available.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1451. *Proposed by Anice Seybold, North Central College, Naperville, Illinois*

A student makes an error in breaking a fraction into partial fractions. He writes

$$\frac{x^4 - 3x^3}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}.$$

He then clears of fractions and substitutes in succession -1, 1, and 2 as values of x in order to obtain three equations to solve for A, B, and C. Another student correctly carries out the indicated division and uses the remainder, $-x^2-3x+2$, correctly. He writes

$$\frac{-x^2 - 3x + 2}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}.$$

Both students get the same values for A, B, and C. How does this happen?

E 1452. *Proposed by N. A. Court, University of Oklahoma*

Find two positive integers such that their sum will be a factor of their product.

E 1453. *Proposed by José Gallego-Diaz, University of Puerto Rico*

Let A be the sum of the digits of a natural number N, let $B=A+N$, let A' be the sum of the digits of the number B, and let $C=B+A'$. Find A if the digits of C are those of A in reverse order.

(Dedicated to the memory of Victor Thébault.)

E 1454. *Proposed by Leonard Carlitz, Duke University*

If Δ, R, r denote the area, circumradius, and inradius of a triangle with sides a_1, a_2, a_3, show that (1) $(R+r)^2 \geqq \Delta\sqrt{3}$, (2) $(a_1a_2a_3)^2 \geqq (4\Delta/\sqrt{3})^3$, (3) $(R\sqrt{3})^3 \geqq a_1a_2a_3$, with equality only when the triangle is equilateral.

E 1455. *Proposed by M. T. L. Bizley, London, England*

Let O be the origin, X the point $(p, 0)$, and Y the point $(0, p)$, where p is a positive prime number. The triangle OXY is divided into p triangles (of equal area) by the lines joining O to the points $(p-r, r)$ for $r=1$ to $r=p-1$. The inte-

riors of the outermost two of these triangles clearly contain no lattice points (*i.e.*, points whose coordinates are both integers). Prove that the interiors of the remaining $p-2$ triangles all contain equal numbers of lattice points.

SOLUTIONS

Two Related Inequalities

E 1421 [1960, 593]. *Proposed by* (1) *S. P. Franklin and G. A. Hutchison, University of California at Los Angeles, and* (2) *W. R. Becker, New York City*

(1) Prove, for any integer $n>1$, $n/(n-1)>\{n^2/(n^2-1)\}^n$.

(2) Prove, for all $a>0$ and $b>0$, $\{(a+1)/(b+1)\}^{b+1}\geqq(a/b)^b$.

Solution by Joe Lipman, Harvard University. By elementary calculus $(a+1)^{b+1}/a^b$, considered as a function of a, has an absolute minimum when $a=b$. This is tantamount to (2), which reduces to (1) when $b=n-1$ and $a=n$.

Also solved by A. N. Aheart, R. H. Anglin, Leon Bankoff, P. R. Chernoff, D. B. Coleman, A. E. Danese, W. G. Dotson, Jr., F. J. Duarte, D. L. Faass, David Friedman, Michael Goldberg, L. D. Goldstone, R. E. Greene, S. H. Greene, R. L. Helmbold, V. E. Hoggatt, J. E. Homer, Jr., A. S. Howard, Erwin Just, Peter Marks, D. C. B. Marsh, C. S. Ogilvy, Thomas Porsching, L. A. Ringenberg, R. E. Shafer, Richard Sinkhorn, D. R. Sondergeld, Guy Torchinelli, W. C. Waterhouse, Alan Wayne, C. C. Yalavigi, David Zeitlin, and the proposers. Late solutions by D. A. Breault, N. K. Govil, and M. A. Malik.

Many solvers obtained (1) from the Bernoulli inequality $(1+x)^n>1+nx$, $x>-1$ and $\neq 0$, by setting $x=-1/n^2$. Marsh obtained (1) by considering, for $n>1$, the n numbers $1-1/n, 1, \cdots, 1$; these are positive and not all equal, whence their geometric mean, $(1-1/n)^{1/n}$, is strictly less than their arithmetic mean, $1-1/n^2$. Duarte and Hoggatt gave noncalculus proofs of (2). Friedman and Zeitlin showed more generally that $[(a+c)/(b+c)]^{b+c}\geqq(a/b)^b$, $a>0$, $b>0$, $c\geqq 0$.

Polynomial Solutions of $f(x^2)+f(x)f(x+1)=0$

E 1422 [1960, 593]. *Proposed by R. W. Kilmoyer, Jr., Lebanon Valley College*

Find polynomials $f(x)$ such that $f(x^2)+f(x)f(x+1)=0$.

Solution by T. K. Cook and C. F. Pinzka, University of Cincinnati. We note that $f(z)=0$ implies $f(z^2)=0$ and $f((z-1)^2)=0$. Since the set of zeros of $f(x)$ is finite and closed under the transformation $z\to z^2$, each z must lie at the origin or on the unit circle of the complex plane. Closure under the transformation $z\to(z-1)^2$ restricts the possible zeros to 0 and 1. Thus $f(x)$ is of the form $ax^m(x-1)^n$, and substitution in the functional equation leads to $f(x)\equiv 0$ or $f(x)=-x^n(x-1)^n$, $n=0, 1, 2, \cdots$.

Also solved by A. N. Aheart, Ken Alles, E. J. Barbeau, Jr., Alan Beal, J. L. Brown, Jr., Leonard Carlitz, W. G. Dotson, Jr., Robert Farrell and John Wood (jointly), J. F. Foley, Michael Goldberg, R. E. Greene, S. H. Greene, Erwin Just and Norman Schaumberger (jointly), A. G. Konheim, Joe Lipman, Peter Marks, D. C. B. Marsh, R. W. Means, R. A. Melter, Marvin Mielke, Morris Morduchow, Jack Silver, Richard Sinkhorn, Wu Ta-Sun, Guy Torchinelli, W. C. Waterhouse, C. C. Yalavigi, and the proposer.

Many of the submitted solutions were incomplete. Carlitz remarked that the only polynomial solutions of $f(x^2)+f(x)f(x-1)=0$ are $f(x)\equiv 0$ and $f(x)=-(x^2+x+1)^n$, $n\geqq 0$.

Closed Self-intersecting Curves

E 1423 [1960, 593]. *Proposed by Aboulghassem Zirakzadeh, University of Colorado*

Consider a plane closed curve possessing only ordinary points and a finite number of double points. Assign a positive direction to the curve and, starting from an ordinary point A, trace the curve in the given positive direction. Assign number 1 to the first double point met, 2 to the second double point met, and so on until you come back to A. Prove that of the two integers assigned to any double point, one is always even and the other always odd.

Solution by W. C. Waterhouse, Harvard University. A proof is given in Selection 10 of *The Enjoyment of Mathematics*, by H. Rademacher and O. Toeplitz (translated into English by H. Zuckerman), and is roughly as follows.

We want to show that between successive passages through a given double point an even number of double points are passed through. Call the part of the curve traced (itself a closed curve) B, and the rest of the curve (also a closed curve) C. All double points of B are certainly passed through twice, and we need consider only the intersections of B and C. But C can be replaced by a regular curve without changing its intersections with B, and then the Jordan Curve Theorem shows that there are an even number of intersections of B with C.

Also solved by E. J. Barbeau, Jr., Brother Joseph Heisler, Michael Goldberg, L. D. Goldstone, Peter Marks, D. C. B. Marsh, J. C. Mathews, and Krishna Sarati.

Some of the above solutions were questionable. The problem seems to have arisen in an observation of Gauss, and it has an intimate connection with knot theory.

Complete Sequences

E 1424 [1960, 593]. *Proposed by V. E. Hoggatt and Charles King, San Jose State College*

A sequence $\{W_i\}$ of positive integers is *complete* if for each positive integer N there exists a subsequence $\{W_{i_j}\}_{j=1}^{k}$ of $\{W_i\}$ such that $N=\sum_{j=1}^{k} W_{i_j}$.

(1) Show that the sequence $\{F_i\}$ of Fibonacci numbers $(F_{i+2}=F_{i+1}+F_i,$ $F_1=F_2=1)$ is complete.

(2) Show that if any one member of the sequence $\{F_i\}$ is deleted, the sequence is still complete.

(3) Show that if two members of the sequence $\{F_i\}$ are deleted, the sequence becomes incomplete.

[*Note.* The problem has been reworded more felicitously than originally.]

Solution by Jack Silver, Montana State University. Let $\{W_i\}$ be obtained from $\{F_i\}$ by deleting one member from $\{F_i\}$. Let N be the first positive integer not representable as the sum of a subsequence of $\{W_i\}$. Either $\sum_{i=1}^{n} F_i$ or $\sum_{i=1}^{n+1} F_i-F_j$, for some $j<n+1$, equals $\sum_{i=1}^{n} W_i$, whence $\sum_{i=1}^{n} W_i \geqq \sum_{i=1}^{n+1} F_i -F_{n+1}=\sum_{i=1}^{n} F_i=F_{n+2}-1$. Then, if n is the greatest k such that $F_k \leqq N$,

$\sum_{i=1}^{n} W_i \geqq N$. Let L be the g.l.b. of the sums of all subsequences of $\{W_i\}$ whose sums exceed N. Say $L=\sum_{j=1}^{k} W_{i_j}$, where the terms of the series are arranged in ascending order of magnitude. If $W_{i_1}=1$, delete it to get $\sum_{j=2}^{k} W_{i_j}=L-1\geqq N$. Otherwise we have $N>W_{i_1}-1\geqq 1$, and, by the defining property of N, $W_{i_1}-1 =\sum_{r=1}^{m} W'_{i_r}$ where $W'_{i_r}<W_{i_j}$ for all r and $j\geqq 2$, whence $\sum_{r=1}^{m} W_{i_r}+\sum_{j=2}^{k} W_{i_j}= L-1\geqq N$. Since in both cases we obtain expansions contradicting the defining properties of L and N, (2) follows.

Since (2) implies (1), (1) also follows.

If both F_k and F_r, $k<r$, are deleted, then $\sum_{i=1}^{r-2} W_i=\sum_{i=1}^{r-1} F_i-F_k=F_{r+1}-1 -F_k<F_{r+1}-1=W_{r-1}-1$; thus W_{r-1} cannot be expanded in the required way, and (3) follows.

Also solved by J. L. Brown, Jr., Michael Goldberg, John Jordan, Erwin Just, Joe Lipman, Peter Marks, D. C. B. Marsh, Marvin Mielke, Donna J. Seaman, R. E. Shafer, R. P. Tapscott, Guy Torchinelli, W. C. Waterhouse, and the proposers.

The identity $\sum_{i=1}^{n} F_i=F_{n+2}-1$, employed in the above proof, is easily established by induction. Brown obtained necessary and sufficient conditions for a sequence of positive integers to be complete, and for a complete sequence to remain complete after the deletion of an arbitrary member. The proposers pointed out that the sequence $\{2^i\}$ is complete, but the sequence $\{L_i\}$ of Lucas numbers ($L_{i+2}=L_{i+1}+L_i$, $L_1=1$, $L_2=3$) is incomplete. Deletion of any one member from the first sequence renders it incomplete; addition of the number 2 to the Lucas sequence renders it complete

Square within a Triangle

E 1425 [1960, 593]. *Proposed by D. J. Newman, Brown University*

If a square lies within a triangle, prove that the area of the square does not exceed half the area of the triangle.

Solution by the proposer. By a variational procedure we can see that the largest square in a given triangle has all four vertices on the sides of the triangle. It is trivial to show that three vertices are so located; that the fourth is follows from the easily proven fact that a square not having all four vertices on the sides of the triangle has an infinitesimal motion sending it into the interior of the triangle.

So let the side s of the square lie completely on the side c of the triangle. Draw the altitude h to c; it is cut by the square into lengths $h-s$ and s. By similar triangles we have $(h-s)/s=h/c$, and so $1/s=1/c+1/h$. Hence $1/s^2 =(1/c+1/h)^2\geqq 4/ch$, which is to say that the area of the square $=s^2\leqq ch/4$ $=$ half the area of the triangle.

Also solved by Leon Bankoff, William Chapco, T. R. Curry, Guy Di Antonio, Jane Evans, Michael Goldberg, L. D. Goldstone, Ned Harrell, S. L. Hunt, A. R. Hyde, Erwin Just, L. M. Kaplan, Peter Marks, D. C. B. Marsh, Marvin Mielke, Jack Silver, C. C. Yalavigi, and David Zeitlin.

In the inequality $(1/c+1/h)^2\geqq 4/ch$ we have equality when and only when $c=h$, so the area of the square can actually equal half the area of the triangle only when, for appropriate lettering of the triangle, $c=h$ and angles A and B are nonobtuse. Several solvers pointed out that in the problem the word "square" can be replaced by the word "parallelogram," for the one can be carried into the other by an affine transformation, which multiplies all areas by a constant factor. The result

of the problem follows from the stronger theorem: "If F is a convex curve surrounding a closed set K of area A then there is a parallelogram inscribed in F of area $\frac{1}{2}A$. Moreover there is no parallelogram of area strictly larger than $\frac{1}{2}A$ inscribed in F if and only if F is a triangle." See this MONTHLY, vol. 67, March 1960, C. M. Fulton and S. K. Stein, "Parallelograms Inscribed in Convex Curves"

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4947. *Proposed by D. J. Newman, Yeshiva University*

Let $a_{mn}=\exp(-m-n/e^{2m})$. Prove that the infinite matrix $A=(a_{mn})$ represents a bounded operator on Hilbert space. That is, it takes vectors $x_1, x_2, \cdots$ for which $\sum x_n^2<\infty$ into vectors $y_1, y_2, \cdots$ for which $\sum y_n^2<\infty$.

4948. *Proposed by Lawrence Glasser, Carnegie Institute of Technology*

Show that the volume of any sphere in Hilbert space is zero.

4949. *Proposed by Richard Bellman, the RAND Corporation*

Consider the problem of maximizing the functional

$$J(\lambda_1, \cdots, \lambda_n) = \int_0^T \left(\sum_{i=1}^n a_i\lambda_i(t) \right) \exp \left(\int_t^T \sum_{i=1}^n b_i\lambda_i(s)ds \right) dt,$$

over all functions $\lambda_i(t)$ satisfying the conditions $\lambda_i(t) \geqq 0$, $\sum_{i=1}^n \lambda_i(t)=1$, $0\leqq t\leqq T$. Show that the solution can be obtained in terms of the solution of the differential equation $du/dt=\max_{1\leq i\leq n} (a_i+b_iu)$, $u(0)=0$, and thus determine the solution.

4950. *Proposed by T. S. Nanjundiah, Central College, Bangalore, India*

Evaluate

$$\prod_{n=1}^{\infty}{}' \frac{n^3+p^3}{n^3-p^3}, \qquad \sum_{n=1}^{\infty}{}' \frac{n^2}{n^6-p^6}, \qquad p=1, 2, \cdots,$$

the accent indicating that $n=p$ is to be omitted.

The product is known for the case $p=1$. See Bromwich, *Infinite Series* (1926), p. 313, Ex. 18; *Collected Papers of Ramanujan*, (1927), p. 322, Qn. 261.

4951. *Proposed by J. F. Heyda, General Electric Co., Cincinnati, Ohio.*

Solve the nonlinear integral equation

$$mx + b = \int_0^x \frac{dt}{\sqrt{\{P(t) - P(x)\}}}, \qquad x \geqq 0, m \geqq 0, b \geqq 0,$$

for $P(0) - P(x)$, where $P(x)$ is monotone decreasing.

4952. *Proposed by Lawrence Shepp, Princeton University*

Banach has shown (*Opérations Linéaires*, pp. 29–34) that there exists a method which satisfies certain natural axioms whose domain is the set of bounded sequences. His proof is unnecessarily long because he uses the functional:

$$P(S_n) = \operatorname*{glb}_{\{i_1, i_2, \cdots, i_k\}} \limsup_n \sum_{j=1}^{k} \frac{S_{n+i_j}}{k} \text{ for bounded } \{S_n\},$$

where the same proof works for the simpler

$$q(S_n) = \operatorname*{glb}_k \limsup_n \sum_{j=1}^{k} \frac{S_{n+j}}{k}.$$

Prove:

$$P(S_n) = q(S_n) = \lim_k \limsup_n \sum_{j=1}^{k} \frac{S_{n+j}}{k}.$$

SOLUTIONS

A Function Schlicht in the Upper Half-plane

4890 [1960, 187]. *Proposed by H. S. Shapiro, New York University*

Let $f(t)$ be positive, integrable and strictly decreasing in $(0, \infty)$, and suppose that the Fourier cosine transform of f is strictly decreasing in $(0, \infty)$. Then $F(z) = \int_0^\infty f(t) e^{izt} dt$ is regular and schlicht in the upper half-plane.

Indications of proof by the proposer. An elementary topological argument reduces the problem to showing that $F(z)$ cannot take the same value at two distinct points of the real axis. (Note that analyticity of $F(z)$ follows from uniform convergence by a familiar argument.) Now, if $F(x_1) = F(x_2)$, the Fourier cosine transform $\phi_c(x)$ of f must take the same value at x_1 as at x_2. This implies, since ϕ_c is even and decreasing in $(0, \infty)$, that $x_1 = -x_2 \neq 0$. But also the sine transform $\phi_s(x)$ must take equal values at x_1 and x_2, *i.e.*,

$$\phi(x_1) = \phi(x_2) = \phi(-x_1) = -\phi(x_1)$$

and so $\phi(x_1) = 0$. But because $f(t)$ is $\downarrow$, its sine transform can vanish only at $x = 0$ (Titchmarsh, *The Fourier Integral*, Theorem 123; he proves $\phi_s \geqq 0$, but the strong statement follows if one assumes f is strictly decreasing).

A and B, where A^2+B^2 is prime

4893 [1960, 294]. *Proposed by Joe Lipman, University of Toronto*

Let A, B be positive integers, A odd, B even, and let $A^2+B^2=p$, a prime,

(a) Show that, except perhaps for sign, A and B are respectively the real and the imaginary parts of S, where

$$S = \sum_{x=0}^{\frac{1}{2}(p-1)} \exp\{\tfrac{1}{2}\pi i \operatorname{ind}(x^4 + a^4)\},$$

with a any nonzero, fixed quadratic residue (mod p).

(b) Let R be any fixed quadratic residue (mod p) and N any fixed non-residue. Prove

$$A = \sum_{x=0}^{\frac{1}{2}(p-1)} \left(\frac{x^4 + R}{p}\right), \qquad B = \sum_{x=1}^{\frac{1}{2}(p-1)} \left(\frac{x^4 + N}{p}\right)$$

where the summands are Legendre symbols.

Solution by Leonard Carlitz, Duke University. (a) Let g be a primitive root (mod p), where $p\equiv 1$ (mod 4) and let C_i denote the set of numbers (mod p), $\{g^{i+4r}\}$, $(r=0, 1, \cdots, \frac{1}{4}(p-5))$. Also let (i, j) denote the number of solutions of

$$\alpha_i + 1 \equiv \alpha_j \pmod{p} \qquad (\alpha_i \in C_i,\ \alpha_j \in C_j).$$

Gauss (*Untersuchungen über höhere Arithmetik, Theorie der biquadratischen Reste I*, pp. 511–533) determined the numbers (i, j). For $p\equiv 1$ (mod 8) he found

$$16(0, 0) = p - 6A - 11, \qquad 16(0, 1) = p + 2A - 4B - 3,$$
$$16(0, 2) = p + 2A - 3, \qquad 16(0, 3) = p + 2A + 4B - 3;$$

while for $p\equiv 5$ (mod 8),

$$16(0, 0) = p + 2A - 7, \qquad 16(0, 1) = p + 2A + 4B + 1,$$
$$16(0, 2) = p - 6A + 1, \qquad 16(0, 3) = p + 2A - 4B + 1.$$

On the other hand

$$S = \sum_{\substack{x=0 \\ x^4 \not\equiv -1}}^{\frac{1}{2}(p-1)} i^{\operatorname{ind}(x^4+1)} = 1 + 2 \sum_{\substack{\alpha\in C_0 \\ \alpha \not\equiv -1}} i^{\operatorname{ind}(\alpha+1)}.$$

When $p\equiv 1$ (mod 8) we get

$$S = 1 + 2\{(0, 0) + (0, 1)i - (0, 2) - (0, 3)i\} = -A - Bi.$$

When $p\equiv 5$ (mod 8) we get

$$S = 1 + \tfrac{1}{8}\{(0, 0) + (0, 1)i - (0, 2) - (0, 3)i\} = A + Bi.$$

In either case we have $p=A^2+B^2$, $A\equiv 1$ (mod 4).

Since

$$\sum_{x=0}^{\frac{1}{2}(p-1)} i^{\operatorname{ind}(x^4+a^4)} = 1 + \frac{1}{2}\sum_{x=1}^{p-1} i^{\operatorname{ind}(x^4+a^4)} = 1 + \frac{1}{2}\sum_{x=1}^{p-1} i^{\operatorname{ind}(a^4x^4+a^4)}$$

$$= 1 + \frac{1}{2}\sum_{x=1}^{p-1} i^{\operatorname{ind}(x^4+1)} = \sum_{x=0}^{\frac{1}{2}(p-1)} i^{\operatorname{ind}(x^4+1)},$$

we may assume $a=1$; in fact, it is not necessary to assume that a is a quadratic residue of p.

(b) Put

$$\psi(r) = \sum_{x=0}^{\frac{1}{2}(p-1)} \left(\frac{x^4+r}{p}\right),$$

so that

$$2\psi(r) - 1 = \sum_{x=0}^{p-1}\left(\frac{x^4+r}{p}\right) = \sum_{y=0}^{p-1}\left(1+\left(\frac{y}{p}\right)\right)\left(\frac{y^2+r}{p}\right)$$

$$= -1 + \sum_{y=0}^{p-1}\left(\frac{y}{p}\right)\left(\frac{y^2+r}{p}\right);$$

where we have used the formula

$$\sum_{y=0}^{p-1}\left(\frac{y^2+r}{p}\right) = -1 \qquad (r \not\equiv 0 \pmod{p}).$$

Thus

$$(*) \qquad 2\psi(r) = \sum_{y=0}^{p-1}\left(\frac{y}{p}\right)\left(\frac{y^2+r}{p}\right).$$

We have next

$$4\sum_{r=0}^{p-1}\psi^2(r) = \sum_{x,y=0}^{p-1}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)\sum_{r=0}^{p-1}\left(\frac{x^2+r}{p}\right)\left(\frac{y^2+r}{p}\right)$$

$$= (p-1)\sum_{x^2\equiv y^2}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right) - \sum_{x^2\not\equiv y^2}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)$$

$$= p\sum_{x^2\equiv y^2}\left(\frac{xy}{p}\right) - \sum_{x,y}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right) = 2p(p-1).$$

Since $\psi(rc^2) = (c/p)\psi(r)$, this reduces to

$$(**) \qquad 2(p-1)(\psi^2(R)+\psi^2(N)) = 2p(p-1),\ \psi^2(R)+\psi^2(N) = p.$$

Since

$$\psi(1) = \sum_{y=1}^{\frac{1}{2}(p-1)} \left(\frac{y}{p}\right)\left(\frac{y^2+1}{p}\right)$$

and $y^2+1\equiv 0$ for exactly one y in $1\leqq y\leqq \frac{1}{2}(p-1)$, it follows that $\psi(1)\equiv\frac{1}{2}(p-3)$ $\equiv 1 \pmod 2$.

The formula (**) with $\psi(r)$ defined by (*) is due to Jacobsthal (*Journal für die reine und angewandte Mathematik*, vol. 132 (1907), pp. 238–245).

Also solved by Emma Lehmer and the proposer.

An Inequality for Hermitian Matrices

4894 [1960, 294]. *Proposed by Olga Taussky, California Institute of Technology*

Let A, B be two positive definite Hermitian matrices which can be transformed simultaneously by unitary transformation to diagonal forms of similarly ordered numbers. Let x be any vector of complex numbers. Show that $(Ax, x)(Bx, x)\leqq(ABx, x)(x, x)$, and discuss the case of equality. Two sets of n positive numbers, $\{a_i\}$, $\{b_i\}$ are called similarly ordered if $(a_i-a_k)(b_i-b_k)\geqq 0$ for all i, $k=1, \cdots . n$.

Solution by Marvin Marcus, University of British Columbia. It will be seen that the positive definite condition is unnecessary. We assume that A and B are commutative Hermetian matrices with eigenvalues $\lambda_1\geqq\cdots\geqq\lambda_n$ and $\mu_1\geqq\cdots\geqq\mu_n$ respectively. The conditions of the problem imply that there exists an orthonormal basis of common eigenvalues of A and B, $e_1, \cdots, e_n$, such that $Ae_i=\lambda_i e_i$, $Be_i=\mu_i e_i$, $i=1, \cdots, n$. For any vector x let p_x and q_x be respectively the smallest and largest integers i for which $\sigma_i=|(x, e_i)|^2\neq 0$. We prove the

THEOREM. *$(x, x)(ABx, x)-(Ax, x)(Bx, x)\geqq 0$ with equality if and only if either $\lambda_{p_x}=\lambda_{q_x}$ or $\mu_{p_x}=\mu_{q_x}$. If equality holds for an x for which $p_x=1$ and $q_x=n$ then either A or B is a multiple of the identity.*

Proof. We compute

$$\begin{aligned}(x, x)(ABx, x) &- (Ax, x)(Bx, x)\\ &= \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\lambda_i\mu_i - \sum_{i=1}^{n}\sigma_i\lambda_i\sum_{i=1}^{n}\sigma_i\mu_i\\ &= \sum_{i,j=1}^{n}(\sigma_i\sigma_j\lambda_j\mu_j - \sigma_i\sigma_j\lambda_i\mu_j) = \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i\sigma_j(\lambda_i-\lambda_j)(\mu_i-\mu_j).\end{aligned}$$

Since $\sigma_i\geqq 0$ and $(\lambda_i-\lambda_j)(\mu_i-\mu_j)\geqq 0$ the inequality follows (actually this is the Tchebycheff inequality). The case of equality can occur if and only if $\sigma_i\sigma_j(\lambda_i-\lambda_j)(\mu_i-\mu_j)=0$ for all $1\leqq i<j\leqq n$. Setting $i=p_x$, $j=q_x$ and noting that $\sigma_{p_x}\sigma_{q_x}>0$ we have at least one of the possibilities $\lambda_{p_x}=\lambda_{q_x}$ or $\mu_{p_x}=\mu_{q_x}$. On the

other hand if either of these possibilities obtain then for any $1 \leqq i < j \leqq n$ either (a): $i < p_x$ or $j \geqq q_x$ and $\sigma_i \sigma_j = 0$, or (b): $p_x \leqq i < j \leqq q_x$ and $(\lambda_i - \lambda_j)(\mu_i - \mu_j) = 0$ follows from the ordering of the eigenvalues. Hence we conclude that

$$\sum_{i,j=1}^{n} \sigma_i \sigma_j (\lambda_i - \lambda_j)(\mu_i - \mu_j) = 0$$

and the equality holds. The last statement follows from the fact that a Hermitian matrix with all eigenvalues equal must be a multiple of the identity.

Also solved by A. F. Kaupe, Jr., R. F. Rinehart, J. E. Potter, and the proposer.

The Laplace Transform of $\log^2 x$

4895 [1960, 295]. *Proposed by E. S. Keeping, University of Alberta*

γ being Euler's constant, prove that

$$\int_0^\infty (\log x)^2 e^{-x} dx = \tfrac{1}{6}\pi^2 + \gamma^2.$$

Note by T. C. Brown, Reed College, Portland, Oregon. This is problem 3766, for which solutions are found in this MONTHLY 1938, 57–58. Later [1958, 695], M. E. Levenson extended the result to a recursion formula for $\int_0^\infty (\log x)^{n+1} e^{-x} dx$.

Also solved by A. N. Aheart, A. R. Bradley, D. A. Breault, J. W. Brown, R. G. Buschman, L. Carlitz, P. R. Chernoff, P. L. Chassin, A. E. Danese, G. Di Antonio, A. B. Farnell, James Foster, Roberto Frucht, Todd Gitlin, M. L. Glasser, George Glauberman, S. H. Greene, S. W Greenhouse, Hur-bun Hou, M. S. Klamkin, T. V. Lakshminarasimhan, Joe Lipman, Y. Matsuoka John Moon and Ted Petrie, C. S. Ogilvy, J. E. Potter, G. E. Raynor, M. B. Ritterman, P. G. Rooney, P. R. Sanders, R. E. Shafer, George Shortley, F. C. Smith, Dmitri E. Thoro, Harry Weingarten, J. S. White, David Zeitlin, and the proposer—many of whom referred to the earlier problem which your Editor had overlooked.

Singularities on the Unit Circle

4896 [1960, 295]. *Proposed by H. S. Shapiro, New York University*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where the a_n lie in the interior of a Jordan curve Γ in the complex plane, the origin being exterior to Γ. Prove that there exist singularities $z_1, \cdots, z_k$ $(k \geqq 1)$ on the unit circle, and positive integers $n_1, \cdots, n_k$ such that $z_1^{n_1} \cdots z_k^{n_k} = 1$.

Solution by D. J. Newman, Yeshiva University. We need first the

LEMMA. If Γ is a Jordan curve with 0 in its exterior, then there is a polynomial $P(z)$ with $P(0) = 0$ and $\operatorname{Re} P(z) > 1$ for all z in the interior of Γ.

The result follows easily from this lemma, namely $P(a_n)$ is bounded and so $\sum P(a_n) z^n$ is convergent for $|z| < 1$. Since $\operatorname{Re} P(a_n) > 1$ it follows that $\operatorname{Re} P(a_n) z^n \to \infty$ as $z \to 1^-$ and so $z = 1$ is a singularity of $\sum P(a_n) z^n$; hence, for some $k > 0$, $\sum a_n^k z^n$ has a singularity at $z = 1$. But $\sum a_n b_n z^n$ can have singularities only at $z_1 z_2$ where z_1 is a singularity of $\sum a_n z^n$, z_2 a singularity of $\sum b_n z^n$, hence $\sum a_n z^n$

must have k singularities on $|z|=1$ such that $1=z_1z_2 \cdots z_k$ and this is the result. (These z_i need not be distinct.)

Proof of lemma. Let a be a large positive number, then $1+a/z$, being analytic inside and on Γ, can be approximated by polynomials. So let $Q(z)$ be such that $|1+a/z-Q(z)|<1$ for z inside Γ. Choose $P(z)=zQ(z)$, then $P(0)=0$ and $\operatorname{Re} P(z) \geqq \operatorname{Re}(z+a-|z|)$ throughout the interior of Γ. If a is large enough, however, the right side is >1.

Also solved by J. L. Ullman and the proposer.

Polynomial in Two Variables

4897 [1960, 295]. *Proposed by D. J. Newman, Yeshiva University*

Let $F(x, y)$ be such that: for any fixed y. $F(x, y)$ is a polynomial in x, and for any fixed x, $F(x, y)$ is a polynomial in y. Prove $F(x, y)$ is a polynomial in x and y.

Editorial Note. The solution is given in a Mathematical Note by F. W. Carroll, *A polynomial in each variable separately is a polynomial,* this MONTHLY, vol. 68, 1961, p. 42. This note was submitted to the Editors before the present problem appeared in this department. Morris Morduchow treated the same problem in: *On surface-fitting in three variables,* Journal of Applied Physics, 20 (1949), pp. 390–392.

Independent solutions submitted by L. Carlitz, P. R. Chernoff, G. Di Antonio, N. J. Fine, Harley Flanders, Todd Gitlin, A. P. Hillman, Joe Lipman, Christopher Metz, J. V. Whittaker, and the proposer. Late solution by J. L. Brown, Jr.

Number of Sequences of Special Type

4898 [1960, 295]. *Proposed by R. C. Reed, University College of the West Indies*

A sequence of n digits, each digit being a 1, 2 or 3, will be called an A_n-sequence. The six permutations of 123 are the only proper A_3-sequences, and for $n>3$ a proper A_n-sequence is defined recursively as follows: an A_n-sequence $(n>3)$ is proper if, and only if, it can be obtained from some proper A_{n-1}-sequence by deleting a digit of the A_{n-1}-sequence and putting in its place the other two digits (in either order). Thus the sequences 213, 2233, 31233 are examples of proper A_n-sequences.

Show that the number of distinct proper A_n-sequences is $\frac{3}{4}\{3^{n-1}-2-(-1)^n\}$.

Solution by A. P. Hillman, University of Santa Clara. Define an A_n-sequence to be an S_n if each of 1, 2, 3 appears an odd number of times when n is odd, and if at least two of 1, 2, 3 are present and each appears an even number of times when n is even. If $n>3$, an S_n has one of 1, 2, 3 appearing at least twice. One of these multiple appearances must be next to a different digit. If these distinct adjacent digits are replaced by the third digit, an S_{n-1} results. Induction now shows that an A_n-sequence is proper if and only if it is an S_n.

From one S_n many S_{n+1}'s may be obtained by the given rule. Of these, just three may be obtained as follows: Choose one of 1, 2, 3; place it in front of the S_n; and change the first of the non-chosen digits appearing in the S_n to the other

nonchosen digit. For example, 123 leads to 1133, 2323, and 3223. On the other hand, by the inverse process, corresponding to any S_{n+1} there is a unique S_n with the exception of the six S_{n+1}'s of the form $abcc \cdots c$ if n is even. Hence, this new operation applied to all S_n results without repetition in all S_{n+1} if n is odd, and in all but six of the S_{n+1}'s if n is even. It follows that the number p_n of S_n's satisfies

$$p_{2k} = 3p_{2k-1}, \qquad p_{2k+1} = 3p_{2k} + 6.$$

The desired formula follows immediately by induction.

Also solved by W. J. Blundon, Donald W. Brown, E. N. Gilbert, John B. Kelly, James Singer, and the proposer.

Editorial Note. In terms of the greatest integer function the number of proper A_n-sequences may be written as $3[3^{n-1}/4]$. Several solvers pointed out that the desired number, when n is odd, equals the coefficient of x^n in the expansion of $\sinh^3 x$; when n is even, it is the coefficient of x^n in $\cosh^3 x$ except that this includes the three improper sequences having a single digit repeated.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematical Methods of Operations Research. By Thomas L. Saaty. McGraw-Hill, New York, 1959. $10.00.

The subject of operations research is insufficiently mature to be treated axiomatically, as one can treat probability theory or thermodynamics. The subject can hardly be treated even systematically—since each professional would have a different system. Most books on operations research use the approach of case studies of specific projects with the apparent aim of getting the general idea across. This book is refreshingly different in that an attempt has been made to discuss in considerable detail some of the new branches of applied mathematics which have developed because of and for O.R. There are chapters on linear and quadratic programming, game theory, queueing theory, and optimization. There are also chapters which develop the necessary ideas in probability, statistics, and elementary analysis. The book should be excellent as a text for a senior or first-year graduate course.

WILLIAM VIAVANT
University of Oklahoma

Reduktionstheorie des Entscheidungsproblems im Prädikatenkalkül der Ersten Stufe. By János Surányi. Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1959. 216 pp. About $6.25.

This excellent work is the first systematic presentation of all the main reductions of the decision problem for the predicate calculus. A *reduction class* is a class of formulas such that, for every formula A of the predicate calculus, there can be effectively constructed a formula B in the class such that either A and B are both satisfiable or neither is satisfiable. The following, in particular, are shown to be reduction classes: (1) The class of closed formulas of the form $\forall_{x_1}\forall_{x_2}\forall_{x_3}\mathbf{M}_1$ & $\forall_{y_1}\forall_{y_2}\exists_{y_3}\mathbf{M}_2$, where the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are constructed using only truth functions, individual variables, one-place predicate variables, and at most one two-place predicate variable; (2) The class of closed prenex formulas with a prefix of the form $\exists\forall\exists\forall^2$ in which one-place and at most four two-place predicate variables appear; (3) The class of closed prenex formulas with a prefix of the form $\exists\forall\exists\forall^n$ in which only a single two-place predicate variable appears.

This last reduction class is obtained through the use of arithmetic methods, with the help of the Löwenheim-Skolem theorem. As the author himself noted, the use of the axiom of choice in the proof of that theorem can be avoided by more complex constructions.

Surányi also sketches Kalmár's reduction of the decision problem to the case of arbitrary finite domains (using the Skolem-Herbrand theorem, proved in the appendix) and his proof of Church's theorem on the recursive unsolvability of the general decision problem.

An introductory chapter contains all the information on the predicate calculus which would be necessary for any mathematician who is not primarily a logician to understand the remainder of the book. Before each complicated proof is attempted, the idea underlying it is presented. False starts are sometimes made, followed by explanations of why they won't succeed. The proofs themselves are remarkably complete and very clear. This is an altogether admirable production.

JOSEPH S. WHOLEY
Harvard University

Differential Geometry. By Erwin Kreyszig. University of Toronto Press, Toronto, 1959. xiv+352 pp. $8.50.

An introduction to the differential geometry of curves and surfaces in three-dimensional Euclidean space is set forth here, and the presentation affords excellent preparation for Riemannian geometry of n dimensions. Full use of the tensor calculus is made, and the concepts of tensor analysis are developed as needed in a simple and natural manner. For instance, the Christoffel symbols are not introduced until they are needed (p. 140) to portray the Gauss formulae. This is typical of the author's concern for clarity.

In the preface, the author says that he has tried "to present the whole subject-matter in the simplest possible form consistent with the needs of mathematical rigour." This reviewer is so impressed with the success achieved toward the stated goal, that he would recommend the book for students who are interested in tensor methods and who may not have the benefit of an instructor.

After a useful introductory chapter on preliminaries the second chapter covers the usual theory of space curves but with more than usual lucidity and rigor. The next three chapters introduce foundations of tensor calculus along with surface theory. Pertinent remarks indicate that surface area introduces difficulties not encountered in finding arc length. Chapter six on mappings is an excellent one—unusually complete with introduction of the Bergman metric. In the seventh chapter on absolute differentiation and parallel displacement one sees again how the author prepares the reader by giving continuity, a preview of what is to follow, and by displaying the dominant concepts in a striking way. The last chapter on special surfaces includes minimal surfaces and a treatment on modular surfaces of analytic functions.

One does not find here long lists of exercises at the ends of chapters. Rather, a few problems are strategically located to improve understanding, and the solutions of these appear at the end of the book.

This is a free translation of the author's *Differentialgeometrie* which appeared in the series Mathematik und ihre Anwendungen in Physik und Technik (series A, Vol. 25).

C. E. SPRINGER
The University of Oklahoma

Plane Trigonometry. By Nathan O. Niles. Wiley, New York, 1959. xi+234 pp. \$3.95.

This textbook contains twelve chapters pertaining to the following topics in the order indicated: fundamental concepts, trigonometric functions of angles, the right triangle, trigonometric functions of real numbers, fundamental identities, variations and graphs of the trigonometric functions, trigonometric functions of composite angles, logarithms, oblique triangles, inverse trigonometric functions, trigonometric equations, vectors and complex numbers.

Radians are introduced in Chapter 1 and used throughout the text. Another excellent feature of this text is its treatment of the trigonometric functions of real numbers. The discussion of Chapter 4 gives many details and is followed in later chapters by frequent reminders of the two interpretations of the argument of a trigonometric function. It should be noted that functions of the general angle are discussed in Chapter 2 and that the special definitions of the functions of an acute angle are emphasized in Chapter 3. Slide-rule solutions of triangles are discussed in detail.

The following tables are included: four-place values of trigonometric functions of angles in degrees and radians; four-place values of trigonometric functions of real numbers or angles in radians and degrees; four-place logarithms

of numbers from 1–10; four-place logarithms of trigonometric functions of angles in degrees.

The exposition of the text is at the right level for the student. Several illustrations of each topic, diagrams, and answers for odd-numbered problems help to make it a useful text for the student.

This book is adequate for a three-hour, one-semester course in trigonometry. Instructors who wish to put less emphasis upon the solution of triangles will find sufficient material for a two-hour course stressing analytic trigonometry. The terminology and coverage of analytic trigonometry are consistent with recommendations of recent national committees and provide the student with a good foundation for subsequent courses in mathematics.

EDITH R. SCHNECKENBURGER
University of Buffalo

Plane Trigonometry. By A. W. Goodman. Wiley, New York, 1959. Including tables, xvii+267 pp. $4.50. Without tables, xvii+197 pp. $3.75.

This textbook contains an excellent development of the traditional subject matter of trigonometry. A notable feature of the text is its emphasis upon definition, theorem, and proof. In Chapter 0 we find discussions of mathematical symbols and of the meaning of *axiom* and *theorem.* In subsequent chapters definitions are carefully labeled and stated with precision. Statements requiring proof, such as the many formulas of trigonometry, are called theorems and proofs are given. The duality principle pertaining to identities is a new theorem which provides an illustration of one way of extending mathematics.

The book emphasizes solution of triangles as well as analytic trigonometry. The author discusses the trigonometric functions of an acute angle in Chapter 1 so that the student can observe the process of modifying a definition to fit a new situation when he studies the trigonometric functions of a general angle. The following topics follow Chapter 1 in the order indicated: logarithms, logarithmic solution of right triangles, trigonometric functions of a general angle, elementary trigonometric identities, oblique triangles, addition formulas, radian measure, graphs of trigonometric functions, trigonometric equations, inverse trigonometric functions, areas, vectors, complex numbers.

The following tables are available: squares and square roots; values of trigonometric functions to four places; mantissas of common logarithms of numbers to five decimal places; logarithms of trigonometric functions to five decimal places.

The discussion of objective at the beginning of each chapter, clarity and level of exposition, and illustrative problems make this a text which can be read by the student with understanding. The informal style of the author and the format of the book will be appreciated by students. Answers are given for the odd-numbered problems. The *Preface to the Student* contains good suggestions for studying mathematics.

There is sufficient material for a three-hour, one-semester college course for

students who have had no previous work in trigonometry. In view of the rapidly changing curriculum in secondary schools it is probable that many instructors will prefer to use the text for a two-hour, one-semester course in analytic trigonometry or for a trigonometry unit in a freshman course covering various subjects. The book is also suitable for a high school trigonometry course.

The student who uses this text will acquire insight into the nature of mathematics and how it has developed, along with a knowledge of trigonometry.

EDITH R. SCHNECKENBURGER
University of Buffalo

Theory and Solution of Ordinary Differential Equations. By Donald Greenspan. Macmillan, New York, 1960. viii+148 pp. $5.50.

If you have grown weary teaching the traditional "cook-book" course in differential equations, this is the book to consider. The usual content of a differential equations course—minus applications— is presented in a rigorous setting, requiring the student to exhibit understanding of point set theory and elementary real variable theory (in the text and in the variety of exercises). Two proofs of the fundamental existence theorem are presented—one employing the classical Picard iterative technique and the other an elegant approach through the use of functional analysis. In general, the book has a systematic and well-written approach to its subject. In particular, the Frobenius method for solution in series is admirably treated.

I have some reservations, however. I gather the impression from certain topics scantily outlined (special functions, Laplace transform, Sturm-Liouville theory) that the author is pressed for space. Yet he presents linear differential equations of order two in one chapter and then generalizes them practically word for word to those of order n in the next chapter. Also, it is disturbing to find the author recalling the properties of determinants, but neglecting to state theorems involving interchange of limit operations. I might add that the latest editions (Agnew, Churchill, Zygmund) should be cited in the bibliography.

Nevertheless, there is a great deal to praise in this book and it should prove a welcome relief for those who are interested in more than routine solution of differential equations without resort to treatises.

ARTHUR E. DANESE
Union College, Schenectady, N. Y.

A Primer of Real Functions. By Ralph P. Boas, Jr. The Mathematical Association of America (Carus Monograph No. 13), 1960. xi+189 pp.

This excellent two-chapter little book treats a variety a specialized subjects in lively manner. The subject matter, for the most part, is selected from the foundations of analysis and ranges from important pathological examples to fundamental theorems and some of their applications. The first chapter presents some basic notions inherent to the foundations of analysis, and the last chapter treats some important properties of various classes of functions. Of

particular interest here is the treatment of the linear function and convex functions. The entire presentation is (as the author states in his preface) informal and includes some sprinkled bits of wit and philosophy.

PASQUALE PORCELLI
Louisiana State University

Calculus of Functions of One Argument with Analytic Geometry and Differential Equations. By Edward J. Cogan, Robert Z. Norman and Gerald L. Thompson. Prentice-Hall, Englewood Cliffs, N. J., 1960. x+587 pp. $8.75.

This book is suitable for an introductory course in the study of functions of one variable. Ideas of set theory and logic are introduced early; parts of analytic geometry and trigonometry are presented as they are needed. Material intended for a first semester includes differentiation and integration of exponential and logarithmic functions and some work in differentiation equations. The rest of the book includes trigonometric and inverse trigonometric functions, methods of integration, improper integrals, infinite series and differential equations. The discussion of vectors, postponed until the last chapter, is brief. There is no work on multiple integration. No tables are included but practice in their use is provided through references to Cogan and Norman: *Handbook of Calculus, Difference and Differential Equations.*

This book is well organized and well written; it has many illustrative examples and many problems. The concepts involved are clearly presented. Since students will later use books with standard notation, one questions the advantage of some of the symbols used; *e.g.*, $(x^{\frac{1}{2}}[x^2])(4)$ for $f[g(4)]$, with $f(y)=y^{\frac{1}{2}}$ and $y=g(x)=x^2$, and the use of ln, sin, and $\int f$ to replace $\ln x$, $\sin x$ and $\int f(t)dt$.

FLORENCE M. MEARS
The George Washington University

Advanced Algebra, Part I. By E. A. Maxwell. Cambridge University Press, New York, 1960. ix+311 pp. $2.75.

This book deals with the usual topics of advanced algebra,—polynomials and related equations and inequalities, simultaneous equations, determinants, complex numbers, partial fractions, graphs of rational functions, permutations and combinations, the binomial theorem, summation of finite series, and infinite series. Partial fractions are studied in logical detail for both repeated linear and quadratic factors of the denominator. Methods for summing generalized arithmetic and harmonic series are provided. The binomial series lead to expansions of rational functions in power series, these, in turn, to exponential and logarithmic series. To discuss the latter series the author employs elementary calculus but of necessity states the basic assumptions about convergent series; the detailed discussion might better be postponed to a calculus text.

The book is carefully prepared and concisely written. The author emphasizes the logical structure of algebra as much as he believes consistent with the mathe-

matical maturity of the students for whom the text is intended. His illustrative material contains techniques for solving routine problems and develops methods which the intelligent student can adapt to the solution of numerous challenging examples. The reader must attend closely both to the theory and the illustrative examples if he is to solve the problems of the text. For such a reader the book contains much of immediate interest and provides valuable training for mathematical growth.

HELEN G. RUSSELL
Wellesley College

Geometry. By Charles F. Brumfiel, Robert E. Eicholz, and Merrill E. Shanks. Addison-Wesley, Reading, Mass., 1960. xi+288 pp. $4.75.

The authors present a fairly elementary but mathematically sound (after Hilbert) version of Euclidean geometry. This book is possibly definitive and deserves the most careful consideration.

Chapter 1 is a summary of the student's previous experience with geometry. Chapter 2, Logic, introduces terms of elementary logic, and nature of proof. The vocabulary of logic is not employed much in the sequel. In Chapters 3–6, the major innovations become manifest, and merits of the text's construction appear. The language is formidable, so that one viewing this text for adoption should work most of the problems.

However, the main body of theorems, chapters 7–13, follow nicely. Space Geometry, chapter 14, is not much more than a list of space postulates and theorems, and should be accessible to students who had mastered the course thus far. By now it must be evident that the book is not intended for weak classes. Chapter 15, Analytic Geometry, approximates in content that of a good intermediate algebra, although it is more sound.

The authors of this important and valuable text should be praised. Still, it has defects, mostly of writing, which could prevent extensive adoption. One hopes that it will have enough commercial success to carry it through at least one revision.

Possibly the major contribution of this text will be in helping to revitalize secondary-school geometry.

E. L. WALTERS
William Penn Senior High School
York, Pennsylvania

Dynamic Programming and Markov Processes. By Ronald A. Howard. Technology Press of M.I.T. and Wiley, New York, 1960. viii+136 pp. $5.75.

The motivation of this monograph is best described in the author's own words: "The systems engineer or operations researcher is often faced with devising models for operational systems. The systems usually contain both probabilistic and decision-making features, so that we should expect the resultant model to be quite complex and analytically intractable. This has indeed been the case

for the majority of models . . . proposed. The exposition of dynamic programming by Richard Bellman [*Dynamic Programming*, Princeton University Press, 1957, Chapter XI] gave hope to those engaged in the analysis of complex systems, but this hope was diminished by the realization that more problems could be formulated by this technique than could be solved. Schemes that seemed quite reasonable often ran into computational difficulties . . . not easily circumvented."

The author develops an analytic structure for a decision-making system based on the Markov process as system model. The method of optimization is an iterative technique similar to dynamic programming. Beginning with a description of discrete-time Markov processes, the model is generalized to include economic rewards and the decision process itself. Decision processes with simple probabilistic structures are solved by the policy-iteration method and detailed applications are given to problems of taxicab operation, baseball, and automobile replacement. The succeeding chapters deal with more complicated probabilistic structures (multiple-chain processes), the effect of discounting future rewards, and the analysis of continuous-time Markov processes. The development is swift, concise and lucid. This book is a welcome addition to the growing literature on sequential problems.

H. KAUFMAN
McGill University

Arithmetic. By Fred Marer, Samuel Skolnik, Orda E. Lewis. Little, Brown, Boston, 1960. viii+246 pp.

According to the preface, "This material was first organized for a course initiated at the request of many departments, and of students who felt the lack of proficiency in the fundamentals of arithmetic." (One can only infer that these students are at Los Angeles City College, the institution of two of the authors.)

In this uninspiring book no attempt is made to give the reader a concept of the beauty and pattern of the real number system, or to present any basic principles of operation which eliminate the many special cases. Instead, the first seventeen chapters are filled with the traditional "bag-of-tricks" rules for adding, subtracting, multiplying, and dividing whole numbers and fractions (common, improper, and decimal). Other brief chapters introduce the metric system, proportion, mensuration, and square root. The exercises are prolific but exceedingly simple and unchallenging. Several of the many rules are poorly worded and misleading; for example, to change to percent a fraction whose denominator is 100, one is instructed to "replace the denominator of 100 by the percent sign, %."

It is the opinion of this reviewer that a remedial arithmetic course based on a book of this caliber has no place in an institution of higher education.

VIOLET HACHMEISTER LARNEY
State University of New York,
College of Education, Albany

Mathematical Methods for Digital Computers. Edited by A. Ralston and H. S. Wilf. Wiley, New York, 1960. xi+293 pp. $9.00.

This volume is composed of six main parts: (1) Generation of Elementary Functions, (2) Matrices and Linear Equations, (3) Ordinary Differential Equations, (4) Partial Differential Equations, (5) Statistics, (6) Miscellaneous Methods. Each part is divided into one or more chapters, each of which is an independent entity written by someone who is quite familiar with the subject and who is well known to both the mathematical and the digital computing fraternities. With the exception of the first chapter, the chapters are organized in the same manner, namely: (1) Problem Description, (2) Mathematical Discussion, (3) Calculational Procedure, (4) Flow Chart, (5) Description of Flow Chart, (6) Sample Problem, (7) Subroutines, (8) Memory Requirements, (9) Estimation of Running Time, and (10) References.

This reviewer's experience with beginning graduate students has indicated the efficiency of the above format in that the students were able to program and run problems on a computer with very little assistance using only the material in this book. The editors are to be commended for resisting what must have been a great temptation—using ALGOL for the descriptions!

This is a book that should be in the hands of everyone who intends to do any serious work on a digital computer.

FREDERICK WAY, III
Case Institute of Technology

Analytic Geometry and Calculus. By Abraham Schwartz. Holt, Rinehart and Winston, New York, 1960. xi+864 pp. $9.50.

The first goal set by the author in writing this book was that "the student must be able to read his textbook and learn from it." The liberal use of examples and numerous remarks combined with a reasonably precise presentation are noticeable features of the book which support this goal. Examples are used both to motivate and to illustrate the concepts under discussion. The remarks warn of possible mistakes, help clarify subtle points, or explicitly include other observations which are usually only added by the teacher but might sometimes be overlooked.

The student is assumed to have completed trigonometry and such topics from algebra as mathematical induction, binomial theorem, and elementary theory of equations. Material on inequalities and absolute values is included as an appendix. The study of analytic geometry is delayed until after extensive chapters on differentiation and integration. Some calculus can then be used to study conic sections and curve sketching. Vectors are also used to study analytic geometry. In fact, an important aspect of the book is its thorough use of vector analysis.

The book does not define and prove theorems about limits and continuity until Chapter 7 (p. 375). In this way, the student has a chance to become

familiar with the concepts in an intuitive way and to develop confidence and appreciation before examining the underlying theory. When unable to give a rigorous proof effectively, the author at least gives a plausible argument and observes why it is inadequate. Often, each hypothesis in a theorem is discussed to show the necessity for such a condition. Thus the student is gradually introduced to mathematical sophistication in an informal way.

In the opinion of the reviewer, this text will have wide acceptance by both teachers and students.

JAMES H. MCKAY
Michigan State University Oakland

Intermediate Algebra. By Roy Dubisch, Vernon E. Howes and Steven J. Bryant. Wiley, New York, 1960. xii+286 pp. $4.50.

Intermediate algebra has traditionally emphasized the teaching of techniques for solving algebraic equations and transforming various algebraic expressions. In recent years it has been urged in many quarters that more attention should be paid to algebraic structure, that is to say to the development of algebra as an abstract system from a basic set of axioms. The book under review offers a reasonable compromise between these two points of view. Algebraic techniques and manipulations constitute the authors' primary objective. On the other hand the axiomatic structure of algebra and the role of the axioms in the proof of algebraic theorems has been neither completely neglected nor unduly stressed. The book is written as an introduction to mathematics rather than as a collection of tricks for solving problems. The introduction of notations usually met only in more advanced mathematics will prepare the student for their use in later work. In this category are the use of the summation sign in connection with the progressions, double subscript notation in the discussion of determinants and two by two matrices in connection with complex numbers. A function is carefully defined and the terms domain and range introduced. The chapter on exponents and logarithms stresses the exponential and logarithmic functions and their properties rather than logarithmic computation. The latter is not omitted however. References for further study, a short table of four-place logarithms and answers to odd-numbered problems are included.

D. C. MURDOCH
University of British Columbia

Digital Computer Principles. By Wayne C. Irwin. Van Nostrand, Princeton, N. J., 1960. vi+321 pp. Trade $8.00, Text $6.75.

Well-written, easy-to-understand description of fundamentals upon which modern computers are designed. Aiming at the beginner, the author does a good job of introducing the arithmetic and logic which underlay design. He then shows how the logical functions are mechanized, how information is stored and how arithmetic is carried out. He next treats control, input-output equipment, and programming. Concluding with a discussion of present hardware trends, he

has given us a well-rounded introduction into what is commonly called the hardware concepts of digital computers.

For the potential user, the mathematician or programmer, this book would not be a good primer. However, it can serve effectively to round out the background of the user who has mastered the application of computers and is now motivated to learn more about the how and why of the hardware.

M. A. SHADER
IBM Corporation

Probability, an Introduction. By Samuel Goldberg. Prentice-Hall, Englewood Cliffs, N. J., 1960. ix+322 pp. $5.95.

Few tasks in teaching mathematics are more demanding than that of teaching a mathematically sound course in probability to students with a background of only two years of high school algebra. Such a course necessarily must be the combination of two courses: one on set theory and one on probability. The teacher of this course must indeed rejoice that with this fine text a valuable and much needed aid has become available.

Throughout the book the author shows a high degree of awareness of the difficulties commonly encountered by beginning students of probability. He skillfully makes use of all conceivable devices to facilitate understanding. These include: in the first chapter on sets, each definition is illustrated by numerous examples, many of which point to a later application to probability; proofs of theorems in the algebra of sets by means of membership tables are immediately followed by verifications by means of Venn diagrams, thus enabling the student to develop his intuition, which is so necessary at this stage of his development.

Chapter 2 carefully develops probability theory for finite sample spaces. To avoid the difficulties customarily encountered by students when combinatorial problems and probability are introduced almost simultaneously, the discussion of permutations, combinations (although the latter term is not used in the text outside the preface) and the binomial theorem is deferred until Chapter 3, where they are developed in complete detail. It appears somewhat inconsistent, however, that an introduction to mathematical induction and the use of summation signs, both used extensively in the text, are not also included.

In Chapter 4 random variables are defined as functions on sample spaces, and probability distributions, means, standard deviations, joint probability functions, covariance, and correlation are discussed. To keep the text suitable for a semester course, the author has wisely restricted the discussion of examples of probability distributions to that of the binomial case which constitutes the fifth and last chapter of the text. It includes a brief introduction to testing statistical hypotheses and ends with what certainly should whet any student's appetite for further study: the complete solution of a realistic decision problem.

The text includes 360 well-chosen problems, with answers (in many cases

complete solutions) to half of these provided at the end of the book.

Mathematics or statistics departments offering a basic course in probability using the language and notation of sets should find this text ideally suited to their needs.

H. L. ALDER
University of California, Davis

BRIEF MENTION

Basic Theorems in Matrix Theory, National Bureau of Standards, Applied Mathematics Series No. 57. By Marvin Marcus. Order from Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C., 1960. iv+27 pp. 15¢.

This amazingly inexpensive pamphlet is the finest collection of theorems concerning finite matrices that it has ever been your reviewer's privilege to see. No attempt whatsoever is made to give proofs; this is merely a collection of properties, definitions and theorems. Nevertheless, it is a valuable adjunct and, at its low price, should be in the hands of every member of a class in theory of matrices.

Mathematical Methods in the Social Sciences. By Kenneth J. Arrow, Samuel Karlin and Patrick Suppes. Stanford University Press, Stanford, California, 1960. viii+365 pp. $8.50.

The proceedings of the first Stanford symposium on mathematical methods in the social sciences, held in the summer of 1959, contains nine papers on economics, four papers on management science, and ten papers on psychology.

The Solution of Equations in Integers. By A. O. Gelfond, translated from the Russian by Leo F. Boron. P. Noordhoff-Groningen, Holland, 1960. 72 pp. $1.00.

A fine book that a high school student, a college student, and a professor may all enjoy.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Ball State Teachers College: Dr. J. M. Egar, Texas Agricultural and Mechanical College, has been appointed Associate Professor; Dr. Harry Langman, Ohio Northern University, has been appointed Visiting Professor for 1960–61.

Brown University: Professor Iacapo Barsotti, University of Pittsburgh, has been appointed Professor; Professor Emeritus T. H. Hildebrandt, University of Michigan, has been appointed Visiting Professor; Associate Professor Katsami Nomizu, Catholic University, has been appointed Associate Professor; Dr. R. D. M. Accola, Harvard University, has been appointed Assistant Professor; Dr. Leon Greenberg has been promoted to Assistant Professor.

College of William and Mary: Dr. F. W. Weiler, Ohio State University, has been appointed Assistant Professor; Mr. R. D. H. Jones has been appointed Instructor.

Iowa State University: Dr. W. M. Gilbert, Princeton University, has been appointed Associate Professor; Assistant Professor R. K. Meany, Texas Christian University, has been appointed Assistant Professor; Assistant Professor J. S. Rue and Mr. T. J. Robinson, University of North Dakota, have been appointed Instructors; Assistant Professor M. F. Ruchte has been promoted to Associate Professor.

Miami University: Dr. C. E. Capel, Westinghouse Research Laboratory, Pittsburgh, Pennsylvania, has been appointed Professor; Dr. Yukihiro Kodama, Defense Academy, Yokosuka, Japan, has been appointed Lecturer; Mr. R. F. DeMar has been promoted to Assistant Professor.

Montclair State College: Mrs. Hildegard Howden, Temple University, and Mr. Edward Urband, Bloomfield College, have been appointed Assistant Professors; Associate Professor P. C. Clifford has been promoted to Professor; Assistant Professor J. A. Schumaker has been promoted to Associate Professor.

Oklahoma State University: Assistant Professor S. M. Harmon, Fresno State College, has been appointed Associate Professor; Assistant Professor R. A. Hultquist, DePauw University, has been appointed Assistant Professor; Assistant Professor G. A. Haddock, Arkansas State Teachers College, has been appointed Instructor.

Pamona College: Mr. J. E. Vought, University of Michigan, has been appointed Instructor; Assistant Professor K. L. Cooke has been promoted to Associate Professor.

Southern Illinois University: Professor J. M. H. Olmsted, University of Minnesota, has been appointed Professor and Chairman of the Department of Mathematics; Assistant Professors Marian A. Moore, and J. C. Wilson have been promoted to Associate Professors.

Stanford University: Professors Hans Samelson, University of Michigan, and Ralph Phillips, University of California, Los Angeles, and Associate Professor John Myhill, University of California, Berkeley, have been appointed Professors; Dr. Donald Ornstein, University of Wisconsin, has been appointed Assistant Professor; Dr. P. L. Duren, Massachusetts Institute of Technology, has been appointed Instructor; Assistant Professor James McGregor has been promoted to Associate Professor; Dr. J. W. Lamperti has been promoted to Assistant Professor.

Tennessee Polytechnic Institute: Professor C. G. Phipps, University of Florida, and Associate Professor W. A. Small, Grinnell College, have been appointed Professors; Major General R. C. Hood, Jr., Duke University, has been appointed Associate Professor.

Wayne State University: Professors Ky Fan, University of Notre Dame, and Seymour Sherman, University of Pennsylvania, have been appointed Professors; Associate Professor Melvin Henriksen, Purdue University, has been appointed Visiting Professor; Associate Professor Felix Haas has been appointed Acting Head of the Department of Mathematics; Associate Professor Samuel Kaplan has been promoted to Professor; Assistant Professor Samuel Goldberg has been promoted to Associate Professor.

University of Missouri: Drs. M. D. George, University of Maryland, and L. J. Lange, Bureau of Standards, Boulder, Colorado, have been appointed Assistant Professors; Professor M. V. SubbaRao, Sri Venkateswara University, Tirupati, India, has been appointed Visiting Professor.

University of Tulsa: Dr. J. F. Evans, Pan American Petroleum Corporation, has been appointed Assistant Professor; Mr. R. R. Kinkade, University of Kansas, and Mr. R. M. McDonald, Douglas Aircraft Company, Tulsa, Oklahoma, have been appointed Instructors.

University of Vermont: Assistant Professor D. E. Moser, University of Massachusetts, has been appointed Associate Professor; Assistant Professor J. A. Izzo has been promoted to Associate Professor.

Brother Thomas Warren, San Joaquin Memorial High School, Fresno, California, has been appointed Chairman of the Department of Mathematics at Cathedral High School, Los Angeles, California.

Mr. W. F. Coulson, University of Minnesota, has been appointed Assistant Professor at the University of Alberta.

Mr. R. E. Cummings, Navy Mine Defense Laboratory, Panama City, Florida, has accepted a position as Mathematician with George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama.

Mr. W. R. Derrick, Oklahoma State University, has accepted a position as Mathematician Programmer with International Business Machines, Endicott, New York.

Associate Professor J. N. Eastham, Cooper Union, has been promoted to Professor.

Mr. Eugene Enrione, University of Miami, has accepted a position as Mathematician with Douglas Aircraft, Los Angeles, California.

Professor H. S. Everett, University of Chicago, has been appointed Professor at East Stroudsburg State College.

Associate Professor W. B. Fulks, University of Minnesota, has been appointed Professor at Oregon State College.

Professor Wallace Givens, Wayne State University, has been appointed Professor at Northwestern University.

Mr. Reid Haywood, Martin Company, Baltimore, Maryland, has accepted a position as Senior Program Analyst with the International Electric Corporation, Paramus, New Jersey.

Mr. R. L. Jacobsen, State University of Iowa, has accepted a position as Actuarial Assistant with Northwestern National Life, Minneapolis, Minnesota.

Mr. Walter James, University of Minnesota, has accepted a position with the Minnesota Mining & Manufacturing Company, St. Paul, Minnesota.

Mr. Bernard Levenson, Bureau of Applied Social Research, Columbia University, has been appointed Lecturer and Research Associate at Johns Hopkins University.

Visiting Assistant Professor C. W. Lytle, Drew University, has been appointed Assistant Professor.

Professor W. S. Massey, Brown University, has been appointed Professor at Yale University.

Mr. T. O. McCarley, Notre Dame University, has been appointed Instructor at Central State College, Oklahoma.

Dr. R. A. McHaffey, Rutgers, The State University, has been appointed Assistant Professor at the University of Massachusetts.

Assistant Professor T. D. Oxley, Jr., Kansas State College, has been appointed Assistant Professor at Drake University.

Mr. J. A. Pavelcak, College of St. Thomas, has been appointed Instructor at Merrimack College.

Mr. S. D. Pratico, Fordham University, has accepted a position in the Applied Programming Department of International Business Machines, Manhatten, New York.

Mr. J. R. Stagner, University of Redlands, has accepted a position as Engineering Technician with United Electrodynamics, Pasadena, California.

Assistant Professor J. P. Van Alstyne, Hamilton College, has been promoted to Associate Professor.

Mr. M. L. Whitaker, Florida State University, has been appointed Associate Professor at Radford College.

Dr. J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, New York, has joined General Dynamics Corporation's General Atomic Division in San Diego, California, as Assistant Chairman of the theoretical physics department of the John Jay Hopkins Laboratory for Pure and Applied Science.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE OCTOBER MEETING OF THE INDIANA SECTION

The annual joint meeting of the Indiana Section of the Mathematical Association of America with the Indiana Academy of Science was held on Friday, October 7, 1960, at Manchester College, North Manchester, Indiana. Professor Kermit Carlson of Valparaiso University presided at the morning session and Professor M. E. Shanks of Purdue University at the afternoon session. The meeting was attended by 43 persons, of whom 26 were members of the Association.

Professor J. C. Polley of Wabash College, Chairman of the Indiana School and College Committee on Mathematics, announced a meeting to be held at Purdue University on October 22 under the aegis of the committee. This meeting, the first of five to be sponsored by the committee during the year, had for its theme the preparation of secondary school teachers. Professor E. E. Moise of Harvard University was to be the principal speaker.

Professor R. C. Buck of the University of Wisconsin, Chairman of the Committee for the Undergraduate Program in Mathematics, delivered the invited hour address entitled, "Crises, Past and Present," a commentary on the changes being wrought in the teaching of mathematics and their causes. The following short papers were presented:

1. *Mathematics and the younger generation*, by Professor H. J. Zassenhaus, University of Notre Dame.

2. *The mathematics program at Rose Polytechnic Institute*, by Professor T. P. Palmer, Rose Polytechnic Institute.

3. *The state of mathematics in the State of California*, by Professor Harley Flanders, Purdue University.

4. *Comments on the liaison between high school and college mathematicians*, by Professor A. E. Hallerberg, Valparaiso University.

5. *Interdepartmental seminar: a new course at DePauw University*, by Professor R. J. Thomas, DePauw University.

P. T. MIELKE, *Secretary*

THE OCTOBER MEETING OF THE IOWA SECTION

A combined meeting of the Iowa Section of the Mathematical Association of America and the National Council of Teachers of Mathematics was held October 10 and 11, 1960, at the State University of Iowa, Iowa City, Iowa. The meeting was a Regional Orientation Conference in Mathematics, sponsored by the National Council of Teachers of Mathematics under the Regional Director, Dr. H. V. Price, State University of Iowa. Some twenty members of the Iowa Section attended the conference.

The following program was presented:

1. *Progress in mathematics and its implications for the secondary school*, by Dr. G. B. Price, Executive Secretary, Conference Board of the Mathematical Sciences.

Changes in presenting mathematical materials today were described as being so great as to be called revolutionary. Three causes of this revolution were: (1) advances made in mathematics as a result of mathematical research; (2) automation revolution-introduction of machines that control machines; (3) introduction of large scale automatic computing machines. Implications for schools were: (1) small high schools cannot normally provide the program and teachers needed; (2) many high school teachers need more training; (3) higher standards should be required of teach-

ers of mathematics—from elementary teachers on up; (4) teachers must re-examine techniques they are now using.

2. *Projects in school mathematics*, by Dr. Kenneth Brown, Specialist in Mathematics, United States Office of Education.

Dr. Brown discussed the school mathematics projects that are underway in various parts of the country. He pointed out that these projects are discussed briefly in "Aids for Mathematics Education: Mathematics—A Universal Language of Modern Civilization: OE-29013," U. S. Department of Health, Education and Welfare, Office of Education, Washington 25, D. C.

3. *University of Illinois program* (*UICSM*), by Miss Grace Wandke, Barrington Consolidated High School, Barrington, Illinois.

4. *University of Maryland program* (*UMMaP*), by Mrs. Ruth Brown, McKistry Junior High School, Waterloo, Iowa.

5. *Ball State program*, by Mr. Wilson Banks, Pleasant Valley-Riverdale High School, Bettendorf, Iowa.

6. *School Mathematics Study Group program*, by Miss Joan Tanzer, Mechanical Arts High School, St. Paul, Minnesota.

7. *Implementing a new mathematics program for secondary schools*, by Dr. W. E. Ferguson, Newton High School, Newtonville, Massachusetts.

Eight basic steps were listed for implementing a new mathematics program: (1) recognition by the school authorities of the need for a new mathematics program; (2) adequate preparation of teachers in the mathematics that is now being taught for the first time in secondary schools; (3) selection of a new program; (4) selection of students for the program; (5) informing parents about the new program; (6) informing other members of the school system about the new program and its implications for the mathematics program, Kindergarten through Grade 12; (7) continuation of teacher preparation for carrying the new program to higher grades and also lower grades; (8) provision for adequate time and compensation for carrying on the new program year after year.

8. *Question and answer*, by a panel consisting of Dr. G. B. Price, Dr. Kenneth Brown, Dr. W. E. Ferguson, and Mr. F. B. Allen, Lyons Township High School and Junior College, La Grange, Illinois.

E. L. CANFIELD, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University on October 28 and 29, 1960. Professor Kathrine C. Mires, Chairman of the Section, presided. There were 160 persons in attendance, including 71 members of the Association.

The following officers were elected for a one-year term: Chairman, Professor J. A. Nickel, Oklahoma City University; Vice-Chairman, Professor R. R. Murphy, Panhandle A and M College; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association and contains expository papers believed to be of particular interest to high school teachers. Research papers are presented at the spring meeting. The following papers were presented:

1. *Time delays in waiting for a traffic light*, by Professor J. A. Nickel, Oklahoma City University.

2. *Completeness in uniform spaces*, by Professor J. C. Mathews, University of Oklahoma.

3. *Extremal elements of convex cones*, by Professor E. K. McLachlan, Oklahoma State University.

4. *Mathematical faces of flexagons*, by Professor C. O. Oakley, Haverford College.

5. *Sums of series*, by Professor W. A. Rutledge, University of Tulsa.

6. *A commentary on normal forms in sentential calculus*, by Mr. L. E. DeNoya, Oklahoma State University.

7. *SMSG mathematics in Junior High*, luncheon address by Miss Veryl Shult, Washington, D. C.

8. *General aims and purposes of the CUPM recommendations for teacher training in mathematics*, by Professor Robert Wisner, Executive Director, Committee on the Undergraduate Program of the Mathematical Association of America.

Professor J. H. Zant of the Oklahoma Committee for the Improvement of Mathematics Instruction, and Mr. F. R. Born of the State Office of Education, were asked to comment on these recommendations. Each college giving teacher training was invited to have an official representative present at this meeting. Dr. Wisner's talks were follows by open discussion and questions from the floor.

R. V. ANDREE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA, Rose Polytechnic Institute, Terre Haute, May 6, 1961.

IOWA, Simpson College, Indianola, April 14, 1961.

KANSAS, Ottawa University, April 15, 1961.

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Fordham University, New York, April 15, 1961

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA

MISSOURI, University of Missouri, Columbia, April 22, 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY

NORTHEASTERN, University of Vermont, Burlington, June 20, 1961.

NORTHERN CALIFORNIA

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, University of Colorado, Boulder, April 28–29, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, March 17–18, 1961.

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.

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THE FOURIER TRANSFORM AND MEAN CONVERGENCE

E. J. McSHANE, University of Virginia

At the undergraduate level the Fourier transform is customarily studied by pointwise convergence, and only absolutely integrable functions are considered. In view of the importance of convergence in the mean and of the invariance of the integral of the absolute square under the Fourier transform, it would seem desirable to have a treatment of the Fourier transform in terms of convergence in the mean, still using only Riemann integrals, since ordinarily no other integral is known to undergraduates. The present paper presents such a treatment of two fundamental theorems, the invariance of the integral of the absolute square and the inversion theorem.

Let us say that a (complex-valued) function f on the real axis is *square-integrable* if f is (Riemann) integrable over every interval and $|f|^2$ is integrable over the whole real axis. The Fourier theory makes no distinction between functions f_1 and f_2 which are equal at so many places that the integral of $|f_1-f_2|^2$ is 0 (or, what amounts to the same thing, the integral of $|f_1-f_2|$ is 0). Hence such functions will be called *equivalent.* Properly speaking, the range and the domain of the transform are not functions, but equivalence classes of functions. But we follow custom in saying that the transformation maps functions on functions, and we "identify" any two equivalent functions.

If f is square-integrable we define its norm to be

$$\|f\| = \left[\int |f(t)|^2 dt\right]^{1/2}.$$

(When the limits of integration are omitted they are understood to be $-\infty$ and ∞.) We assume it known that if f and g are square-integrable so is $f+g$, and $\|f+g\| \leqq \|f\| + \|g\|$. It is also convenient, though by no means necessary, to make use of the inner product

$$(f, g) = \int f(t)\overline{g(t)}\, dt,$$

along with the most elementary properties of the inner product.

If f is square-integrable, for each positive number A we define $g_{[A]}$ to be the function for which

$$g_{[A]}(s) = (2\pi)^{-1/2} \int_{-A}^{A} e^{-ist} f(t) dt \qquad (-\infty < s < \infty). \tag{1}$$

If there is a function g, necessarily square-integrable, such that

$$\lim_{A \to \infty} \|g_{[A]} - g\| = 0,$$

we say that g is the Fourier transform of f (although it might be better to call it the Fourier-Plancherel transform), and we write $g = Tf$. The conjugate Fourier transform $\overline{T}f$ is similarly defined, with e^{ist} in place of e^{-ist}. The connection between this convergence in mean and the pointwise convergence of $g_{[A]}$ to a limit is not easily studied with only the Riemann integral at hand, but the next lemma states a simple and useful relationship.

LEMMA 1. *Let $h_1, h_2, h_3, \cdots$ be a Cauchy sequence of square-integrable functions (that is, assume that $\|h_m - h_n\|$ tends to 0 as m and n tend to ∞.) Let h_0 be defined on the real axis, and let B be a set of real numbers such that no interval contains more than a finite number of points of B. Assume that whenever $[a, b]$ is a closed interval containing no point of B, $h_n(t)$ converges uniformly to $h_0(t)$ on $[a, b]$. Then h_0 is square-integrable, and $\lim_{n\to\infty} \|h_n - h_0\| = 0$.*

Proof. Let A be positive, and let $b_1, \cdots, b_p$ be the points of B between $-A$ and A, in increasing order. If ϵ is positive, there is an n_ϵ such that if m and n both exceed n_ϵ, $\|h_n - h_m\| < \epsilon/2$. Hence for any small positive δ

$$\int_{-A+\delta}^{b_1-\delta} + \int_{b_1+\delta}^{b_i-\delta} + \cdots + \int_{b_p+\delta}^{A-\delta} | h_n(t) - h_m(t) |^2 dt \leqq (\epsilon/2)^2.$$

On each interval of integration $h_m(t)$ tends uniformly to $h_0(t)$, so if we fix n at any integer above n_ϵ we have the same inequality with h_0 in place of h_m. Letting δ tend to 0 gives

$$\int_{-A}^{A} | h_n(t) - h_0(t) |^2 dt \leqq (\epsilon/2)^2,$$

and letting A tend to ∞ yields $\|h_n - h_0\|^2 \leqq (\epsilon/2)^2 < \epsilon^2$. This establishes both conclusions.

For any two real numbers a, b $(b > a)$ we define $f_{a,b}$ to be

$$f_{a,b}(t) = [\operatorname{sgn}(t - a) - \operatorname{sgn}(t - b)]/2;$$

that is,

$$f_{a,b}(t) = \begin{cases} 0 & \text{if } t < a \text{ or } t > b, \\ 1/2 & \text{if } t = a \text{ or } t = b, \\ 1 & \text{if } a < t < b. \end{cases} \tag{2}$$

It is a trivial calculation that its Fourier transform $g_{a,b}$ is

$$g_{a,b}(s) = \frac{i}{\sqrt{(2\pi)}} \frac{e^{-isb} - e^{-isa}}{s} \qquad (s \neq 0). \tag{3}$$

(This has the finite limit $(b - a)/\sqrt{(2\pi)}$ as s tends to 0.) These functions are tools in proving both our theorems.

LEMMA 2. *If f is square-integrable and has a Fourier transform g, then $\|g\|^2 = \lambda\|f\|^2$, where*

$$\lambda = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2}\, dv.$$

Proof. If $[a, b]$ and $[c, d]$ are any two intervals, we compute

$$(g_{a,b}, g_{c,d}) = (2\pi)^{-1}\int s^{-2}\left[e^{-is(b-d)} + e^{-is(a-c)} - e^{-is(b-c)} - e^{-is(a-d)}\right]ds.$$

When we replace the imaginary exponentials by means of Euler's formula we find that the imaginary part of the integrand is an odd function, so its integral from $-\infty$ to ∞ is 0, and the inner product is real. In particular, if $c=a$ and $d=b$, by a trigonometric identity and the change of variable of integration $v=s(b-a)/2$, we obtain

$$\|g_{a,b}\|^2 = \pi^{-1}\int s^{-2}[1 - \cos s(b-a)]ds = \lambda(b-a).$$

If $a \leqq b \leqq c$, then $f_{a,c} = f_{a,b} + f_{b,c}$, and by the linearity of the transformation $g_{a,c} = g_{a,b} + g_{b,c}$. Moreover, because the inner product $(g_{a,b}, g_{b,c})$ is real it is equal to $(g_{b,c}, g_{a,b})$. Hence

$$\begin{aligned}\|g_{a,c}\|^2 &= (g_{a,b} + g_{b,c}, g_{a,b} + g_{b,c}) \\ &= \|g_{a,b}\|^2 + 2(g_{a,b}, g_{b,c}) + \|g_{b,c}\|^2.\end{aligned}$$

For three of these terms we substitute their values as just computed. The left member then eliminates the first and last terms of the right member, and we have $(g_{a,b}, g_{b,c}) = 0$. Finally, if $a \leqq b \leqq c \leqq d$, by the proof just completed both $(g_{a,b}, g_{b,c})$ and $(g_{a,b}, g_{b,d})$ vanish, hence so does their difference $(g_{a,b}, g_{c,d})$.

Next let f be a step-function, with values $c_1, \cdots, c_p$ on the respective intervals (in increasing order) $t_0 < t \leqq t_1, \cdots, t_{p-1} < t \leqq t_p$. For notational simplicity we write f_j to stand for f_{t_{j-1},t_j}; then (except perhaps at the points $t_0, \cdots, t_p$) $f = c_1 f_1 + \cdots + c_p f_p$. Hence, if $g_j = Tf_j$ and $g = Tf$, we have $g = c_1 g_1 + \cdots + c_p g_p$. Multiplying this by its conjugate and integrating yields, in view of the fact that $(g_j, g_k) = 0$ if $j \neq k$, and $(g_j, g_j) = \lambda(t_{j-1} - t_j)$,

$$\begin{aligned}\|g\|^2 &= \lambda\,|c_1|^2(t_1 - t_0) + \cdots + \lambda\,|c_p|^2(t_p - t_{p-1}) \\ &= \lambda\int |f(t)|^2 dt = \lambda\|f\|^2.\end{aligned}$$

Thus the conclusion of the lemma holds when f is any step-function.

Next let f be Riemann integrable (therefore bounded) on an interval $[-K, K]$ and vanish outside that interval; let M be an upper bound for $|f|$. By definition of the Riemann integral there is a sequence of step-functions

$S_1, S_2, \cdots$ vanishing outside $[-K, K]$, with absolute values not greater than M, and such that

$$\lim_{n\to\infty}\int_{-K}^{K} | S_n(t) - f(t) | \, dt = 0.$$

Since $|S_n - f|^2 \leqq 2M(|S_n - f|)$ this implies that $||S_n - f||$ tends to 0. Hence $||S_n - S_m||$ tends to 0 as m and n increase. If we write $G_n = TS_n$, $g = Tf$, by the preceding paragraph $||G_n - G_m||$ also tends to 0 as m and n increase. Also the limiting process defining g is trivial; $g_{[A]}$ has the same value for all A greater than K. For all s,

$$| g(s) - G_n(s) | \leqq (2\pi)^{-1/2}\int_{-K}^{K} | e^{-ist} | \cdot | f(t) - S_n(t) | \, dt,$$

and as n increases this last tends to 0 and is independent of s. Hence $G_n(s)$ tends to $g(s)$ uniformly for all s. By Lemma 1, g is square-integrable and $||g - G_n||$ tends to 0. But then by the triangle inequality and the preceding paragraph

$$||g||^2 = \lim_{n\to\infty} ||G_n||^2 = \lim_{n\to\infty} \lambda ||S_n||^2 = \lambda ||f||^2.$$

Finally let f be any square-integrable function that has a Fourier transform $g = Tf$, and let A be any positive number. Define $f_A(t)$ to be $f(t)$ if $-A \leqq t \leqq A$, 0 otherwise. Then, with the notation of the definition $g_{[A]} = Tf_A$, and by the preceding paragraph $||g_{[A]}||^2 = \lambda ||f_A||^2$. As A increases, by definition of Tf we have $\lim ||g_{[A]} - g||^2 = 0$, so that $||g_{[A]}||^2$ tends to $||g||^2$; and by definition of the integral from $-\infty$ to ∞, $||f_A||^2$ tends to $||f||^2$. Hence $||g||^2 = \lambda ||f||^2$, completing the proof of the lemma.

COROLLARY 1. *Let f be square-integrable. If there is a point set B such that only finitely many points of B lie in any interval of the real axis, and the functions $g_{[A]}(s)$ of equation* (1) *converge uniformly to a limit $g(s)$ on every closed interval that contains no point of B, then g is square-integrable, and is the Fourier transform of f.*

Proof. Use the notation of the last paragraph of the preceding proof. If A goes through any sequence of values tending to ∞, $||f_A - f||$ tends to 0, so the f_A form a Cauchy sequence. By Lemma 2, so do their transforms $g_{[A]}$. The conclusion follows by way of Lemma 1.

COROLLARY 2. *If f is square-integrable and also is absolutely integrable, the transforms $g_{[A]}(s)$ converge uniformly to a continuous limit $g(s)$, and g is the Fourier transform of f.*

Proof. For all s,

$$| g(s) - g_{[A]}(s) | \leqq (2\pi)^{-1/2}\left\{\int_{-\infty}^{\infty} - \int_{-A}^{A} | e^{-ist} | \cdot | f(t) | \, dt\right\} \cdot$$

This is independent of s and tends to 0 as A increases, so $g_{[A]}(s)$ converges uniformly to $g(s)$; since $g_{[A]}$ is continuous, so is g. By Corollary 1, g is square-integrable and $g = Tf$.

If in particular we take $f(t)$ to be 0 if $t<0$ and to be e^{-t} if $t \leqq 0$, by Corollary 2 its Fourier transform g satisfies $g(s) = (2\pi)^{-1/2}(1+is)^{-1}$, as we compute without trouble. Then

$$\|f\|^2 = \int_0^\infty e^{-2t}dt = 1/2,$$

$$\|g\|^2 = (1/2\pi)\int_{-\infty}^\infty (1+s^2)^{-1}ds = 1/2,$$

and so by Lemma 2 we have $\lambda = 1$. Now Lemma 2 becomes

THEOREM 1. *If f is square-integrable and has a Fourier transform, then* $\|Tf\|^2 = \|f\|^2$.

If in (1) we had replaced e^{-ist} by e^{ist} the effect would have been only to replace $g_{[A]}$ by its reflection $\rho g_{[A]}$, for which $\rho g_{[A]}(s) = g_{[A]}(-s)$. This has no effect on the norm, so f has a Fourier transform Tf if and only if it has a conjugate Fourier transform $\overline{T}f$, and in that case $\|\overline{T}f\| = \|Tf\| = \|f\|$.

Our inversion theorem asks nothing more than that f have a Fourier transform.

THEOREM 2. *Let f be square-integrable and have a Fourier transform $g = Tf$. Then g has a conjugate Fourier transform, and $\overline{T}g = f$.*

Proof. For positive A we define, analogously to (1),

$$h_{[A]}(t) = (2\pi)^{-1/2}\int_{-A}^{A} e^{its}g_{a,b}(s)ds. \tag{4}$$

Consider first the function $f_{a,b}$ defined in (2) and its transform $g_{a,b}$ defined in (3). For this, (4) becomes

$$\begin{aligned} h_{[A]}(t) &= (i/2\pi)\int_{-A}^{A}\left[e^{is(t-b)} - e^{is(t-a)}\right]s^{-1}ds \\ &= (1/2\pi)\int_{-A}^{A}\left[\frac{\sin s(t-a)}{s} - \frac{\sin s(t-b)}{s}\right]ds; \end{aligned}$$

the cosine terms have dropped out because $[\cos s(t-a) - \cos s(t-b)]/s$ is odd and its integral from $-A$ to A is 0. By a change of variable in each of the two terms of the right member this becomes

$$h_{[A]}(t) = (1/2\pi)\left[\int_{-A(t-a)}^{A(t-a)}\frac{\sin v}{v}dv - \int_{-A(t-b)}^{A(t-b)}\frac{\sin v}{v}dv\right].$$

By the usual elementary discussion of its graph we show that $(\sin v)/v$ is integrable from 0 to ∞, hence from $-\infty$ to ∞. Let the latter integral be μ; then for each positive ϵ there is an X_ϵ such that if $X > X_\epsilon$ then

$$\left| \int_{-X}^{X} \frac{\sin v}{v} dv - \mu \right| < \epsilon.$$

If we allow negative X also, this yields

$$\left| \int_{-X}^{X} \frac{\sin v}{v} dv - \mu \operatorname{sgn} X \right| < \epsilon \qquad (|X| > X_\epsilon).$$

Let $[c, d]$ be a closed interval not containing either a or b, and let δ be the (positive) distance from the nearer of these two points to the closed interval $[c, d]$. Then for all A such that $A > X_\epsilon/\delta$ we have $|A(t-a)|$ and $|A(t-b)|$ both greater than X_ϵ for all t in $[c, d]$. The preceding inequality and the last equation for $h_{[A]}$ yield

$$| h_{[A]}(t) - \mu[\operatorname{sgn}(t-a) - \operatorname{sgn}(t-b)]/2\pi | < 2\epsilon \qquad (c \leqq t \leqq d).$$

That is (cf. (2)) $h_{[A]}(t)$ tends to $\mu f_{a,b}(t)/\pi$ uniformly on the interval $[c, d]$. Corollary 1 has an obvious dual for the conjugate Fourier transform; this, with B the set consisting of the two points a and b, informs us that $\mu f_{a,b}/\pi$ is the conjugate Fourier transform of $g_{a,b}$. It is easily seen that μ is positive, so that $\|\mu f_{a,b}/\pi\| = (\mu/\pi)\|f_{a,b}\|$. By the dual of Theorem 1 for the conjugate Fourier transform we have $\|\mu f_{a,b}/\pi\| = \|g_{a,b}\|$, while by Theorem 1 applied to $f_{a,b}$ we have $\|g_{a,b}\| = \|f_{a,b}\|$. These together imply that $\mu/\pi = 1$, and so $f_{a,b}$ is the conjugate Fourier transform of $g_{a,b}$.

By the linearity of the transformation, it follows at once that $\overline{T}(TS) = S$ for every step-function S.

Now let ϵ be any positive number. We can find a number K such that

$$0 \leqq \|f\| - \left[\int_{-K}^{K} | f(t) |^2 dt \right]^{1/2} < \epsilon/6;$$

then (as in the proof of Lemma 2) we can find a step-function S vanishing outside the interval $[-K, K]$ such that

$$\left[\int_{-K}^{K} | S(t) - f(t) |^2 dt \right]^{1/2} < \epsilon/6.$$

These together imply $\|S-f\| < \epsilon/3$. We take Fourier transforms and apply Theorem 1; the result is $\|TS-g\| = \|S-f\| < \epsilon/3$.

Let us define g_A to be the function with values

$$\begin{aligned} g_A(s) &= 0 \quad \text{if } s < -A \text{ or } s > A; \\ &= g(s) \quad \text{if } -A \leqq s \leqq A. \end{aligned}$$

Its conjugate Fourier transform is, as in (4), $h_{[A]}$. There is an A_ϵ such that $\|g_A - g\| < \epsilon/3$ whenever $A > A_\epsilon$, by definition of the integral. From this and the preceding paragraph, $\|g_A - TS\| < 2\epsilon/3$ whenever $A > A_\epsilon$. We take conjugate Fourier transforms and use Theorem 1 and the remark after it to obtain $\|\overline{T}g_A - \overline{T}(TS)\| < 2\epsilon/3$. But since S is a step-function, $\overline{T}(TS) = S$, and by definition $\overline{T}g_A$ is $h_{[A]}$; so we have $\|h_{[A]} - S\| < 2\epsilon/3$. Since $\|S - f\| < \epsilon/3$, it follows that $\|h_{[A]} - f\| < \epsilon$ whenever $A > A_\epsilon$. That is, by definition $\overline{T}g = f$.

A MODEL OF QUASI-EUCLIDEAN SPACE

A. R. AMIR-MOÉZ, Purdue University, AND A. L. FASS, Queens College

A quasi-unitary space is a linear space over the complex field with a quasi-inner product defined for it. In case the field is the field of real numbers we call the space quasi-Euclidean. Before we give the model we shall discuss a simple form of quasi-inner product and quasi-product of matrices. We also discuss a way of estimating sum of squares of some elements of a matrix minus the sum of squares of other elements by defining and using quasi-singular values of a matrix.

1. Definitions and notations. Let V be an n-dimensional unitary space, with an orthonormal base $\{\alpha_1, \cdots, \alpha_n\}$. Let $\xi = (x_1, \cdots, x_n)$ and $\eta = (y_1, \cdots, y_n)$ be two n-tuples of complex numbers, so that $\xi, \eta \in V$. Then we define

$$(\xi, \eta)_k = -\sum_{i=1}^{k} x_i\bar{y}_i + \sum_{i=k+1}^{n} x_i\bar{y}_i$$

to be the quasi-inner product of order k of ξ and η. For the general case see [4], page 150.

Now let $A = (a_{ij})$ and $B = (b_{ij})$ be two n-by-n matrices with complex elements. We define the quasi-product of order k of A and B, written $(AB)_k$, to be the matrix $C = (c_{lm})$ where c_{lm} is the quasi-inner product of the lth row of A and the conjugate of the mth column of B.

Let J_k be a diagonal matrix with -1 for its first k diagonal elements and $+1$ for the rest. Then it is easy to see that

$$(\xi, \eta)_k = (J_k\xi, \eta) = (\xi, J_k\eta),$$

and $(AB)_k = AJ_kB$.

A set of n vectors $\{\beta_1, \cdots, \beta_n\}$ will be called a quasi-orthonormal base of order k if

$$(\beta_i, J_k\beta_i) = -1, \ i = 1, \cdots, k, \qquad (\beta_i, J_k\beta_i) = 1, \ i = k+1, \cdots, n,$$
$$(\beta_i, J_k\beta_j) = 0, \ i \neq j.$$

If the rows of a matrix A form a quasi-orthonormal base of order k, the matrix will be called quasi-unitary of order k, denoted by U_k. The columns of a quasi-unitary matrix form also a quasi-orthonormal base of order k [2].

2. Quasi-orthogonal sets. Let $\{\xi_1, \cdots, \xi_n\}$ be a linearly independent set of vectors, with $(\xi_i, \xi_j)_k = 0$, $i \neq j$. Then the set may be ordered in such a way that

$$(\xi_i, J_k\xi_i) < 0, \ i = 1, \cdots, k, \qquad (\xi_i, J_k\xi_i) > 0, \ i = k+1, \cdots, n.$$

This theorem for the real case is Sylvester's law of inertia ([4], p. 158); but this proof uses a different method. We first state the following:

LEMMA. *If $\{\xi_1, \cdots, \xi_n\}$ is linearly independent, and $(\xi_i, J_k\xi_j) = 0$ for all $i \neq j$, then $(\xi_i, J_k\xi_i) \neq 0$, $i = 1, \cdots, n$.*

The proof is quite simple and will be omitted.

Now to prove the theorem, let $\{\xi_1, \cdots, \xi_n\}$ satisfy the hypothesis above and assume $(\xi_i, J_k\xi_i) < 0$ for $i = 1, \cdots, s$. If $\zeta = \sum_{i=1}^{s} c_i\xi_i$, then

$$(\zeta, J_k\zeta) \leqq 0, \tag{1}$$

since

$$(\zeta, J_k\zeta) = \left(\sum_{i=1}^{s} c_i\xi_i, \sum_{j=1}^{s} c_jJ_k\xi_j\right) = \sum_{i=1}^{s} c_i\bar{c}_i(\xi_i, J_k\xi_i),$$

with equality in (1) if and only if every c_i is zero. Now we let P_k be the projection onto the subspace spanned by $\{\alpha_1, \cdots, \alpha_k\}$ *i.e.*, if $\xi = (x_1, \cdots, x_n)$, then $P_k\xi = (x_1, \cdots, x_k, 0, \cdots, 0)$, and let $Q_k = I - P_k$. Consider any set of scalars $c_1, \cdots, c_s$ such that

$$c_1P_k\xi_1 + \cdots + c_sP_k\xi_s = 0.$$

Let $\eta = c_1\xi_1 + \cdots + c_s\xi_s$. Then $\eta = P_k\eta + Q_k\eta = Q_k\eta$, so that

$$(\eta, J_k\eta) \geqq 0. \tag{2}$$

Thus by (1) and (2), $(\eta, J_k\eta) = 0$, and $c_1 = \cdots = c_s = 0$. Thus $\{P_k\xi_1, \cdots, P_k\xi_s\}$ is linearly independent, and $s \leqq k$.

In a similar way, if there are t vectors in the set for which $(\xi_i, J_k\xi_i)$ is positive, then $t \leqq n-k$. But since no $(\xi_i, J_k\xi_i)$ is zero, we must have $s = k$, and $t = n-k$.

3. A model of quasi-Euclidean space. Let R_n be a real n-dimensional Euclidean space with an orthonormal base. We consider the set $\{S\}$, where

$$S = a\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} a_i x_i + d,$$

with a, d, $a_1, \cdots, a_n$ real numbers, and $(x_1, \cdots, x_n)$ a point of R_n. Then

$$S = \tfrac{1}{2}(a+d)\left(\sum_{i=1}^{n} x_i^2 + 1\right) + \tfrac{1}{2}(a-d)\left(\sum_{i=1}^{n} x_i^2 - 1\right) - 2\sum_{i=1}^{n} a_i x_i.$$

Thus to S corresponds an ordered $(n+2)$-tuple: $[\frac{1}{2}(a+d), \frac{1}{2}(a-d), a_1, \cdots, a_n]$. We may consider for such $(n+2)$-tuples the quasi-inner product of order one. This allows an interesting geometric application. To an element $S \in \{S\}$ for which $a, a_1, \cdots, a_n$ are not all zero, corresponds the n-sphere $s=0$. If the radius of this sphere is r, then $(S, S)_1 = 0$ if and only if $r=0$. Next, if $(S_1, S_1) \neq 0$ and $(S_2, S_2) \neq 0$, then

$$\frac{(S_1, S_2)_1}{(S_1, S_1)_1^{1/2}(S_2, S_2)_1^{1/2}} = \cos\theta,$$

where θ is the angle between the spheres. In addition it is easily verified that if for

$$S_1 = a_1\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} a_{1i}x_i + d_i,$$

$$S_2 = a_2\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} a_{2i}x_i + d_2$$

we have $(S_1, S_1) = (S_2, S_2) = 0$ and $a_1 = a_2$, then $(S_1 - S_2, S_1 - S_2)^{\frac{1}{2}}$ is the Euclidean distance between the points $S_1 = 0$ and $S_2 = 0$.

4. Examples of quasi-unitary matrices. Consider the translation $x_i = X_i + h_i$, $i = 1, \cdots, n$ on the space R_n of 3. This induces a linear transformation on $\{S\}$ whose matrix is

$$\begin{pmatrix} 1+\frac{1}{2}\sum h_i^2 & -\frac{1}{2}\sum h_i^2 & -h_1 \cdots -h_n \\ \frac{1}{2}\sum h_i^2 & 1-\frac{1}{2}\sum h_i^2 & -h_1 \cdots -h_n \\ -h_1 & h_1 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \vdots & \vdots & \\ -h_n & h_n & \end{pmatrix},$$

which is quasi-unitary of order one.

It is also true that, to an element of the conformal group of transformations on R_n, there corresponds a matrix of the form pU_1, where p is a real number and U_1 is quasi-unitary of order one. For example, to the inversion $\mathfrak{s}(\xi) = (p\xi)/|\xi|^2$,

$\xi \neq 0$, $\xi \in R_n$, there corresponds the transformation on S with matrix

$$\begin{pmatrix} \frac{1}{2}(1+p^2) & \frac{1}{2}(1-p^2) & 0 & \cdots & 0 \\ -\frac{1}{2}(1-p^2) & -\frac{1}{2}(1+p^2) & 0 & \cdots & 0 \\ 0 & 0 & p & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & p \end{pmatrix}.$$

5. Quasi-singular values of a matrix. Let $A=(a_{ij})$ be any n-by-n matrix. Then A^*J_kA and AJ_kA^* are both Hermitian. Let $\lambda_1 \geqq \cdots \geqq \lambda_n$ be the characteristic values of A^*J_kA. Then $\lambda_1, \cdots, \lambda_n$ are called left quasi-singular values of A. The characteristic values $\mu_1 \geqq \cdots \geqq \mu_n$ of AJ_kA^* are called right quasi-singular values of A. In general left and right quasi-singular values are not the same.

If $\{\alpha_1, \cdots, \alpha_n\}$ is the original base in V, then $A\alpha_i=(a_{i1}, \cdots, a_{in})$. Concerning the elements a_{ij} of the matrix A we have:

THEOREM 1.

$$\lambda_n \leqq -\sum_{j=1}^{k} |a_{ij}|^2 + \sum_{j=k+1}^{n} |a_{ij}|^2 \leqq \lambda_1.$$

More generally,

THEOREM 2.

$$\lambda_n + \cdots + \lambda_{n-l+1} \leqq \sum_{p=1}^{l}\left(-\sum_{j=1}^{k} |a_{i_pj}|^2 + \sum_{j=k+1}^{n} |a_{i_pj}|^2\right) \leqq \lambda_1 + \cdots + \lambda_l,$$

for any integers $i_1 < \cdots < i_l \leqq n$.

Similar theorems may be stated and proved for the numbers $\mu_1, \cdots, \mu_n$ and the terms in the columns of the matrix A. The proofs in all cases are like those in [1] and [5].

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A PARITY RELATION PARTITIONS ITS FIELD DISTINCTLY

FRANK HARARY, University of Michigan

1. Introduction. A classical result which pervades all of mathematics states that an equivalence relation partitions its field. In this note, we obtain a stronger statement by observing that a weaker hypothesis implies the same result. The result is well known to several mathematicians, but does not appear to have been published.

A *relation* is a set of ordered couples. An *equivalence relation* is reflexive, symmetric, and transitive. Consider a relation R whose field is S. The following are the conventional definitions of the three properties of an equivalence relation, where x, y, z are any elements of S: (1) R is *reflexive* if xRx; (2) R is *symmetric* if xRy implies yRx; (3) R is *transitive* if xRy and yRz imply xRz.

A comprehensive and clear presentation of several different kinds of reflexivity is contained in Copi [2]. We now call (3) "generally transitive"; this is the usual definition of transitivity as given in such authoritative books as [1] and [7]. But there is another definition of transitivity in the mathematical literature. It appears in the books on graph theory by König [5] and on symbolic logic by Lewis and Langford [6].

R is *distinctly transitive* if whenever x, y, z are distinct elements of S for which xRy and yRz, then xRz. It is clear that every generally transitive relation is distinctly transitive, but not conversely. Of course, R is an equivalence relation if and only if it is reflexive, symmetric, and distinctly transitive. It was shown in [3] that these three properties constitute a "very independent" axiom system. We now state the above-mentioned classical result precisely:

THEOREM E. *A relation R over a field S is an equivalence relation if and only if there is a partition of S into subsets S_i such that for any two elements x, $y \in S$, xRy if and only if x and y are members of the same subset S_i.*

2. Parity relations. We now introduce a type of relation which resembles an equivalence relation in its symmetric and transitive aspects, but differs from it with regard to reflexivity.

An *irreflexive relation* R is one in which no element of S is in the relation to itself. A *parity relation* is irreflexive, symmetric, and distinctly transitive. We shall say that a relation R *partitions its field distinctly* if there exists a partition of S into subsets S_i such that for any two distinct elements x, $y \in S$, xRy if and only if x and y lie in the same subset.

THEOREM. *A parity relation partitions its field distinctly.*

We defer the proof to the next section, where it will be readily demonstrated in terms of some concepts from graph theory.

COROLLARY. *A relation is symmetric and distinctly transitive if and only if it partitions its field distinctly.*

Proof. The "only if" part of the corollary is an immediate consequence of the proof of the theorem. The "if" part is obvious since the relation among elements of a set S of "being in the same subset" is certainly an equivalence relation. Hence, it is necessarily symmetric and distinctly transitive.

Theorem E follows at once from this corollary.

3. Some graphical concepts. We find it convenient to prove the theorem of the preceding section by exploiting known elementary results from the theory of directed graphs, or more briefly "digraphs;" see [5] or [4].

A *digraph* is an irreflexive relation. We call the elements of its field the *points* of the digraph and the ordered pairs in the relation are its *lines*, or *directed lines*. There are three possible ways [4] of defining connectedness for digraphs, called strong, unilateral, and weak connectedness. We require only the latter concept. A digraph is *weakly connected* or, more briefly, *weak*, if for any partition of its set of points into two nonempty subsets, there is a line joining a point of one subset with a point of the other. A (directed) *path* in a digraph D is a collection of n distinct points a_i together with the directed lines $\overrightarrow{a_1a_2}$, $\overrightarrow{a_2a_3}$, $\cdots$, $\overrightarrow{a_{n-1}a_n}$; this is a path *from* a_1 *to* a_n. A *semipath* consists of n distinct points together with exactly one from each of the following pairs of lines: $\overrightarrow{a_1a_2}$ or $\overrightarrow{a_2a_1}$, $\overrightarrow{a_2a_3}$ or $\overrightarrow{a_3a_2}$, $\cdots$, $\overrightarrow{a_{n-1}a_n}$ or $\overrightarrow{a_na_{n-1}}$. Obviously a digraph is weak if and only if any two points are joined by a semipath. A *weak component* of a digraph D is a maximal weakly connected subgraph. Clearly, the weak components of a digraph partition its set S of points, and two points are in the same weak component of D if and only if they are joined by a semipath. We are now ready to prove the theorem.

Proof of theorem. Let R be a parity relation whose field is S. We shall also use the letter R for the digraph of this relation. Let S_1, S_2, $\cdots$ be the subsets of points in the weak components of R. Then these subsets S_i constitute a partition of S. Let x and y be any two distinct elements of S. If xRy, then the digraph R contains the directed line $\overrightarrow{xy}$. Thus x and y are certainly in the same weak component.

Conversely, if x and y are distinct points in the same weak component S_i then there exists a semipath joining them. By the symmetry of relation R, we find that there also exists in this weak component a directed path from x to y. Applying the hypothesis of distinct transitivity, it follows that the line $\overrightarrow{xy}$ lies in this weak component; hence, xRy.

The consequence of this theorem is that the distinct partitioning of a set is independent of reflexivity. Thus, if we have a relation R which is neither reflexive nor irreflexive, there still results a distinct partitioning of its field provided R is symmetric and distinctly transitive.

An example of a parity relation among people is provided by: x is a sibling of y. For by definition this relation is symmetric and distinctly transitive, but irreflexive.

A *graph* is an irreflexive symmetric relation. A *complete graph* is one in which any two distinct points lie in the relation to each other. The graphical interpretation of a parity relation is the following: *R is a parity relation if and only if each of its weak components is a complete graph.*

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AN OPTIMUM SHAPE FOR FAIRING THE EDGE OF AN ELECTRODE

GORDON RAISBECK, Bell Telephone Laboratories

1. Introduction. Application of conformal mapping to electromagnetic theory. Boundary-value problems involving the two-dimensional Laplace equations, which arise in many physical problems, can often be solved with the aid of conformal mapping. For suppose that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

over some domain. Then we can define the complementary function $\psi(x, y)$ such that $\chi=\phi+i\psi$ is an analytic function of $z=x+iy$. Suppose also that $z=f(w)$, $w=u+iv$, is an analytic function. Then $\chi[f(w)]=\phi[f(w)]+i\psi[f(w)]$ is also analytic, and hence

$$\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}=0.$$

If the function $z=f(w)$ is a 1-1 mapping over some domain of z and w, its inverse transforms solutions of Laplace's equation in the xy-plane into solutions in the uv-plane, with equipotential lines being transformed into equipotential lines and so on. This technique is useful because it enables us to transform problems involving a complicated boundary into problems involving a simple boundary such as a straight line or a circle, for which special solutions of the Laplace equation are available.

Consider, for example, a region bounded by two straight lines which join at an angle α (Fig. 1)

$$w = u + iv = re^{i\theta}, \qquad r \geqq 0, \qquad 0 \leqq \theta \leqq \alpha \leqq 2\pi,$$

and let us seek a solution to the electrostatic potential problem $\phi(u, v)$ which is zero on the boundary.* The mapping function $z = w^{\pi/\alpha}$ maps the region in the w-plane into

$$z = (re^{i\theta})^{\pi/\alpha} = r^{\pi/\alpha}e^{i\pi\theta/\alpha} = \rho e^{i\psi}, \qquad \rho \geqq 0, \qquad 0 \leqq \psi \leqq \pi,$$

i.e., into the upper half plane. An example of a simple potential function which vanishes on the boundary is $\phi(x, y) = y$ which is the potential of a uniformly charged plane at $y = 0$.

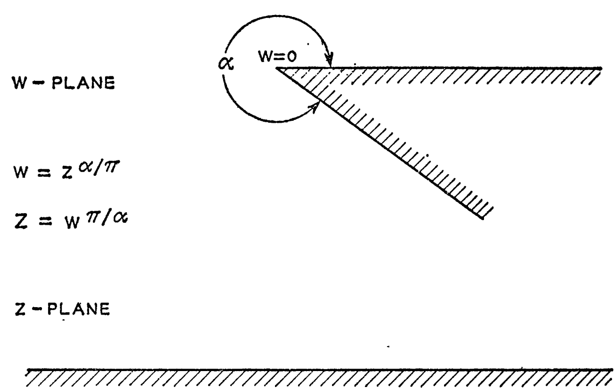

FIG. 1

In this case,

$$\phi(u, v) = y(u, v) = \operatorname{Im}(u + iv)^{\pi/\alpha} = (u^2 + v^2)^{\pi/2\alpha} \sin\left(\frac{\pi}{\alpha} \tan^{-1} \frac{v}{u}\right)$$

is a solution meeting the boundary conditions.

It is often convenient to retain the whole analytic function $\chi(w) = \phi(w) + i\psi(w)$, instead of only the real part $\phi(w)$. For example, if ϕ is an electrostatic potential and $\mathbf{E}$ the corresponding electric field vector, then $\mathbf{E} = -\operatorname{grad} \phi$. But

$$\frac{d\chi}{dw} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial w} + i\left[\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial w}\right].$$

Eliminating ψ by the Cauchy-Riemann equations,

* We are really imagining a three-dimensional problem, where all parameters are independent of the third orthogonal space-variable; the uv-plane is a cross-section.

$$\frac{\partial \phi}{\partial u} = \frac{\partial \psi}{\partial v}, \qquad \frac{\partial \phi}{\partial v} = -\frac{\partial \psi}{\partial u},$$

we find

$$\frac{d\chi}{dw} = \frac{\partial \phi}{\partial u} - i\frac{\partial \phi}{\partial v} = \overline{\operatorname{grad} \phi},$$

where the bar denotes the complex conjugate. Hence $\mathbf{E} = \overline{d\chi/dw}$. If the boundary is an equipotential, the surface charge density is

$$\rho = |\mathbf{D}| = \epsilon |\mathbf{E}| = \epsilon |d\chi/dw|,$$

where $\mathbf{D}$ is the electric displacement vector and ϵ is the dielectric constant of the medium surrounding the conductor. In the present case:

$$\begin{aligned} \chi &= -ix + y = -iz = -iw^{\pi/\alpha}, \\ d\chi/dw &= (-\pi i/\alpha)w^{\pi/\alpha-1}, \\ \rho &= \epsilon |d\chi/dw| = (\pi\epsilon/\alpha)(u^2 + v^2)^{\pi/(2\alpha)-1/2}. \end{aligned}$$

(Although $\chi = -iz$ in the present illustrative example, this is not true in general.)

It is clear from this example that if $\alpha > \pi$, *i.e.*, if the corner is convex, the charge density is unbounded in the neighborhood of the corner. In a practical case this leads to such phenomena as corona discharge.

In the case where the figure represents the cross-section of a uniform cylindrical (not necessarily circular) electrical transmission line with perfectly conducting boundaries, we can represent a solution of Maxwell's equations by

$$\mathbf{D}/\epsilon = \mathbf{E} = -\operatorname{grad} \phi e^{i\omega t - \gamma\zeta}, \ \mu\mathbf{H} = \mathbf{B} = \sqrt{(\mu\epsilon)} \operatorname{grad} \psi e^{i\omega t - \gamma\zeta},$$

where

$\mathbf{D}$ = electric displacement vector,
$\mathbf{E}$ = electric intensity vector,
$\mathbf{H}$ = magnetic intensity vector,
$\mathbf{B}$ = magnetic flux-density vector,
μ = permeability of the medium,
ϵ = permittivity of the medium (*i.e.*, absolute dielectric constant),
ω = frequency in radians per second,
$\gamma^2 = -\omega^2\mu\epsilon$,
ζ = a coordinate perpendicular to the section, *i.e.*, along the axis of the transmission line, and
t = time.

From the above we can deduce that the surface current density at the bounding surfaces is

$$|\mathbf{j}| = \sqrt{(\epsilon/\mu)} |\operatorname{grad} \phi| = (\pi/\alpha)\sqrt{(\epsilon/\mu)}(u^2 + v^2)^{\pi/(2\alpha)-1/2}.$$

Consider, for example, a region bounded by two straight lines which join at an angle α (Fig. 1)

$$w = u + iv = re^{i\theta}, \qquad r \geqq 0, \qquad 0 \leqq \theta \leqq \alpha \leqq 2\pi,$$

and let us seek a solution to the electrostatic potential problem $\phi(u, v)$ which is zero on the boundary.* The mapping function $z = w^{\pi/\alpha}$ maps the region in the w-plane into

$$z = (re^{i\theta})^{\pi/\alpha} = r^{\pi/\alpha}e^{i\pi\theta/\alpha} = \rho e^{i\psi}, \qquad \rho \geqq 0, \qquad 0 \leqq \psi \leqq \pi,$$

i.e., into the upper half plane. An example of a simple potential function which vanishes on the boundary is $\phi(x, y) = y$ which is the potential of a uniformly charged plane at $y = 0$.

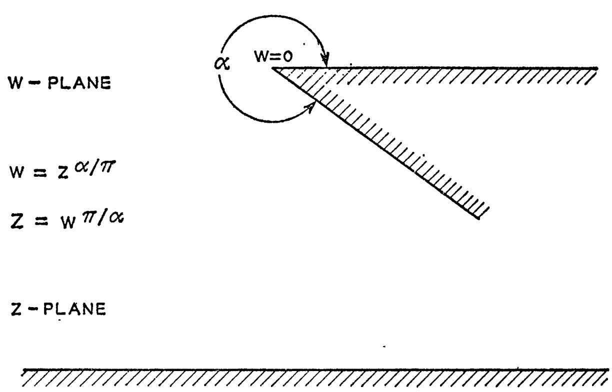

FIG. 1

In this case,

$$\phi(u, v) = y(u, v) = \operatorname{Im}(u + iv)^{\pi/\alpha} = (u^2 + v^2)^{\pi/2\alpha} \sin\left(\frac{\pi}{\alpha}\tan^{-1}\frac{v}{u}\right)$$

is a solution meeting the boundary conditions.

It is often convenient to retain the whole analytic function $\chi(w) = \phi(w) + i\psi(w)$, instead of only the real part $\phi(w)$. For example, if ϕ is an electrostatic potential and $\mathbf{E}$ the corresponding electric field vector, then $\mathbf{E} = -\operatorname{grad}\phi$. But

$$\frac{d\chi}{dw} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial w} + i\left[\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial w}\right].$$

Eliminating ψ by the Cauchy-Riemann equations,

* We are really imagining a three-dimensional problem, where all parameters are independent of the third orthogonal space-variable; the uv-plane is a cross-section.

$\pi\theta_1, \pi\theta_2, \cdots, \pi\theta_n$ (see Fig. 2). For simplicity, let all the angles be equal. Recalling that the angle between the semi-infinite straight sides is α, we find that $\pi\theta_1 + \cdots \pi\theta_n = n\pi\theta = \alpha - \pi$, $\theta = (\alpha - \pi)/n\pi$. Letting

$$P_n(z) = \prod_{j=1}^{n} (z - z_j),$$

we find

$$w = \int [P_n(z)]^{(\alpha/\pi - 1)/n} dz.$$

If as before $\chi(z) = z$, then

$$\frac{d\chi}{dw} = \frac{dz}{dw} = [P_n(z)]^{(1-\alpha/\pi)/n}.$$

The region on the real z-axis corresponding to the multiple corner is the region where the (real) zeros of $P_n(z)$ occur. Inasmuch as the exponent of P_n in the above equation is negative, the value of $|d\chi/dw|$ is unbounded near each of these zeros. However, we can still pursue the idea of making $|d\chi/dw|$ as nearly constant as possible, in some sense, and see where it leads.

In some intuitive sense the Tschebyscheff* polynomials are most nearly constant over an interval; for in an interval $(-1, 1)$ containing their zeros, the maximum absolute value of each is unity, and this is attained once between each pair of zeros. No other polynomials of the same degree and same mean-square magnitude have their extreme excursion confined to such narrow limits. Further amplification of the line of reasoning is unnecessary, for the result amply justifies the questionable logic. Following this heuristic notion, let

$$P_n(z) = T_n(z) = \tfrac{1}{2}[\{z + \sqrt{(z^2 + 1)}\}^n + \{z - \sqrt{(z^2 - 1)}\}^n].$$

The representation chosen is particularly convenient, for

$$\lim_{n\to\infty} [T_n(z)]^{1/n} = z + \sqrt{(z^2 - 1)}, \qquad \operatorname{Im} z > 0.$$

We can therefore look more carefully at the mapping function

$$w = \int^z [z + \sqrt{(z^2 - 1)}]^a dz \qquad (a = \alpha/\pi - 1)$$

$$= \begin{cases} \tfrac{1}{2}[z^2 - z\sqrt{(z^2 - 1)} + \log\{z - \sqrt{(z^2 - 1)}\}] & (a = -1), \\ \dfrac{[z + \sqrt{(z^2 - 1)}]^a[z - a\sqrt{(z^2 - 1)}]}{1 - a^2} & (-1 < a < 1), \\ \tfrac{1}{2}[z^2 + z\sqrt{(z^2 - 1)} - \log\{z + \sqrt{(z^2 - 1)}\}] & (a = +1). \end{cases}$$

* I am indebted to Bruce P. Bogert for suggesting the use of Tschebyscheff polynomials.

We choose the branch of $\sqrt{(z^2-1)}$ which is positive real for real $z>1$, negative real for real $z<-1$, and which has positive imaginary part for z in the upper half plane. It is clear from direct calculation that for $-1<z<+1$,

$$\left| d\chi/dw \right| = \left| dw/dz \right|^{-1} = \left[z^2 + \left\{\sqrt{(1-z^2)}\right\}^2\right]^{-\alpha/2} = 1.$$

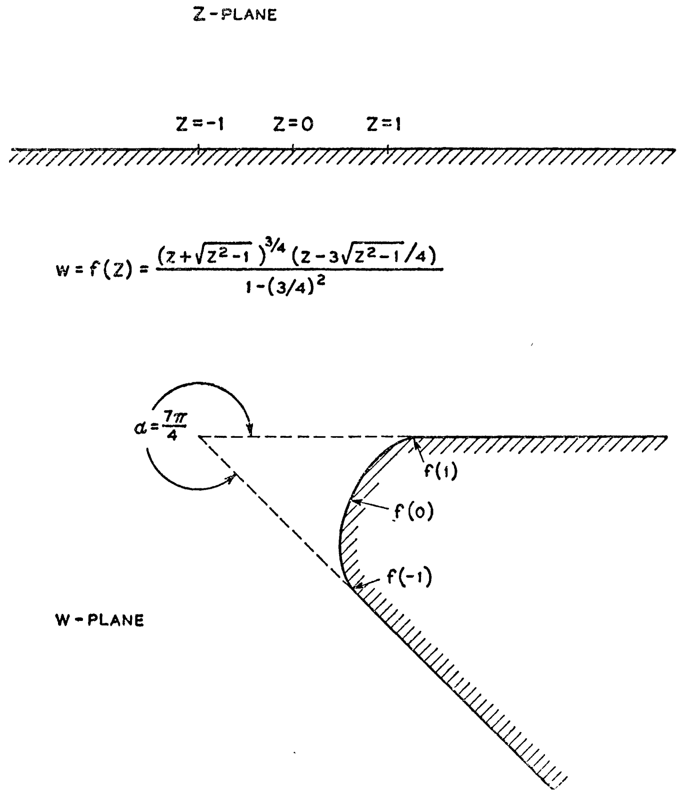

FIG. 3
AN EXAMPLE OF MAPPING FUNCTION FOR $\alpha=7\pi/4$

In fact, we can deduce that for the general case $-1<a<1$, *i.e.*, $0<\alpha<2\pi$,

$$1 < z < \infty \quad \text{(real } z\text{)} \quad \text{maps into } w = \rho e^{i0}, \quad (1-a^2)^{-1} < \rho < \infty;$$

$$-\infty < z < -1 \quad \text{(real } z\text{)} \quad \text{maps into } w = \rho e^{i\alpha}, \quad \infty > \rho > (1-a^2)^{-1};$$

$$-1 < z < 1 \quad \text{(real } z\text{)} \quad \text{maps into a hypocycloid}$$

$$w = \frac{-(1+a)\cos(1-a)\theta + (-1+a)\cos(1+a)\theta}{2(a^2-1)} + i\,\frac{(a-1)\sin(1+a)\theta + (a+1)\sin(1-a)\theta}{2(a^2-1)},$$

where $z=\cos\theta$. For $a=\pm 1$,

$$1 < z < \infty \quad (z \text{ real}) \quad \text{maps into } w = \rho e^{i0}, \quad \tfrac{1}{2} < \rho < \infty;$$
$$-\infty < z < -1 \; (z \text{ real}) \quad \text{maps into } w = e^{i0} \mp \tfrac{1}{2}\pi i, \quad \infty > \rho > \tfrac{1}{2};$$
$$-1 < z < 1 \quad (z \text{ real}) \quad \text{maps into a cycloid } w = \tfrac{1}{4}[1 + \cos 2\theta \mp i(2\theta - \sin 2\theta)].$$

An example of the map, for $\alpha = 7\pi/4$, $a=3/4$, is shown in Figure 3. In geometric terms, the result is always two straight lines emanating with no change in direction from the extremities of a single leaf of a cycloid or a hypocycloid.

4. Uniqueness of the mapping function. It is interesting to note that, except for a bilinear transformation in the z-plane which maps the upper half plane into itself, and translations and rotations in the w-plane, the mapping found above is unique. Inasmuch as the proof is less interesting than the fact, it will only be sketched here.

Let $w=f(z)$ be a mapping which maps the upper half plane into a region bounded by portions of the lines

$$w = re^0, \; r > r_1, \qquad w = re^{i\alpha}, \; r > r_2$$

and some arc joining them, such that

$$f(\infty) = \infty, \qquad f(1) = r_1, \qquad f(-1) = r_2 e^{i\alpha},$$

and

$$|df/dz| = k \qquad \text{for } -1 < z < 1.$$

Let

$$g(z) = \int^z \{z + \sqrt{(z^2 - 1)}\}^a dz$$

as above. Because straight lines are analytic arcs, f and g are both regular for real z, $1<|z|<\infty$, as well as for z with positive real part. Consideration of $g^{-1}[f(z)]$ in the neighborhood of $z=\infty$ shows that it must have a simple pole with real residue. Hence $h(z)=f'(z)/g'(z)$ is analytic in the upper half plane, at ∞, and on the real axis except for $-1<z<1$. It can be extended by analytic continuation, thus: $h(z), =\overline{h(\bar{z})}$ to include the whole plane save a cut on the real axis $-1<z<1$. Furthermore, since f is a mapping function, $f'(z)$ is never zero, and hence neither is $h(z)$. Hence by the maximum modulus theorem, $|h(z)|$ attains its maximum and minimum values on the boundary $-1<z<1$. But over this range

$$|h(z)| = |f'(z)/g'(z)| = k/1.$$

Hence $h(z)$ is a constant, and $f(z)$ is a linear function of $g(z)$, the function already determined.

For $a=\pm 1$, the above argument needs a trifling modification. In this case, the half plane is mapped by f or by g on to the inside or outside of a semi-infinite

parallel-sided strip, and for the above argument to work the widths of the strips must be the same.

5. Other properties of the mapping function. Continuation across the boundary. The boundary described above is by no means an ultimate boundary. It is in fact made up of three analytic arcs joined at three point singularities (including one at ∞). The mapping function can be extended across any one of the three arcs.

The extensions across the straight lines are not very interesting, for they lead to simple geometric reflections. However, the continuation across the cycloid does lead to an interesting result. Suppose, for example, we make a cut in the z-plane from $+1$ to $+\infty$ and from -1 to $-\infty$ along the real axis, and extend the mapping across $(-1, +1)$. Formally the integrations giving w as a function of z are still correct, provided we choose the appropriate branch of the square root. If we cross the axis and return to a real value to z, $z>1$, from below, clearly we take the other branch. But

$$z - \sqrt{(z^2 - 1)} = \{z + \sqrt{(z^2 - 1)}\}^{-1}.$$

Hence, the effect is to change a into $(-a)$ in the above representation. Following this argument through shows that the whole z-plane with the above two slits is mapped into the whole w-plane with two slits making an angle α (the slits being the straight portions of the boundary already found). Such a map will have applications in potential problems involving two sheets nearly joined at the edge, or fluid-flow problems from some orifices.

6. A general property of polynomial expansions. The above analysis is related in an interesting way to the theory of general polynomial expansions. Suppose we are given a schlicht, bounded, simply-connected domain B and a set of polynomials $P_n(z)$ such that every function $f(z)$ regular on B can be represented in an expansion of the form $f(z) = \sum b_n P_n(z)$ convergent at every point of B. Then, with further restrictions about the absolute scale of $P_n(z)$, the nature of the domain B, how and whether the expansion converges on the boundary of B, and so forth, Walsh and others [1], [2] have shown that in some sense

$$\lim_{n\to\infty} [P_n(z)]^{1/n} = g(z),$$

where $g(z)$ is the exterior mapping function which maps the exterior of B into the exterior of the unit circle. If B is the line segment from $(-1, 0)$ to $(+1, 0)$, then

$$\lim_{n\to\infty} [P_n(z)]^{1/n} = g(z) = z + \sqrt{(z^2 - 1)}. \tag{1}$$

For this case, Szegö [3] has proved the following:

Let $\{P_n(x)\}$, $n=1, 2, \cdots$, *be a set of polynomials of degree n respectively, orthonormal in* $(-1, 1)$ *with a weighting factor* $w(x)$ *such that*

$$\int_{-\pi}^{\pi} \left| \log \{\sin \theta \, w(\cos \theta)\} \right| d\theta$$

exists. Then for all x not in the real interval $(-1, 1)$,

$$P_n(x) \simeq \frac{[x + \sqrt{(x^2 - 1)}]^n}{\sqrt{(2\pi)} D[x - \sqrt{(x^2 - 1)}]},$$

where

$$D(u) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log [\sin \theta \, w(\cos \theta)] \frac{1 + ue^{i\theta}}{1 - ue^{i\theta}} d\theta \right\}, \tag{2}$$

and furthermore $D(u) \neq 0$ *for* $|u| < 1$, *and* $D(0)$ *is real and greater than zero. The branch of the square root is that which approaches x as* $|x| \to \infty$.

Obviously this implies (1) for the case at hand, and actually somewhat more. Hence in the heuristic argument aired earlier, the polynomials of Legendre, for example, might have been used instead of those of Tchebyscheff, with the same final result. The reader may satisfy himself, for example, by looking in the cited work of Szegö, that (1) is not trivial for Legendre or many other polynomial sets. In spite of the fact that many different choices of polynomials would suffice to find the desired conformal map, the heuristic choice of Tchebyscheff polynomials is upheld: they are the best choice because they make $D(u)$ [eq. (2)] constant.

7. Conclusions and applications. In a certain sense, the mapping described above answers the question, what is the best way to shape the corner of a conductor to relieve electrical stresses at sharp edges? It can be applied directly to the shaping of high-voltage bus bars and connected straps, to the shaping of supports located near high-voltage conductors and to the shaping of the edges of a conductor in a strip transmission line. It may also be applied to the design of contours of channels carrying fluid flow where reduction of turbulence may be desired, such as a bend in a canal or the joint between a conical and a cylindrical portion of a projectile or missile. In cases where the ideal conditions of the mathematical problem are not completely valid, the choice of a hypocycloidal curve to fair a corner may still give a better result than a circular arc, or may lead more easily to a suitable empirical solution.

I am indebted to H. O. Pollak, B. P. Bogert, and A. Pal for their counsel and assistance.

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THE ERDÖS INEQUALITY AND OTHER INEQUALITIES FOR A TRIANGLE

A. OPPENHEIM, University of Malaya

1. It is well known that, if P is a point in a triangle ABC whose distances are x, y, z from the vertices and p, q, r from the sides, a remarkable inequality connects the six numbers x, y, z, p, q, r:

$$x + y + z \geqq 2(p + q + r) \tag{1}$$

with equality if and only if the triangle is equilateral and P is its center. This inequality was found by Erdös* and proved by Mordell and Barrow [1].

I have obtained many other inequalities connecting these distances, *e.g.*,

$$px + qy + rz \geqq 2(qr + rp + pq), \tag{2}$$

$$yz + zx + xy \geqq 4(qr + rp + pq), \tag{3}$$

$$xyz \geqq 8pqr, \tag{4}$$

$$2^{-1}(p^{-1} + q^{-1} + r^{-1}) \geqq x^{-1} + y^{-1} + z^{-1}. \tag{5}$$

For an internal point P equality can only occur when the triangle is equilateral and P is its center. There is also equality at the vertices but not on the sides.

It will be seen that (1), (2), (5) are equivalent in the sense that each can be obtained from the others by using well-known geometrical transformations. I shall prove (2) directly and then use the following transformations: (i) isogonal conjugates, (ii) inversion, (iii) reciprocation. For simplicity I shall take P as the center for inversion and for reciprocation.

Before beginning the proofs I note that in *Mathematical Reviews*, vol. 21, 1960, p. 162 it is stated that A. Florian has proved (in my notation) the inequality

$$x^k + y^k + z^k \geqq 2^k(p^k + q^k + r^k) \tag{6}$$

for $|k| \leqq 1$ with equality as in (1).

That (6) holds for $0 < k < 1$ is certainly true (as I shall prove): for $k = 0$, (6) is an identity: for $-1 \leqq k < 0$, (6) is false as my inequality (5) indicates. In fact for $-1 \leqq k < 0$ the signs of inequality in (6) must be reversed. So too for the other inequality of Florian:

$$x^k + y^k + z^k > 2(p^k + q^k + r^k) \qquad (|k| > 1). \tag{7}$$

It is true for $k > 1$: for $k < -1$ the sign of inequality must be reversed.

2. Proof of (2) **and** (4). We begin with the elementary inequality

* Erdös first proposed his inequality as a problem in this MONTHLY in 1935. In 1937 two proofs were published, one by Mordell and one by Barrow. Barrow indeed proves a stronger inequality which I propose, in another note, to extend in a remarkably simple way.

$$ax \geqq bq + cr, \tag{8}$$

equality holding if and only if P is on the altitude from A. If h_1 is this altitude, then

$$a(x + p) \geqq ah_1 = 2\Delta = ap + bq + cr, \tag{9}$$

whence (8) follows at once.

From (8), $px \geqq (b/a)pq + (c/a)pr$ with equality if and only if P is on the altitude from A or if P is on BC. Hence

$$\sum px \geqq \sum \left(\frac{b}{c} + \frac{c}{b}\right) qr \tag{10}$$

with equality if and only if P is a vertex or if (the triangle being acute-angled) P is the orthocenter.

But $b/c + c/b \geqq 2$ by the theorem of the means so that (10) yields

$$px + qy + rz \geqq 2(qr + rp + pq) \tag{11}$$

for any point P in a triangle ABC. Equality holds when P is a vertex or when the triangle is equilateral and P is its center. This proves (2).

To prove (4) apply the inequality of the means directly to (8). Then $ax \geqq 2(bcqr)^{\frac{1}{2}}$ and two similar inequalities. Multiplication yields

$$xyz \geqq 8pqr, \tag{12}$$

with equality as in (2). This proves (4).

3. I now prove the inequality

$$(px)^k + (qy)^k + (rz)^k \geqq 2^k((qr)^k + (rp)^k + (pq)^k) \tag{13}$$

for $0 < k < 1$, equality occurring as for (11). To prove (13), I apply to (8) the well-known elementary inequality

$$\left(\frac{u + v}{2}\right)^k \geqq \frac{u^k + v^k}{2} \qquad (u > 0, v > 0, 0 < k < 1), \tag{14}$$

equality occurring if and only if $u = v$. Thus (8) yields

$$(px)^k \geqq 2^{k-1}\{(b/a)^k(pq)^k + (c/a)^k(pr)^k\}, \tag{15}$$

whence

$$\sum (px)^k \geqq 2^{k-1} \sum \left\{\left(\frac{b}{c}\right)^k + \left(\frac{c}{b}\right)^k\right\} (qr)^k, \tag{16}$$

with equality as for (10). From (16) by the argument which led from (10) to (11) we obtain (13).

4. Isogonal conjugates. If P and P' are such that AP, AP' are equally in-

clined to the bisectors of A; BP, BP' equi-inclined to the bisectors of B; CP, CP' to those of the angle C, then P and P' are isogonal conjugates. If accented letters refer to P', then

$$pp' = qq' = rr', \qquad p'\sum aqr = 2\Delta qr, \qquad p\sum aq'r' = 2\Delta q'r'. \tag{17}$$

$$x'\sum aqr = 2\Delta px, \qquad x\sum aq'r' = 2\Delta p'x'. \tag{18}$$

The proof is simple. In trilinear coordinates the isogonal property yields

$$\frac{p'}{1/p} = \frac{q'}{1/q} = \frac{r'}{1/r} = \frac{2\Delta}{\sum a/p} = \frac{2\Delta pqr}{\sum aqr}$$

whence (17) follows. As for (18) it is enough to note that

$$x \sin A = (q^2 + r^2 + 2qr \cos A)^{1/2},$$
$$x' \sin A = (q'^2 + r'^2 + 2q'r' \cos A)^{1/2} = (r^2 + q^2 + 2rq \cos A)^{1/2} 2\Delta pqr/qr\sum aqr,$$

so that $x'\sum aqr = 2\Delta px$, which is the first equation in (18).

It will be observed that if P is an *internal* point of the triangle so also is P' and vice-versa. Hence *any homogeneous inequality (or equality) connecting x, y, z, p, q, r for any internal point P leads to another homogeneous inequality (or equality) merely by the substitution*

$$S: \quad (x, y, z; p, q, r) \to (px, qy, rz; qr, rp, pq). \tag{19}$$

Thus (11) yields $\sum qrpx \geqq 2\sum rppq$ or $\sum x \geqq 2\sum p$, the Erdös inequality: and conversely the Erdös inequality yields (11). But (12) is clearly self-dual for this transformation.

If we apply S to (13) it yields the inequality

$$\sum x^k \geqq 2^k \sum p^k \qquad (0 < k < 1) \tag{20}$$

given by Florian. (My knowledge of Florian's result derives from *Mathematical Reviews* cited above: his paper is not accessible to me.)

5. Inversion. I consider next inversion and, for simplicity, inversion with respect to the internal point P itself. Here we obtain a new triad A', B', C' corresponding to A, B, C: P is internal to the triangle $A'B'C'$. Using accents to denote the distances of P relating to the triangle $A'B'C'$ it is clear that

$$\begin{aligned} xx' &= K^2, & yy' &= K^2, & zz' &= K^2 \\ p'yz &= K^2p, & q'zx &= K^2q, & r'xy &= K^2r \end{aligned} \tag{21}$$

where K is the radius of inversion.

(For $p' \cdot B'C' = 2\Delta B'PC' = PB' \cdot PC' \cdot B'C'/$(diameter of circle $B'PC'$): diameter $= K^2/p$.)

It follows therefore that *inequalities (or equalities) which involve x, y, z, p, q, r homogeneously remain true under the substitution denoted by*

$$V:\quad (x, y, z; p, q, r) \to (yz, zx, xy; px, qy, rz). \tag{22}$$

Apply V to the Erdös inequality: we obtain the new inequality

$$yz + zx + xy \geqq 2(px + qy + rz) \tag{23}$$

which, combined with (11), yields another inequality

$$yz + zx + xy \geqq 4(qr + rp + pq). \tag{24}$$

Thus (1), (24), (12) show that the three elementary symmetric functions of x, y, z are at least equal to the corresponding elementary symmetric functions of $2p$, $2q$, $2r$.

V applied to (11) gives rise to a new inequality

$$xyz(p + q + r) \geqq 2\sum qryz \tag{25}$$

or $2^{-1}\sum q^{-1}r^{-1} \geqq \sum p^{-1}x^{-1}$.

Apply S to (25): we obtain (after obvious reduction)

$$p^{-1} + q^{-1} + r^{-1} \geqq 2(x^{-1} + y^{-1} + z^{-1}) \tag{26}$$

which is enough to show that (6), so far as $k = -1$ is concerned, must have the sign of inequality reversed.

6. Reciprocation. I consider next reciprocation and again for simplicity reciprocation with respect to P. We obtain a triangle $A''B''C''$; P is still an internal point. The distances of P from the sides of $A''B''C''$ are inversely proportional to its distances from the vertices of ABC: its distances from A'', B'', C'' are inversely proportional to its distances from the sides of ABC.

It follows that *inequalities (or equalities) involving* x, y, z, p, q, r *homogeneously remain valid under the substitution*

$$R:\quad (x, y, z, p, q, r) \to (p^{-1}, q^{-1}, r^{-1}; x^{-1}, y^{-1}, z^{-1}). \tag{27}$$

As an example apply R to the Erdös inequality: we obtain (26) at once.

As another example apply R to (20): we obtain $\sum p^{-k} \geqq 2^k \sum x^{-k}$ or

$$2^{-k}\sum p^{-k} \geqq \sum x^{-k} \qquad (0 < k < 1) \tag{28}$$

which shows that Florian's inequality (6) needs reversal of sign of inequality when the k of (6) is negative.

7. Corresponding to any one inequality we obtain a set of six inequalities (which may coincide) by applying these transformations. The set which arises from the Erdös inequality is the following:

$$\begin{array}{ll} \sum x \geqq 2\sum p, & \sum px \geqq 2\sum qr, \\ \sum yz \geqq 2\sum px, & \sum 1/p \geqq 2\sum 1/x, \\ \sum 1/qr \geqq 2\sum 1/px, & \sum 1/px \geqq 2\sum 1/yz. \end{array}$$

On the other hand the inequality $xyz \geqq 8pqr$ gives rise to no new inequality, while, in contrast, the inequality

$$xyz \geqq (q+r)(r+p)(p+q) \tag{29}$$

(not proved here) gives rise to a set of three inequalities.

8. Other inequalities can be obtained by applying the transformations to such inequalities as $ax \geqq bq+cr$ (if we make the necessary changes for a, b, c) or by inverting or reciprocating with respect to points other than P.

Two invariants of the group of transformations may be noted:

$$\text{(i) } J = pqr/xyz, \qquad \text{(ii) } H = a/p + b/q + c/r. \tag{30}$$

9. I state also a surprising theorem I have found about a triangle G and a point P in its plane (not on its sides).

Let G_1 be the triangle formed by the feet of the perpendiculars from P on the sides of the triangle G. Form G_2 from G_1 and P in the same way. Likewise construct G_3 from G_2 and P. Then the triangle G_3 so constructed is necessarily similar to the original triangle G.

This theorem, which arises incidentally from the group of transformations considered, extends also to plane polygons: the nth polygon constructed being similar to the original polygon.

10. The inequality (2) extends at once to n dimensions: by the same proof as in Section 2 I find that

$$\sum_{i=1}^{n+1} p_i x_i \geqq 2 \sum_{1 \leqq i < j \leqq n+1} p_i p_j; \tag{31}$$

equality occurs at a vertex or, if the "tetrahedron" is regular, at the center: there is no other case of equality.

Corresponding to (4) is the inequality

$$x_1 x_2 \cdots x_{n+1} \geqq n^{n+1} p_1 p_2 \cdots p_{n+1}. \tag{32}$$

Other inequalities, not very easy to write down, arise by inversion and reciprocation.

11. I am indebted to Dr. Diananda and Dr. Guha for pointing out to me that (3), (4), (5) and also reciprocation are to be found in Fejes-Tóth [2]. To Professor Pedoe I express my thanks for helpful criticism.

I conclude by stating without proof an inequality much stronger than (3):

$$\sum yz \geqq \sum (p+q)(p+r). \tag{33}$$

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A DEVELOPMENT OF A SERIES STUDIED BY H. W. GOULD

M. T. L. BIZLEY, Bacon and Woodrow, London, AND A. W. JOSEPH,
Wesleyan and General Assurance Society, Birmingham, England

1. **Introduction.** The purpose of this note is to draw attention to certain properties of the series

$$(1) \qquad \beta_t \equiv \beta_{t,p}(x) \equiv 1 + \binom{t+p}{1}x + \cdots + \binom{t+rp}{r}x^r + \cdots,$$

and of allied series, and to develop some of their consequences. The analysis will lead to an explicit solution of all the $m+n$ roots of $x(1+\theta)^{m+n}-\theta^n=0$. The series $\int_0^x \beta_p dx$, which will be denoted by ϕ, or $\phi_p(x)$, plays an important part in the theory. Thus

$$(2) \qquad \phi \equiv \phi_p(x) \equiv \sum_{j=1}^{\infty} \frac{1}{jp-j+1}\binom{jp}{j}x^j = \sum_{j=1}^{\infty} \frac{1}{j}\binom{jp}{j-1}x^j$$

$$(3) \qquad \phi' = \beta_p.$$

2. **A property of β_t and some consequences.** The series (1) and (2) are of the type studied by H. W. Gould ([3], [4]), who has shown ([3], eqs. (7), (9)) that in our notation, and for all real values of t:

$$(4) \qquad \beta_t = \frac{(1+\phi)^{t+1}}{1+\phi-p\phi},$$

$$(5) \qquad x(1+\phi)^p = \phi,$$

$$(6) \qquad (1+\phi)^t = \sum_{j=0}^{\infty}\binom{t+pj}{j}\frac{t}{t+pj}x^j,$$

where $|x| < |(p-1)^{p-1}/p^p|$.

Writing $t=0$ in (4) we have

$$(7) \qquad \beta_0 = \frac{1+\phi}{1+\phi-p\phi}$$

and hence,

$$(8) \qquad \beta_t = (1+\phi)^t\beta_0.$$

Thus β_t is a geometric progression, for successive values of t, the common ratio $1+\phi$ being closely allied to a particular term in the progression, *viz.*, β_p, by (3). From (8), with $t=p-1$, we have

$$\beta_{p-1} = \frac{\beta_p}{1+\phi} = \frac{\phi'}{1+\phi}.$$

Integrating, we obtain

$$F \equiv F_p(x) \equiv \sum_{j=1}^{\infty} \frac{1}{j}\binom{jp-1}{j-1} x^j = \log(1+\phi)$$

since both F and ϕ vanish when $x=0$; *i.e.*,

$$e^F = 1 + \phi. \tag{9}$$

Similarly from (8) with $t=p-2$, we have

$$\beta_{p-2} = \frac{\beta_{p-1}}{1+\phi} = \frac{\phi'}{(1+\phi)^2}.$$

Integrating, we obtain

$$\psi \equiv \psi_p(x) \equiv \sum_{j=1}^{\infty} \frac{1}{j}\binom{jp-2}{j-1} x^j = \frac{-1}{1+\phi} + 1$$

since both ψ and ϕ vanish when $x=0$; hence

$$(1+\phi)(1-\psi) = 1, \tag{10}$$

$$e^{-F} = 1 - \psi, \tag{11}$$

and from (5) and (10),

$$x = \psi(1-\psi)^{p-1}. \tag{12}$$

Also from (7) and (10),

$$\beta_0 = \frac{1+\phi}{1-(p-1)\phi} = \frac{1}{1-p\psi}. \tag{13}$$

These relations were deduced by Bizley [1] as a special case of a more general investigation, by combinatorial arguments based upon the geometrical properties of lattice paths.

3. Generating function of β_t $(t \geqq 0)$. Define

$$\begin{aligned}
[\beta]_t &= \beta_t && (t \geqq 0),\\
&= \beta_t - 1 && (-p+1 \leqq t < 0),\\
&= \beta_t - 1 - \binom{t+p}{1} x && (-2p+2 \leqq t < -p+1),\\
&= \beta_t - 1 - \binom{t+p}{1} x - \binom{t+2p}{2} x^2 && (-3p+3 \leqq t < -2p+2),
\end{aligned}$$

and so on. Thus $[\beta]_t$ is that part of β_t which starts with the first term $\binom{t+rp}{r}x^r$, where $t+rp \geqq r$.

Consider $s+x/s^{p-1}$ for positive x,s. This takes the value ∞ when $s=0$, drops

to a minimum when $s^p = (p-1)x$ and rises again to ∞ when $s=\infty$. If $x<(p-1)^{p-1}/p^p$ it will be found that the minimum value of $(s+x/s^{p-1})$ falls below unity, and thus there are then two and only two real positive roots of the equation $s+x/s^{p-1}=1$ or $s^p-s^{p-1}+x=0$. Denote them by α_1 and $\alpha_2(\alpha_2<\alpha_1)$. (From (12), one of these roots is $(1-\psi)$ and it will be seen later that, in fact, $1-\psi=\alpha_1$).

Now it is easy to verify that, provided $s+x/s^{p-1}<1$,

$$E \equiv 1/\{1-(s+x/s^{p-1})\} = \sum_{t=-\infty}^{\infty} [\beta]_t s^t. \tag{14}$$

Alternatively $E=-s^{p-1}/(s^p-s^{p-1}+x)=\sum_{r=1}^{p} A_r/(1-\gamma_r s)$, where A_r is a constant and $\gamma_1, \cdots, \gamma_p$ are the roots of $x\theta^p-\theta+1=0$ and where, to fix ideas, we will take $\gamma_1=1/\alpha_1$, $\gamma_2=1/\alpha_2$ (whence $\gamma_1<\gamma_2$).

If $|\gamma_r|s<1$ for $\alpha_2<s<\alpha_1$ then $A_r/(1-\gamma_r s)$ must be expanded as $A_r(1+\gamma_r s+\gamma_r^2 s^2+\cdots)$. If $|\gamma_r|s>1$ for $\alpha_2<s<\alpha_1$ then $A_r/(1-\gamma_r s)$ must be expanded as

$$-A_r\left(\frac{1}{\gamma_r s}+\frac{1}{\gamma_r^2 s^2}+\cdots\right).$$

We can place γ_1 and γ_2 into the proper categories because $\gamma_1 s=s/\alpha_1<1$ for $\alpha_2<s<\alpha_1$ and $\gamma_2 s=s/\alpha_2>1$ for $\alpha_2<s<\alpha_1$. We can also place the roots γ_r, $r\neq 1, 2$, into the proper categories since from (14) and the fact that $[\beta]_t=\beta_t=\beta_0(1+\phi)^t$ for $t\geqq 0$, there can be only one root γ of $x\theta^p-\theta+1=0$ such that $|\gamma|s<1$ for $\alpha_2<s<\alpha_1$, and that root γ_1 is $1+\phi$.

It should be noticed that the condition $|x|<(p-1)^{p-1}/p^p$, which allows E to be expanded, is also the condition that the series β_t and ϕ should be convergent.

4. **Roots of** $x(1+\theta)^{m+n}-\theta^n=0$. Let $p=(m+n)/n$, where m and n are positive integers, and let ω be an nth primitive root of unity. Writing

$$\nu_r = \phi_p(\omega^r x^{1/n}) = \sum_{j=1}^{\infty} \frac{1}{jp+1}\binom{jp+1}{j}\omega^{rj}x^{j/n}, \tag{15}$$

we have from (5), $\omega^r x^{1/n}(1+\nu_r)^p=\nu_r$ and hence $x(1+\nu_r)^{m+n}=\nu_r^n$. Similarly $x(1+\mu_r)^{m+n}=\mu_r^m$, where, with q denoting $(m+n)/m$,

$$\mu_r = \phi_q(\omega'^r x^{1/m}) = \sum_{j=1}^{\infty} \frac{1}{jq+1}\binom{jq+1}{j}\omega'^{rj}x^{j/m} \tag{16}$$

(ω' being an mth primitive root of unity).

Therefore $x[1+(1/\mu_r)]^{m+n}=(1/\mu_r)^n$. Thus each of the quantities $\nu_0, \nu_1, \cdots, \nu_{n-1}, 1/\mu_0, 1/\mu_1, \cdots, 1/\mu_{m-1}$ satisfies the equation $x(1+\theta)^{m+n}-\theta^n=0$. Since it is clear from (15) and (16) that these quantities are all distinct, it follows that they are the $m+n$ roots of the equation.

As an example, in the case of the quintic $x(1+\theta)^5-\theta=0$, the roots are:

$$\nu_0 = \phi_5(x) = x + \frac{10}{1.2}x^2 + \frac{15.14}{1.2.3}x^3 + \frac{20.19.18}{1.2.3.4}x^4 + \cdots,$$

$$1/\mu_0 = [\phi_{5/4}(x^{1/4})]^{-1} = x^{-1/4}[1 + \phi_{5/4}(x^{1/4})]^{-5/4},$$

which from (6) is equal to $x^{-1/4}\Phi_{5/4}(x^{1/4})$, where

$$\Phi_{5/4}(x^{1/4}) = 1 - \frac{5}{4}x^{1/4} - \frac{1}{2}\cdot\frac{5}{4}\cdot\frac{1}{4}x^{1/2} - \frac{1}{3}\cdot\frac{5}{4}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}}{1.2}x^{3/4} - \cdots,$$

and

$$1/\mu_1 = [\phi_{5/4}(ix^{1/4})]^{-1} = -ix^{-1/4}\Phi_{5/4}(ix^{1/4}),$$
$$1/\mu_2 = [\phi_{5/4}(-x^{1/4})]^{-1} = -x^{-1/4}\Phi_{5/4}(-x^{1/4}),$$
$$1/\mu_3 = [\phi_{5/4}(-ix^{1/4})]^{-1} = ix^{-1/4}\Phi_{5/4}(-ix^{1/4}).$$

The trinomial equation $z^{\lambda+\mu}+Az^{\mu}+AB=0$ has been solved by Frame [2] in a form which has similarities to our solution of the equation $x(1+\theta)^{m+n}-\theta^n=0$. However, in general our equation is not transformable into Frame's form, although if $n=1$ it becomes (with $1+\theta=z$)

$$z^{m+1} - \frac{1}{x}z + \frac{1}{x} = 0.$$

This is Frame's form with $\lambda=m$, $\mu=1$, $A=-1/x$, $B=-1$.

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A METHOD FOR COMPUTING THE REAL ROOTS OF DETERMINANTAL EQUATIONS

JAMES LUCIEN HOWLAND, University of Ottawa, Ottawa, Canada

1. The determinantal equations to be considered are of the form

$$\det(A - \lambda B) = 0, \tag{1}$$

where A, B are real, symmetric, $n \times n$ matrices. These equations arise in the study of undamped small vibrations of a mechanical system about a position of stable equilibrium ([6] Chs. VII, VIII). In this application, A and B are both positive definite. It is shown below that general polynomial equations with real coefficients may be cast in this form, with matrices A and B which are not necessarily definite.

2. Consider first the case in which the matrix B is definite and the roots of (1) are, therefore, necessarily real. For real values of λ the matrix

$$K(\lambda) = A - \lambda B \tag{2}$$

is symmetric. The characteristic value-vector problem for (2) defines characteristic values μ_i and normalized characteristic vectors y_i for which

$$Ky_i = \mu_i y_i \tag{3}$$

and it follows at once that

$$\det(A - \lambda B) = \pi\mu_i, \tag{4}$$

so that λ is a root of (1) when, and only when, one of the functions $\mu_i(\lambda)$ defined by the problem (3) is zero. Moreover, for any real value of λ, the μ_i and the associated vectors y_i may be computed by the Jacobi method of iterated rotations [3][4]. The n roots of (1) are obtained by solving the n equations

$$\mu_i(\lambda) = 0. \tag{5}$$

The Newton-Raphson iteration ([2] Sec. 3.312) may be applied to solve these equations. This application is based upon the formula

$$d\mu_i/d\lambda = -(y_i B y_i) \tag{6}$$

due to Jacobi [4] (see also [1], p. 283) and leads to the iteration

$$\lambda_{k+1} = \phi(\lambda_k), \tag{7}$$

where

$$\phi(\lambda) = (y_i A y_i)/(y_i B y_i). \tag{8}$$

Experiments with a number of cases for which $n \leqq 10$ indicate that, with $\lambda_1 = 0$, convergence of the iteration (7) to ten places is ordinarily obtained in three steps. The amount of computation involved is formidable, however, as the com-

plete solution requires approximately $3n$ applications of the diagonalization procedure.

If $U(\lambda)$ is the orthogonal matrix which diagonalizes $K(\lambda)$, the numerator and denominator of (8) are corresponding diagonal elements of the matrices $U^{-1}AU$ and $U^{-1}BU$ respectively. Thus, during the diagonalization procedure, it is convenient to transform A and B separately and obtain the elements of (8) as by-products of this procedure.

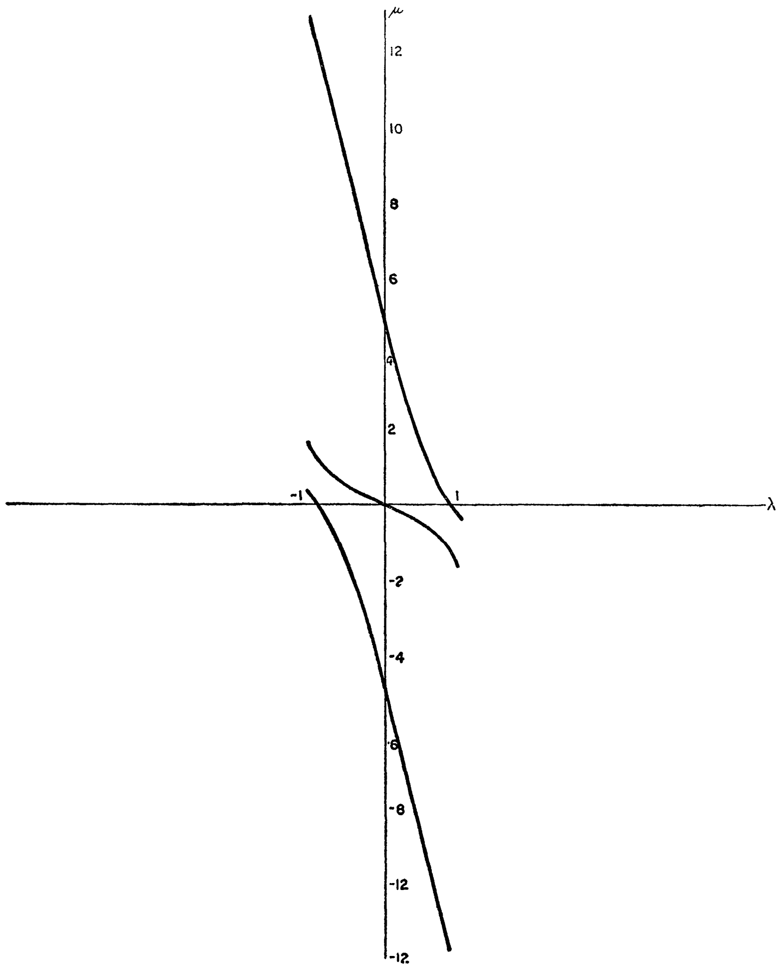

FIG. 1

Formula (6) shows that the functions $\mu_i(\lambda)$ are monotonic, and hence, admit exactly one root apiece. Further information concerning the functions $\mu_i(\lambda)$ may be obtained from the formulae for their second derivatives, which may be written (see [1], *loc. cit.*)

$$\frac{d^2\mu_i}{d\lambda^2} = -2\sum_{k\neq i}\frac{q_{ik}^2}{\mu_i-\mu_k}, \quad q_{ik} = \frac{(y_i(BA-AB)y_k)}{\mu_i-\mu_k}.$$

As one might expect, the $\mu_i(\lambda)$ are linear functions of λ, and the y_i are independent of λ, whenever A and B commute.

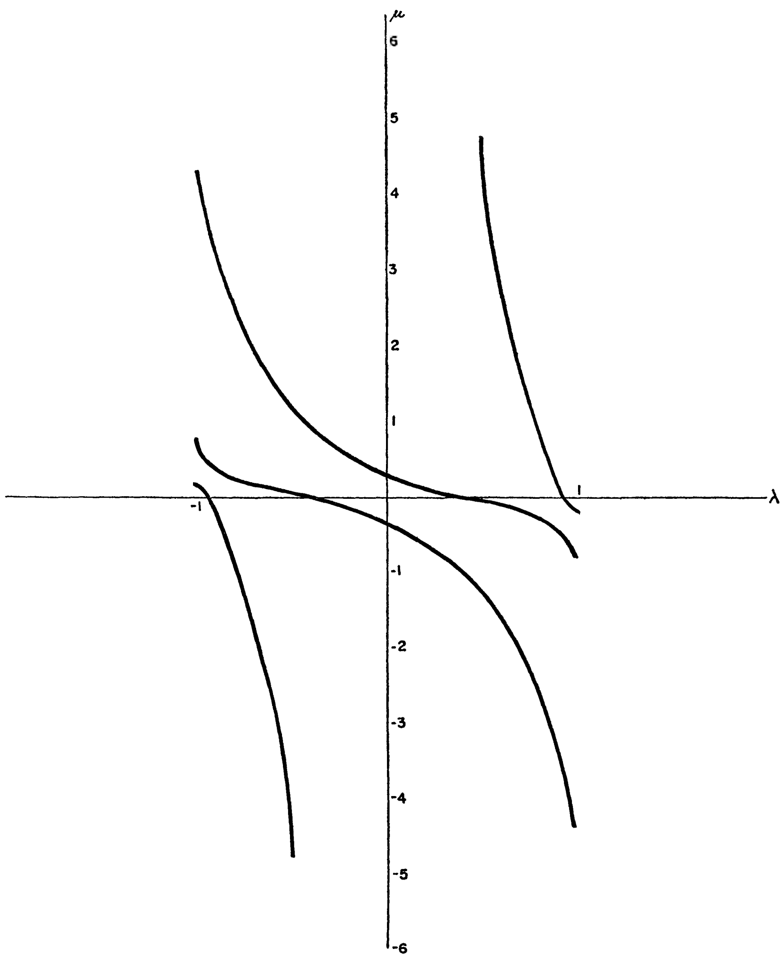

FIG. 2

Figures 1 and 2 are sketches of the functions μ_i obtained by writing the Chebyshev polynomials of the first kind, of degrees three and four, respectively, in the form $T_n(x)\alpha\ \det(A-xB)$, where B is positive definite (see [5], p. 26). In these cases the elements of $AB-BA$ are somewhat larger than those of either A or B, and the plots indicate the corresponding nonlinear character of these functions.

3. The procedure by which an arbitrary polynomial with real coefficients may be written in the form (1) is illustrated by the case of the polynomial

$$P_4(\lambda) = \lambda^4 - a_1\lambda^3 - a_2\lambda^2 - a_3\lambda - a_4 = 0 \qquad (a_4 \neq 0). \tag{9}$$

The companion matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which has the property that $\det(M-\lambda I) \equiv P_4(\lambda)$, may be symmetrized by premultiplication with the nonsingular symmetric matrix

$$B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & 0 \\ a_3 & a_4 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

For, in this case

$$A = BM = \begin{pmatrix} a_1^2 + a_2 & a_1a_2 + a_3 & a_1a_3 + a_4 & a_1a_4 \\ a_1a_2 + a_3 & a_2^2 + a_4 & a_2a_3 & a_2a_4 \\ a_1a_3 + a_4 & a_2a_3 & a_3^2 & a_3a_4 \\ a_1a_4 & a_2a_4 & a_3a_4 & a_4^2 \end{pmatrix} \tag{11}$$

is clearly symmetric, so that, with these definitions of A and B, the equation (9) is written in the form (1).

4. Formulas (10) and (11) make possible the construction of numerical examples in which neither A nor B is definite. Indeed, any polynomial with a pair of complex conjugate roots must give rise to precisely this situation. The case of the quartic $\lambda^4 - 1 = 0$ has been studied both analytically and numerically. In this case

$$AB - BA = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

and $\det(K - \mu I) \equiv (\mu - 1 + \lambda)(\mu - 1 - \lambda)(\mu^2 - \lambda^2 - 1)$ so that the loci of the characteristic values μ_i consist in two straight lines $\mu = 1 - \lambda$, $\mu = 1 + \lambda$, together with the hyperbola $\mu^2 - \lambda^2 - 1 = 0$.

Numerical results obtained in this case give values of the μ_i corresponding to the two lines and the two branches of the hyperbola. Convergence to the roots ± 1 was obtained in two iterations, as was expected. In the case of the hyperbola, convergence to the minimum (maximum) point was obtained in three or four iterations, where the denominator of (8) vanished, bringing the iteration to a halt. Thus the procedure would appear to be effective in evaluating the real roots of a polynomial, regardless of the existence of complex roots.

5. More efficient methods than that proposed are available for the solution of determinantal equations in the case in which B, or A, is definite. Thus the proposed method, while believed to be new, is of theoretical interest only, at least in this case. The method may find some practical application, however, as a stable, accurate process for evaluating the real roots of polynomial equations.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

ON THE IDENTITIES OF DIRECT PRODUCTS OF CERTAIN ALGEBRAS

ADIL YAQUB,* Purdue University

1. Introduction. In a recent paper [2], the author proved that every strictly functionally complete algebra $\mathfrak{U}$ of more than one element has the property that all the strict identities of $\mathfrak{U}$ are consequences of a certain *finite* subset of these identities. In this present note, we generalize this result to a direct product $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ of the strictly functionally complete algebras $\{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ where $\{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ form an independent set of algebras. Indeed, we shall prove the following

* Now at the University of California, Santa Barbara.

THEOREM. *Let* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$ *be an independent set of strictly functionally complete algebras, and let the order of* $\mathfrak{U}_i = M_i \geqq 2$ $(i=1, \cdots, n)$. *Let* $\mathfrak{U} = \mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ *be the direct product of* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$. *Then there exists a finite set I of strict identities of* $\mathfrak{U}$ *with the property that every strict identity of* $\mathfrak{U}$ *is a consequence of the identities in I.*

2. Preliminaries. In preparation for the proof of the above theorem, we shall first recall the basic concepts involved (see [1], [2]).

Let $\mathfrak{U} = (U, p, \cdots)$ be a universal algebra with primitive operations, $p, \cdots$. Let $U = \{\cdots, x, \cdots\}$.

A (k-ary) $\mathfrak{U}$-*function* $f(x_1, \cdots, x_k)$ is a composition, via the primitive operations, of indeterminate symbols $x_1, \cdots, x_k$ over the set U together with a (possibly empty) set of constants† (=fixed$\in U$).

An $\mathfrak{U}$-function is called a *strict* $\mathfrak{U}$-*function* if it involves no constants. If $\mathfrak{U}$-functions $f(x, \cdots)$ and $g(x, \cdots)$ represent the same mapping we speak of an $\mathfrak{U}$-*identity* $f(x, \cdots) = g(x, \cdots)$. If both f and g are strict $\mathfrak{U}$-functions we speak of a *strict* $\mathfrak{U}$-*identity*. $\mathfrak{U}$ is *finite*, of order n, if U is a class of n elements. $\mathfrak{U}$ is said to be (*functionally*) *complete*—respectively (*functionally*) *strictly complete*—if U is finite and if each mapping of the set $U \times \cdots \times U$ into U may be expressed as some $\mathfrak{U}$-function—respectively as some *strict* $\mathfrak{U}$-*function.*

We now proceed to define the concept of *independence* (see [1]). Let Sp $= (n_1, n_2, \cdots)$ be a given finitary species, and let $o_1, o_2, \cdots$ denote the primitive operation symbols of this species, where o_i is n_i-ary, $o_i = o_i(x_1, x_2, \cdots, x_{n_i})$. An *expression* $\phi(x, \cdots)$ of species Sp is one or more indeterminate symbols $x, \cdots$ composed via the operation symbols o_i.

In the various universal algebras $\mathfrak{U}_1, \mathfrak{U}_2, \cdots$ of species Sp we shall usually use the same symbols $o_i, \cdots$ to denote the respective primitive operations, *e.g.*, $\mathfrak{U}_1 = (U_1, o_1, o_2, \cdots)$, $\mathfrak{U}_2 = (U_2, o_1, o_2, \cdots)$, etc. Thus, in each $\mathfrak{U} = (U, o_1, o_2, \cdots)$, every expression $\phi(x, \cdots)$ becomes a particular $\mathfrak{U}$-function if we let $x, \cdots$ range over U. We denote this $\mathfrak{U}$-function by $\phi(x, \cdots)(\text{mod } \mathfrak{U})$, or $\phi(x, \cdots)(\mathfrak{U})$. Similarly, $\phi_1 = \phi_2(\text{mod } \mathfrak{U})$, also written $\phi_1 \equiv \phi_2(\text{mod } \mathfrak{U})$, denotes that the above is an identity of the algebra $\mathfrak{U}$, when the expressions ϕ_1, ϕ_2 are "interpreted" mod $\mathfrak{U}$.

Let $\{\mathfrak{U}_i\} = \{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ be a finite set of algebras of the same species Sp. We shall say that $\{\mathfrak{U}_i\}$ satisfies the *Chinese residue condition* or $\{\mathfrak{U}_i\}$ is an *independent* set of algebras if, corresponding to each set of expressions $\phi_1, \cdots, \phi_n$, there exists an expression E such that $E = \phi_i$ (mod $\mathfrak{U}_i$), for $i = 1, \cdots, n$.

For examples and criteria of independence see [1].

3. The main theorem. For the proof of the main theorem (see introduction), we need the following result which was proved in [1] and which we shall state as a lemma.

LEMMA 1. *Suppose* $\tilde{\mathfrak{U}} = \{\mathfrak{U}_i\} = \{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ *is a set of algebras* $\mathfrak{U}_i$, *each of*

† A constant may also be defined as a "0-ary" function.

which is strictly functionally complete and of order >1, *and where the algebras* $\mathfrak{U}_i$ *are of the same species* Sp. *Suppose that* $\{\mathfrak{U}_1, \cdots, \mathfrak{U}_n\}$ *form an independent set of algebras. Suppose that* $\overline{\mathfrak{U}}$ *is an algebra of species* Sp. *A sufficient* (*and necessary*) *condition for* $\overline{\mathfrak{U}}$ *to be subdirectly representable in* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *is that* $\overline{\mathfrak{U}}$ *satisfies all strict identities which are common to the algebras* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *or, what is the same, that* $\overline{\mathfrak{U}}$ *satisfies all strict identities of the algebra* $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ (*direct product*).

This is essentially a reformulation of Theorem 6.1 (A) of [1].

Although in the above lemma we assumed that *each* strict identity of $\mathfrak{U}_1 \times \mathfrak{U}_2 \times \cdots \times \mathfrak{U}_n$ is also an identity of $\overline{\mathfrak{U}}$, in the proof of this lemma only a *finite* number of strict $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$-identities were assumed to be $\overline{\mathfrak{U}}$-identities (see [1]). Let this *finite* set of strict $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$-identities be denoted by I. It is now easily seen that the above lemma can be strengthened as follows:

LEMMA 1′. *Suppose* $\tilde{\mathfrak{U}}$; $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$; $\overline{\mathfrak{U}}$, *are as in Lemma* 1. *A sufficient* (*and necessary*) *condition for* $\overline{\mathfrak{U}}$ *to be subdirectly representable in* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *is that* $\overline{\mathfrak{U}}$ *satisfies all strict identities of* $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ *in the finite set* I.

We are now in a position to prove the following

THEOREM 1. *Let* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$ *be a finite independent set of algebras, each of which is strictly functionally complete and of order* >1 *and let all the algebras* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$ *be of the same species* Sp. *Let* $\mathfrak{U} = \mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ *be the direct product of* $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$. *Then the strict identities of* $\mathfrak{U}$ *have the above finite set* I *of strict* $\mathfrak{U}$-*identities as a finite basis, i.e., every strict identity of* $\mathfrak{U}$ *is a consequence of the identities in* I.

Proof. Let B be any algebra of species Sp which satisfies set I, and let L be any strict identity of $\mathfrak{U}(=\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n)$. Theorem 1 will be proved by showing that L is also a strict identity of B. Now, by Lemma 1′, B is subdirectly representable in $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$, *i.e.*,

(1) $B \cong$ subdirect product of subdirect powers $\mathfrak{U}_1^{(N_1)}, \cdots, \mathfrak{U}_n^{(N_n)}$ of $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$.

But, since L is an identity of $\mathfrak{U}_i$ $(i=1, \cdots, n)$, therefore, L is also an identity of the subdirect power $\mathfrak{U}_i^{(N_i)}$ of $\mathfrak{U}_i$ (see [1], Th. 1.1), $(i=1, \cdots, n)$. Hence L is an identity of the direct product of $\mathfrak{U}_1^{(N_1)}, \cdots, \mathfrak{U}_n^{(N_n)}$. Therefore, *a fortiori* L is an identity of the subdirect product of $\mathfrak{U}_1^{(N_1)}, \cdots, \mathfrak{U}_n^{(N_n)}$. Hence, by (1) above, L is an identity of B. This proves the theorem.

In conclusion, I wish to express my indebtedness and gratitude to Professor A. L. Foster for his generous counsel.

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AN APPLICATION OF THE GENERATING FUNCTION TO DIFFERENTIAL EQUATIONS

T. A. Newton, Washington State University

The calculus discussed here is based upon a power series transform for sequences which was introduced by Laplace in 1812 [4], a year before he introduced his now well-known integral transform. This series transform has been extensively studied as a *generating function* of a given sequence (see Erdélyi *et al.* [1], pp. 228–282). A discussion of its applications in the theory of finite differences is found in Jordan ([3], pp. 20–44). Goldberg ([2], pp. 189–207) discusses the application of this transform to the solution of linear difference equations with constant coefficients. First, we will briefly outline some of the basic theory of this transform and its inverse. We will then show how the inverse transform can be applied to formalize the finding of the power series expansion of the solution of a linear differential equation in a manner that seems to be new.

Let $\mathfrak{F}$ denote the set of complex valued functions, defined and holomorphic on some open connected subset (depending upon the individual function) of the complex plane containing the origin. Let $\mathfrak{R}$ denote the set of equivalence classes of $\mathfrak{F}$ so defined that two elements of $\mathfrak{F}$ are equivalent if their restrictions to some neighborhood of zero are identical. For convenience sake, we will not distinguish between an element of $\mathfrak{F}$ and the equivalence class to which it belongs. Now let $\mathcal{S}$ denote the set of all sequences $a=\{a_n\}$ such that $\overline{\lim}\sqrt[n]{|a_n|}$ is finite. We then consider the mapping $G: \mathcal{S}\to\mathfrak{R}$ for which $u=G(a)$ if

$$u(t) = \sum_{n=0}^{\infty} a_n t^n$$

for $|t| < (\lim \sqrt[n]{|a_n|})^{-1}$. The function $u=G(a)$ is called the *generating function* of the sequence a. It follows from the theory of the Taylor series that this mapping $G: \mathcal{S}\to\mathfrak{R}$ is one-to-one; and for the inverse mapping, G^{-1}, $a=G^{-1}(u)$ if and only if

$$a_n = u^{(n)}(0)/n! \tag{1}$$

for $n=0, 1, \cdots$. It is immediate that for $a\in\mathcal{S}$, $a=G^{-1}(G(a))$; while for $u\in\mathfrak{R}$, $u=G(G^{-1}(u))$.

It follows immediately from their definition that the mappings G and G^{-1} are linear. That is, for α and β constants: if $a\in\mathcal{S}$, $b\in\mathcal{S}$, then $(\alpha a+\beta b)\in\mathcal{S}$ and

$$G(\alpha a + \beta b) = \alpha G(a) + \beta G(b); \tag{2}$$

if $u\in\mathfrak{R}$, $v\in\mathfrak{R}$, then $(\alpha u+\beta v)\in\mathfrak{R}$ and

$$G^{-1}(\alpha u + \beta v) = \alpha G^{-1}(u) + \beta G^{-1}(v). \tag{3}$$

Define the mapping $E^m: \mathcal{S}\to\mathcal{S}$, $m=0, 1, \cdots$, by $(E^m a)_n = a_{n+m}$ for $n=0, 1, \cdots$, where $E^1=E$. It can be established by induction that if $U_m=G(E^m a)$, then

$$U_m(t) = t^{-m}[u(t) - a_0 - ta_1 - \cdots - t^{m-1}a_{m-1}] \tag{4}$$

for $t \neq 0$, and $m = 1, 2, \cdots$.

Upon checking the above reference to Goldberg ([2], pp. 189–207), one finds that (2), (3), and (4) are the basis for the operational calculus which makes the generating function applicable for the solution of difference equations. As with the Laplace transform, a table of generating functions is a valuable tool for such applications. Any table of power series expansions can be used as a starting point for such a table, and further entries with their derivations can be found in the references cited above.

We shall now show how the inverse transform, $G^{-1}\colon \mathcal{R} \to \mathcal{S}$, is applicable to the solution of differential equations.

It follows from the Cauchy product theorem for power series that if $u \in \mathcal{R}$, $v \in \mathcal{R}$, then $uv \in \mathcal{R}$ and if $a = G^{-1}(u)$, $b = G^{-1}(v)$, then

$$(G^{-1}(uv))_n = \sum_{j=0}^{n} a_j b_{n-j} \tag{5}$$

for $n = 0, 1, \cdots$. We might well call the sequence defined by (5) the *convolution* of the sequences a and b, and if we were to denote it by $a*b$, we would then have $G(a)G(b) = G(a*b)$.

It follows from (1) that if $u \in \mathcal{R}$, then $u^{(1)} \in \mathcal{R}$, and

$$(G^{-1}(u^{(1)}))_n = \frac{u^{(n+1)}(0)}{n!} = (n+1)\frac{u^{(n+1)}(0)}{(n+1)!} = (n+1)(EG^{-1}(u))_n$$

for $n = 0, 1, \cdots$. We can now establish by induction that $u^{(m)} \in \mathcal{R}$ and

$$(G^{-1}(u^{(m)}))_n = (n+1)(n+2)\cdots(n+m)(E^m G^{-1}(u))_n \tag{6}$$

for $m = 1, 2, \cdots, n = 0, 1, \cdots$. Finally, we combine (5) and (6) to conclude that if $u \in \mathcal{R}$, $v \in \mathcal{R}$, then $uv^{(m)} \in \mathcal{R}$, and if $a = G^{-1}(u)$, $b = G^{-1}(v)$, then

$$(G^{-1}(uv^{(m)}))_n = \sum_{j=0}^{n} a_j(n-j+1)(n-j+2)\cdots(n-j+m)b_{n-j+m} \tag{7}$$

for $n = 0, 1, \cdots$ and $m = 1, 2, \cdots$.

Consider now the differential equation $(1+t^2)v'' + 2tv' - 2v = 0$. If we define $u_0(t) = (1+t^2)$, $u_1(t) = 2t$, $u_2(t) = -2$, and apply (3), we get

$$G^{-1}(u_0 v'') + G^{-1}(u_1 v') + G^{-1}(u_2 v) = 0. \tag{8}$$

Substituting into (7) with $m = 2$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$, and $a_j = 0$ for $j > 2$, it follows that if $G^{-1}(v) = b$, then

$$(G^{-1}(u_0 v''))_0 = 2b_2, \quad (G^{-1}(u_0 v''))_1 = 2 \cdot 3b_3,$$

$$(G^{-1}(u_0 v''))_n = (n+1)(n+2)b_{n+2} + (n-1)nb_n \qquad \text{for } n \geqq 2.$$

Likewise, we find

$$(G^{-1}(u_1v'))_0 = 0,$$
$$(G^{-1}(u_1v'))_n = 2nb_n \quad \text{for } n \geqq 1,$$
$$(G^{-1}(u_2v))_n = -2b_n \quad \text{for } n \geqq 0.$$

Substituting into (8) and simplifying we get the difference system

$$b_2 = b_0,\ b_3 = 0, \qquad (n+1)b_{n+2} + (n-1)b_n = 0, \qquad \text{for } n \geqq 2.$$

It follows that b_0 and b_1 are arbitrary, $b_{2k+1}=0$, and $b_{2k}=[(-1)^{k+1}/(2k-1)]b_0$ for $k=1, 2, \cdots$. That is,

$$v(t) = b_0 + b_1t + b_0\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} t^{2k}\right] = b_1t + b_0(1 + t\cdot\tan^{-1} t).$$

The above steps are precisely those met in the usual text-book solution of the differential equation (8) when one assumes a solution of the form

$$v(t) = \sum_{n=0}^{\infty} b_nt^n.$$

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AN ALGEBRAIC PROOF OF KIRCHHOFF'S NETWORK THEOREM*

A. Nerode and H. Shank, The University of Chicago

1. Introduction. Weyl [10] gave probably the first complete proof of the existence and uniqueness of currents in a resistive network subject to Ohm's law and Kirchhoff's voltage and current laws (Eckmann [2, 3], Roth [7, 8]). Much earlier Kirchhoff [5] gave an explicit expression for these currents in terms of maximal trees of the correlated graph. There are a number of papers on this theorem scattered through the literature, notably those of Ahrens [1] and Franklin [4]; and there has been considerable interest recently in its possible use in connection with computation of solutions to network problems (MacWilliams [6]). We give a simple linear algebraic proof of this ancient theorem, and incidentally obtain for graphs an expression for the projection of a real 1-chain on the space of 1-cycles in terms of maximal trees.

2. Preliminaries. A (finite oriented) graph G consists of a finite set V (of *vertices*) and a subset G of $V\times V$ (of *branches*) such that if $(v_1, v_2)\in G$ then

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$(v_2, v_1) \notin G$. A *subgraph* G_1 of G consists of a subset G_1 of G together with the set V_1 of all vertices occurring on branches of G_1. The space C_1 is the real inner product space having orthonormal basis G; the space C_0 is the real inner product space having orthonormal basis V. We denote inner product by $(\, , \,)$. The space Z_1 of 1-*cycles* is the null space of the linear transformation ∂: $C_1 \to C_0$ having $\partial(v_i, v_j) = v_j - v_i$. Vertices v_i and v_j are *connected* if there is a $c \in C_1$ with $\partial c = v_j - v_i$. G is *connected* if every pair of its vertices is connected. (Note that if $\partial c = v_j - v_i$ there is a c_1 with $\partial c_1 = v_j - v_i$ and $(c_1, b) = 0$ or $(c_1, b) = \pm 1$ for each branch b.) B_1, the space of 1-*coboundaries*, is the orthogonal complement of Z_1. A *forest* is a graph whose space of 1-cycles is 0-dimensional; a *tree* is a connected forest. (Rephrasing the last parenthetical observation, in a tree there is a unique c with $\partial c = v_j - v_i$, and its nonzero coefficients are all ± 1.) Throughout this note R: $C_1 \to C_1$ is a linear transformation such that for each branch b, $Rb = r_b b$ for some positive real r_b. If F is a maximal subforest of G, set $W_F = 1$ if $F = G$, $W_F = \prod_{b \in G-F} r_b$ otherwise. If $b \notin F$, $F \cup \{b\}$ contains a unique 1-cycle in which b occurs with coefficient $+1$; let $\overline{F}$: $C_1 \to C_1$ be the linear transformation such that $\overline{F}b$ is this unique 1-cycle if $b \notin F$, $\overline{F}b = 0$ otherwise.

3. Results

THEOREM. *Let G be a connected graph. Define a linear transformation S: $C_1 \to C_1$ by $S = \sum_T W_T \overline{T}$, where the summation is over all maximal trees of G. Then RS is self-adjoint.*

LEMMA 1. *Suppose T is a maximal tree. Then if $b_1, \cdots, b_k$ constitute all complementary branches, $\{\overline{T}b_1, \cdots, \overline{T}b_k\}$ is a basis for Z_1.*

Proof. It is clear that the $\overline{T}b_i$ are linearly independent. If z is any 1-cycle, we show that $z = \sum_i (z, b_i)\overline{T}b_i$. For $b_j \notin T$, $(z - \sum_i (z, b_i)\overline{T}b_i, b_j) = (z, b_j) - \sum_i (z, b_i)(\overline{T}b_i, b_j) = (z, b_j) - \sum_i (z, b_i)\delta_{ij} = (z, b_j) - (z, b_j) = 0$. Hence $z - \sum_i (z, b_i)\overline{T}b_i$ is a 1-cycle of T, therefore 0.

COROLLARY. *For any 1-cycle z, $\overline{T}z = z$; for any coboundary b, $\overline{T}^* b = 0$ (where $\overline{T}^*$ denotes the adjoint of $\overline{T}$).*

LEMMA 2. *All nonzero coefficients of $\overline{T}b$ are ± 1.*

Proof. If $b = (v_1, v_2)$, $\partial(\overline{T}b - b) = \partial\overline{T}b - \partial b = -\partial b = v_2 - v_1$.

LEMMA 3. *Suppose T is a maximal tree, that b_i, b_j are branches with $(\overline{T}b_i, b_j) \neq 0$. Then $U = (T - \{b_j\}) \cup \{b_i\}$ is a maximal tree.*

Proof. The only nonzero 1-cycle of $U \cup \{b_j\}$ contains both b_i and b_j. Since U does not contain b_j, it does not contain any nonzero 1-cycle. Maximality follows from Lemma 1.

LEMMA 4. *Suppose T, U, b_i, b_j are as in Lemma 3. Then*

$$(\overline{T}b_i, b_j) = (\overline{U}b_j, b_i), \qquad r_j W_T = r_i W_U.$$

LEMMA 5. *Let X_{ij} be the set of all maximal trees T containing b_j such that $(\overline{T}b_i, b_j) \neq 0$. Then for $i \neq j$,*

$$\sum_{T \in X_{ij}} W_T\, r_j(\overline{T}b_i, b_j) = \sum_{U \in X_{ji}} W_U\, r_i(\overline{U}b_j, b_i).$$

Proof of theorem. For $i \neq j$,

$$\begin{aligned}(RSb_i, b_j) &= \sum_T W_T\, r_j(\overline{T}b_i, b_j) = \sum_{T \in X_{ij}} W_T\, r_j(\overline{T}b_i, b_j) = \sum_{U \in X_{ji}} W_U\, r_i(\overline{U}b_j, b_i) \\ &= \sum_U W_U\, r_i(\overline{U}b_j, b_i) = (Sb_j, Rb_i) = (S^*Rb_i, b_j).\end{aligned}$$

Thus $RS = S^*R = (RS)^*$ and RS is self-adjoint.

COROLLARY. *Let N be the number of maximal trees of G. Then $(1/N)\sum_T \overline{T}$ is the orthogonal projection of C_1 on Z_1.*

Proof. By the corollary to Lemma 1, $(1/N)\sum_T \overline{T}$ is the identity on Z_1; by the theorem, it is self-adjoint. Thus for a 1-coboundary b,

$$\left(\frac{1}{N}\sum_T \overline{T}b, b_i\right) = \left(b, \frac{1}{N}\sum_T \overline{T}b_i\right) = 0.$$

COROLLARY. *For $c \in C_1$, there is exactly one $z \in Z_1$ with $Rz - c \in B_1$. Moreover, $z = (1/\Delta)R^{-1}S^*c$, where $\Delta = \sum_T W_T$.*

Proof. For any such z, $S^*(Rz - c)$ is in B_1 and hence in the null space of each term of $S^* = \sum_T W_T\overline{T}^*$. Since each $\overline{T}$ is the identity on Z_1 we have

$$S^*c = S^*Rz = RSz = R\sum_T W_T\overline{T}z = R\sum_T W_T z = R\Delta z = \Delta Rz,$$

from which existence, uniqueness, and the asserted formula follow. (For a discussion of N and Δ see Trent [9].)

The first and second sentences of this corollary are respectively the previously mentioned theorems of Weyl and Kirchhoff. In network terminology, G is a network with resistance r_b on branch b and voltage source (c, b) on branch b; (z, b) is the current resulting in branch b (see also Eckmann [2] or Roth [7]). By taking inner products with a branch b, one obtains

$$(z, b) = \frac{1}{\Delta}\sum \frac{W_T}{r_b}(c, \overline{T}b),$$

where the summation extends over all maximal trees T such that $b \in G - T$. This is the original form of Kirchhoff's result. The conditions $z \in Z_1$ and $Rz - c \in B_1$ express Ohm's law and Kirchhoff's voltage and current laws.

We omit consideration of other coefficient fields; for example, complex numbers in the case of lumped parameter networks ([2], [7]).

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THE RECURRENCE THEOREM

FRED B. WRIGHT,* Tulane University of Louisiana

The purpose of this note is to give a completely self-contained account of the recurrence theorem of ergodic theory. This theorem was first proved by Poincaré, and, in modern form, asserts that if T is a measure-preserving transformation on a finite measure space and if E is any measurable set, then almost every point of E returns to E infinitely often under application of T. More generally, if all of the transformations T, T^2, T^3, $\cdots$, are incompressible, the same conclusion holds for any measure space. In 1947, Halmos [1] showed that if T is incompressible, one-one, and if T^{-1} is measurable, then all powers of T are incompressible. In 1959, Taam [2] succeeded in removing the restrictions on T, by carefully analyzing the already quite involved combinatorial proof of Halmos. Independently, in 1959 the author of this note, in the course of an investigation of the properties of endomorphisms of Boolean algebras, discovered a very simple proof of the recurrence theorem which circumvents the necessity of proving that powers are incompressible. This proof appears midway in a paper [3] which introduces considerable machinery whose function is more general than a proof of the recurrence theorem, and consequently it appears more difficult than it is, in fact. We show here that this theorem is completely on the surface.

Throughout, let X be a set, let $\mathcal{S}$ be a σ-algebra of subsets of X, and let $\mathcal{J}$ be a σ-ideal of $\mathcal{S}$. The triple $(X, \mathcal{S}, \mathcal{J})$ will be called a *measurability space.* A

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mapping T of X into itself will be called a *measurable* transformation if $E \in \mathcal{S}$ implies $T^{-1}E \in \mathcal{S}$. If T is a measurable transformation and if $E \in \mathcal{S}$, the set $E - \bigcup_{j=1}^{\infty} T^{-j}E$ is the set of all those points in E which never return to E, and the set $E - \bigcap_{m=0}^{\infty} \bigcup_{j=m+1}^{\infty} T^{-j}E$ is the set of those points of E which do not return infinitely often to E. The measurable transformation T is called *recurrent* if $E - \bigcup_{j=1}^{\infty} T^{-j}E \in \mathcal{J}$ for all $E \in \mathcal{S}$, and is called *infinitely recurrent* if $E - \bigcap_{m=0}^{\infty} \bigcup_{j=m+1}^{\infty} T^{-j}E \in \mathcal{J}$ for all $E \in \mathcal{S}$.

A measurable transformation T is called *incompressible* if $E \subset T^{-1}E$ implies $T^{-1}E - E \in \mathcal{J}$. We remark that T is incompressible if and only if $T^{-1}E \subset E$ implies $E - T^{-1}E \in \mathcal{J}$. For, let $E \in \mathcal{S}$ and let $F = X - E$. Then $E \subset T^{-1}E$ if and only if $T^{-1}F \subset F$, and $E - T^{-1}E = T^{-1}F - F$, $T^{-1}E - E = F - T^{-1}F$.

A set $E \in \mathcal{S}$ is called a *wandering set* if the sequence $E, T^{-1}E, T^{-2}E, \cdots$, is disjoint. A measurable transformation is called *conservative* if every wandering set is in $\mathcal{J}$.

The Recurrence Theorem. *Let T be a measurable transformation of a measurability space $(X, \mathcal{S}, \mathcal{J})$. Then the following are equivalent*: (1) *T is incompressible*; (2) *T is conservative*; (3) *T is recurrent*; (4) *T is infinitely recurrent.*

Proof. For any $E \in \mathcal{S}$, let $E^* = \bigcup_{j=1}^{\infty} T^{-j}E$. We first prove that (1) implies (3). Let $E \in \mathcal{S}$, and let $A = E \cup E^*$. Then $T^{-1}A = E^* \subset A$, and since T is incompressible, $A - T^{-1}A \in \mathcal{J}$. But $A - T^{-1}A = E - E^*$.

Next, (3) implies (2). Suppose E is a wandering set; then E and E^* are disjoint, so that $E - E^* = E$. If T is recurrent, then $E = E - E^* \in \mathcal{J}$, so that T is conservative.

Now we show that (2) implies (1). If E is any set in $\mathcal{S}$, then $E - E^*$ is a wandering set. If T is conservative, $E - E^* \in \mathcal{J}$. But then if $T^{-1}E \subset E$, we have $E^* = T^{-1}E$, and hence $E - T^{-1}E \in \mathcal{J}$. Thus T is incompressible.

Finally, we prove that (4) and (3) are equivalent. First observe that if $B = \bigcap_{m=0}^{\infty} T^{-m}E^*$, then $B \subset E^*$ and $E - E^* \subset E - B$. Thus if T is infinitely recurrent, it is recurrent. Conversely, assume (3). By what has been shown, we know that (1) holds; we use both (1) and (3). It is clear that $E - B = (E - E^*) \cup [E \cap \bigcup_{m=1}^{\infty} (T^{-m}E^* - T^{-(m+1)}E^*)]$. Since T is recurrent, $E - E^* \in \mathcal{J}$. Since $T^{-1}(T^{-m}E^*) = T^{-(m+1)}E^* \subset T^{-m}E^*$ for each $m \geqq 0$, and since T is incompressible, $T^{-m}E^* - T^{-(m+1)}E^* \in \mathcal{J}$. Since $\mathcal{J}$ is a σ-ideal, then $E - B \in \mathcal{J}$. This completes the proof.

We must remark that in the proof of the equivalence of (1), (2), and (3), *no use was made of any property whatsoever of the family $\mathcal{J}$.*

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WEDDERBURN'S THEOREM AND A THEOREM OF JACOBSON

I. N. HERSTEIN, Cornell University

In teaching an undergraduate class in modern algebra (whose students, although very bright, were mostly sophomores and so had little algebraic knowledge or technique) the author was faced with the problem of presenting to the class two theorems which long had been among his favorites, namely, Wedderburn's theorem on finite division rings and (in the division ring case) Jacobson's theorem that a ring in which $x^{m(x)}=x$ for all x is commutative. In order to do so he devised the proofs presented here; these proofs may be of interest to others confronted with similar problems.

The only facts needed to follow the proofs and which are not absolutely trivial are:

1. The multiplicative group of a finite field is cyclic.
2. If F is a finite field, $\alpha\neq 0\in F$ then there exist λ, $\mu\in F$ so that $1+\lambda^2-\alpha\mu^2=0$.

Of course, many proofs of Wedderburn's theorem exist. In fact, we presented Wedderburn's original proof [2] and the slight twist, in its "punch-line," introduced by Witt [3]. Other proofs, to name but a few, are those of Artin [1] and Zassenhaus [4]. The proof here is closest in spirit to that of Artin, but seems to be both shorter and more elementary. Moreover, no use is made either of counting or of nontrivial number theory. It is of interest that the proof finally hinges on the fact that the quaternions over a finite field do not form a division ring. This is equivalent to making use of fact (2) in the introduction.

We begin with

LEMMA 1. *Let D be a division ring of characteristic $p>0$ with center Z, and P the prime field with p elements contained in Z. Suppose $a\in D$, $a\notin Z$ is such that $a^{p^n}=a$ for some $n>0$. Then there exists an $x\in D$ such that*

(1) *$xax^{-1}\neq a$,*

(2) *$xax^{-1}\in P(a)$, the field obtained by adjoining a to P.*

Proof. Define the mapping $\delta: D\to D$ by $\delta(x)=xa-ax$ for all $x\in D$. $P(a)$ is a finite field, and has, say, p^m elements. These all satisfy the equation $u^{p^m}=u$. By a trivial verification we immediately have that $\delta^p(x)=xa^p-a^px$, $\delta^{p^m}(x)=xa^{p^m}-a^{p^m}x=xa-ax=\delta(x)$ for all $x\in D$. Thus $\delta^{p^m}=\delta$.

Now if $\lambda\in P(a)$, $\delta(\lambda x)=(\lambda x)a-a(\lambda x)=\lambda(xa-ax)=\lambda\delta(x)$ since λ commutes with a. Thus if I denotes the identity map on D, $\delta\lambda I=(\lambda I)\delta$ for all $\lambda\in P(a)$. Now the polynomial $u^{p^m}-u$ considered over $P(a)$ has all its p^m roots as the elements of $P(a)$. Thus $u^{p^m}-u=\prod_{\lambda\in P(a)}(u-\lambda)$. Thus since δ commutes with all λI, $\lambda\in P(a)$,

$$0=\delta^{p^m}-\delta=\prod_{\lambda\in P(a)}(\delta-\lambda I).$$

If for every $\lambda\neq 0\in P(a)$, $\delta-\lambda I$ annihilates no nonzero element in D, then since

$\delta(\delta-\lambda_1 I)\cdots(\delta-\lambda_k I)=0$ we would get that $\delta=0$, that is that $xa-ax=0$ for all $x\in D$, forcing $a\in Z$ contrary to hypothesis. Thus there is a $\lambda\neq 0\in P(a)$ and an $x\neq 0\in D$ so that $(\delta-\lambda I)x=0$; that is $xa-ax=\lambda x$ and so $xax^{-1}=a+\lambda\neq a\in P(a)$, proving the lemma.

COROLLARY. *In Lemma* 1 $xax^{-1}=a^i\neq a$ *for some integer* i.

Proof. Let a be of order s, then in the field $P(a)$ all the roots of the polynomial u^s-1 are $1, a, a^2, \cdots, a^{s-1}$. Since xax^{-1} is in $P(a)$ and is a root of this polynomial, $xax^{-1}=a^i$ follows.

We first prove the

THEOREM (Wedderburn). *A finite division ring is a field.*

Proof. Let D be a finite division ring. We assume the theorem to be true for division rings with fewer elements than D.

We first remark that if $a, b\in D$ are such that $b^ta=ab^t$ but $ab\neq ba$, then $b^t\in Z$. For consider $N(b^t)=\{x\in D \mid xb^t=b^tx\}$. $N(b^t)$ is a subdivision ring of D, so if it were not D it would be commutative. Since $a, b\in N(b^t)$ and these do not commute, it must then be that $N(b^t)=D$.

Pick $a\in D$, $a\notin Z$ such that a minimal positive power of a falls in Z. Clearly this minimal power is a prime, r. By the corollary to Lemma 1, there is an $x\in D$ such that $xax^{-1}=a^i\neq a$. Since r is a prime, using the little Fermat theorem, $x^{r-1}ax^{-(r-1)}=a^{i^{r-1}}=a^{1+ru}=aa^{ru}=\lambda a$ where $a^{ru}=\lambda\in Z$. Since $x\notin Z$, $x^{r-1}\notin Z$ by the minimal nature of r; thus by the remark in the paragraph above, $x^{r-1}a\neq ax^{r-1}$, so that $\lambda\neq 1$. Let $b=x^{r-1}$. Thus $bab^{-1}=\lambda a$; consequently $a^r=ba^rb^{-1}=(bab^{-1})^r=\lambda^ra^r$, forcing $\lambda^r=1$. We claim that if $y\in D$ is such that $y^r=1$ then $y=\lambda^i$, for in the field $Z(y)$ there are at most r roots of the polynomial t^r-1 and these are already given by the powers of λ. Now $b^r=\lambda^rb^r=(a^{-1}ba)^r=a^{-1}b^ra$; thus $b^ra=ab^r$, $ba\neq ab$ which implies that $b^r\in Z$. The multiplicative group of Z is cyclic and is generated by an element γ. Thus $a^r=\gamma^n$, $b^r=\gamma^m$. If $n=kr$ then $(a/\gamma k)^r=1$, which would make $a/\gamma k=\lambda^i$ and would lead to $a\in Z$. Thus $r\nmid n$; similarly $r\nmid m$. Let $a_1=a^m$, $b_1=b^n$. Thus $a_1^r=a^{mr}=\gamma^{mn}=b^{nr}=b^r$ and

$$a_1b_1=\mu b_1a_1(\mu\neq 1\in Z) \tag{1}$$

($\mu\neq 1$ since $r\nmid n, m$ so $a_1, b_1\notin Z$) and $\lambda^r=1$. Computing $(b_1^{-1}a_1)^r$ using (1) we arrive at

$$(b_1^{-1}a_1)^r=\mu^{-(1+2+\cdots+(r-1))}b_1^{-r}a_1^r=\mu^{-r(r-1)/2}.$$

If r is odd then since $\mu^r=1$, we have that $(b_1^{-1}a_1)^r=1$. But then $b_1^{-1}a_1=\lambda^i$; and this implies that $a_1b_1=b_1a_1$, a contradiction. Hence if r is odd the theorem is proved.

If $r=2$, then since $\mu^2=1$, $\mu\neq 1$ we have $\mu=-1$. The characteristic must also then be different from 2. Also $\alpha=a_1^2=b_1^2\in Z$, $a_1b_1=-b_1a_1\neq b_1a_1$. In Z there are elements ξ, η so that $1+\xi^2-\alpha\eta^2=0$. But then $(a_1+\xi b_1+\eta a_1b_1)^2=\alpha(1+\xi^2-\alpha\eta^2)$

$=0$. Being in a division ring this yields $a_1+\xi b_1+\eta a_1b_1=0$. Thus

$$0 \neq 2a_1^2 = a_1(a_1 + \xi b_1 + \eta a_1 b_1) + (a_1 + \xi b_1 + \eta a_1 b_1)a_1 = 0.$$

This contradiction finishes the proof.

We now proceed to prove the

THEOREM (Jacobson). *Let D be a division ring such that for every $x\in D$ there exists an integer $n(x)>1$ so that $x^{n(x)}=x$. Then D is commutative.*

Proof. For $a\neq 0\in D$, $a^n=a$, $(2a)^m=2a$. Putting $q=(n-1)(m-1)+1$ we have $q>1$, $a^q=a$, $(2a)^q=2a$, so that $(2^q-2)a=0$. Thus D has characteristic $p>0$. If P is the prime field with p elements contained in Z, then $P(a)$ has p^h elements, so that $a^{p^h}=a$. So if $a\notin Z$ the conditions of Lemma 1 are satisfied and so there exists a $b\in D$ such that (1) $bab^{-1}=a^u\neq a$. Suppose $b^{p^k}=b$ and consider

$$W = \left\{ x \in D \,\middle|\, x = \sum_{i,j}^{p^h,p^k} p_{ij}a^kb^j,\ p_{ij} \in P \right\}.$$

W is finite, is closed under addition, and by virtue of (1) is closed under multiplication. Thus W is a finite ring and being a subring of the division ring D the two cancellation laws hold so it is a finite division ring. But then it is commutative. Since a, $b\in W$ this forces $ab=ba$, contrary to $a^ub=ba$. This proves the theorem.

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A NOTE ON THE GENERALIZED WILSON'S THEOREM

L. CARLITZ, Duke University

It is familiar that if $m\geqq 1$ and W_m denotes the product of the integers $\leqq m$ and prime to m then ([2], Ths. 47, 59)

$$W_m \equiv \pm 1 \pmod{m}, \tag{1}$$

where the lower sign occurs provided $m=1, 2, 4, p^r, 2p^r$ $(r\geqq 1)$ and p is a prime >2. (For other references see [1], Ch. 3.)

In this note we prove the following related result.

Let p be a fixed prime >3 and let P_m denote the product of the integers $\leqq m$ and prime to p. Then if $p^r\mid m$, $r\geqq 1$, we have

$$P_m \equiv ((p-1)!)^{m/p} \pmod{p^{r+2}} \qquad (p > 3). \tag{2}$$

$$\prod_{h=1}^{p^r} f(h) \equiv 1 \pmod{p^{r+3}}.$$

This completes the proof of (2).

As a special case of (2) we note that

$$P_{p^r} \equiv ((p-1)!)^{p^{r-1}} \pmod{p^{r+2}} \qquad (p > 3,\ r \geqq 1). \tag{8}$$

Since

$$Q_r = \frac{p^r!}{p^{r-1}!(p!)^{p^{r-1}}} = \frac{P_r}{((p-1)!)^{p^{r-1}}},$$

(8) is equivalent to

$$Q_r \equiv 1 \pmod{p^{r+2}} \qquad (p > 3,\ r \geqq 1). \tag{9}$$

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A NOTE ON MULTIPLE SERIES OF POSITIVE TERMS

JOSEPH W. ANDRUSHKIW, Seton Hall University

1. Introduction. It is well known that, if $f(x)$ is a positive function which decreases monotonely towards zero in the interval $0 \leqq x < \infty$, the following theorem holds:

THEOREM 1. *The sequence* $u_n = \sum_{i=0}^{n} f(i) - \int_0^n f(x)dx$ *is convergent and* $\lim u_n = C$, *where* $0 < C < f(0)$.

From the above theorem follow two corollaries:

COROLLARY 1. *The sequence* $u_n(p, q) = \sum_{i=pn}^{qn} f(i) - \int_{pn}^{qn} f(x)dx$ *is convergent to zero if* $0 < p < q$.

COROLLARY 2. *The series* $\sum_{i=0}^{\infty} f(i)$ *converges or diverges with the integral* $\int_0^\infty f(x)dx$ (Maclaurin's integral test).

Theorem 1 and Corollary 1 hold even if $\int_0^\infty f(x)dx$ is divergent. If, in spite of the divergence of the last integral, $\lim_{n\to\infty} \int_{pn}^{qn} f(x)dx = L$ exists, also $\lim_{n\to\infty} \sum_{i=pn}^{qn} f(i) = L$; *e.g.* $\lim_{n\to\infty} \sum_{i=pn}^{qn} 1/i = \log(q/p)$.

If $f(x, y)$ is a positive montonely decreasing function of both variables there are in the literature few generalizations of Corollary 2. Some of them assume that the function $f(x, y)$ is positive outside a certain closed curve in the xy-coordinate plane and diminishes steadily in value as the point (x, y) recedes

from the origin in any direction ([1], p. 359; [2], p. 196). Other authors ([3], p. 86; [4], p. 415) state, in a rather vague way, that the double series converges or diverges with the double integral or discuss only a sufficient condition for convergence ([5], p. 84).

Considering the function $f(x, y)=(x+1)^{-1}e^{-(x+1)y}$ we check that the function is positive and monotonely decreasing in the domain $0\leqq x<\infty$, $0\leqq y<\infty$, and that the integral $\int_0^\infty\int_0^\infty (x+1)^{-1}e^{-(x+1)y}dxdy$ exists. However, the double series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)^{-1}e^{-(i+1)j}$$

is divergent because of the divergence of its zero-th row, $\sum_{i=0}^{\infty}(i+1)^{-1}$. This shows that the convergence of the double integral is not a sufficient condition for the convergence of the double series.

In this note we will prove a theorem analogous to Theorem 1 and hence we will develop a necessary and sufficient condition for the convergence of the double series. Then the results will be generalized for functions in any finite number of variables.

2. Integral test for convergence of double series. Denote by $f(x, y)$ a function which has the following properties (A): (A_1) It is integrable and positive in the domain $0\leqq x<\infty$, $0\leqq y<\infty$; (A_2) $f(x_1, y_1)\geqq f(x_2, y_2)$ if $x_1<x_2$, $y_1<y_2$; (A_3) $\lim f(x, y)=0$ if $x+y\to\infty$. The properties A_1 and A_2 lead immediately to the inequality

$$(1)\quad \sum_{i=p+1}^{p+n}\sum_{j=q+1}^{q+m}f(i,j)\leqq\int_p^{p+n}\int_q^{q+m}f(x,y)dxdy\leqq\sum_{i=p}^{p+n-1}\sum_{j=q}^{q+m-1}f(i,j),\quad p\geqq 0,\ q\geqq 0.$$

Since

$$(2)\qquad \sum_{i=0}^{n}\sum_{j=0}^{m}f(i,j)=\sum_{i=1}^{n}\sum_{j=1}^{m}f(i,j)+\sum_{i=1}^{n}f(i,0)+\sum_{j=1}^{m}f(0,j)+f(0,0),$$

$$(3)\qquad \sum_{i=1}^{n}f(i,0)\leqq\int_0^n f(x,0)dx,\qquad \sum_{j=1}^{m}f(0,j)\leqq\int_0^m f(0,y)dy,$$

the left side of the inequality (1) yields

$$(4)\quad \sum_{i=0}^{n}\sum_{j=0}^{m}f(i,j)-\int_0^n\int_0^m f(x,y)dxdy\leqq\int_0^n f(x,0)dx+\int_0^m f(0,y)dy+f(0,0).$$

Similarly, from the decomposition

$$(5)\qquad \sum_{i=0}^{n}\sum_{j=0}^{m}f(i,j)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f(i,j)+\sum_{i=0}^{n-1}f(i,m)+\sum_{j=0}^{m-1}f(n,j)+f(n,m),$$

and since

$$\sum_{i=0}^{n-1} f(i, m) \geqq \int_0^n f(x, m)dx, \qquad \sum_{j=0}^{m-1} f(n, j) \geqq \int_0^m f(n, y)dy, \tag{6}$$

it follows, after application of the right side of the inequality (1), that

$$\begin{aligned} 0 &< f(n, m) + \int_0^n f(x, m)dx + \int_0^m f(n, y)dy \\ &\leqq \sum_{i=0}^{n}\sum_{j=0}^{m} f(i, j) - \int_0^n\int_0^m f(x, y)dxdy. \end{aligned} \tag{7}$$

Now, let us assume that $\int_0^\infty f(x, 0)dx$ and $\int_0^\infty f(0, y)dy$ are convergent, and let

$$u_n = \sum_{i=0}^{n}\sum_{j=0}^{n} f(i, j) - \int_0^n\int_0^n f(x, y)dxdy.$$

LEMMA. *Let $f(x, y)$ be a function satisfying the conditions (A) and such that $\int_0^\infty f(x, 0)dx$ and $\int_0^\infty f(0, y)dy$ exist. Given an $\epsilon > 0$, there exists an integer $\mu > 0$ such that for $m > \mu$, $\sum_{i=0}^\infty f(i, m) < \epsilon$, and an integer $\nu > 0$ such that for $n > \nu$, $\sum_{j=0}^\infty f(n, j) < \epsilon$.*

Since $\int_0^\infty f(x, 0)dx$ is convergent so is $\sum_{i=0}^\infty f(i, 0)$ and, consequently, $\sum_{i=0}^\infty f(i, m)$. Then, there exists an integer N such that $\sum_{i=N}^\infty f(i, 0) < \frac{1}{2}\epsilon$; therefore, also, $\sum_{i=N}^\infty f(i, m) < \frac{1}{2}\epsilon$ for all positive m. Because of property A_3 an integer $\mu > 0$ can be found such that for $m > \mu$, $f(0, m) < \frac{1}{2}(\epsilon/N)$. Consequently, $f(i, m) < \frac{1}{2}(\epsilon/N)$ and $\sum_{i=0}^{N-1} f(i, m) < \frac{1}{2}\epsilon$. Finally, it follows that

$$\sum_{i=0}^{\infty} f(i, m) = \sum_{i=0}^{N-1} f(i, m) + \sum_{i=N}^{\infty} f(i, m) < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

The second part of the lemma can be proved in similar way.

Now, consider the difference $u_{n+p} - u_n$

$$\begin{aligned} &= \sum_{i=0}^{n+p}\sum_{j=0}^{n+p} f(i,j) - \int_0^{n+p}\int_0^{n+p} f(x, y)dxdy - \sum_{i=0}^{n}\sum_{j=0}^{n} f(i,j) + \int_0^n\int_0^n f(x, y)dxdy \\ &= \left(\sum_{i=1}^{n}\sum_{j=n+1}^{n+p} f(i,j) - \int_0^n\int_n^{n+p} f(x, y)dxdy\right) \\ &\quad + \left(\sum_{i=n+1}^{n+p}\sum_{j=1}^{n+p} f(i,j) - \int_n^{n+p}\int_0^{n+p} f(x, y)dxdy\right) + \sum_{i=n+1}^{n+p} f(i, 0) + \sum_{j=n+1}^{n+p} f(0, j). \end{aligned} \tag{8}$$

Since, according to (1), the expressions in the parentheses are nonpositive, it follows that

$$u_{n+p} - u_n \leqq \sum_{i=n+1}^{n+p} f(i, 0) + \sum_{j=n+1}^{n+p} f(0, j). \tag{9}$$

On the other hand we observe that

$$(10)\qquad u_{n+p} - u_n = \left(\sum_{i=0}^{n} \sum_{j=n}^{n+p} f(i,j) - \int_0^n \int_n^{n+p} f(x, y)dxdy \right)$$
$$+ \left(\sum_{i=n}^{n+p} \sum_{j=0}^{n+p} f(i,j) - \int_n^{n+p} \int_0^{n+p} f(x, y)dxdy \right) - \sum_{i=0}^{n} f(i, n) - \sum_{j=0}^{n+p} f(n, j),$$

and, since the quantities in the parentheses are nonnegative, we have

$$(11)\qquad -\left(\sum_{i=0}^{n} f(i, n) + \sum_{j=0}^{n+p} f(n, j) \right) \leqq u_{n+p} - u_n.$$

From (11) and the lemma it follows that there exists an integer $\nu > 0$ such that, for $n > \nu$,

$$(12)\qquad -\epsilon \leqq u_{n+p} - u_n.$$

Further, the convergence of $\int_0^\infty f(x, 0)dx$ and $\int_0^\infty f(0, y)dy$ assures the existence of an integer $\nu^* > 0$ such that, for $n > \nu^*$, the right side of (9) is less than ϵ. Thus we obtain

$$(13)\qquad \left| u_{n+p} - u_n \right| < \epsilon, n > \max(\nu, \nu^*), p \text{ any positive integer,}$$

and, hence, a theorem analogous to Theorem 1.

THEOREM 2. *If $f(x, y)$ is a function satisfying the conditions (A) and the integrals $\int_0^\infty f(x, 0)dx$ and $\int_0^\infty f(0, y)dy$ are convergent, then the sequence*

$$u_n = \sum_{i=0}^{n} \sum_{j=0}^{n} f(i, j) - \int_0^n \int_0^n f(x, y)dxdy$$

is convergent, and $\lim u_n = C$, *where*

$$0 < C < \int_0^\infty f(x, 0)dx + \int_0^\infty f(0, y)dy + f(0, 0).$$

The following corollary is a consequence of convergence of the sequence u_n.

COROLLARY 3. *If $f(x, y)$ is a function satisfying the conditions (A) and the integrals $\int_0^\infty f(x, 0)dx$ and $\int_0^\infty f(0, y)dy$ converge, then the sequence*

$$u_n(p, q) = \sum_{i=pn}^{qn} \sum_{j=pn}^{qn} f(i, j) - \int_{pn}^{qn} \int_{pn}^{qn} f(x, y)dxdy,$$

where $0 < p < q$, converges to zero.

The $\lim_{n\to\infty} \int_{pn}^{qn}\int_{pn}^{qn} f(x, y)dxdy$ may exist, although $\int_0^\infty \int_0^\infty f(x, y)dxdy$ does not.

COROLLARY 4. *If* $\lim_{n\to\infty} \int_{pn}^{qn}\int_{pn}^{qn} f(x, y)dxdy = L$, *then* $\lim_{n\to\infty} \sum_{i=pn}^{qn} \sum_{j=pn}^{qn} f(i, j) = L$. For example,

$$\lim_{n\to\infty} \sum_{i=pn}^{qn} \sum_{j=pn}^{qn} (x+y)^{-2} = \log\left[\frac{(p+q)^2}{4pq}\right].$$

Finally, if, also, $\int_0^\infty\int_0^\infty f(x, y)dxdy$ exists, we obtain from Theorem 2 the following criterion for convergence of double series.

THEOREM 3. *The series $\sum_{i=0}^\infty \sum_{j=0}^\infty f(i, j)$, where $f(x, y)$ is a function possessing the properties (A), converges if and only if the integrals $\int_0^\infty\int_0^\infty f(x, y)dxdy$, $\int_0^\infty f(x, 0)dx$, and $\int_0^\infty f(0, y)dy$ converge.*

3. Generalization. Let $f(x_1, \cdots, x_w)$ be a function which possesses the following properties (B): (B_1) It is positive and integrable in the domain $0 \leqq x_k < \infty$, $k=1, \cdots, w$; (B_2) $f(x_1, \cdots, x_w) \geqq f(y_1, \cdots, y_w)$ if $x_k < y_k$, $k=1, \cdots, w$; (B_3) $\lim f(x_1, \cdots, x_w) = 0$ if $\sum_{k=1}^w x_k \to \infty$.

Further we let:

$$J = J(\alpha_1, \beta_1; \cdots; \alpha_w, \beta_w) = \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_w}^{\beta^w} f(x_1, \cdots, x_w)dx_1 \cdots dx_w, \tag{14}$$

$$S = S(\alpha_1, \beta_1; \cdots; \alpha_w, \beta_w) = \sum_{i_1=\alpha_1}^{\beta_1} \cdots \sum_{i_w=\alpha_w}^{\beta_w} f(i_1, \cdots, i_w). \tag{15}$$

The properties B_1 and B_2 yield the inequality

$$\begin{aligned} S(\alpha_1+1, \beta_1; \cdots; \alpha_w+1, \beta_w) &\leqq J(\alpha_1, \beta_1; \cdots; \alpha_w, \beta_w) \\ &\leqq S(\alpha_1, \beta_1 - 1; \cdots; \alpha_w, \beta_w - 1). \end{aligned} \tag{16}$$

If in $f(x_1, \cdots, x_w)$ some $w-r$ variables are replaced by the corresponding lower limits of the integral J and the remaining variables are $x_{k_1}, \cdots, x_{k_r}$, then the function $g(x_{k_1}, \cdots, x_{k_r})$ in r variables is obtained. Similarly, by $G(x_{k_1}, \cdots, x_{k_r})$ we denote the function in r variables if some $w-r$ variables in $f(x_1, \cdots, x_w)$ are replaced by the corresponding upper limits of the integral J. Taking this into account, we set

$$j_r = j_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \int_{\alpha_{k_1}}^{\beta_{k_1}} \cdots \int_{\alpha_{k_r}}^{\beta_{k_r}} g(x_{k_1}, \cdots, x_{k_r})dx_{k_1} \cdots dx_{k_r} \tag{17}$$

$$J_r = J_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \int_{\alpha_{k_1}}^{\beta_{k_1}} \cdots \int_{\alpha_{k_r}}^{\beta_{k_r}} G(x_{k_1}, \cdots, x_{k_r})dx_{k_1} \cdots dx_{k_r} \tag{18}$$

and similarly for the series

$$s_r = s_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \sum_{i_{k_1}=\alpha_{k_1}}^{\beta_{k_1}} \cdots \sum_{i_{k_r}=\alpha_{k_r}}^{\beta_{k_r}} g(i_{k_1}, \cdots, i_{k_r}), \tag{19}$$

$$S_r = S_r(\alpha_{k_1}, \beta_{k_1}; \cdots; \alpha_{k_r}, \beta_{k_r}) = \sum_{i_{k_1}=\alpha_{k_1}}^{\beta_{k_1}} \cdots \sum_{i_{k_r}=\alpha_{k_r}}^{\beta_{k_r}} G(i_{k_1}, \cdots, i_{k_r}). \tag{20}$$

There are C_r^w r-tuple integrals (and series) formed in this way. On setting

$$u(n_1, \cdots, n_w; p, q) = S(pn_1, qn_1; \cdots; pn_w, qn_w) \\ - J(pn_1, qn_1; \cdots; pn_w, qn_w) \tag{21}$$

and because of the identities

$$\begin{aligned} S(pn_1, qn_1; \cdots; pn_w, qn_w) &= S(pn_1 + 1, qn_1; \cdots; pn_w + 1, qn_w) \\ &\quad + \sum s_{w-1} + \cdots + \sum s_1 + f(pn_1, \cdots, pn_w) \\ &= S(pn_1, qn_1 - 1; \cdots; pn_w, qn_w - 1) \\ &\quad + \sum S_{w-1} + \cdots + \sum S_1 + f(qn_1, \cdots, qn_w), \end{aligned} \tag{22}$$

where each summation is taken over C_r^w r-tuple series $(r=1, \cdots, w-1)$, we obtain from (16),

$$\begin{aligned} 0 &< f(qn_1, \cdots, qn_w) + \sum J_1 + \cdots + \sum J_{w-1} \\ &\leqq u(n_1, \cdots, n_w; p, q) \leqq f(pn_1, \cdots, pn_w) + \sum j_1 + \cdots + \sum j_{w-1}. \end{aligned} \tag{23}$$

Since the lemma can also be generalized, we obtain from (23) a theorem analogous to Theorem 1:

THEOREM 4. *If all the r-tuple integrals* $j_r(0, \infty; \cdots; 0, \infty)$, $r=1, \cdots, w-1$, *are convergent, then* $\lim u(n, \cdots, n; 0, 1) = C$, *where*

$$0 < C < \sum j_1 + \cdots + \sum j_{w-1} + f(0, \cdots, 0).$$

Hence we have the following corollaries:

COROLLARY 5. *If* $0<p<q$ *and all the r-tuple integrals* $j_r(0, \infty; \cdots; 0, \infty)$, $r=1, \cdots, w-1$, *are convergent, then* $\lim u(n, \cdots, n; p, q) = 0$.

COROLLARY 6. *If* $\lim J(pn, qn; \cdots; pn, qn) = L$ *then also*

$$\lim S(pn, qn; \cdots; pn, qn) = L.$$

Finally, if the integral $J(0, \infty; \cdots; 0, \infty)$ also converges we obtain

THEOREM 5. *The series* $S(0, \infty; \cdots; 0, \infty)$ *is convergent if and only if the integral* $J(0, \infty; \cdots; 0, \infty)$ *and all the r-tuple integrals* $j_r(0, \infty; \cdots; 0, \infty)$, $r=1, \cdots, w-1$, *converge.*

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A CHARACTERIZATION OF PSEUDO-CHAIN MAPPINGS IN MAYER COMPLEXES

NORMAN LEVINE,* University of Pittsburgh

1. Introduction. Let $K=\{C_p(K), \partial_p^K\}$ and $L=\{C_p(L), \partial_p^L\}$ be two Mayer complexes† and for each integer p, f_p: $C_p(K)\to C_p(L)$ a homomorphism. Such a system of homomorphisms is called a mapping from K to L and is denoted by $f: K\to L$. A mapping $f: K\to L$ is called a chain-mapping if and only if $f_{p-1}\partial_p^K =\partial_p^L f_p$ for every integer p and such mappings have been studied extensively. Their main virtue is that cycles are transformed into cycles and boundaries are transformed into boundaries. A chain mapping $f: K\to L$ then induces in a natural way a homomorphism f_p^*: $H_p(K)\to H_p(L)$ of the p-homology groups of K and L respectively.

2. Pseudo-chain mappings. Let f: $K\to L$ be a mapping from K to L, *i.e.*, for each integer p, f_p: $C_p(K)\to C_p(L)$ is a homomorphism. Let $Z_p(K)$ and $Z_p(L)$ be the groups of p-cycles in $C_p(K)$ and $C_p(L)$, respectively, and $B_p(K)$ and $B_p(L)$ the groups of p-boundaries in $C_p(K)$ and $C_p(L)$ respectively.

DEFINITION 1. *A mapping $f: K\to L$ will be termed a pseudo-chain mapping if and only if for each integer p,* (1) $f_p\{Z_p(K)\}\subset Z_p(L)$ *and* (2) $f_p\{B_p(K)\}\subset B_p(L)$.

In Section 4, we will obtain a characterization of such mappings in terms of a homomorphism from $C_p(K)$ to $C_{p-1}(L)$ to be introduced in Section 3.

Example 1. A pseudo-chain mapping is not necessarily a chain-mapping. For let K and L be abstract simplicial complexes consisting of simplexes (ab), (a), (b) and (xy), (x), (y), respectively. Suppose G is any abelian group. $C_p(K, G)$ and $C_p(L, G)$ will denote the group of p-chains of K over G and L over G, respectively. Define f_1: $C_1(K, G)\to C_1(L, G)$ as follows: $f_1\{g[ab]\}=g[xy]$, where $[ab]$ and $[xy]$ denote oriented 1-simplexes. Define f_0: $C_0(K, G)\to C_0(L, G)$ as follows: $f_0\{g_1[a]+g_2[b]\}=g_1[y]+g_2[x]$. The reader can easily verify that $f: K\to L$ is a pseudo-chain mapping. But it is not a chain mapping, for

$$\begin{aligned}\partial_1^L f_1 g[ab] &= \partial_1^L g[xy] = g[y]-g[x],\\ f_0\partial_1^K g[ab] &= f_0\{g[b]-g[a]\} = g[x]-g[y].\end{aligned}$$

Example 2. Let K and L be abstract simplicial complexes consisting of simplexes (abc), (ab), (bc), (ca), (a), (b), (c) and (xy), (yz), (zx), (x), (y), (z), respectively. Let G be any abelian group. Define f_2: $C_2(K, G)\to G_2(L, G)$ to be trivial and f_1: $C_1(K, G)\to C_1(L, G)$ as follows:

$$f_1\{g_1[ab]+g_2[bc]+g_3[ca]\} = g_1[xy]+g_2[yz]+g_3[zx]$$

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* Now at The Ohio State University.

† For every integer p let C_p be an additive abelian group and ∂_p: $C_p\to C_{p-1}$ a homomorphism satisfying the condition $\partial_{p-1}\partial_p\equiv 0$, i.e., for $c_p\in C_p$, then $\partial_{p-1}\partial_p c_p$ is the identity of C_{p-2}. A collection $\{C_p, \partial_p\}$ satisfying these conditions is called a Mayer Complex.

and f_0: $C_0(K, G) \to C_0(L, G)$ as follows:

$$f_0\{g_1[a] + g_2[b] + g_3[c]\} = g_1[x] + g_2[y] + g_3[z].$$

The reader will easily verify that f: $K \to L$ is a mapping, that $f_p\{Z_p(K, G)\} \subset Z_p(L, G)$ for all p, but $f_1\{B_1(K, G)\} \not\subset B_1(L, G)$.

Example 3. Let K and L be abstract simplicial complexes consisting of simplexes (ab), (bc), (ca), (a), (b), (c) and (xy), (yz), (x), (y), (z) respectively. G will be the coefficient group. Define f: $K \to L$ as follows:

$$f_1\{g_1[ab] + g_2[bc] + g_3[ca]\} = g_1[xy] + g_2[yz],$$
$$f_0\{g_1[a] + g_2[b] + g_3[c]\} = 0.$$

The reader can verify that f: $K \to L$ is a mapping, and that $f_p\{B_p(K, G)\} \subset B_p(L, G)$ for all p, but $f_1\{Z_1(K, G)\} \not\subset Z_1(L, G)$.

Thus Examples 2 and 3 show that the conditions (1) $f_p\{Z_p(K, G)\} \subset Z_p(L, G)$ and (2) $f_p\{B_p(K, G)\} \subset B_p(L, G)$ are independent.

3. The Δ-function. Let $K = \{C_p(K), \partial_p^K\}$ and $L = \{C_p(L), \partial_p^L\}$ be Mayer complexes and f: $K \to L$ a mapping, *i.e.*, for each p, f_p: $C_p(K) \to C_p(L)$ is a homomorphism.

DEFINITION 2. $\Delta(f_p)$: $C_p(K) \to C_{p-1}(L)$ as follows: $\Delta(f_p) = f_{p-1}\partial_p^K - \partial_p^L f_p$ for all integers p.

$\Delta(f_p)$ is clearly a homomorphism from $C_p(K)$ to $C_{p-1}(L)$.

THEOREM 1. *Let K and L be Mayer complexes and f: $K \to L$ a mapping. Then* $\Delta(f_p)\{Z_p(K)\} \subset B_{p-1}(L)$.

Proof. Let $z_p^K \in Z_p(K)$. Then

$$\begin{aligned}\Delta(f_p)\, z_p^K = (f_{p-1}\partial_p^K - \partial_p^L f_p) z_p^K &= f_{p-1}\partial_p^K z_p^K - \partial_p^L f_p z_p^K \\ &= f_{p-1}0 - \partial_p^L f_p z_p^K = \partial_p^L f_p(-z_p^K) \in B_{p-1}(L).\end{aligned}$$

COROLLARY 1. *Let K and L be Mayer complexes and f: $K \to L$ a mapping. Then $\Delta(f_p)$ induces the trivial homomorphism $\Delta(f_p)_*$: $H_p(K) \to H_{p-1}(L)$ on the homology groups of K and L respectively.*

Proof. Let $[z_p^K] \in H_p(K)$, *i.e.*, $[z_p^K]$ is the class of p-cycles homologous to z_p^K. Then $\Delta(f_p)_*\{[z_p^K]\} = [\Delta(f_p)z_p^K] = [b_{p-1}^L]$ (by Th. 1) $= [0]$.

THEOREM 2. *Let K, L, M be Mayer complexes and f: $K \to L$ and g: $L \to M$ be mappings. Then* $\Delta(g_p f_p) = g_{p-1}\Delta(f_p) + \Delta(g_p)f_p$.

Proof.

$$\Delta(g_p f_p) = g_{p-1}f_{p-1}\partial_p^K - \partial_p^M g_p f_p$$

$$
\begin{aligned}
&= g_{p-1}f_{p-1}\overset{K}{\partial}_p - g_{p-1}\overset{L}{\partial}_p f_p + g_{p-1}\overset{L}{\partial}_p f_p - \overset{M}{\partial}_p g_p f_p \\
&= g_{p-1}\{f_{p-1}\overset{K}{\partial}_p - \overset{L}{\partial}_p f_p\} + \{g_{p-1}\overset{L}{\partial}_p - \overset{M}{\partial}_p g_p\} f_p \\
&= g_{p-1}\Delta(f_p) + \Delta(g_p) f_p.
\end{aligned}
$$

THEOREM 3. *Let K and L be Mayer complexes and $f\colon K\to L$ a mapping. Then* $\partial^L_{p-1}\Delta(f_p)+\Delta(f_{p-1})\partial^K_p=0$.

Proof.

$$
\begin{aligned}
\overset{L}{\partial}_{p-1}\Delta(f_p) + \Delta(f_{p-1})\overset{K}{\partial}_p &= \overset{L}{\partial}_{p-1}\{f_{p-1}\overset{K}{\partial}_p - \overset{L}{\partial}_p f_p\} + \{f_{p-2}\overset{K}{\partial}_{p-1} - \overset{L}{\partial}_{p-1} f_{p-1}\}\overset{K}{\partial}_p \\
&= \overset{L}{\partial}_{p-1} f_{p-1}\overset{K}{\partial}_p - \overset{L}{\partial}_{p-1} f_{p-1}\overset{K}{\partial}_p = 0.
\end{aligned}
$$

THEOREM 4. *Let K, L, M be Mayer complexes and $f\colon K\to L$ and $g\colon L\to M$ be mappings. Then* $\Delta(g_{p-1})\Delta(f_p)+\partial^M_{p-1}\Delta(g_pf_p)=g_{p-2}\partial^L_{p-1}f_{p-1}\partial^K_p+\partial^M_{p-1}g_{p-1}\partial^L_pf_p$.

Proof.

$$
\begin{aligned}
&\Delta(g_{p-1})\Delta(f_p) + \overset{M}{\partial}_{p-1}\Delta(g_p f_p) \\
&\quad = (g_{p-2}\overset{L}{\partial}_{p-1} - \overset{M}{\partial}_{p-1} g_{p-1})(f_{p-1}\overset{K}{\partial}_p - \overset{L}{\partial}_p f_p) + \overset{M}{\partial}_{p-1}\{g_{p-1}f_{p-1}\overset{K}{\partial}_p - \overset{M}{\partial}_p g_p f_p\} \\
&\quad = g_{p-2}\overset{L}{\partial}_{p-1} f_{p-1}\overset{K}{\partial}_p - \overset{K}{\partial}_{p-1} g_{p-1} f_{p-1}\overset{K}{\partial}_p + \overset{M}{\partial}_{p-1} g_{p-1}\overset{L}{\partial}_p f_p + \overset{M}{\partial}_{p-1} g_{p-1} f_{p-1}\overset{K}{\partial}_p \\
&\quad = g_{p-2}\overset{L}{\partial}_{p-1} f_{p-1}\overset{K}{\partial}_p + \overset{M}{\partial}_{p-1} g_{p-1}\overset{L}{\partial}_p f_p.
\end{aligned}
$$

THEOREM 5. *Let K, L, M be Mayer complexes and $f\colon K\to L$ and $g\colon L\to M$ be mappings. Then* $g_{p-2}\Delta(f_{p-1})\partial^K_p+\partial^M_{p-1}\Delta(g_pf_p)+\Delta(g_{p-1})\Delta(f_p)=\partial^M_{p-1}\Delta(g_p)f_p$.

Proof. By Theorem 4 it suffices to show that

$$
g_{p-2}\Delta(f_{p-1})\overset{K}{\partial}_p - \overset{M}{\partial}_{p-1}\Delta(g_p)f_p = -\, g_{p-2}\overset{L}{\partial}_{p-1} f_{p-1}\overset{K}{\partial}_p - \overset{M}{\partial}_{p-1} g_{p-1}\overset{L}{\partial}_p f_p.
$$

This easy verification is left to the reader.

4. A characterization of pseudo-chain mappings.

LEMMA 1. *Let $f\colon K\to L$ be a mapping, where K and L are Mayer complexes. Then $f_p\{Z_p(K)\}\subset Z_p(L)$ if and only if $\Delta(f_p)\{Z_p(K)\}=0$.*

Proof. Sufficiency. Let $\Delta(f_p)\{Z_p(K)\}=0$. Take $z^K_p\in Z_p(K)$ and we assert that $f_pz^K_p\in Z_p(L)$. Now

$$
\overset{L}{\partial}_p f_p \overset{K}{z}_p = f_{p-1}\overset{K}{\partial}_p \overset{K}{z}_p - \Delta(f_p)\overset{K}{z}_p = f_{p-1}0 - \Delta(f_p)\overset{K}{z}_p = 0.
$$

Thus $f_pz^K_p\in Z_p(L)$.

Necessity. Let $z^K_p\in Z_p(K)$. Assert $\Delta(f_p)z^K_p=0$. Now

$$
\Delta(f_p)\overset{K}{z}_p = (f_{p-1}\overset{K}{\partial}_p - \overset{K}{\partial}_p f_p)\overset{K}{z}_p = f_{p-1}\overset{K}{\partial}_p \overset{K}{z}_p - \overset{K}{\partial}_p f_p \overset{K}{z}_p = 0.
$$

LEMMA 2. *Let $f: K \to L$ be a mapping. Then $f_p\{B_p(K)\} \subset B_p(L)$ if and only if $\Delta(f_p)\{C_p(K)\} \subset B_{p-1}(L)$.*

Proof. Sufficiency. Suppose $\Delta(f_p)\{C_p(K)\} \subset B_{p-1}(L)$ and take $b_p^K \in B_p(K)$. Then $f_p b_p^K = f_p \partial_{p+1}^K c_{p+1}^K = \Delta(f_{p+1})c_{p+1}^K + \partial_{p+1}^L f_{p+1} c_{p+1}^K$. Thus $f_p b_p^K \in B_p(L)$.

Necessity. Let $f_p\{B_p(K)\} \subset B_p(L)$ and take $c_p^K \in C_p(K)$. Then $\Delta(f_p)c_p^K = (f_{p-1}\partial_p^K - \partial_p^L f_p)c_p^K = f_{p-1}\partial_p^K c_p^K - \partial_p^L f_p c_p^K$. But $f_{p-1}\partial_p^K c_p^K \in B_{p-1}(L)$ and hence $\Delta(f_p)c_p^K \in B_{p-1}(L)$.

THEOREM 6. *Let $f: K \to L$ be a mapping. Then $f: K \to L$ is a pseudo-chain mapping if and only if* (1) $\Delta(f_p)\{Z_p(K)\} = 0$ and (2) $\Delta(f_p)\{C_p(K)\} \subset B_{p-1}(L)$.

The proof follows immediately from the above two lemmas.

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LINEAR TRANSFORMATIONS OF SEQUENCES

A. V. BOYD, Aberdeen, Scotland

Given a sequence $\{s_m\}$ $(m=1, 2, \cdots)$ of real numbers we define s'_m for $m \geqq n$ by $s'_m = \alpha_1 s_m + \cdots + \alpha_n s_{m-n+1}$ where the α_i are real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$.

Kubota [3] has shown that the convergence of $\{s'_m\}$ to a finite limit s implies the convergence of $\{s_m\}$ to s if and only if $f(z) \equiv \alpha_1 z^{n-1} + \cdots + \alpha_n$ has no zeros in $|z| \geqq 1$. Conditions for this have recently been investigated in [1] and [2]. A simple sufficiency condition will here be deduced from the following result obtained by Parodi [4] by an easy application of Rouché's theorem on the zeros of a function:

If $\lambda_1, \cdots, \lambda_n$ are positive numbers such that $\lambda_1 + \cdots + \lambda_n = 1$ and $M = \max_{i=1,\cdots,n} |a_i/\lambda_i|^{1/i}$ then all the roots of $z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = 0$ lie in $|z| \leqq M$.

THEOREM. *If $\alpha_1 s_m + \cdots + \alpha_n s_{m-n+1}$ tends to a finite limit s as m tends to infinity, where the α_i are real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$ and $|\alpha_1| > |\alpha_2| + \cdots + |\alpha_n|$, then s_m tends to s as m tends to infinity.*

Proof. Clearly $\alpha_1 \neq 0$ so that the numbers $a_i = \alpha_{i+1}/\alpha_1$, $(i=1$ to $n-1)$, are finite. Also, if we put $|\alpha_1| - |\alpha_2| - \cdots - |\alpha_n| = \epsilon(n-1)|\alpha_1|$, then $\epsilon > 0$. Next, if we put $\lambda_i = |a_i| + \epsilon$, $(i=1$ to $n-1)$, then the λ_i are all positive, and

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} = (|\alpha_2| + \cdots + |\alpha_n|)/|\alpha_1| + (n-1)\epsilon = 1.$$

On applying Parodi's theorem with n replaced by $n-1$ we have

$$M = \max_{i=1,\cdots,n-1} \{|a_i|/(|a_i| + \epsilon)\}^{1/i} < 1,$$

so that all the roots of

$$\frac{f(z)}{\alpha_1} \equiv z^{n-1} + \frac{\alpha_2}{\alpha_1} z^{n-2} + \cdots + \frac{\alpha_{n-1}}{\alpha_1} z + \frac{\alpha_n}{\alpha_1} = 0$$

lie in $|z| \leqq M < 1$. It follows from Kubota's result stated above that if $\{s'_m\}$ converges to a finite limit then $\{s_m\}$ converges to the same limit.

Remarks. Dr. D. Borwein has drawn my attention to the fact that as a by-product of his work on power series, Schur [5; Satz 17] found that the zeros of the $(n-1)$th degree polynomial $c_1 z^{n-1} + \cdots + c_{n-1}z + c_n$ lie in $|z| < 1$ if and only if all the determinants

$$\Delta_{r+1} = \begin{vmatrix} c_1 & 0 & \cdots 0 & c_n & c_{n-1} & \cdots c_{n-r} \\ c_2 & c_1 & \cdots 0 & 0 & c_n & \cdots c_{n-r+1} \\ \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots\cdot \\ \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots\cdot \\ \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots\cdot \\ c_{r+1} & c_r & \cdots c_1 & 0 & 0 & \cdots c_n \\ c_n & 0 & \cdots 0 & c_1 & c_2 & \cdots c_{r+1} \\ c_{n-1} & c_n & \cdots 0 & 0 & c_1 & \cdots c_r \\ \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots\cdot \\ \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots\cdot \\ \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots\cdot \\ c_{n-r} & c_{n-r+1} & \cdots c_n & 0 & 0 & \cdots c_1 \end{vmatrix}$$

are strictly positive for $r=0$ to $n-2$. By Kubota's result this therefore provides a set of necessary and sufficient conditions for the convergence of $\{s'_m\}$ to a finite limit to imply the convergence of $\{s_m\}$ to the same limit. When $n=3$ and $\alpha_1 \neq 0$ these conditions reduce, as do those found in [1], to $\alpha_2 < \frac{1}{2}$ and $2\alpha_1 + \alpha_2 > 1$. In this case the condition $|\alpha_1| > |\alpha_2| + |\alpha_3| = |\alpha_2| + |1 - \alpha_1 - \alpha_2|$ is satisfied if and only if $\alpha_1 > \frac{1}{2}$, $\alpha_2 < \frac{1}{2}$ and $\alpha_1 + \alpha_2 > 1$. The condition derived here is therefore a sufficient but not a necessary one. For $n \geqq 4$, however, it is much easier to apply than Schur's conditions or than those derived by the method indicated in [2].

The simplicity of the above condition also suggests that it should be possible to find a proof of the theorem depending entirely on real-variable arguments.

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MARKOV CHAINS AS RANDOM INPUT AUTOMATA*

A. S. DAVIS, University of Oklahoma

By making use of certain matrices which have direct probabilistic interpretations (thus avoiding eigenvalues), Kemeny and Snell [1] have recently given a unified and very meaningful presentation of the theory of finite Markov chains. In the present note I wish to indicate a somewhat different matrix-theoretic approach. Its significant feature is that it establishes a connection between Markov chains and automata theory. The connection is based on a theorem which is reminiscent of Birkhoff's result concerning doubly stochastic matrices [2] and which leads to a simple proof of the fundamental theorem (generalized) of regular finite Markov chains.

A *finite automaton* (or *finite deterministic system with input*) consists of a set X of *states* together with a set $\{\Gamma_\theta\}$ of mappings Γ_θ: $X \rightarrow X$ which determine the behavior of the system for each value of the *input parameter* θ. We shall call such a system (X, $\{\Gamma_\theta\}$) a *box* for short. (This is suggested by E. F. Moore's use of the term "black box" [3], viz., a box plus a function which assigns to each state a value from some set of "outputs.") A box of order n (*i.e.*, with n states) can have at most n^n different input values. There exist exactly $2^{n^n}-1$ different boxes of order n. (Thus we require that neither X nor $\{\Gamma_\theta\}$ be void.)

We think of a box as moving through its states in discrete units of time. Given an initial state and a sequence of input values, the sequence of states through which a box moves is completely determined. An elevator in an n-story building is a box whose states are the different floors and whose input is provided by the call-buttons on each floor and the control buttons within. These ideas are more fully discussed in [4].

A *finite Markov chain* consists of a finite set X of states together with a probability rule (p_{ij}) which specifies, for each pair x_i, $x_j \in X$, the probability that the system moves to state x_j, given that it is in state x_i. The system (X, (p_{ij})) will be called a *chain* for short. Given an initial state, the behavior of a chain is completely (but stochastically) determined by the p_{ij}. The reader is referred to [5] for examples and for a brief survey of the Kemeny and Snell approach.

Consider now a box whose consecutive input values are selected randomly and independently by a sampling device of some kind. The box (X, $\{\Gamma_\theta\}$) together with the probability distribution $q(\theta)$ over the input values will be called a *box with random input*. Its behavior is exactly that of a Markov chain. More remarkable, however, is that the converse is also true.

Every box with random input realizes a chain of the same order. Conversely, for every chain of order n, a box of order n with random input can be constructed as a realization. For this construction, no more than n^2-n+1 input values need be used.

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Actually, this is a corollary to a more general result which in a moment we shall state and prove (in slightly different language). Here we note a possible application. Suppose an investigator has before him a rather complex system which he discovers to be a finite Markov chain. He suspects that the system is actually deterministic (in some relatively simple way) and that its probabilistic behavior is due to external sources of random influence of which he may or may not be aware. The above theorem says that his conjecture is not at all a logical impossibility and that the number of input values may be reasonably few. (In any case, there can be no more than $n \log_2 n$ bits of imput variation.)

At this point it is convenient to use matrix terminology. Just as with each chain we associate an $n \times n$ matrix (p_{ij}), with each box we can associate a set of $n \times n$ matrices D_θ, one for each mapping Γ_θ, in the following manner: If θ is held fixed, the mapping $\Gamma_\theta: X \to X$ describes a chain in which all probabilities are 0 or 1. D_θ is just the corresponding $n \times n$ matrix of (degenerate) probabilities. But now we wish to generalize to rectangular matrices.

An $m \times n$ matrix (p_{ij}) is *stochastic* if all entries are nonnegative and each row sums to 1. Hence the totality of all $m \times n$ stochastic matrices forms a convex set of dimension $m(n-1)$. Contained in this set are the $m \times n$ *deterministic matrices* of the form $D(j_1, \cdots, j_m)$, wherein the ith row has 1 in column j_i and zeros elsewhere. A deterministic matrix of the form $D(k, \cdots, k)$ will be called a *star*.

THEOREM 1. *An $m \times n$ matrix M is stochastic if and only if it is a barycenter of $m \times n$ deterministic matrices*:

$$M = q_1 D_1 + \cdots + q_r D_r \left(\sum_i q_i = 1, \; q_i \geqq 0 \right).$$

In this expression, r need be no greater than $m(n-1)+1$.

Remark. By specializing to square matrices and reinterpreting in terms of chains and boxes with random input (given by the distribution $q_1, \cdots, q_r$), we obtain the formulation which originally suggested this theorem.

Proof. That a barycenter is stochastic follows immediately from the fact that we are working in a convex set which contains the D_i. Suppose now that $M = (p_{ij})$ is stochastic. We assert that

$$M = \sum_{j_1 \cdots j_m} p_{1j_1} \cdots p_{mj_m} D(j_1, \cdots, j_m), \tag{1}$$

where the j_i run independently from 1 to n, and that the sum of the coefficients is 1. To obtain the entry in the ath row and bth column, we add up the coefficients of the matrices $D(j_1, \cdots, j_{a-1}, b, j_{a+1}, \cdots, j_m)$ over all j_i $(i \neq a)$. Since for each i, $\sum_{j_i} p_{ij_i} = 1$, we get precisely p_{ab}, as desired. For the same reason, the sum of all coefficients is 1, so that M is indeed a barycenter of deterministic matrices. That their number may be reduced to $m(n-1)+1$ follows from the fact that the

convex set is of dimension $m(n-1)$.

Equation (1) will be referred to as *the representation of M in expanded form.* Let the terms be numbered so that the stars come first, *i.e.*, so that $D_k = D(k, \cdots, k)$ for $k=1, \cdots, n$.

Now specialize once more to square matrices. We note that under the rule for multiplying $n \times n$ deterministic matrices,

$$D(a_1, \cdots, a_n)D(b_1, \cdots, b_n) = D(b_{a_1}, \cdots, b_{a_n}),$$

stars are preserved: $D(a_1, \cdots, a_n)D_k = D_k$ and $D_kD(b_1, \cdots, b_n) = D_{b_k}$, for $k=1, \cdots, n$. This fact, together with the representation of a stochastic matrix in expanded form, gives us

THEOREM 2. *Let a square stochastic matrix M be called* **semiregular** *if for some t a star appears in the representation of* M^t *in expanded form. Let the* **degree of regularity** *be the maximum number of stars that appear (for any power t). Then, if M is semiregular, its powers converge to a matrix whose rows are all identical and whose degree of regularity is the same as that of M.*

Proof. Let S_t be the sum of the coefficients of the stars in $M^t = q_1^{(t)}D_1 + \cdots + q_{n^n}^{(t)}D_{n^n}$. Since M is semiregular, $S_t \neq 0$ for some t. We might as well assume $S_1 \neq 0$. We shall show that $\lim_{t\to\infty} S_t = 1$, thus that $\lim_{t\to\infty} M^t = q_1^* D_1 + \cdots + q_n^* D_n (\sum_i q_i^* = 1)$. That the $q_i^{(t)}$ form nondecreasing sequences bounded above (by 1), and hence that these converge to the q_i^* for $1 \leqq i \leqq n$, will be evident from the way stars are preserved under multiplication. Consider $M^{t+1} = MM^t$:

$$\begin{aligned} &\left[\sum_{i=1}^{nn} q_i D_i\right]\left[\sum_{j=1}^{nn} q_j^{(t)} D_j\right] \\ &= \sum_{i=1}^{nn} q_i D_i \sum_{j=1}^{nn} q_j^{(t)} D_j + \sum_{i=1}^{nn} q_i D_i \sum_{j=n+1}^{nn} q_j^{(t)} D_j + \text{other terms} \\ &= \sum_{j=1}^{n} q_j^{(t)} D_j + \sum_{i=1}^{n} q_i \sum_{j=n+1}^{nn} q_j^{(t)} D_{i(j)} + \text{other terms}, \end{aligned}$$

where $1 \leqq i(j) \leqq n$, by the product rule for stars. Collecting the coefficients of the stars in sight, we get $S_{t+1} \geqq S_t + S_1(1-S_t)$. Then $1 - S_{t+1} \leqq (1-S_1)(1-S_t) \leqq (1-S_1)^{t+1}$. Thus if $S_1 \neq 0$, then $1 - S_t \to 0$, as asserted.

Note that a semiregular matrix can be recognized by the fact that one of its powers has a column with nonvanishing entries. Semiregular matrices are not associated with ergodic chains unless the degree of regularity is n, in which case Theorem 2 is just the fundamental theorem for regular chains.

Addendum: Within the context of the theory of convex polyhedra, Birkhoff's theorem is also an easy consequence of Theorem 1. By definition, the set of all $n \times n$ doubly stochastic matrices is the intersection of the set of $n \times n$ stochastic matrices with the set of their transposes. By Theorem 1 (and its "transpose"),

this is the intersection of two $n(n-1)$-dimensional convex polyhedra in n^2-dimensional space, hence is itself a convex polyhedron (of dimension $(n-1)^2$). The vertices are just the $n\times n$ permutation matrices. For if M is doubly stochastic, but not a permutation matrix, then M contains at least four elements p_{ab}, p_{ac}, p_{db}, p_{dc}, such that $\min(p_{ab}, 1-p_{ab}, \cdots, 1-p_{dc})=\epsilon>0$. Construct M_1 and M_2 by replacing these four elements in M with $p_{ab}+\epsilon$, $p_{ac}-\epsilon$, $p_{db}-\epsilon$, $p_{dc}+\epsilon$ and $p_{ab}-\epsilon$, $p_{ac}+\epsilon$, $p_{db}+\epsilon$, $p_{dc}-\epsilon$, respectively. Then $M=\frac{1}{2}M_1+\frac{1}{2}M_2$ lies between two doubly stochastic matrices and so is not a vertex. In short, the set of $n\times n$ doubly stochastic matrices is a convex polyhedron whose vertices are the $n\times n$ permutation matrices. This is Birkhoff's theorem.

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CORRECTION

A paper by S. Chowla and F. B. Correia (this MONTHLY, vol. 67, 1960, p. 884) pointed out the falsity of the conjecture $p_1\cdot p_2\cdot\cdots\cdot p_{\pi(N)}\sim e^N$ made by E. Trost in his book *Primzahlen, Bd. II* (Basel-Stuttgart, 1953). Professor Lowell Schoenfeld has pointed out that this error was noted by him in his review of this book (Bull. Amer. Math. Soc., vol. 62, 1956, pp. 54–57).

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A PROOF OF THE EULER-FERMAT THEOREM

ROBERT WEINSTOCK, Oberlin College

1. Introduction. The object here is to present a proof of the Euler-Fermat theorem—namely, *if* $(x, m)=1$, *then* $x^{\phi(m)}-1\equiv 0 \pmod m$—that is based upon (i) the Fermat theorem (special case of the foregoing when m is prime) and (ii) a standard method for determining solutions of a polynomial congruence $f(x)\equiv 0 \pmod{p^k}$ from knowledge of the solutions of $f(x)\equiv 0 \pmod{p^{k-1}}$. Here, as usual, $\phi(m)$ is the number of positive integers $\leqq m$ and relatively prime to m.

Throughout we set

$$(1) \qquad f(x) = x^{p-1} - 1, \qquad f'(x) = (p-1)x^{p-2},$$

the latter being the derivative of the former; p denotes an arbitrary prime. We may thus state (and we accept as given) the Fermat theorem in the form

$$(2) \qquad f(x) \equiv 0 \pmod{p} \text{ provided that } (x, p) = 1.$$

The standard method (ii) alluded to above reads as follows:

If y is any solution of $f(x) \equiv 0 \pmod{p^{k-1}}$, *then*

$$(3) \qquad z = y + tp^{k-1}, \quad \textit{where} \quad f'(y)t \equiv -\left[f(y)/p^{k-1}\right] \pmod{p},$$

satisfies $f(z) \equiv 0 \pmod{p^k}$, *provided that* $f'(y) \not\equiv 0 \pmod{p}$. (See [1], pp. 96–97.)

2. Proof for $m = p^r$. Since $\phi(p^r) = p^{r-1}(p-1)$, the Euler-Fermat theorem for $m = p^r$ assumes the form $f(x^{p^{r-1}}) \equiv 0 \pmod{p^r}$ for all x satisfying $(x, p) = 1$. By the Fermat theorem (2) this holds for $r = 1$; we suppose that it holds for $r = k-1$ and proceed to demonstrate that it then holds for $r = k$ ($k \geqq 2$). By the mathematical-induction principle, its validity for all positive integers r then follows.

With u arbitrary to within $(u, p) = 1$, our assumption is that $y = u^{p^{k-2}}$ is a solution of $f(x) \equiv 0 \pmod{p^{k-1}}$. According to (1), the second of (3) reads

$$(4) \qquad (p-1)y^{p-2}t \equiv -\left[(y^{p-1} - 1)/p^{k-1}\right] \pmod{p}$$

—with $(u, p) = 1$, and consequently $(y, p) = 1$, insuring that $f'(y) \not\equiv 0 \pmod{p}$. Since $(p-1) \equiv -1 \pmod{p}$ and $y^{p-1} \equiv 1 \pmod{p}$, by (2), multiplication of both sides of (4) by $-y$ yields $t \equiv \left[(y^p - y)/p^{k-1}\right] \pmod{p}$. Thus, according to (3), we have that $z = y + \left[(y^p - y)/p^{k-1}\right]p^{k-1} = y^p = u^{p^{k-1}}$ satisfies $f(z) \equiv 0 \pmod{p^k}$; that is, $f(u^{p^{k-1}}) \equiv 0 \pmod{p^k}$, which was to have been demonstrated.

3. Proof for arbitrary m. Passage from the case $m = p^r$ to general $m = \prod_{j=1}^{n} p_j^{r_j}$, where $p_1, \cdots, p_n$ are distinct primes, is effected through (i) $\phi(m) = \prod_{j=1}^{n} \phi(p_j^{r_j})$ and (ii) if $a \equiv b \pmod{p_j^{r_j}}$ for all $j = 1, \cdots, n$, then $a \equiv b \pmod{m}$. Indeed, if ϕ_j denotes $\phi(p_j^{r_j})$, then $x^{\phi_j} \equiv 1 \pmod{p_j^{r_j}}$ $j = 1, \cdots, n$, provided that $(x, m) = 1$, by Section 2. We proceed, for each $j = 1, \cdots, n$, to raise the jth of these n congruences to the power $\left[\phi(m)/\phi(p_j^{r_j})\right]$, from which we obtain

$$x^{\phi(m)} \equiv 1 \pmod{p_j^{r_j}}.$$

By (ii) we therefore have $x^{\phi(m)} \equiv 1 \pmod{m}$ whenever $(x, m) = 1$, which is the Euler-Fermat theorem.

4. Remark. The result achieved in Section 2 is closely related to the theorem, proved in [1], page 97, to the effect that $x^{p-1} - 1 \equiv 0 \pmod{p^r}$ has precisely $p-1$ roots for each positive integer r. These roots are identified in Section 2 as $u^{p^{r-1}}$ for $u = 1, \cdots, p-1$.

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THE GENERAL MONGÉ EQUATION AND ITS EXTENSION

L. V. ROBINSON, Wright-Patterson Air Force Base

In the general nonlinear partial differential equation of Mongé,

$$
\begin{gathered}
Rr + Ss + Tt + U(rt - s^2) = V \\
p \equiv \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r \equiv \frac{\partial^2 z}{\partial x^2}, \quad s \equiv \frac{\partial^2 z}{\partial x \partial y}, \quad t \equiv \frac{\partial^2 z}{\partial y^2},
\end{gathered} \tag{1}
$$

it will be assumed here that R, S, T, U, and V are functions of x, y, z, p, and q. By making use of the identities,

$$
\begin{aligned}
pdx + qdy - dz &= 0, \qquad & (rt - s^2)dx + sdq - tdp &= 0, \\
rdx + sdy - dp &= 0, \qquad & (rt - s^2)dy + sdp - rdq &= 0, \\
sdx + tdy - dq &= 0,
\end{aligned} \tag{2}
$$

and combinations of these, an equation of the Mongé type is derived; and, in terms of quantities used, total differential equations are established for finding whatever integrals may exist. These integrals then are given by (7) and (8). Although these total differential equations can be derived from the pair ordinarily found in textbooks on the subject, they have the advantage also that the usual test for integrability may be applied. Still another advantage inherent in the method offered here is that it may be extended to equations involving more than the two independent variables x and y appearing in (1).

Although the right pair of (2) is obtained from the second and third of the left members—and the equations are therefore not independent—it will be seen that certain advantages are gained by including them in arriving at (1) and intermediate integrals. The new approach offered here consists in introducing auxiliary functions of x, y, z, p, and q—namely, P, Q, X, and Y, where $PQXY \neq 0$. Then, after (2), excluding the first in the left members, are multiplied, respectively, by $RSVY$, $STVX$, $QSTU$, $PRSU$, and added, the results may be put in the form,

$$
\begin{aligned}
&S(VYdx - PUdq)Rr + [U(PRdp + QTdq) + V(TXdx + RYdy)]Ss \\
&+ S(VXdy - QUdp)Tt + S(QTdx + PRdy)U(rt - s^2) = S(RYdp + TXdq)V.
\end{aligned} \tag{3}
$$

Now, this equation is equivalent to equation (1), provided that $S \neq 0$ was assumed and

$$
\begin{aligned}
VYdx - PUdq &= VXdy - QUdp = QTdx + PRdy = RYdp + TXdq \\
&= S^{-1}[U(PRdp + QTdq) + V(TXdx + RYdy)].
\end{aligned} \tag{4}
$$

In (4), it is possible to define Q_1, Q_2, W_1, and W_2 in such a way that

$$
\begin{gathered}
QT - VY = PQ_1, \quad PU + TX = -Q_2Y, \quad PR - VX = QW_2, \\
QU + RY = -W_1X.
\end{gathered} \tag{5}
$$

Recalling now that, by hypothesis, $PQXY \neq 0$, one may conclude that

$$(6)\qquad \begin{aligned} Q_1dx + Rdy + Udq &= 0, & Vdy + W_1dp - Tdq &= 0,\\ Vdx - Rdp + Q_2dq &= 0, & Tdx + W_2dy + Udp &= 0. \end{aligned}$$

These and (3) yield therefore

$$\begin{aligned} S(VYdx - PUdq) &= U(PRdp + QTdq) + V(TXdx + RYdy)\\ &= PU(Vdx + Q_2dq) + QTUdq + VTXdx - VY(Q_1dx + Udq)\\ &= V(PU + TX - Q_1Y)dx + U(QT + PQ_2 - VY)dq\\ &= -(Q_1 + Q_2)(VYdx - PUdq). \end{aligned}$$

That is to say $Q_1+Q_2=-S$. Also, from (6), it follows that

$$\begin{aligned} Q_1dxdp &= -(Rdy + Udq)dp = -(Rdp)dy - (Udp)dq\\ &= -(Vdx + Q_2dq)dy + (Tdx + W_2dy)dq\\ &= (Tdq - Vdy)dx + (W_2 - Q_2)dydq = W_1dxdp + (W_2 - Q_2)dydq. \end{aligned}$$

Then, on equating the coefficients of $dxdp$ and $dydq$, one sees that

$$Q_1 = W_1, \qquad Q_2 = W_2, \qquad S = -(W_1 + W_2).$$

When Q_1 and Q_2 are replaced by W_1 and W_2, respectively, in (6), it readily follows from the eliminant of the differential terms that $(RT+UV-W_1W_2)^2=0$. Then, inasmuch as

$$W_1 + W_2 = -S, \qquad W_1W_2 = RT + UV,$$

W_1 and W_2 must be the roots of the quadratic, $W^2+SW+(RT+UV)=0$.

It is worthy of note at this point that, when it is remembered that $Q_1=W_1$ and $Q_2=W_2$ in (5), P, Q, X, and Y in (5) are replaced by dp, dq, dx, and dy, respectively, in (6). Hence, if a matrix μ be defined as

$$(7)\qquad \left\| \begin{matrix} -R & W_2 & V & 0\\ W_1 & -T & 0 & V\\ U & 0 & T & W_2\\ 0 & U & W_1 & R \end{matrix} \right\| \equiv \mu,$$

equations (5) and (6) may now be put into the forms,

$$(8)\qquad \mu\cdot\left\| \begin{matrix} dp\\ dq\\ dx\\ dy \end{matrix} \right\| = 0, \qquad \mu\cdot\left\| \begin{matrix} P\\ Q\\ X\\ Y \end{matrix} \right\| = 0.$$

In seeking intermediate integrals or solutions of (1), it is frequently necessary to add to the first four of equations (8) the identity $dz=pdx+qdy$. It is readily

shown also that, from (4) and (5),

$$\frac{R}{Qdx - Xdq} = \frac{T}{Ydp - Pdy} = \frac{U}{Xdy - Ydx} = \frac{V}{Qdp - Pdq} = \frac{W_1}{Ydq - Qdy} = \frac{W_2}{Pdx - Xdp}.$$

The extension of this method consists, of course, in replacing (2) by their more general counterparts, where the number of independent variables is greater than two, and following through the above reasoning.

To consider a particular case, suppose the given equation is

$$R_1u_{xx} + R_2u_{yy} + R_3u_{zz} + S_1u_{yz} + S_2u_{xz} + S_3u_{xy} = V \qquad u_x \equiv \frac{\partial u}{\partial x}, \qquad u_{xy} \equiv \frac{\partial^2 u}{\partial x \partial y}, \text{ etc.,} \tag{9}$$

where R_1, R_2, R_3, S_1, S_2, S_3, and V are functions of x, y, z, u_x, u_y, and u_z. The method proposed therefore depends upon the assumption that (9) is derived from a linear combination of the equations,

$$\begin{aligned} u_xdx + u_ydy + u_zdz &= du, \\ u_{xx}dx + u_{xy}dy + u_{xz}dz &= du_x, \\ u_{xy}dx + u_{yy}dy + u_{yz}dz &= du_y, \\ u_{xz}dx + u_{yz}dy + u_{zz}dz &= du_z. \end{aligned} \tag{10}$$

Suppose therefore that the second, third, and fourth of equations (10) are multiplied by R_1, MR_2, NR_3, respectively, and added. The result is

$$\begin{aligned} R_1u_{xx}dx &+ MR_2u_{yy}dy + NR_3u_{zz}dz \\ &+ (R_1dy + MR_2dx)u_{xy} \\ &+ (MR_2dz + NR_3dy)u_{yz} + (R_1dz + NR_3dx)u_{xz} \\ &= R_1du_x + MR_2du_y + NR_3du_z. \end{aligned}$$

By multiplying (9) by dx and comparing the two results, one concludes that

$$\begin{aligned} dx = Mdy = Ndz, \qquad & S_1dx = MR_2dz + NR_3dy, \\ S_2dx = R_1dz + NR_3dx, \qquad & S_3dx = R_1dy + MR_2dx, \\ Vdx = R_1du_x + & MR_2du_y + NR_3du_z. \end{aligned} \tag{11}$$

From these equations, the two pairs,

$$\begin{aligned} (S_2 - NR_3)dx &= R_1dz, \qquad & (S_3 - MR_2)dx &= R_1dy, \\ dx &= Ndz, & dx &= Mdy, \end{aligned}$$

then require that

$$R_2M^2 - S_3M + R_1 = 0, \qquad R_3N^2 - S_2N + R_1 = 0.$$

From these equations M and N may be obtained therefore, but these are subject to a restriction imposed by a third equation obtained from the first line of the equations in (11). This is, the values of M and N used must satisfy

$$R_2\left(\frac{M}{N}\right)^2 - S_1\left(\frac{M}{N}\right) + R_3 = 0.$$

In general, two values of M and N can therefore be used in (11) in seeking possible integrals of (9).

Admittedly, equation (9) is a very simple one to be used as an example. Perhaps it suffices however, to justify the claim made in the last sentence of the first paragraph.

A NOTE ON SUMS OF POWERS OF INTEGERS

DAVID ALLISON, University of Cape Town, South Africa

The identity

$$\sum_1^n r^3 = \left(\sum_1^n r\right)^2 \tag{1}$$

is well known, and the question naturally arises, for what positive integers k, m, p, q does the identity

$$\left(\sum_1^n r^k\right)^p = \left(\sum_1^n r^m\right)^q \tag{2}$$

hold? Clearly no cases of interest will be lost by supposing that $p>q$ and that p and q are coprime; and with these assumptions it is easy to show that (1) is the only solution of (2).

The proof depends on the well-known facts that $\sum_{r=1}^n r^m$ is a polynomial in n, of degree $m+1$, and that the coefficient of n^{m+1} is $1/(m+1)$. Suppose then that (2) holds, *i.e.*, that

$$\left(\frac{n^{k+1}}{k+1} + \cdots\right)^p = \left(\frac{n^{m+1}}{m+1} + \cdots\right)^q,$$

with p and q as described above. Then $p(k+1) = q(m+1)$, $(k+1)^p = (m+1)^q$. Thus $(m+1)^q = (p/q)^q(k+1)^q$, whence $(k+1)^{p-q} = (p/q)^q$, and the R.H.S. of this is an integer because the L.H.S. is. Thus $q=1$, and $(k+1)^{p-1} = p$. But $k+1 \geqq 2$; therefore $2^{p-1} \leqq p$, so that $p<3$. Since $p>q$, we have $p=2$, $k=1$, $m=3$.

Reference

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THE REMAINDER TERM IN NUMERICAL INTEGRATION FORMULAS

E. B. LEACH, Case Institute of Technology

Numerical integration formulas give a linear combination of function values on a finite set of points to approximate a definite integral. The coefficients are chosen so that the formula is exact whenever the function f is a polynomial of sufficiently small degree. The operation may evidently be written as a Stieltjes integral having integrand f, whose integrator has jump discontinuities corresponding to the coefficients of the formula. The exact value of the integral is similar in form, so the remainder may be written as a Stieltjes integral:

$$R(f) = -\int_a^b f(x)dG_0(x). \tag{1}$$

The limits a and b are chosen so that $[a, b]$ includes the range of the definite integral and the points involved in the numerical formula. G_0 is chosen so that $G_0(a)=0$. If $R(f)=0$, when $f(x)\equiv 1$, then $G_0(b)=0$, and integration by parts* gives:

$$R(f) = \int_a^b f'(x)G_0(x)dx. \tag{2}$$

Suppose m is the largest integer such that $R(f)=0$ for every polynomial of degree $\leqq m$. We may define functions G_i, $i\leqq m$ inductively by:

$$G_i(x) = -\int_a^x G_{i-1}(t)dt. \tag{3}$$

We may then integrate (2) by parts repeatedly (this time within the context of Riemann integration) to obtain,† if $i\leqq m$:

$$R(f) = \int_a^b f^{(i+1)}(x)G_i(x)dx. \tag{4}$$

In proving (4) inductively we use the fact that $R(f)=0$, if $f=1, x, \cdots, x^m$. Then if (4) is true for any integer $j<m$, and we set $f(x)=x^{j+1}/(j+1)!$, (4) gives $0=G_{j+1}(b)$, and so the outside terms in the integration by parts formula are zero.

It sometimes happens that $G_m(x)$ does not change sign in $[a, b]$. In this case, the mean value theorem permits an estimate that may be more convenient:

$$R(f) = f^{m+1}(\xi)\int_a^b G_m(x)dx. \tag{5}$$

It is not obvious that $G_i(x)$ does not change sign even in simple particular

* We assume that f is sufficiently smooth to justify these manipulations.

† For a different derivation of this form of the remainder, see [1], p. 161 ff.

cases. In particular cases it is usually easy to observe the number of sign changes of $G_1(x)$ in $[a, b]$. But by the mean value theorem $G_{i-1}(x)$ has at least one more sign change than $G_i(x)$, since $G_i(a) = G_i(b) = 0$. So $G_1(x)$ has at least $m-1$ more sign changes than $G_m(x)$, and if $G_1(x)$ has only $m-1$ sign changes, then $G_m(x)$ has no sign changes.

Consider for example the approximation of $\int_0^3 f(x)dx$ by the value $\frac{3}{8}[f(0)+3f(1)+3f(2)+f(3)]$. The kernel G_0 chosen to satisfy (1) is:

$$G_0(x) = \begin{cases} 0, & \text{if } x = 0, \text{ or } x = 3; \\ \frac{3}{8} - x, & \text{if } 0 < x < 1; \\ \frac{3}{2} - x, & \text{if } 1 \leqq x \leqq 2; \\ \frac{21}{8} - x, & \text{if } 2 < x < 3. \end{cases}$$

It is evident that G_0 is an odd function of $x - \frac{3}{2}$. The succeeding G_i's are alternately even and odd functions of $x - \frac{3}{2}$, and need be calculated only to the midpoint of the interval. $G_1(x)$, calculated by (3) is

$$G_1(x) = \begin{cases} -\frac{3}{8}x + \frac{1}{2}x^2, & 0 \leqq x \leqq 1; \\ \frac{9}{8} - \frac{3}{2}x + \frac{1}{2}x^2 & \text{if } 1 \leqq x \leqq 2. \end{cases}$$

In particular, $G_1(1) = \frac{1}{8}$ and $G_1(\frac{3}{2}) = 0$. Then G_1 changes sign twice in (0, 3) (see Fig. 1). Since $m = 3$ in this case, G_3 does not change sign, and the estimate (5) is valid.

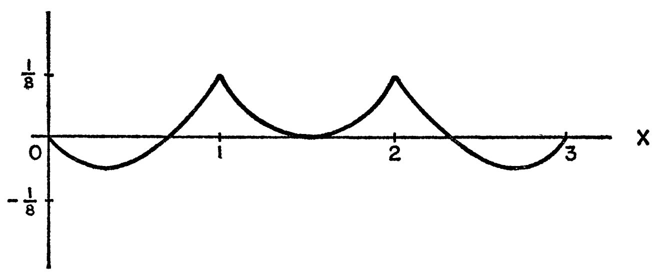

FIG. 1

Hildebrand ([1], p. 73) gives a list of Newton-Cotes formulas for $\int_0^{nh} f(x)dx$, using $f(0), f(h), \cdots, f(nh)$, for $1 \leqq n \leqq 8$. In these cases, the number of sign-changes of G_1 is $m-1$, with the exception of the six-point formula,* $n=5$. In the case of Weddle's rule, six sign changes appear in G_1, and so G_1 could have two sign changes. In this case G_6 really does have two sign changes, and an error estimate of the form (5) is not possible. A somewhat more complicated estimate is given by Hildebrand ([1], p. 160).

* It is known that the remainder estimate (5) is valid for all of the Newton-Cotes formulas. However, the proof is not especially easy (see [2], p. 154 ff., for the case n even and [3] for the case n odd).

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ELEMENTARY UNIQUENESS THEOREMS FOR DIFFERENTIAL EQUATIONS

D. A. KEARNS, Merrimack College, North Andover, Mass.

A first course in differential equations probably is not the place to carry through detailed proofs of existence theorems—nor is it necessary to do so since, for the most part, only equations whose solutions can be exhibited are considered. Nevertheless, it is desirable to have uniqueness theorems available to the student in order that he might make use of the many results which depend upon the unique determination of functions by differential equations and initial conditions.

Our purpose is to present such theorems for first and second order linear equations which are elementary in nature and which do not depend upon any particular procedure for the construction of solutions.

The method we discuss uses an inequality which is well known (R. E. Bellman, *Stability Theory of Differential Equations*, New York, 1953) but its derivation is included for the sake of completeness. The method itself is used extensively in the book mentioned for different purposes.

We will assume once and for all the conditions necessary for the existence of the integrals which appear in the calculations.

LEMMA. *If $u(t)$ and $v(t)$ are nonnegative, if k is a positive constant, and if*

$$u(t) \leqq k + \int_0^t u(x)v(x)dx, \tag{1}$$

then,

$$u(t) \leqq k \exp\left(\int_0^t v(x)dx\right). \tag{2}$$

Proof. From (1) it follows that

$$\frac{u(t)v(t)}{k + \int_0^t u(x)v(x)dx} \leqq v(t).$$

By integrating from 0 to t, we have

$$\ln\left(k + \int_0^t u(x)v(x)dx\right) - \ln k \leqq \int_0^t v(x)dx.$$

Then (2) follows immediately.

We first consider the homogeneous, linear, first-order equation,

$$y'(t) + a(t)y(t) = 0, \tag{3}$$

with initial condition $y(0)=y_0$. Integrating (3) from 0 to t, we obtain

$$y(t) = y_0 - \int_0^t a(x)y(x)dx.$$

Therefore,

$$|y(t)| \leqq |y_0| + \int_0^t |a(x)| |y(x)| dx.$$

This last inequality is of the form (1) so that we may apply (2) and write

$$|y(t)| \leqq |y_0| \exp\left(\int_0^t |a(x)| dx\right). \tag{4}$$

This result enables us to prove the following theorem.

THEOREM 1. *If $a(t)$ is continuous, then if for $t \geqq 0$ there exists a continuous solution of the differential equation*

$$u'(t) + a(t)u(t) = f(t) \tag{5}$$

satisfying

$$u(0) = u_0, \tag{6}$$

that solution is unique.

Proof. Suppose that there are two solutions, $u_1(t)$ and $u_2(t)$, satisfying (5) and (6). Then the function $y(t)=u_1(t)-u_2(t)$ satisfies (3) and the initial condition $y(0)=0$. By (4), $|y(t)| \leqq 0$, which implies $y(t) \equiv 0$. Therefore $u_1(t) \equiv u_2(t)$.

In discussing the second order equation, it will simplify calculations to consider the "normal" form,

$$y'' + a(t)y = 0. \tag{7}$$

Not much generality is sacrificed since the transformation $y=v\exp(-\frac{1}{2}\int a_1(t)dt)$ will reduce $y''+a_1(t)y'+a_2(t)y=0$ to an equation of the form (7).

We will assume a continuous solution of (7) satisfying the initial conditions $y(0)=y_0$, $y'(0)=y_1$. Now,

$$y(t) - ty_1 - y_0 = \int_0^t\left(\int_0^x y''(u)du\right)dx = -\int_0^t\left(\int_0^x a(u)y(u)du\right)dx.$$

Integrating the last iterated integral by parts, we obtain,

$$\int_0^t\left(\int_0^x a(u)y(u)du\right)dx = t\int_0^t a(u)y(u)du - \int_0^t xa(x)y(x)dx.$$

Changing the variable of integration from u to x in the first integral on the right hand side of this equation, we may write

$$\int_0^t\left(\int_0^x a(u)y(u)du\right)dx = \int_0^t (t - x)a(x)y(x)dx.$$

Equation (7) is then equivalent to

$$y(t) = y_0 + ty_1 - \int_0^t (t - x)a(x)y(x)dx.$$

Hence, for $t \geqq 0$,

$$| y(t) | \leqq | y_0 | + t | y_1 | + \int_0^t (t - x) | a(x) | \, | y(x) | dx.$$

If $0 \leqq t \leqq 1$,

$$| y(t) | \leqq | y_0 | + | y_1 | + \int_0^t (t - x) | a(x) | \, | y(x) | dx,$$

and therefore, from the lemma,

$$| y(t) | \leqq (| y_0 | + | y_1 |) \exp\left(\int_0^t (t - x) | a(x) | dx\right). \tag{8a}$$

On the other hand, if $t > 1$,

$$| y(t) | \leqq t(| y_0 | + | y_1 |) + t\int_0^t | a(x) | \, | y(x) | dx,$$

which we may write as

$$\frac{| y(t) |}{t} \leqq | y_0 | + | y_1 | + \int_0^t x | a(x) | \frac{| y(x) |}{x} dx.$$

An application of the lemma then gives us

$$| y(t) | \leqq t(| y_0 | + | y_1 |) \exp\left(\int_0^t x | a(x) | dx\right). \tag{8b}$$

On the basis of (8a) and (8b) and in a manner entirely analogous to the first order case, we may now prove the following theorem.

THEOREM 2. *If $a(t)$ is continuous and if for $t \geqq 0$ there is a continuous solution of the equation $u''(t) + a(t)u(t) = f(t)$ satisfying $u(0) = u_0$ and $u'(0) = u_1$, then that solution is unique.*

A MECHANICAL DEVICE FOR FINDING THE REAL ROOTS OF THE CUBIC

MORTON J. HELLMAN, Rutgers, The State University

The real roots of the reduced cubic

$$x^3 + ax + b = 0 \tag{1}$$

are given graphically by the intersection of

$$y = x^3, \tag{2}$$

$$y = -ax - b. \tag{3}$$

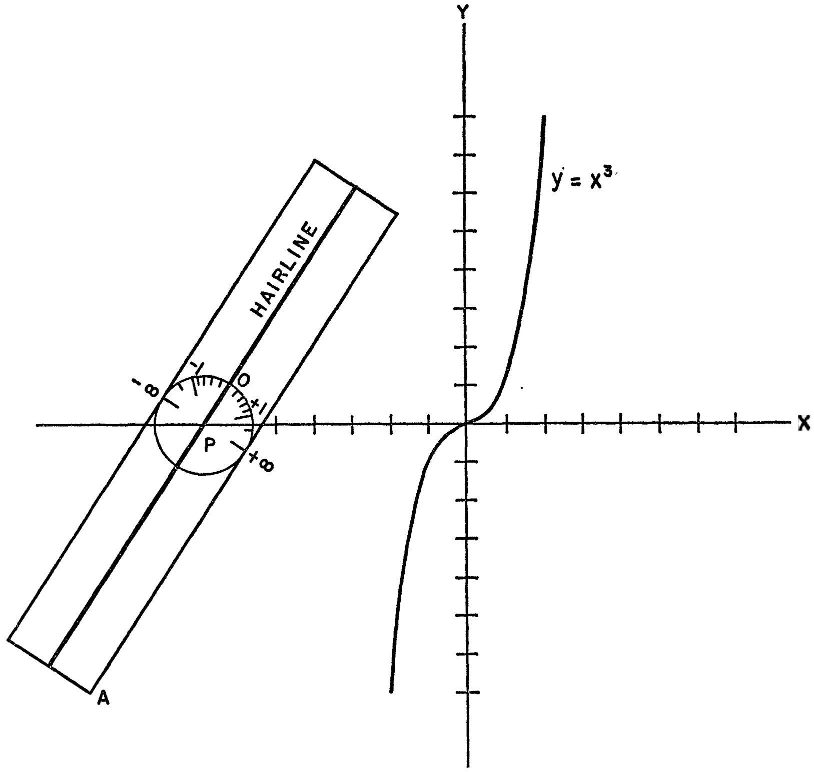

FIG. 1

The mechanical device is designed as follows (Fig. 1): On a plastic strip or one of other suitable material the curve $y=x^3$ is etched. Using a transparent strip, part A is constructed with a pin at its center P which slides along the slotted X-axis; also, A is free to rotate about P. The etchings along the circumference of the circle give the slope of the hairline relative to the X-axis for any position of the hairline other than the vertical. The X and Y axes are calibrated in some convenient units, for example from -10 to $+10$.

Typical operation of the device is as follows: Let $a=-2$, $b=5$. Since $-a$ is the slope of the hairline, rotate A until the circumference marking shows a slope of 2; then slide A parallel to itself until the hairline passes through the point $(0, -5)$; check the slope reading, and make any final adjustments to be sure the readings give slope 2 and Y-intercept -5. Then read off the abscissa of the real intersection of (2) and (3).

To insure that the intersection of (2) and (3) always falls within a readable portion of the graph for large numerical values of slope or Y-intercept, or for small values of both, proceed as follows:

If the roots of (1) are multiplied by m, the resulting equation is

$$x^3 + am^2x + bm^3 = 0,$$

which is equivalent graphically to the intersection of (2) and

$$y = - am^2x - bm^3. \tag{4}$$

For example, if $|-a|$ is too large, multiply it by a suitable fraction, such as $\frac{1}{2}$, and the new line is $y=-\frac{1}{4}ax-\frac{1}{8}b$ with a slope $\frac{1}{4}$ of the original slope and Y-intercept $\frac{1}{8}$ of the previous Y-intercept. This procedure will bring the intersection of the two curves nearer to the two axes. When the point of intersection of (2) and (4) is graphically estimated, multiplication of the resulting roots by $1/m$, 2 in this case, will yield the real roots of (1).

ON PERMUTATIONS AND COMBINATIONS*

S. K. CHATTERJEA, Bangabasi College, Calcutta, India

1. In an earlier note Mullin,† gave the following canonic generators for $\Psi(n)$ and $\Phi(n)$:

$$\Psi(n + 1) = 2\Psi(n), \qquad \Phi(n + 1) = (n + 1)\Phi(n) + 1, \tag{1}$$

where

$$\Psi(n) = \sum_{r=0}^{n} {}^nC_r, \quad {}^nC_r = \frac{n!}{(n - r)!r!}, \quad \Phi(n) = \sum_{r=0}^{n} {}^nP_r, \quad {}^nP_r = \frac{n!}{(n - r)!}.$$

He also proved the following results

$$\Phi(n) \sim (n!)\cdot e, \qquad \Phi(n) \sim n^n e^{1-n}\sqrt{(2\pi n)}, \tag{2}$$

where $n \gg 1$ and $e=2\cdot 718 \cdots$. Here we should like to present some canonic generators for finite series involving ${}^nC_r$ and some asymptotic expressions for finite series involving ${}^nP_r$.

2. In most textbooks on college algebra we notice the following properties of the binomial coefficients:

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* Through an oversight on the part of the Editor, this note appears without having been proofread by the author.

† A. A. Mullin, Three theorems on permutations, this MONTHLY, vol. 64, 1957, pp. 669–670.

$$\Psi_1(n) \equiv \sum_{r=0}^{n} (2r+1)\,{}^nC_r = (n+1)2^n,$$

$$\Psi_2(n) \equiv \sum_{r=0}^{n} (r+1)\,{}^nC_r = (n+2)2^{n-1},$$

$$\Psi_3(n) \equiv \sum_{r=1}^{n} r\cdot{}^nC_r = n2^{n-1},$$

$$\Psi_4(n) \equiv \sum_{r=0}^{n} {}^nC_r/(r+1) = (2^{n+1}-1)/(n+1),$$

$$\Psi_5(n) \equiv \sum_{r=1}^{n} r\cdot\frac{{}^nC_r}{{}^nC_{r-1}} = \tfrac{1}{2}n(n+1),$$

$$\Psi_6(n) \equiv \sum_{r=0}^{n} ({}^nC_r)^2 = (2n!)/(n!)^2.$$

We therefore easily obtain the following canonic generators for $\Psi_i(n)$, where $i=1,\cdots,6$.

$$\begin{aligned}
\Psi_1(n+1) &= 2\Psi_1(n) + 2^{n+1},\\
\Psi_2(n+1) &= 2\Psi_2(n) + 2^n,\\
\Psi_3(n+1) &= 2\Psi_3(n) + 2^n,\\
(n+2)\Psi_4(n+1) &= 2(n+1)\Psi_4(n) + 1 \quad \text{or } \Psi_4(n+1) \sim 2\Psi_4(n), \quad \text{where } n \gg 1,\\
n\Psi_5(n+1) &= (n+2)\Psi_5(n) \quad \text{or } \Psi_5(n+1) \sim \Psi_5(n), \quad \text{where } n \gg 1,\\
(n+1)\Psi_6(n+1) &= 2(2n+1)\Psi_6(n) \quad \text{or } \Psi_6(n+1) \sim 4\Psi_6(n), \quad \text{where } n \gg 1,
\end{aligned}$$

We also notice that $2\Psi_2(n+1)-2^2\Psi_2(n)=\Psi_1(n+1)-2\Psi_1(n)$.

3. Now we consider the expression

$$\Phi'(n) \equiv \sum_{r=0}^{n} (-)^{n-r}\cdot{}^nP_r = n!\left[\frac{(-)^n}{n!} + \frac{(-)^{n-1}}{(n-1)!} + \cdots - \frac{1}{1!} + 1\right].$$

Then

$$\Phi'(n) \sim (n!)e^{-1} \quad \text{when} \quad n \gg 1. \tag{3}$$

Further using Stirling's formula we derive from (3), $\Phi'(n) \sim n^n e^{-(n+1)}\sqrt{(2\pi n)}$, when $n \gg 1$. Again we get from (2) and (3)

$$\Phi(n) \pm \Phi'(n) \sim (n!)\cdot(e \pm e^{-1}), \quad \text{when} \quad n \gg 1. \tag{4}$$

From (4) we easily obtain the following asymptotic expressions

$$\sum_{r=0}^{n/2} {}^nP_{2r} \sim (n!)(e+e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is even};$$

$$\sum_{r=1}^{(n+1)/2} {}^nP_{2r-1} \sim (n!)(e + e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is odd;}$$

$$\sum_{r=0}^{(n-1)/2} {}^nP_{2r} \sim (n!)(e - e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is odd;}$$

$$\sum_{r=1}^{n/2} {}^nP_{2r-1} \sim (n!)(e - e^{-1})/2 \quad \text{when} \quad n \gg 1 \text{ and } n \text{ is even.}$$

4. We again notice that $r\cdot{}^{n-1}P_{r-1} = {}^nP_r - {}^{n-1}P_r$. Putting $r=1, \cdots, n$ and adding, we have (assuming that ${}^{n-1}P_n \equiv 0$)

$$\begin{aligned}\sum_{r=1}^{n} r\cdot{}^{n-1}P_{r-1} &= \sum_{r=1}^{n} {}^nP_r - \sum_{r=1}^{n} {}^{n-1}P_r,\\ &= \sum_{r=1}^{n} {}^nP_r - \sum_{r=1}^{n-1} {}^{n-1}P_r = \Phi(n) + \Phi(n-1).\end{aligned} \tag{5}$$

Thus when $n \gg 1$ we have from (2) and (5):

$$\theta(n) \equiv \sum_{r=1}^{n} r\cdot{}^{n-1}P_{r-1} \sim \{(n-1)!\}\cdot(n-1)e.$$

Using Stirling's formula, we get $\theta(n) \sim \sqrt{(2\pi)}(n-1)^{n+\frac{1}{2}}\cdot e^{2-n}$. Lastly, to find a canonic generator for $\theta(n)$, we consider (1) and (5) to obtain $\theta(n) = (n-1)\Phi(n-1) + 1$. Now

$$\{\theta(n+1) - 1\}/n = \Phi(n) = n\Phi(n-1) + 1 = \{n/(n-1)\}\{\theta(n) - 1\} + 1.$$

Therefore $(n-1)\theta(n+1) = n^2\theta(n) - 1$, which is a canonic generator for $\theta(n)$.

INVERSION OF JACOBIAN MATRICES*

LOUIS BRAND, University of Houston

1. Jacobian matrices. In a transformation of coordinates

$$x^i = f^i(y^1, y^2, \cdots, y^n), \qquad i = 1, \cdots, n,$$

the n^2 derivatives $\partial x^i/\partial y^j$ may be computed directly and arranged in a square array, the *Jacobian matrix* $(\partial x^i/\partial y^j)$. If the n functions f^i do not satisfy a functional relation their Jacobian matrix is nonsingular and admits a reciprocal. Using the summation convention

$$\frac{\partial x^i}{\partial y^r}\frac{\partial y^r}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_j, \qquad i, j = 1, \cdots, n;$$

* Presented to the Texas Section of the Association, Austin, April 17, 1959.

hence if we agree to write Jacobian matrices so that the uppermost index denotes a row,

$$\left(\frac{\partial y^i}{\partial x^j}\right)\left(\frac{\partial x^i}{\partial y^j}\right) = I_n, \tag{1}$$

where I_n is the unit matrix of order n. Therefore the n^2 derivatives $\partial y^i/\partial x^j$ may be computed by inverting the matrix $(\partial x^i/\partial y^j)$.

2. Orthogonal transformations. The inversion of the Jacobian matrix $(\partial x^i/\partial y^j)$ is very simple in the important case of an orthogonal transformation of coordinates. Then we can always write the Jacobian matrix as the product MD of an orthogonal matrix and a diagonal matrix:

$$\left(\frac{\partial x^i}{\partial y^j}\right) = MD. \tag{2}$$

Here D is the diagonal matrix with the positive elements $d_1, \cdots, d_n$, where

$$d_i^2 = \left(\frac{\partial x^1}{\partial y^i}\right)^2 + \cdots + \left(\frac{\partial x^n}{\partial y^i}\right)^2,$$

and M is the orthogonal matrix obtained by dividing the columns of $(\partial x^i/\partial y^j)$ by $d_1, \cdots, d_n$ respectively. The inverted Jacobian matrix is now

$$\left(\frac{\partial y^i}{\partial x^j}\right) = (MD)^{-1} = D^{-1}M^{-1} = D^{-1}M', \tag{3}$$

where M' is the transpose of M and D^{-1} is the diagonal matrix with the elements $1/d_1, \cdots, 1/d_n$.

Consider for example the change from rectangular to spherical coordinates:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Then with $x^i = x, y, z$, $y^i = r, \theta, \phi$, we have $d_1 = 1$, $d_2 = r$, $d_3 = r\sin\theta$, and the matrix

$$\begin{pmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix}.$$

Therefore the matrix

$$\begin{pmatrix} r_x & r_y & r_z \\ \theta_x & \theta_y & \theta_z \\ \phi_x & \phi_y & \phi_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/(r\sin\theta) \end{pmatrix}\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

from which all nine derivatives on the left may be read off.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

SUMMARY OF CONTENT OF SMSG COURSES

Each of the chairmen of writing teams for the SMSG mathematics courses, grades 7 through 12, was asked to prepare a brief summary statement of the content of the new course and ways in which the new course differs from the traditional. These statements are given in the following paragraphs.

Grades 7 and 8. Traditional mathematics courses for grades 7 and 8 include a review of the operations with whole numbers, fractions, and decimals. Percent is introduced, usually in terms of the three cases of percents, each of which is treated separately after various manipulations with percents, including fractional and decimal equivalents of percents. The traditional courses also have rather extensive treatments of percent applications such as commission, simple interest, discount, and insurance. A study of measurement has had an important place, but again much of this is a review of work done in earlier grades and little or none of it is new from a mathematical point of view.

While the new SMSG courses provide for review of the fundamentals of arithmetic, this review has been placed in a new setting with emphasis on number systems. Number systems are treated from an algebraic viewpoint, not only to deepen the student's understanding of arithmetic but also to prepare him for the algebra which is to come. The work on fractions is introduced by defining a fraction as a numeral for the rational number a/b such that $b(a/b)=a$, $b\neq 0$. The grade 8 text starts with an informal treatment of coordinates and equations, and includes a brief introductory chapter on probability. Some of the probability problems were written by biologists associated with the Biological Sciences Curriculum Study, and in a chapter on the lever an attempt has been made to use language consistent with that of the Physical Science Study Committee physics course. Percent applications have a place in the new courses, as do other social applications, for example through governmental statistics in the chapter on graphs and in probability.

The new courses give more than one third of the time to geometry, which is a very considerable change in emphasis from the traditional. Geometric ideas are introduced, first of all, from a nonmetric point of view and then, after a careful treatment of measurement, students are led gradually to a study of properties of triangles and other geometric figures, plane and solid, through an informal deductive approach. Although there is no attempt to give a system of postulates for the geometry, properties are identified on an intuitive or inductive basis and then these properties are used to draw conclusions about, or to prove, other properties. In the chapter on drawings and constructions, instruments in addition to the classical ones are introduced and the student is also provided with experience in sketching figures, especially three-dimensional figures. A grade 8 chapter on nonmetric geometry which comes just before the study of the measurement of volumes and surface areas is, in its topological approach, one of the greatest innovations.

Grade 9. The SMSG ninth-grade course, *First Course in Algebra*, differs from conventional texts in the following ways. It is based upon structure properties of the real number system. This development of algebra is interesting, meaningful, and mathematically sound. It helps bring out the nature of mathematics and strengthens the student's algebraic techniques by relating them to basic ideas. Definitions and properties

are carefully formulated and there is some work with simple proofs. The reading material, which is an important part of the course, is designed to help the student discover ideas. The number line and the simpler language of sets are used to help express the ideas. Inequalities are treated along with equations.

However, as its title suggests, the SMSG ninth-grade course covers essentially the same material as does a conventional first-year algebra text. It teaches the student how to perform the fundamental operations with real numbers and with variables and how to do the usual algebraic manipulations, including factoring, powers and roots, and work with polynomials and fractional expressions. It shows how to solve equations up through quadratic equations in one variable and linear equations in two variables. Graphs of linear and quadratic functions are treated. There is much experience in solving word problems.

Grade 10. The SMSG geometry text differs from conventional ones in content, postulational scheme and manner of treatment.

1. No artificial distinction is made between plane and solid geometry, and a considerable amount of the latter is included. Also, an introduction to analytic plane geometry is provided.
2. The postulate system is a modification of Birkhoff's and is complete. Real numbers are used freely throughout the text, both in the theory and in problems.
3. Accuracy in the statement and use of postulates, definitions and theorems is emphasized.

On the other hand, the text is still basically a treatment of synthetic Euclidean geometry, covering the usual topics: congruence, similarity, parallelism and perpendicularity, area, circles, and construction with straight-edge and compass. There is a main sequence of proved theorems, some minor stated theorems with proofs left as exercises for the students, and a long list of "originals." The basic postulates, definitions and theorems are motivated by appeal to intuition, and many practical and computational problems are given.

Grade 11. The SMSG eleventh-grade text differs from traditional texts in the following important ways:

1. The SMSG text makes much greater demands on the student's ability to learn by reading carefully worded expositions. The writers believe that the development of this ability is essential for success in college mathematics.
2. The study of number systems is stressed as the basis for all understanding of both elementary and advanced mathematics.
3. The idea that algebra is a logical structure built on a relatively small number of fundamental principles is emphasized throughout the text.
4. Presentations which lead the student to make certain predetermined "discoveries" are used where appropriate.
5. Proof is emphasized throughout in order that the student may gain some idea of the nature of a valid mathematical argument.
6. The function concept is developed spirally throughout the text.
7. Coordinate geometry is introduced earlier than usual and is used as a tool in the development of subsequent sections, notably those on trigonometry.
8. The presentation of logarithms reflects contemporary usage which requires more understanding of logarithms and exponential functions and relatively less emphasis on logarithmic computations.
9. The treatment of trigonometry emphasizes identities, equations and graphs of the trigonometric functions more than the computations required in the solutions of triangles.
10. Vectors are developed as a mathematical system and are applied to the solution of a wide variety of problems.

The writers hope that through the studying of this text the student will acquire some ability to handle unforeseen and unforeseeable problems.

The SMSG eleventh-grade text is similar to conventional texts in these respects.

1. The test begins with a review and extension of the basic skills of first year algebra. This review is included in the initial study of number systems.

2. The content is essentially the same as that found in conventional courses in trigonometry and college algebra.

3. Practical applications are given about the same amount of attention as in conventional texts. It was not possible to increase appreciably the number of applications without making unwarranted assumptions about the student's understanding of related fields.

4. The exposition makes use of many illustrative examples and drawings.

5. There is an abundant supply of exercises which have been carefully graded as to difficulty.

Grade 12. The subject matter of *Elementary Functions* is basically conventional. It includes such topics from the theory of equations as the remainder and factor theorems and the usual methods for finding rational roots. The student will find the laws of exponents and logarithms and the rules for changing the base. The chapter on circular functions contains the familiar addition and subtraction formulas and their consequences, identities and equations, and inverse trigonometric functions. An appendix treats the solution of triangles. Emphasis is laid on graphs. However, each of these topics is treated with some novelty and in a new spirit.

Elementary Functions applies the concept of mapping to polynomial, exponential, logarithmic and circular functions. Effective use is made of the ideas of composition and inversion. The treatment of tangents is intuitive, elementary and rigorous. It permits the introduction of Newton's method, and applications to maximum-minimum problems and to graphing. The treatment prepares for calculus without trespassing upon it. The explanation of exponentials and logarithms is novel and unusually clear and thorough. Trigonometry is freshly developed in line with the mapping idea. The style is informal. Explanations are full and concrete, and they convey the spirit of mathematical thinking.

OFFERINGS FOR FRESHMEN

BANCROFT H. BROWN, Dartmouth College

In 1958 the Committee on the Undergraduate Program of the Mathematical Association of America brought out an "Outline of Recommended Courses" (The Duren Report).

The first recommendation is: "A student who enters college meaning to take a (technical) course in mathematics must be ready without further preparation to take a course in calculus and analytic geometry."

With that statement those of us at Dartmouth responsible for the elementary courses agree wholeheartedly. The emphasis is on the word "calculus." The statement "and analytic geometry" is somewhat ambiguous. The elements of this subject are actually in the secondary course. Many colleges give only lip service to this. The analytic geometry of Salmon, of Loney, and of Fine, a discipline, or perhaps a fine art, studied for its own sake, is a luxury, a pretty-pretty which we can't afford.

The Duren Report defines in detail two courses, Calculus I and Calculus II.

Dartmouth had pretty well anticipated these recommendations by several years, and after some five years of experience, we are able to offer definite comments. The content of I and II together is good, but Calculus I could be improved. The Duren Report makes the clear distinction that the analytic geometry in Calculus I shall be of affine character only. This sounds well in theory, but in practice such a course is too dull, too abstract, and too monotonous. We think we should occasionally go Euclidean, and introduce the sine and cosine, complex numbers, and the solution of the second order linear homogeneous differential equation with constant coefficients. Our experience shows that this does not involve a breathless pace.

More generally, this is why we think the Duren Calculus I needs substantial changes. Many of our students who later will major in the social sciences take Calculus I and follow this with a course on "Elementary Mathematics of Sets," or "Finite Mathematics." Suppose we follow the Duren Report literally. They will have experience with exponential functions, but none with oscillatory functions. It may be true that the social sciences have made little use of oscillatory functions in the past, but can we guarantee that this will be true in the future? It seems to us that the first course (and for many this will be the only course in the calculus) should be as broad as possible. Admittedly you have to steer between breadth and superficiality. We think it can be done.

The difficult problems of honors sections and advanced placement cannot be considered in this brief review. At Dartmouth this year we had 35 freshmen, who had had the equivalent of a college semester or more of calculus whom we put in advanced placement; and we put 60 more men in honors groups. There are no easy answers; these are local problems.

One curious new development should be pointed out, and labeled for the menace that it is. It has become increasingly prevalent for secondary schools to give their seniors brief courses from two weeks up to two months "to show what the calculus is all about." This isn't at all for advanced placement. The idea, apparently laudable but absolutely fallacious, is that this will facilitate the change from high school to college. The truth is that nothing hurts a boy so much as to enter college thinking he knows all about the calculus, believing that he is way ahead of his class and can coast the first month or two. The Day of Reckoning is just as sure as death and taxes. Every college welcomes with open arms the student who can be put in advanced placement; but this half-way stuff is a headache for everyone.

At Dartmouth, nearly three-quarters of the freshman class elect the calculus (regular course, honors, or advanced placement). Of the remaining quarter, some lack ability, many have plenty of ability, but lack interest. Any "requirement," any concerted effort to bring this group into the calculus course would not be to their best interest and would inevitably weaken the course. (In fact, if by some auspicious use of the crystal ball we could persuade the lowest 7 per cent of the present calculus group to stay away, every one would be happier.) Should this remaining 25 per cent, or any substantial part of it elect any mathe-

matics? If it is a technical course, my answer is no. Specifically, I have no faintest interest in patching together assorted topics from college algebra, trigonometry, solid geometry, and analytic geometry. But this 25 per cent must never be sold short; it contains at least 50 per cent of the creative genius of the entering class. Can a nontechnical course be devised for these men which is not a travesty on mathematics or science?

I think we have found an affirmative answer in our elective course "Inquiries in Mathematics," which has been taken by some 2700 students in the last 15 years. The course features number, time, and space. "Number" includes such topics as the number system, numbers to other bases, Fermat's Minor Theorem and its converse, Fechner's Law, and the paradoxes of scoring systems. "Time" includes a study of several calendars, including the highly complicated, and apparently arbitrary Jewish Calendar. "Space" is primarily concerned with cartography. Other topics have at various times included the application of Kepler's Laws to sputniks, Euler's theory of unicursal net-works, and a systematic survey of organized professional gambling. The emphasis is on understanding what the problem is, and then inventing and applying a mathematical technique for its solution.

We think the course has been successful. We don't urge others to adopt it—in fact we have never bothered to write a text. It is a local course, and perhaps I may say, an individual course. But we feel that any college of liberal arts does well to consider that its students who are not interested in science may profit from some unconventional course based largely on scientific curiosity.

SOME COMMENTS ON TEACHING OF THE CALCULUS IN SECONDARY SCHOOLS

GERALD R. RISING, Norwalk Public Schools, Norwalk, Connecticut

In an educational note about college freshmen who have studied the calculus in high school (this MONTHLY, vol. 66, 1959, pp. 584–586), Professor J. H. Neelley of Carnegie Institute of Technology has directed a number of criticisms at secondary teaching which merit serious consideration. These criticisms may be applied to two different types of high school courses: (1) the full-year analytic geometry and calculus course designed usually to meet the requirements of College Entrance Examination Board Advanced Standing in Mathematics; and (2) courses which introduce limits and differentiation and integration of polynomial functions with applications in units ranging from a few weeks to a full semester in length. As a high school teacher of both types of course and as a university teacher of elementary and intermediate calculus, I believe that I can speak with some authority about these programs.

It has been indicated that secondary schools are teaching only mechanical application of formulas without understanding. I believe that the students who have successfully passed the advanced placement type of course in mathematics must know much more than this. The CEEB examination is a fine and sophisticated instrument. Students who have scored high must have done as well as

control groups of college students at good universities. There should be no question about others bypassing this course at the college level. Students who have some background in calculus may well be placed in honors sections in college but should not be considered for further advancement.

The advanced-placement classes which I have seen in New York and in Connecticut have been excellent. College texts are used. In every case with which I am familiar the teacher assigned to this section is the strongest teacher in the department. This may be compared with the staffing often utilized in colleges whereby a graduate student overburdened with his own course work is asked to teach this course. If I were a student, I know which of these teachers I would choose; and my own experience with transfers into my evening college classes from day school and with students pleading for vacation help supports this choice.

There are weak teachers at all levels. I suggest, however, that we cannot retreat from curricular advances because of the weaknesses of individual teachers. I have had students return to visit me who tell me of the rote teaching in their college classes and the failure of teachers on even the professorial level to communicate ideas, but this does not destroy my faith in the general caliber of college instruction. I believe rather that both secondary and college teachers could gain from greater exchange of ideas and teaching methods. I believe, too, that both could benefit greatly from closer inspection of some of the really fine teaching going on in elementary schools.

Certainly the answer to these high-quality students is not, as is so often suggested, more of the same work they have done in secondary schools. This is like the method too often used early in the history of dealing with bright youngsters whereby they were assigned ten more problems of each type than their average neighbor. The students who are in such accelerated college-level programs in high school know their basic mathematics. If they did not, they would not be in such a program. There are today many students going on to college with deficiencies in mathematics as well as in other areas. I do not believe that these are the two to ten per cent taking part in accelerated mathematics programs. From my experience with these children I would gladly match them with second-semester college freshmen.

Here, because I believe that it is appropriate, I will describe the mathematics program which I organized at Greece Olympia High School in Rochester, New York, a program similar to that offered in many fine secondary schools over the country. A student can elect: elementary algebra (in grade 9), synthetic plane geometry with some analytics (10), intermediate algebra and trigonometry integrated (11), and half-year courses in advanced algebra and solid geometry concurrently with a full-year experimental course devoted to many of the non-analysis areas so often discussed as "modern mathematics" (12). Thus a non-accelerated student can elect five years of high school mathematics without infringing on the college domain (unless you recall that secondary schools have taken over both college algebra and trigonometry within the last two decades).

To take advanced placement about five per cent of the students take seventh- and eighth-grade mathematics in the seventh grade, push down the courses listed above one year, and take advanced-placement analytic geometry and calculus in their senior year. It is my contention that students completing this program are in an excellent position to enter sophomore college mathematics courses. This type of student is progressing successfully in some of the best universities.

While it is true that some colleges like Dartmouth and Notre Dame which have unusual freshman programs are not in a good position to accept the advanced-placement students, still the vast majority of schools are teaching the traditional course using any one of a dozen texts which are substantially the same. Mathematics teachers on the secondary level should encourage students to find out the type of course offerings in schools at which the students hope to matriculate. In one case I discouraged a student from accelerating in mathematics because of this difficulty. He was able to devote this extra time to another field. Unfortunately such early decisions are very difficult to make.

It is somewhat more difficult to defend the introduction to calculus given in a semester or less in secondary schools. I, too, disagree with what some schools are offering. I do believe, however, that this type of unit can be of real service to the college program. Students can spend a substantial amount of time on two topics which most college teachers will agree deserve more time than they are able to allot to them. These two topics are limits and the definition of the derivative. Rather than gloss over these topics superficially in a lesson or two, as is so often done in college courses, the secondary program offers the possibility of spending two or three weeks on each topic. I do not believe that it is necessary to teach any formulas in calculus in a secondary program, the definition providing the only basis for taking derivatives. By solid teaching of these topics the secondary teacher can help remove what I believe to be two of the toughest hurdles to mathematics students in the college program.

These students should certainly not be allowed to skip a semester of calculus. It is appropriate for them to dig more deeply into elementary calculus in the sort of honors program offered at some colleges, following approaches like that of Artin or Hardy.

Here are some specific recommendations to the college critics of secondary teaching:

(1) Communicate directly with the secondary teacher or principal of students you think are poorly prepared. In many cases you will find that the real fault lies within the admissions office of your own college. All too often even the best colleges are accepting students who do not deserve to go on to higher education.

(2) Compare your criticisms of the secondary programs with the graduate school's criticisms of your own. We all have the tendency to blame the teacher farther down the line, forgetting that at no level and with no student can we expect absolute mastery.

(3) Try to understand the real point of the accelerated programs. They are designed to give the talented student an opportunity to go farther and more deeply into mathematics on the college level.

(4) Realize that you are not alone in having to revise your program because of the changing education at lower levels. All along the line revisions are forcing higher grade levels to adapt to the changing background of the students. Unfortunately this is a cumulative process, higher grades being forced to revise several times. You at the college level will be hardest hit by this process, but you should also be able to reap the most benefit from it.

A COOPERATIVE PROGRAM FOR TEACHER EDUCATION LEADING TO THE B.A. AND B.S. DEGREES

WILLIAM H. EDSON AND J. W. BUCHTA, University of Minnesota

How to combine the specialization required for beginning graduate work in mathematics and science with the breadth of preparation that is desirable for secondary school teaching is a matter of general concern. Degree requirements of liberal arts colleges are frequently set with graduate school preparation in mind, whereas colleges of education have usually looked to the job of the secondary school teacher in specifying their curriculums.

The College of Education and the College of Science, Literature, and the Arts at the University of Minnesota are working together in the second year of an administrative arrangement that enables students to earn concurrently the B.S. degree granted by the College of Education and the B.A. degree granted by the Arts College. Students graduating under the plan have the course work prerequisite to graduate school admission and are eligible for the teaching certificate in Minnesota.

Under the cooperative plan, students meet *all* of the degree requirements of each college. These include the courses in psychology, professional education and practice teaching in the College of Education, the distribution requirements in language, science, social science, English, and humanities, and the major and minor requirements in the Arts College. Since the liberal arts courses specified by the College of Education at Minnesota occupy the greater fraction of a student's time, it is possible for many, by proper planning, to satisfy the requirements of the two colleges by attending one quarter or a summer and a quarter beyond the four years. The time will depend on the major, the planning of a program early in the college career, and the credit load a student is able to carry.

Although several minor procedural modifications concerning advising, registration, and record keeping were necessary, only two changes affecting degree requirements were made. First, the liberal arts college agreed to accept 13 quarter credits of educational psychology and educational philosophy toward its general elective requirements. Second, the colleges agreed that the residence

requirement in either school could be satisfied by three quarters of registration after admission to the upper division (junior year).

Under the plan students are enrolled in the College of Science, Literature, and the Arts for the first two years, as are all students preparing to teach secondary academic subjects. At the beginning of the junior year those desiring the combined program apply for admission to the Upper Division of the College of Science, Literature, and the Arts and for admission to the College of Education. If approved by both colleges they register in the Upper Division of the liberal arts college during their junior year and in the College of Education in their senior year.

The combined plan has not been designed to replace the regular programs in either of the colleges but rather to attract some additional people to teaching in all fields, especially in science and mathematics, in our secondary schools and to encourage graduate work in the teaching subject area. In this, the plan appears to be successful.

Mathematics for Parents

The parents of junior high school students in the Westport, Connecticut, public schools have expressed considerable interest in the public school program of mathematics since 1958. Their interest has increased steadily, and last year parents began to ask for some way to find out more about the new curriculum. The main reason for this interest seems to be a desire on their part to help their children with the practice work at home.

As a result of this interest, a course for parents is being conducted on Thursday evenings from 7:30 P.M. to 9:30 P.M. through the school's adult education program. The teacher of this class is the Westport mathematics supervisor. Each adult in the course has a seventh grade text (SMSG), 1960 edition, and the plan is to go as far and as fast as the class will permit. The class consists of 24 adults, quite heterogeneous in background. The interest is high and more people have expressed a desire to join the group.

RAY WALCH, Mathematics Supervisor, Westport, Connecticut, Public Schools

A Survey of New Programs in Mathematics

Studies in Mathematics Education, A Brief Survey of Improvement Programs for School Mathematics has recently been prepared by the mathematics department of Scott, Foresman and Company and is available for teachers. The pamphlet contains a concise summary of the work of the various study groups in mathematics and may be especially useful for inservice workshops and teacher training classes. This pamphlet is an extension and revision of an earlier pamphlet prepared in 1959. Descriptions of several programs have been added in this revision. The price of the pamphlet is 50 cents.

California Mathematics and Science Teachers to Meet

The California Mathematics Council and the California Science Teachers Association are planning a joint spring meeting for April 28–30, 1961. The California Mathematics Council is affiliated with the National Council of Teachers of Mathematics and the California Science Teachers Association with the National Science Teachers Association. Meetings of mathematics and science teachers at the secondary school level are not too frequent and California is to be congratulated on promoting this kind of cooperative action. Over 2000 teachers took part in the annual meeting of the California Mathematics Council held in Monterey in December.

Modernized Methods of School Mathematics Teaching

A new approach to the teaching of mathematics in school, to replace the outworn methods of the past, was planned by a working party organized and financed by the O.E.E.C. Office for Scientific and Technical Personnel, which recently met in Zagreb and Dubrovnik in Yugoslavia. This new approach included improved methods of presentation, designed to appeal to the interest of the pupil and at the same time to stress the connections between the various branches of mathematics.

The working party, which consisted of twenty leading mathematicians from O.E.E.C. countries and the United States, included representatives of universities, secondary schools and institutions training secondary school teachers. European countries represented in the group were Belgium, Denmark, France, Germany, Italy, Sweden, Switzerland, the United Kingdom and Yugoslavia.

The modernized curriculum recommended includes new and improved contents for courses in algebra, geometry and analysis, and better integration of subject matter in algebra itself and between algebra and other branches of mathematics. It is designed to aid national authorities in preparing new textbooks and planning regular and experimental courses in mathematics.

This endeavor to substitute a new and livelier spirit in the teaching of mathematics for uninspiring traditional methods forms part of the O.S.T.P. program for stimulating co-operative action among member countries in educating sufficient scientists, engineers and technicians to meet their increasing needs for these specialized skills.

(News Release by OEEC Information Division, October 3, 1960).

Other NSF Visiting Lecturer Programs

Secondary Schools. In addition to the support given by National Science Foundation to MAA for a Visiting Lecturer Program to Secondary Schools, NSF supports visiting lecturer programs to secondary schools sponsored on a national basis by the American Institute of Biological Sciences, American Chemical Society, and the American Institute of Physics.

Colleges. The National Science Foundation supports a visiting lecturer program to colleges which is sponsored by the Society for Industrial and Applied Mathematics, as well as the MAA program reported last month. Other NSF college programs of visiting lecturers are sponsored by the American Anthropological Association; American Society for Engineering Education; Society of American Foresters; American Geological Institute; American Institute of Physics; American Physiological Society; American Psychological Association; Society of Wood Science and Technology, and American Meteorological Society.

Foreign Scientists. The National Science Foundation also sponsors lecturer programs in which the lecturers are scientists from foreign countries. There are nine such programs, including one under the auspices of the American Mathematical Society. John W. Green, secretary of the Society, is director of the program for mathematics. These programs for visiting lecturers are also sponsored by the American Astronomical Society; American Institute of Biological Sciences; American Chemical Society; Engineers Joint Council; American Geological Institute; American Meteorological Society; American Institute of Physics, and American Psychological Association.

Academies of Science. Through a different program of the National Science Foundation, support is given to educational programs in the sciences sponsored by state academies of science, sometimes in cooperation with other local scientific groups, including sections of the Mathematical Association of America. While many of the academies of

science do not have a section on mathematics, others do. Quite a number of their programs which provide for visiting lecturers from colleges and universities to the secondary schools in the states do include mathematics, but the mathematics lecturers are often not very numerous. Among the academy of science programs for 1960–61 which include visiting lecturers as a part of the program, are those sponsored by the state academies in Arizona, Arkansas, Colorado-Wyoming, Indiana, Iowa, Maryland, Minnesota, Montana, Nebraska, New Mexico, Ohio, Texas, Utah, Virginia, and West Virginia. In addition, under the NSF state academies of science project, a grant has been made to the Mathematics Speakers' Bureau of metropolitan New York for visiting lecturers to high schools in their area. The grant is made directly to Cooper Union, New York, since the sponsoring group is not incorporated. The program of the Speakers' Bureau is directed by Professor James N. Eastham of Cooper Union. The regional representatives for the MAA Visiting Lecturer Program to Secondary Schools have been asked to offer full cooperation to the academies of science in their regions so that the programs under MAA auspices and those under the academies can both be strengthened and so that there will not be undesirable duplication.

Supplement to CCSSO Purchase Guide

An updating of the Purchase Guide of 1959 in the form of a 64-page supplement was published in February 1961 by the Council of Chief State School Officers. The supplement was organized on the same plan as their 1959 comprehensive guide, of which some 43,000 copies were distributed without cost to state and local school systems throughout the country. The guide is intended to provide advice on the purchase of new materials for mathematics, science and the modern foreign languages, and to assist teachers and administrators in modernizing course content in curriculum of science and mathematics and in foreign languages.

Science and Mathematics Students Honored at IBM Junior Science Symposium

Three hundred outstanding science and mathematics secondary school students in the metropolitan New York area attended the IBM Junior Science Symposium in October. Co-sponsors with International Business Machines Corporation were the Science Manpower Project of Columbia Teachers College and Columbia University. Dr. Edward Teller, currently Professor of Physics-at-Large at the University of California speaking on "Geometry of Space and Time" at the opening lecture of the symposium, stated, "At the beginning of the century ideas about the world in which we live underwent a radical change. The strangeness of these ideas has prevented their spread. These new concepts should be communicated to our children in high school when their minds are most receptive."

Six of the students, with records of particularly outstanding achievements, presented papers during the symposium on original research they have conducted:

Matrices and determinants, James Pepe, Xaverian High School, Brooklyn, New York.

The use of the digital computer in the investigation of Fibonacci numbers, Harry Saal, Midwood High School, Brooklyn, and Columbia University.

The Stern-Gerlach experiment, Martin Breidenbach, Pascack Valley Regional High School, Hillsdale, New Jersey.

Crystals and their growth, Frank J. Traina, St. Augustine Diocesan High School, Brooklyn, New York.

Isolation and identification of antibiotic-producing microorganisms, Kathleen McGarrity, Aquinas High School, Bronx, New York.

A technique for the quatitative determination of free serum amino acids, Bernard Rappoport, New Dorp High School, Staten Island, New York.

In addition to Dr. Teller, other noted scientists who spoke during the course of the four-day symposium included: Lipman Bers, Institute of Mathematical Sciences, New York University; C. S. Wu, Physics Department, Columbia University; and, M. J. Kopac, Professor of Biology, New York University.

Dean Mina S. Rees of Hunter College conducted a panel discussion on "Tomorrow's Opportunities in Science."

Students were selected by their schools on the basis of their interests and accomplishments in science and mathematics. The symposium is by invitation only and is a distinct honor for those attending. Honored students represented public, private, and parochial secondary schools. The purpose of the symposium is to advance and promote science and mathematics at the secondary school level.

The four-day symposium included a trip to Poughkeepsie, where principal IBM research and product development laboratories and a manufacturing center for medium- and large-scale computers are located. It is the home of STRETCH, a system capable of 75 billion computations a day.

(From an IBM News Release).

CORRECTION

We regret that in the report of the Kentucky Conference of College Science and Mathematics Staff Members (this MONTHLY, vol. 68, 1961, p. 61), the name of Jerrold W. Zacharias was misspelled.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1456. *Proposed by J. F. Darling, Woodstown, N. J.*

Prove that in a triangle with sides a, b, c and semiperimeter s,

$$a^2 + b^2 + c^2 \geqq (36/35)(s^2 + abc/s),$$

with equality only if the triangle is equilateral.

E 1457. *Proposed by Aaron Herschfeld, Pennsylvania State University, Hazleton, Pennsylvania*

Show that the set of numbers $J_m = m^2+1$, $m = 1, 2, \cdots$, contains an infinity of composite $J_N = J_mJ_n$. In fact, for arbitrary m, find two pairs of corresponding integers n, N.

E 1458. *Proposed by Hans Schwerdtfeger, McGill University*

Let $x_1 < \cdots < x_n$ be n points on the x-axis. Let P be a point in the (x, y)-plane with ordinate different from zero. If d_i is the distance of P from x_i show that

$$\sum_{j=1}^{n} a_j d_j^2 = \begin{Bmatrix} 1 \text{ if } n = 3 \\ 0 \text{ if } n > 3 \end{Bmatrix},$$

where $a_i = 1/f'(x_i)$, $f(x) = (x - x_1) \cdots (x - x_n)$.

E 1459. *Proposed by Azriel Rosenfeld, Yeshiva University*

Let $S(m, N)$ be the statement: "There exist m primes not exceeding N which are consecutive terms of an arithmetic progression." For example, $S(3, N)$ is true for $N \geqq 7$ (use 3, 5, 7); $S(5, N)$ is true for $N \geqq 29$ (use 5, 11, 17, 23, 29). Prove that $S(7, N)$ is false for $N < 900$, and that $S(11, N)$ is false for $N < 10{,}000$.

E 1460. *Proposed by Masao Arai, Jiyu Gakuen, Tokyo, Japan*

Let x, y, n be positive integers and let $\chi(n)$ denote the number of pairs of integers x, y satisfying $(x, n) = (y, n) = (x+y, n) = 1$, $x < n$, $y < n$. Find a formula for $\chi(n)$.

SOLUTIONS

The Soap Contest

E 1426 [1960, 692]. *Proposed by C. F. Pinzka, University of Cincinnati*

Professor E. P. B. Umbugio is trying to supplement his meager academic salary by entering soap contests. One such contest requires the contestants to find the number of paths in the following array which spell out the word MATHEMATICIAN:

M
M A M
M A T A M
M A T H T A M
M A T H E H T A M
M A T H E M E H T A M
M A T H E M A M E H T A M
M A T H E M A T A M E H T A M
M A T H E M A T I T A M E H T A M
M A T H E M A T I C I T A M E H T A M
M A T H E M A T I C I C I T A M E H T A M
M A T H E M A T I C I A I C I T A M E H T A M
M A T H E M A T I C I A N A I C I T A M E H T A M

Umbugio has counted 1587 paths which originate from one of the first five rows. With the deadline for submitting entries approaching, he is distraught to say the least. Help the Professor out by finding the number of paths with a minimum of computation.

I. *Solution by J. F. Leetch, Ohio State University.* One may count paths "backwards" from the N. In counting the left half of the array, including the center column, there are two choices for each backward step. Thus, this portion yields 2^{12} paths. Doubling this number and subtracting the center column, which was counted twice, yields $2^{13}-1=8191$ paths.

II. *Solution by D. A. Moran, University of Illinois.* Every path must originate on a "boundary M" and terminate at the unique N. The paths lying completely in the left half of the array (including the vertical path) correspond one-to-one with the words of twelve letters each of which is chosen from the pair (H, V), where H stands for "horizontal" and V for "vertical." The number of these is $2^{12}=4096$. The paths lying completely in the right half of the array (excluding the vertical path) are similarly found to number $2^{12}-1=4095$. Thus the total number of distinct paths is 8191, since every path lies completely in one half of the array.

Also solved by B. C. Anderson, R. H. Anglin, Leon Bankoff, S. D. Barcun, D. Y. Barrer, Alan Beal, J. D. Beyer, A. M. Broshi, M. L. Cantor, Robert Carlos, J. W. Clawson, R. J. Cormier, Dennis Couzin, Rufus Crane, H. H. Crapo, G. S. Cunningham, H. T. David, R. J. Distler, K. P. Dressler, Underwood Dudley, O. E. Eason, S. J. Einhorn, S. H. Eisman, J. S. Elston, Jeanne G. English, E. T. Frankel, J. E. Freund, W. W. Funkenbusch, José Gallego-Diaz, Michael Goldberg, L. D. Goldstone, H. W. Gould, R. E. Greenwood, Stan Guberud, E. W. Harrington, Jr., Charles Hatfield, Frank Hawthorne, R. L. Helmbold, L. J. Huber, Erwin Just, William Kantor, L. F. Klosinski, David Lash, You-Freng Lin, M. V. Mahoney and P. L. Renz (jointly), E. W. Marchand, D. C. B. Marsh, Glenn Mayfield, Ann Miller, Walter Penney, Frank Perez, J. L. Pietenpol, I. N. Presson, E. V. Price, J. L. Purdy, Alvah Raymond, Robin Robinson, L. R. Ruch, T. A. Schoen, Arnold Singer, G. E. Smith, W. B. Stovall, Jr., Eric Sturley, J. D. Vineyard, J. A. Ward, K. P. Yanosko, and the proposer. Late solutions by H. C. Dixon, Jr., Anthony Hug, J. B. Muskat, Adrian Peterson, and Mary Agnes Racki.

Bankoff pointed out that the Professor evidently failed to notice that the number of letters in MATHEMATICIAN and the sum of the digits in E 1426 are both equal to 13! Several solvers called attention to the similar Problems 256, 257, 258 in Dudeney's *Amusements in Mathematics* and Problems 30 and 38 in his *Canterbury Puzzles.*

A Further Triangle Inequality

E 1427 [1960, 692]. *Proposed by F. Leuenberger, Zuoz, Switzerland*

If a_1, a_2, a_3 are the sides and h_1, h_2, h_3 the altitudes of a triangle T, show that

$$\sqrt{3}\sum_{1}^{3} a_i \geqq 2\sum_{1}^{3} h_i,$$

with equality if and only if T is equilateral. (Dedicated to the memory of Victor Thébault.)

Solution by Leonard Carlitz, Duke University. We have first

$$\left(\sum a_i\right)^2 \geqq 3\sum a_1a_2 = (3a_1a_2a_3/2\Delta)\sum h_i = 6R\sum h_i,$$

where Δ is the area and R the circumradius of the triangle, with equality only in the case of an equilateral triangle. Now, as a special case of a theorem on

convex polygons (see Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und in Raum*, p. 6), we have also $3R\sqrt{3} \geqq \sum a_i$, with equality only in the case of an equilateral triangle. Consequently

$$\sqrt{3}(\sum a_i)^2 \geqq 2(\sum a_i)(\sum h_i),$$

so that $\sqrt{3}\sum a_i \geqq 2\sum h_i$, with equality only in the case of an equilateral triangle.

Also solved by A. N. Aheart, Leon Bankoff, T. R. Curry, Jane Evans, José Gallego-Diaz, L. D. Goldstone, Earl High, William Cantor, D. C. B. Marsh, Norman Schaumberger, P. D. Thomas, Dale Woods, and the proposer. Late solutions by H. E. Bray, C. P. Donahoo, Jr., Bob Snell, and Guy Torchinelli.

An Infinite Product

E 1428 [1960, 693]. *Proposed by Aaron Lieberman, The G. C. Dewey Corporation, N. Y.*

Evaluate

$$\prod_0^\infty [1 + (1/2)^{2^n}].$$

I. *Solution by Leon Bankoff, Los Angeles, Calif.* It is easily verified by induction that

$$\prod_0^n [1 + (1/2)^{2^n}] = 2[1 - (1/2)^{2^{n+1}}].$$

Consequently

$$\lim_{n\to\infty} \prod_0^n [1 + (1/2)^{2^n}] = 2.$$

II. *Solution by D. W. Bailey, University of Oregon.* For $|z| < 1$,

$$\prod_0^\infty [1 + z^{2^n}] = \sum_0^\infty z^n = 1/(1 - z).$$

Also solved by A. N. Aheart, John Avila, Alan Beal, P. B. Bennett and D. P. Kelly and A. L. Sulton (jointly), R. C. Bollinger, A. M. Broshi, W. J. Carpenter, P. R. Chernoff, F. H. Cleveland, Dennis Couzin, Gus Di Antonio, Underwood Dudley, Ragnar Dybvik, S. J. Einhorn, S. H. Eisman, P. G. Engstrom, Jane Evans, F. A. Faucher, J. E. Freund, Stuart Friedman, José Gallego-Diaz, Seymour Geisser and Clifford Patlak (jointly), Michael Goldberg, L. D. Goldstone, Juris Hartmanis, G. A. Heuer, J. E. Homer, Jr. and David Zeitlin (jointly), Erwin Just, William Kantor, J. H. Kaplan, P. G. Kirmser, L. F. Klosinski, A. G. Konheim, Sidney Kravitz, D. C. B. Marsh, Marvin Mielke, R. P. Miller, Otto Mond, C. S. Ogilvy, Walter Penney, J. L. Pietenpol, R. L. Renz, V. M. Sakhave, Norman Schaumberger, George Senge, Arnold Singer, Sister Kenneth Kolmer, Eric Sturley, W. C. Waterhouse, Lawrence Zalcman, and the proposer. Late solutions by D. A. Breault, J. B. Muskat, Dmitri Thoro, and Guy Torchinelli.

Several solvers pointed out that the problem is not new. It is found in Knopp, *Theory of Functions II*, Ex. 4b, p. 21; Ahlfors, *Complex Analysis*, Ex. 2, p. 155; Bromwich, *Infinite Series* (2nd

ed.), Ex. 6, p. 114; Knopp, *Theory and Application of Infinite Series*, Ex. 85b, p. 228; Apostol, *Mathematical Analysis*, Ex. 12–39(d), p. 388.

A Definite Integral

E 1429 [1960, 693]. *Proposed by H. S. Brock, David Taylor Model Basin, Washington, D. C.*

If

$$f(\theta) = \frac{1 + a^2b - 2ab\cos\theta + 2b\cos 2\theta - a(1+b)\cos 3\theta}{1 + a^2 - 2a\cos 3\theta},$$

where $-1<a<1$, show that $\int_0^\pi f(\theta)d\theta=\pi$.

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. Since

$$V(n, k) = \int_0^\pi \frac{\cos n\theta d\theta}{1 + a^2 - 2a\cos k\theta} = \begin{cases} \pi a^m/(1-a^2), & \text{if } n = mk, \\ 0, & \text{if } k \nmid n, \end{cases}$$

(see Bromwich, *Introduction to the Theory of Infinite Series*, 2nd ed. (1926), Prob. 26, p. 529), it follows that

$$\begin{aligned}\int_0^\pi f(\theta)d\theta &= (1 + a^2b)V(0,3) - 2abV(1,3) + 2bV(2,3) - a(1+b)V(3,3) \\ &= [(1+a^2b)\pi/(1-a^2)] - [a(1+b)\pi a/(1-a^2)] = \pi.\end{aligned}$$

Also solved by José Gallego-Diaz, L. D. Goldstone, L. A. MacColl, and D. C. B. Marsh.

Most of the solutions employed contour integration about the unit circle. It is to be noted that the value of the integral is independent of both parameters in the integrand. The proposer said the integral arose in connection with a recently developed theorem in the theory of elasticity.

Double Roots

E 1430 [1960, 693]. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development*

What is the highest order of multiplicity a root can have for the equation

$$x(x-1)(x-2)\cdots(x-n+1) = \lambda?$$

Solution by D. C. B. Marsh, Colorado School of Mines. Since the polynomial $p(x)\equiv x(x-1)(x-2)\cdots(x-n+1)$ has all of its zeros real and distinct, so does $p'(x)$. For $p(x)-\lambda$ to have a zero of multiplicity m, $p'(x)$ must have this zero with multiplicity $m-1$. Therefore any root of $p(x)=\lambda$ can have multiplicity no greater than 2. That 2 is possible is shown by the case $\lambda=p(z)$, where z is a zero of $p'(x)$.

Also solved by D. W. Bailey, Alan Beal, Leonard Carlitz, P. R. Chernoff, Underwood Dudley, José Gallego-Diaz, Seymour Geisser and Clifford Patlak (jointly), Michael Goldberg, R. T. Hood, Erwin Just, William Kantor, A. G. Konheim, R. W. Means, J. L. Pietenpol, P. L. Renz, and W. C. Waterhouse. Late solution by Guy Torchinelli.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problem and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4933 [1960, 927]. CORRECTION. *Proposed by I. N. Herstein, Cornell University*

Suppose a ring R is the set-theoretic union of a finite number of commutative fields having the same unit element; prove that R must then be a commutative field.

4953. *Proposed by O. P. Aggarwal and Irwin Guttman, McGill University*

Show that

$$\iint_S e^{-(x^2+y^2)/2}dxdy = \iint_T e^{-(x^2+y^2)/2}dxdy,$$

where S is the square $0 \leqq (x, y) \leqq a$ and T is the triangle bounded by the coordinate axes and the line $x+y=a\sqrt{2}$.

4954. *Proposed by D. J. Newman, Yeshiva University*

Show that, if $f(x) \geqq 0$ is convex, then

$$\int_0^\infty f^2 \leqq \tfrac{2}{3} \max f \int_0^\infty f,$$

and that $\frac{2}{3}$ is best possible.

4955. *Proposed by T. L. Saaty, Office of Naval Research, Washington, D. C.*

Show that

$$\sum_{k=2}^{n} \binom{n}{k} (1-\beta)^k \beta^{n-k} \sum_{j=1}^{k-1} \frac{1}{j} = \sum_{j=1}^{n-1} \frac{(1-\beta^j)(1-\beta^{n-j})}{j}.$$

4956. *Proposed by Leonard Carlitz, Duke University*

Find all analytic solutions $f(x)$, $g(x)$, of the functional equation

$$f(x+y) = f(x)g(y) + f(y)g(x).$$

4957. *Proposed by H. S. Shapiro, New York University*

Let D be a bounded domain in the complex plane, and let $f(z)$, $\phi_1(z), \cdots,$ $\phi_n(z)$ be linearly independent functions analytic in and on the boundary of D. Then there exists a unique set of complex numbers $\lambda_1, \cdots, \lambda_n$ such that the following integral is a minimum:

$$\iint_D \left| f(z) - \sum_{k=1}^{n} \lambda_k \phi_k(z) \right| dS.$$

4958. *Proposed by A. G. Konheim, IBM Research, Yorktown Heights, N. Y.*

A polynomial $P(x) = \sum_{i=0}^{n} a_{n-i}x^{n-i}$ will be called symmetric if $a_{n-i} = a_i$, $i = 0, 1, \cdots, n$. Prove that a symmetric polynomial over $GF(2)$ is not primitive.

SOLUTIONS

Maximal Chains in a Boolean Algebra

4871 [1959, 816]. *Proposed by Juris Hartmanis, Ohio State University*

Prove that the Boolean algebra of all subsets of the integers has maximal chains whose length is denumerable and has also maximal chains whose length is not denumerable.

Solution by W. M. Perel and R. C. Wherritt, Louisiana State University, New Orleans. Let I be the set of integers. Define $C_0 = \varnothing$, and, for $n \geqq 1$, let $C_{2n-1} = \{p : |p| \leqq n-1\}$ and $C_{2n} = C_{2n-1} \cup \{n\}$. Then $\{C_m : m \geqq 0\} \cup \{I\}$ is a denumer- maximal chain.

Now let f be any one-to-one function taking the rationals onto the integers. For every real number r, define $D_r = \{f(t) : t \text{ rational}, t \leqq r\}$. Then $D_r \subset D_s$ if and only if $r \leqq s$, $D_r \neq D_s$ if $r \neq s$, and $\{D_r : r \text{ real}\} \cup \{\varnothing\} \cup \{I\}$ is a nondenumerable maximal chain. The maximality is equivalent to completeness of the reals in the Dedekind-cut sense.

Also solved by Irma Esrig, J. B. Johnston, Elliott Mendelsohn, and Hans Schneider.

Compact Topological Space

4891 [1960, 188]. *Proposed by I. S. Gál, Yale University*

Let X be a regular topological space and let A be a dense subset of X such that every net with values in A has a nonvoid adherence in X. Show that X is compact. Is regularity necessary?

Solution by the proposer. Let $\{O_i\}$ $(i \in I)$ be an open cover of X which contains no finite subcover. Since X is regular $\{O_i\}$ $(i \in I)$ admits a refinement $\{Q_j\}$ $(j \in J)$ such that * $c(Q_j) \subseteq O_i$ for every $j \in J$ and for some $i = i(j) \in I$. Let the index set I be well ordered and denote by S_k the set $S_k = Q_k - \cup [Q_j : j < k]$. We prove that the interior of S_k is not void for an infinity of indices $k \in J$. For suppose that $S_k^i = \varnothing$ for every $k \geqq \kappa$, that is $c(Q_k) \subseteq c(\cup [Q_j : j < k])$ for every

* For convenience in printing, $c(\ )$ is used to denote closure.

$k \geqq \kappa$, where κ is a finite index. Let $\mathcal{C}$ be the class of those initial segments $K \subseteq J$ for which

$$c(\cup[Q_j: j \in K]) \subseteq c(\cup[Q_j: j \leqq \kappa]).$$

Then $\mathcal{C}$ is nonvoid because $(1, \cdots, \kappa) \in \mathcal{C}$. It can be ordered by inclusion: $K_1 \leqq K_2$ if $K_1 \subseteq K_2$. There is a maximal element in $\mathcal{C}$ namely $\cup[K: K \in \mathcal{C}]$ itself is an element of $\mathcal{C}$. In fact,

$$\cup[Q_j: j \in \cup[K: K \in \mathcal{C}]] \subseteq c(\cup[Q_j: j \leqq \kappa]),$$

and so the same inclusion relation holds for the closure of this union.

We prove that $\cup[K: K \in \mathcal{C}]$ is J. For let $K \in \mathcal{C}$ be a proper subset of J and let k' be the first index not in K. We set $K' = K \cup \{k'\}$ and obtain

$$\begin{aligned} c(\cup[Q_j: j \in K']) &= c(\cup[Q_j: j < k']) \cup c(Q_{k'}) = c(\cup[Q_j: j < k']) \\ &= c(\cup[Q_j: j \in K]) \subseteq c(\cup[Q_j: j \leqq \kappa]). \end{aligned}$$

Hence $K' \in \mathcal{C}$ and K is not maximal. Consequently $\cup[K: K \in \mathcal{C}] = J$ and this implies that $\cup[Q_j: j \in J) \subseteq c(\cup[Q_j: j \leqq \kappa])$. However, on the one hand, $\{Q_j\}$ $(j \in J)$ is a cover of X and so $X = c(Q_1) \cup \cdots \cup c(Q_K)$. On the other hand, $\{c(Q_j)\}$ $(j \in J)$ is a refinement of $\{O_i\}$ $(i \in I)$ and by hypothesis $\{O_i\}$ $(i \in I)$ does not contain a finite subcover. This is a contradiction and the original hypothesis is false. In other words, S_k^t is not void for infinitely many indices $d \in J$. Let D be any infinite set of such indices $d \in J$, having the property that there is no last element in D. Since A is dense in X the sets $S_d \cap A$ are not void. If $a_d \in S_d \cap A$ for every $d \in D$ then the linear net (a_d) $(d \in D)$ has a void adherence in X. Moreover the sets $\cup[S_d \cap A: d \leqq \delta]$ $(\delta \in D)$ form a filter base in A whose adherence in X is void.

The regularity of X is necessary in the following weak sense: There exists a noncompact X and a dense subset A of X such that every net with values in A has a nonvoid adherence in X. As a matter of fact we can choose X to be a Hausdorff space: Let $X = [-1, +1]$ and let $0 \subseteq X$ be open if it can be obtained from a set which is open in the usual sense in X by omitting points of the form $1/n$ $(n = \pm 1, \pm 2, \cdots)$.

Also solved by Wu Ta-Sun, and J. V. Whittaker.

A Convergent Sequence

4899 [1960, 381]. *Proposed by R. C. Buck, University of Wisconsin*

As $(x, y) \to (0, 0)$ let $\lim F(x, y) = 0$. Suppose that $\{s_n\}$ is a real sequence such that $0 \leqq s_{n+m} \leqq (s_n, s_m)$ for all $n, m = 1, 2, \cdots$. Suppose also that $\lim (s_1 + \cdots + s_n)/n = 0$. Show that $\lim s_n = 0$.

Solution by S. A. Andrea, Oberlin College. Let $\epsilon > 0$ be chosen. Choose δ so that $|x| < \delta$, $|y| < \delta$ implies $|F(x, y)| < \epsilon$. Choose n so that $k \geqq n$ implies $(s_1 + \cdots + s_k)/k < \delta/2$. Then we know that $\sum_{p=1}^{k} (s_p + s_{k+1-p})/k < \delta$. This means

that for some p we have $(s_p+s_{k+1-p})<\delta$ which in turn gives us $s_p<\delta$ and $s_{k+1-p}<\delta$. Now we see that $s_{k+1}\leqq F(s_p, s_{k+1-p})<\epsilon$ for all $k\geqq n$, *i.e.*, $\lim s_n=0$.

Also solved by Robert Breusch, T. C. Brown, E. J. Burr, N. J. Fine, A. J. Geurts, George Glauberman, R. B. Kelman, J. G. Mauldon, M. D. Mavinkurve, J. E. Potter, Albert Wilansky, and the proposer.

Editorial Note. The above proof (as, indeed, in all solutions received) assumes either (a) $F(x, y)$ is continuous at (0, 0) or (b) the s_j are positive. Breusch cites the following counter-example to show that the conclusion does not follow unless some additional assumption is made:

$$F(x, y) = \begin{cases} 1 & \text{if } x = y = 0, \\ x + y & \text{otherwise;} \end{cases} \qquad s_k = \begin{cases} 1 & \text{if } k = 2^r, \\ 0 & \text{otherwise.} \end{cases}$$

Sequence with Arbitrarily Slow Convergence

4900 [1960, 382]. Proposed by S. P. Lloyd, Bell Telephone Laboratories

For every measurable subset E of the unit interval let $\{C_n(E)\}$ denote the sequence of numbers

$$C_n(E) = \int_E e^{2\pi inx}dx, \qquad n = 1, 2, \cdots.$$

Show that a sequence $\{C_n(E)\}$ may tend to zero arbitrarily slowly, in the following sense: for every positive nondecreasing sequence $\{a_n\}$, if $\sup_n a_n|C_n(E)|$ is finite for every E then $\sup_n a_n$ is finite. (Problem 4749 [1958, 453], treats the case $a_n=n$.)

Solution by B. J. Pettis, University of North Carolina. Let Σ be a σ-ring of sets, $\{C_n\}$ a sequence of complex countably additive functions on Σ, and $\{a_n\}$ a complex sequence. For each n, set $D_n=a_nC_n$ and let c_n and d_n be the total variations on Σ of C_n and D_n respectively, so that $d_n=|a_n|c_n$. If $\sup_n |D_n(E)| < \infty$ is true for each E in Σ, then by a theorem of Nikodym's (Dunford-Schwartz, *Linear Operators*, p. 309) the supremum K of the set $[|D_n(E)| : n\in\omega, E\in\Sigma]$ is finite. Clearly $4K\geqq d_n$ for every n, so that the l.u.b. δ of $\{d_n\}$ is finite and $\delta\geqq|a_n|c_n$ for every n. If also the lower limit γ of $\{c_n\}$ is positive then $\lim\sup|a_n| \leqq\delta/\gamma$, implying that $\{a_n\}$ is bounded. Since in the problem above $c_n=\int_0^1|e^{2\pi inx}|dx =1$, the desired conclusion is clear.

Also solved by the proposer.

Determinants Constructed from a Given Set of Elements

4902 [1960, 382]. *Proposed by R. A. Melter, University of Missouri*

At most how many different values may an nth order determinant have, if its elements are a given set of n^2 different, nonzero real numbers?

Solution by P. L. Renz, University of Pennsylvania, and J. M. Gorfinkel, Stanford University. There are $(n^2)!$ distinct $n\times n$ matrices which may be formed from n^2 algebraically independent elements. We divide the matrices into equivalence classes of matrices having the same determinant. Each such equivalence class has at least $(n!)^2$ elements as there is a set T of $n!^2$ distinct transformations

obtained by combinations of row and column permutation and transposition leaving the determinant of any $n \times n$ matrix invariant. On the other hand, the elements being algebraically independent, the value of $|(\theta_{ij})|$ completely determines the elements lying in the same row and column as θ_{ij} as those elements which do not multiply θ_{ij} in $|(\theta_{ij})|$. If $|(\theta_{ij})| = |(\phi_{ij})|$ for two of our $n!^2$ matrices, then we may bring the element θ_{11} to the 1, 1 place by transforming (ϕ_{ij}) by an element of T. Then, at worst by a transposition, we bring the elements of the first row and column of (θ_{ij}) into the first row and column respectively of the transformed (ϕ_{ij}) matrix. Continuing in this way, we transform (ϕ_{ij}) into $(\bar{\phi}_{ij})$ by elements of T until its first row and column, with the possible exception of the last two elements of the first column, are placed exactly as those of (θ_{ij}). Then, since $|\bar{\phi}_{ij})| = |(\theta_{ij})|$, it follows by the above that $(\bar{\phi}_{ij}) = (\theta_{ij})$ and thus that (ϕ_{ij}) is one of the $n!^2$ matrices obtained from (θ_{ij}) by transformations of T. Then the number of distinct values that the determinants of our matrices have is the number of distinct matrices divided by the number of matrices in each equivalence class, *i.e.*, $(n^2)!/n!^2$.

Also solved by Roberto Frucht and Michael Goldberg.

A Limit Problem

4905 [1960, 479]. *Proposed by Basil Gordon, Redstone Arsenal, Alabama*

Does the following limit exist:

$$\lim_{x \to 1^-} (1 - x)^{1/2} \prod_{n=0}^{\infty} (1 + x^{4^n})?$$

I. *Solution by Paul Erdös, University of Melbourne, Australia.* The following more general theorem holds: Let $a_1 < a_2 < \cdots$ be a sequence of integers and let $0 < \alpha < 1$. Then

$$\lim_{x \to 1} (1 - x)^{\alpha} \prod_{k=1}^{\infty} (1 + x^{a_k}) = c \tag{1}$$

is impossible for $c \neq 0$ and $c \neq \infty$. (This will imply a negative answer to the proposed problem since a simple calculation shows that the given limit cannot be 0 or ∞.)

To prove the theorem, assume that (1) is true for $c \neq 0, \infty$. Then

$$\limsup a_k/(a_1 + \cdots + a_{k-1}) > 1. \tag{2}$$

Assume that (2) is already proved. Write $\prod_{k=1}^{\infty} (1 + x^{a_k}) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$\lim_{x \to 1} (1 - x)^{\alpha} \sum_{n=0}^{\infty} b_n x^n = c, \qquad (c \neq 0, c \neq \infty).$$

But then by a theorem of Hardy and Littlewood (Proc. London Math. Soc., 1914, p. 174),

$$(3) \qquad \lim_{n\to\infty} \sum_{k=0}^{n} b_k/n^{\alpha} = c', \qquad (c' \neq 0, c' \neq \infty).$$

(In fact, $c' = c/\Gamma(1+\alpha)$.) But (3) is impossible since by (2) there is a fixed $\epsilon > 0$ and arbitrarily large values of $m(=a_1+a_2+\cdots+a_{k_i})$ so that for every $m < n < m(1+\epsilon)$, we have $b=0$, which contradicts (3). Thus to complete the proof we have only to prove (2).

Assume (2) false. Then a simple argument shows $\limsup a_k^{1/k} \leqq 2$ (for if not, then for infinitely many k's, $a_k > (2+\epsilon)^{k-l} a_{k-l}$ for all $l < k$, which implies (2)). But $\limsup a_k^{1/k} \leqq 2$ implies that for every ϵ,

$$(4) \qquad \lim_{x\to 1} (1-x)^{1-\epsilon} \prod_{k=1}^{\infty} (1 + x^{a_k}) = \infty$$

which contradicts (1); thus our statement is proved. ((4) is evident: put $x = 1 - 1/n$, $1 + (1-1/n)^{a_k} = 2 + o(1)$ if $a_k = o(n)$ and the number of $a_k < m$ exceeds $(1+o(1)) \log m/\log 2$ as $m \to \infty$.)

II. *Solution by J. H. Roberts, Duke University.* The limit does not exist. Letting f denote the given function, we have, for $0 < x < 1$,

$$(\mathrm{i}) \qquad f(x)f(x^2) = (1+x)^{1/2},$$

$$(\mathrm{ii}) \qquad \frac{f(x)}{f(x^4)} = \left(\frac{1+x}{1+x^2}\right)^{1/2}.$$

Formula (ii) follows directly from the definition of f, formula (i) follows from the identity:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \prod_{n=0}^{\infty} (1 + x^{2^n}), \qquad |x| < 1.$$

Now by (i), if $\lim_{x\to 1-} f(x) = L$ exists, then $L^2 = 2^{1/2}$ and $L = 2^{1/4}$.

On the other hand, computation shows that if $\log_{10}(d) = -.00001$, $(d = .9999769744)$, then $\log_{10} f(d) > .076 > (\log_{10} 2)/4$, whence $f(d) > 2^{1/4}$. Now by (ii), $0 < x < 1$ implies $f(x^{1/4}) > f(x)$. Thus, for the sequence $x_1 = d$, $x_2 = d^{1/4}, \cdots$, $x_{n+1} = x_n^{1/4}, \cdots$, we have $2^{1/4} < f(x_1) < f(x_2) < \cdots < f(x_n) < \cdots$, with all $x_n < 1$. Thus $2^{1/4}$ cannot be the limit. Therefore the limit does not exist.

Also solved by Y. S. Chow, Sigvard Jacobs, Glenn M. Roe, and the proposer.

Editorial Note. However, $f(x)$ is bounded on $0 \leqq x < 1$. From (ii) we have $f(x) > f(x^4) > \cdots > f(x^{4^n}) > \cdots$. But $x^{4^n} \to 0$ and $f(0) = 1$. Thus $f(x) > 1$ which also, in light of (i) proves $f(x) < \sqrt{(1+x)}$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Matrices. By W. V. Parker and J. C. Eaves. Ronald Press, New York, 1960. viii+195 pp. $7.50.

The first chapter of the book is a short discussion of preliminary ideas, mainly fields. The material following includes elementary properties of matrices and partitioning, singular and nonsingular matrices, equivalence of bilinear forms and matrices, rank, systems of linear equations, congruence, quadratic forms, vector spaces, determinants, invariant factors, characteristic vectors and equations, rational canonical form, minimum function, Hamilton-Cayley theorem, elementary divisors, Jordan normal form, and normal matrices. There are exercises following each section and just before the index an extensive bibliography appears.

This is an introductory text on matrices with abstraction kept to a minimum. It is clearly and carefully written with good detailed explanations and many examples to illustrate the theory. Matrices are introduced through linear forms rather than through linear transformations. In considering this book one should also be aware that a linear transformation is defined in it as a certain kind of matric equation rather than in the usual way as a linear mapping of a vector space. This latter concept is mentioned briefly in Chapter 8, but is not called a linear transformation, and the relations between matrices and linear transformations are not made clear.

HOWARD E. CAMPBELL
Michigan State University

An Introduction to Algebra for College Students. By W. A. Rutledge and Simon Green. Prentice Hall, Englewood Cliffs, N. J., 1960. ix+276 pp. $4.95.

This text is for college students who have had at most one year of high school algebra. It is, as the authors state in their preface, an attempt to utilize the postulational approach in the presentation of the algebra of the real number system.

The treatment of standard topics from Chapter 7 on is very good. The sections on exponents, radicals, functions, and graphing are carefully written.

In early chapters the field postulates are presented and an *attempt* is made to present the calculating techniques of algebra as theorems that may be derived from these postulates. A superficial reading of these first chapters leaves one with the impression that the authors' objectives have been reasonably well achieved, but close inspection discloses several shortcomings. The teacher who recognizes these weaknesses should be able to use the text effectively.

In general, the student is not taken into full partnership in the authors' program of deriving algebraic techniques from the postulates. For example, addition of integers is illustrated by three proofs of special cases (i) $6+(-3)=3$; (ii) $6+(-9)=-3$; (iii) $-4+(-3)=-7$. Then the old rules for addition of integers are stated. It seems quite probable that students will ignore the proofs and cleave to the rules.

Properties of numbers are postulated or tacitly assumed and then later on are proved in exercises. For example after several pages of working with *the* reciprocal and *the* negative of a real number it is proved, in an exercise, that each real number has a *unique* negative and each nonzero real number a *unique* reciprocal.

Subtraction is defined by the agreement that $a-b=a+(-b)$. However the important property that $b+x=a \rightarrow x=a-b$ is not proved. There is no proof that $a(b-c)=ab-ac$. An argument that $-(a-1)$ is the number $(-a)+1$ is clumsily presented with no justification for several of the steps in the argument. One might almost as well instruct students to "take off the parentheses and change signs."

On page 15 subtraction occurs before the operation has been defined. Much of the language is objectionable. One finds the familiar confusion of symbol with concept. Examples of careless language are: "The symbol $-a$ might or might not be a negative number." "The negative of a number need not have the negative symbol." "3/1 is an integer; 6/2 is not an integer by form but has a value which is an integer." "When we express a number with the usual arithmetic symbols, such as 34, we shall call it an *arithmetic* number or call this the *value* of the number." "We define an *exponent* n of the base $a \cdots$. We are going to define the nth root of the number a, if there is one, by the symbol $a^{1/n}$. The fraction $1/n$ serves to define an nth root." "A solution is a known number that will make the equation true."

Although we feel strongly that the text should be carefully rewritten, cleaning up the language and reorienting the axiomatic development, yet in its present form it is certainly superior to the usual cookbook text. In the hands of a competent instructor it could be used as the basis of a satisfactory course.

CHARLES BRUMFIEL
The University of Michigan

The Modern Aspects of Mathematics. By Lucienne Felix. Translated by Julius H. Hlavaty and Fancille H. Hlavaty. Basic Books, New York, 1960. xiii+194 pp. $5.00.

This is a book about mathematics which contains much mathematics. It sets the stage for the advent of the works of the Bourbakists. It describes their attempts to broaden the foundations of logic and mathematics sufficiently to support present-day research and to provide a basis for a unified structure in which all branches of mathematics, both old and new, intuitive and axiomatic,

can find their place and be properly evaluated. There is a summary of modern logic, set theory, modern algebra, and topology. This is a well-written book, spiced with interesting historical items and examples of mathematical theory. It concludes with some remarks on the implications of the concepts described on the pre-college curriculum, and the reader must conclude with the author that a broad training in mathematics is a necessity, at least for the teachers of bright students, both at the secondary and elementary levels. Mathematics teachers at all levels should find this book full of interesting material about their profession.

C. L. SEEBECK, JR.
University of Alabama

Foundations of Modern Analysis (vol. X of *Pure and Applied Mathematics*). By J. Dieudonné. Academic Press, New York, 1960. xiv+361 pp. $8.50.

This unusual, in part revolutionary, book covers parts of analysis which the author believes should be known to all graduate students in mathematics. Chapters I through VII are familiar enough: elements of set theory; real numbers; metric spaces (elementary theory, not including Baire category, but including the Stone-Weierstrass theorem and the Tietze-Urysohn extension theorem for metric spaces); normed spaces; Hilbert spaces; spaces of continuous functions (the influence of Bourbaki is discernible in the author's discussion of "regulated" functions). Chapters VIII and IX, titled *Differential Calculus and Analytic Functions*, respectively, are radical indeed. Derivatives are defined for continuous mappings of a Banach space E into a Banach space F; the derivative when it exists is a bounded linear mapping of E into F. The mean value theorem, primitive functions, Jacobians, higher derivatives all appear, with logically precise definitions, but in disguises that may prove highly effective. The Lebesgue integral is not discussed. Analytic functions are also treated with maximum generality: usually they are functions from open subsets of n-dimensional real or complex space to a real or complex Banach space that admit local power series expansions. A weak (homotopic) form of Cauchy's theorem is established. Chapter X, titled *Existence Theorems*, is also quite abstract. Chapter XI, *Elementary Spectral Theory*, is a standard treatment.

The book is written with great precision, and is provided with a huge number of problems. (Some of these, the great Picard theorem for example, are highly sophisticated.)

The author's style is vigorous, and his views unambiguous. In discussing many traditional topics, his attitude seems to be, never call a spade a spade if you can call it a damned old shovel. For students well grounded in classical analysis, study of this book will undoubtedly be a stimulating and exciting adventure. Whether it will become a standard text for first-year real variables courses, only time will tell.

EDWIN HEWITT
University of Washington

Elementary Mathematical Programming. By Robert W. Metzger. Wiley, New York, 1958. ix+246 pp. $5.95.

This book describes in simple terms some methods used in solving what is generally known as the "linear programming problem," *i.e.*, the problem of finding a vector which minimizes a linear form, subject to certain linear constraints. The author has endeavored to make the concepts and techniques of linear programming accessible to those having little training in mathematics, and in this endeavor he has succeeded rather well. The treatment of mathematical material such as this in a nonmathematical way seems inevitably to require at least the appearance of prolixity, if not the actuality, so that patient, careful reading is necessary for a full understanding of the subjects considered.

After a brief introductory chapter dealing with the history of Operations Research and the areas in which mathematical programming has been found useful, the author in Chapter 2, "Distribution Methods," takes up the so-called "transportation problem" in which, in its simplest form, it is required to ship various amounts of a product from each of m warehouses to satisfy the demands of each of n customers in such a way as to minimize the shipping costs. The "stepping-stone," modified distribution, and Vogel approximation methods of solving this problem are presented in detail. The text here is very amply supplemented by figures. The reader with very little more than a minimum of mathematical preparation should find the mathematical formulation of the three-dimensional problem at the end of this chapter satisfying. Chapter 3 takes up the simplex method of solution of the linear programming problem. The method is illustrated by means of the simple, two-product, manufacturing situation. The introduction here leaves something to be desired and the only hint as to the origin of the term *simplex* is misleading in that it implies that it may derive from the nature of the method, an idea of which the reader, admittedly with the help of an admonition by the author, is shortly disabused. The method is well illustrated and the rules for its use listed. The geometric interpretation of the problem is then given. It would seem that a clearer idea of the problem might be imparted were this interpretation presented at the beginning. The dual problem and the modified simplex method are then treated. Chapter 4 is devoted to approximation methods; Chapter 5 to two applications based on real situations. Chapter 6 (six pages) points out the availability of digital computer programs, specifically for IBM computers, for the solution of larger problems, but is rather misleading to those interested in using the IBM 650 for their problems but who are unfamiliar with its capabilities and those of auxiliary machines invariably found in such a computing center. (One rarely is reduced to interpreting punched-card output manually, and the one and one-half hours mentioned is total time for *all* phases of the processing of the problem.) Chapters 7–10 present in clear fashion problems in production planning, stock splitting, material-handling scheduling, and job and salary evaluation.

There are a few minor defects in exposition and technique, *e.g.*, occasional

failure to italicize technical terms when first mentioned, and at least one definition of the "is when" type, but in general it is a well-written book. There are no exercises.

FRANKLIN S. MCFEELY
Montana State College

A Course in Applied Mathematics. By Derek F. Lawden. The English Universities Press, London, 1960. xv+655 pp. 70/-. (about $7.00).

This excellent work is essentially an introduction to classical mathematical physics. The major subdivisions (together with the pages devoted to each) are as follows: Dynamics (270), Statics (178), Field Theory (193)—mostly electricity and magnetism, and Hydromechanics (87). The minimum mathematical requirement for progress through the book at a satisfactory rate is a good knowledge of advanced calculus, including some vector analysis exclusive of the gradient, divergence, and curl, for as asserted in the preface: "It is also assumed that the reader will be attending a course of lectures in pure mathematics during which he will study the theory of linear differential equations both ordinary and partial, the definition and evaluation of surface and volume integrals and the notation and techniques of vector analysis."

According to a statement on the jacket: "Every effort has been made to assist those students who must depend largely on their own resources while studying the subject." In the reviewer's opinion, the author has succeeded admirably in fulfilling this purpose, and this special suitability for unassisted study is perhaps the cardinal characteristic of the book. Altogether, this work presents many instances of superior expository writing. The rigor level is a cut above the average for textbooks in this field—a circumstance that is conducive to clarity; and the theoretical developments are sufficiently detailed for solo study. The chapters are organized into short integrated units consisting of either a single article or a set of two or three closely related articles followed in either case by illustrative examples. This structure should be welcomed by those readers who by choice or necessity will use short study periods; perhaps in addition it will serve as a deterrent to mental indigestion. The articles are usually of two or three pages each and the illustrative problems are worked out in detail. There are 546 unsolved problems.

For the mathematician interested in theoretical physics this book might serve as a source for quick references at the introductory level. There are a few places where one might wish to introduce a mean-value theorem or some other device for improving the rigor but on the whole the mathematical developments are satisfactory and in some cases, as in the treatment of fields in dielectric and magnetic media, the strategies employed are assuredly time-saving. In conclusion, this work is a valuable addition to the expository literature on mathematical physics.

HOMER V. CRAIG
University of Texas and Boeing Airplane Co.

Calculus. By Wray G. Brady and Maynard J. Mansfield. Little, Brown, Boston, 1960. ix+456 pp. $8.00.

This book "offers a course in modern mathematics via the calculus, and on the other hand, a course in calculus via modern mathematics."

In the first chapter elementary notions of set theory, real inequalities, intervals, bounds, least upper bounds, etc. are introduced. In this discussion the field properties of the real numbers are assumed. In the second chapter, functions (as sets of ordered pairs), limits, and continuity are defined with great care and yet with excellent motivational comments. Proofs usually omitted in elementary books are given for many statements (*e.g.* the boundedness theorem for continuous functions). Throughout the book a strictly up-to-date notation and viewpoint is maintained.

Except for the unusually rigorous exposition and the liberal use of functional equations, chapters III through IX cover the standard material on differentiation and anti-differentiation of real functions of one variable with applications.

Chapter X contains a radical departure from the orthodox elementary treatment of integration. The Riemann-Stieltjes integral is introduced and it is shown that averages, for countable collections, can be expressed as integrals. This is done with the following machinery. The concept set function and measure are introduced with a finite measure and a length measure given as examples. Certain measurable functions, called "traits," (*e.g.*, random variables), and then a generalized concept of distribution function are introduced. Finally infinite averages (of traits) are defined and relations with definite integrals shown. There follows a discussion of standard material on integration with benefit of the more general treatment afforded by use of the Stieltjes integral at the outset. In particular the authors are able correctly to discuss probability distributions in Chapter XIV.

As the above sketch should indicate, the book is a far cry from the material and point of view assumed in most beginning calculus courses, and would be too strong a diet for most freshmen or sophomores. It would, however, be eminently suitable for an honors course for mathematics and science majors.

The book is clearly written (although the introductory material on measure, traits, sums, etc. in Chapter X could profit from more examples and better diagrams), and appears to be free of typographical errors. The authors have written an excellent book with extreme care.

R. J. NELSON
Case Institute of Technology

BRIEF MENTION

Modern Computing Methods, Notes on Applied Science No. 16. National Physical Laboratory. Purchasable through Her Majesty's Stationery Office, 423 Oxford Street, London W 1, England, 1959. vi+129 pp. $1.52.

An excellent introduction to computer-oriented algebraic mathematics and finite

differences, including differential equations. This is apparently the same as the book published in hard-back cover by the Philosophical Library of New York at $8.75.

Progress in Automation, Vol. I. Edited by Andrew D. Booth. Academic Press, New York, 1960. viii+231 pp. $8.50.

Another book from our British cousins, this time devoted mainly to industrial use of computers in control situations.

Advances in Computers, Vol. I. Edited by Franz L. Alt. Academic Press, New York, 1960. 316 pp. $10.00.

This collection of six papers includes General Purpose Programming for Business Applications by C. C. Gotlieb; Numerical Weather Prediction by N. A. Phillips; The Present Status of Automation Translation of Languages by Y. Bar-Hillel; Programming Computers to Play Games by A. L. Samuel; Machine Recognition of Spoken Words by R. Fatehchand; and Binary Arithmetic by G. W. Reitwiesner.

Automatic Language Translation. By Anthony G. Oettinger. Harvard University Press 1960. 380 pp. $10.00.

No mere popular treatise filled with generalities, but a serious well-written account of the Russian translation project at the Harvard computation laboratory. It includes some good remarks on computer programming, particularly as it applies to data processing of languages.

Introduction Mathêmatique a la Mêcanique des Fluides. By M. Caius Jacob. Gauthier-Villars, Paris, 1959. 1286 pp. $10.00.

A hefty tome dealing with both compressible and incompressible fluid mechanics.

Handbuch Der Schulmathematik. By Georg Wolff. Hermann Schroedel Verlag K G, Berlin HANNOVER, Darmstadt Postfach 89, 1960, 296 pp.

Algebra for Commerce and Liberal Arts. By A. K. Bettinger and W. A. Dwyer. Pitman, New York, 1960. 243 pp. $4.00.

A nice job of printing of an extremely routine presentation.

Intermediate Algebra, Alternate Edition. By Lovincy J. Adams. Holt, Rinehart and Winston, New York, 1960. xiii+414 pp. $4.50.

Plane Trigonometry, 4th Edition. By Fred W. Sparks and Paul K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1960. x+308 pp. $5.25. An old favorite.

Ingenious Mathematical Problems and Methods. By L. A. Graham. Dover, New York, 1960. vii+237 pp. $1.45.

There are some charming bits in this paperback collection. Originally published in the "Private Corner for Mathematicians" of the Graham DIAL.

An Introduction to Linear Programming and the Theory of Games. By S. Vajda. Wiley, New York, 1960. 76 pp. $2.25.

A brief but stimulating introduction to Vajda's better-known works.

Espaces Topologiques, Fonctions Multivoques. By Claude Berge. Dunod, Paris, 1959. 272 pp.

Lectures on Ergodic Theory. By Paul R. Halmos. Chelsea, N. Y., 1956. vii+99. $2.95.

A reprint of the Mathematical Society of Japan's 1956 edition of Halmos' 1955 lectures at the University of Chicago.

Engineering Mathematics. By J. Blakey and M. Hutton. Philosophical Library, New York, 1960. 603 pp. $10.00.

This seems to be a somewhat "cook-book" version of Blakey's *University Mathematics* with some additional work on statistics, relaxation methods and Laplace transformations.

The Real Projective Plane, 2nd Edition. By H. M. S. Coxeter. Cambridge University Press, New York, 1960. xi+226 pp. $3.75 (paperback).

Congratulations to Cambridge Press for bringing us a revised edition of Coxeter's excellent 1949 work. This is not merely a reprinting of the previous book. Many changes have been made.

The Theory of Linear Economic Models. By David Gale. McGraw-Hill, New York, 1960. 350 pp. $9.50.

Gale doesn't expect students of economics to know the necessary mathematics when they start his book, but neither does he undertake to teach linear theory without developing mathematics. He develops those portions of vector and matric theory which he will need, and those only—a commendable procedure.

Normed Rings. By M. A. Naimark, translated from the Russian by Leo F. Boron. P. Noordhoff-Groningen, Holland, 1960. xvi+560 pp. $13.00.

There is an amazing amount of information packed into less than 600 pages. The first 154 pages deal with basic ideas from topology and functional analysis, then in succession come: fundamental concepts in propositions in the theory of normed rings, commutative normed rings, representations of symmetric rings, special rings, group rings, rings of operators in Hilbert space, and decomposition of a ring of operators into irreducible rings.

Riemann Surfaces. By Lars V. Ahlfors and Leo Sario. Princeton University Press, Princeton, N. J., 1960. xi+382 pp. $10.00.

This excellent book will be reviewed in the Bulletin of the American Mathematical Society in the forthcoming issue.

Cartesian Geometry of the Plane. By E. M. Hartley. Cambridge University Press, New York, 1960. xi+324 pp. $3.75.

Frankly, I would enjoy teaching a course in analytic geometry from this text. If your school still gives a separate course in analytic geometry, I sincerely recommend that you obtain a copy of this book for consideration.

Grundlagen Der Analysis. By Edmund Landau. Chelsea, New York, 1960. 173 pp. $1.95.

With a complete German-English vocabulary, this old favorite, both of mathematicians and of students who wish to pass a German reading examination without knowing any German, is again available in a modern reprint. The publisher has even translated the preface to save the above students possible difficulty in translation.

The New Mathematics. By Irving Adler. Mentor, New York, 1960. 192 pp. 50¢.

Pick up a copy of this Mentor edition at your corner drugstore and look it over for yourself. It is worthwhile.

Éléments D'Algèbre. By Gaston Julia. Gauthier-Villars, Paris, 1959. viii+207 pp. $7.93.

Applied Boolean Algebra. By Franz E. Hohn. Macmillan, New York, 1960. xx+139 pp. $2.50.

A fine paperback which you will wish to call to the attention of your electrical engineering students. Anyone unfamiliar with the use of Veitch diagrams will want to read Chapter 4. This valuable, practical tool is often overlooked.

Introduction A L'utilisation Pratique de la Transformation De Laplace. By Gustav Doetsch. Gauthier-Villars, Paris, 1959. viii+198 pp. $7.50.

Modern Elementary Statistics, 2nd Edition. By John E. Freund. Prentice-Hall, Englewood Cliffs, N. J., 1960. x+413 pp. $6.95.

One of the better modern books on statistics for those who do not have a calculus background.

The Mathematical Foundations of Quantum Statistics. By A. Y. Khinchin, translated from the Russian by Irwin Shapiro. Graylock Press, Albany, N. Y., 1960. xi+231 pp. $10.00.

Thermodynamics and statics based on a sound statistical background.

Contributions to Probability and Statistics. Edited by Olin, Ghurye, Hoeffding, Madow and Mann. Stanford University Press, Stanford, California, 1960. x+517 pp. $6.50.

This series of essays in honor of Harold Hotelling contains 38 short papers by people who have been closely associated with Professor Hotelling. What more fitting tribute can there be to a great scholar?

Five welcome reprints from Dover Press, New York, 1960:

Algebras and Their Arithmetics. By Leonard Eugene Dickson. xii+241 pp. $1.35.

Statistics Manual. By Crow, Davis and Maxfield. xvii+288 pp. $1.55.

Differential Equations for Engineers. By Philip Franklin. vii+299 pp. $1.65.

The Applications of Elliptic Functions. By Alfred George Greenhill. xi+357 pp. $1.75.

Coordinate Geometry. By L. P. Eisenhart. xi+297 pp. $1.65.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dr. R. W. Bass, Mathematician and Chief Scientist for the American Manufacturing Corporation, Aeroscope Division, Catonsville, Maryland, has been named Maryland's Outstanding Young Scientist of 1960 by Maryland Academy of Sciences.

Harpur College: Associate Professor A. D. Ziebur, Ohio State University, has been appointed Associate Professor; Dr. K. W. Anderson, University of Illinois, and Dr. Lily H. Seshu, University of Toronto, have been appointed Assistant Professors.

Hunter College: Associate Professor I. H. Rose, University of Massachusetts, has been appointed Associate Professor; Assistant Professors A. D. Bradley and Carolyn Eisele have been promoted to Associate Professors; Professor L. S. Hill and Assistant Professor Helen K. Kutman retired September 1, 1960.

Pennsylvania State University: Mr. W. D. Bouwsma, University of Michigan, has been appointed Assistant Professor; Mr. L. F. Boron, St. Vincent's College, has been appointed Instructor; Associate Professors Raymond Ayoub and J. B. Bartoo have been promoted to Professors; Assistant Professors C. C. Faith and Mary L. McCammon have been promoted to Associate Professors; Associate Professor C. C. Faith is spending the academic year at the Institute for Advanced Study; Professor Orrin Frink is spending the academic year at University College, Dublin, Ireland.

Polytechnic Institute of Brooklyn: Dr. J. S. Lomont, International Business Machines, Yorktown Heights, New York, has been appointed Associate Professor; Assistant Professor George Bachman, Rutgers University, has been appointed Assistant Professor; Mr. Clifford Marshall, Miss Sulbha Agarwal, Miss Helen Bowden, and Mrs. Ruth G. Favro have been appointed Instructors; Associate Professor J. C. Scanlon has been promoted to Professor; Mr. R. B. Lowe has been promoted to Assistant Professor.

Purdue University: Associate Professor Harley Flanders, University of California, Berkeley, has been appointed Professor; Associate Professor R. L. Blair, University of Oregon, Assistant Professor Melvin Carter, North Carolina State College, Assistant Professor L. J. Cote, Syracuse University, Dr. R. A. Gambill, General Motors, Indianapolis, Indiana, Dr. J. H. Michael, University of Adelaide, and Assistant Professor G. J. Rieger, University of Maryland, have been appointed Associate Professors; Assistant Professor A. R. Amir-Moez, Queens College, Dr. R. B. Kane and Assistant Professor Rosemarie Stemmler, University of Illinois, and Dr. J. C. Lillo, RIAS, have been appointed Assistant Professors; Mr. E. H. Lehman, North Carolina State College, Mr. L. F. Bruening, Mr. R. E. Hughs, and Mr. Kenzo Seo have been appointed Instructors; Associate Professors Philip Dwinger, Melvin Henriksen, Meyer Jerison, and Stanley Reiter have been promoted to Professors; Assistant Professors A. H. Copeland, Jr., Morris Skibinsky, R. F. Williams and R. E. Zink have been promoted to Associate Professors; Drs. James Hogg, E. M. McNally, L. D. Pyle, and R. F. Scott, have been promoted to Assistant Professors.

Rutgers, The State University: Drs. A. W. Adler, Massachusetts Institute of Technology, R. W. Carroll, University of Nancy, France, B. I. Gross, University of Pennsylvania, Benjamin Muckenhoupt, De Paul University, and Samuel Park, University of Pittsburg, have been appointed Assistant Professors; Mr. R. T. Bumby and Mr. F. M. Sand, Princeton University, Mr. J. H. Oppenheim, University of Illinois, Dr. D. R. Ostberg, University of California, Berkeley, and Mr. Thomas Paley, Skidmore College, have been appointed Instructors; Associate Professors K. G. Wolfson, Joshua Barlaz, and H. J. Zimmerberg have been promoted to Professors; Assistant Professor J. C. E. Dekker has been promoted to Associate Professor; Mr. R. E. Bryan and Mr. R. C. Courter have been promoted to Assistant Professors; Professor M. G. Galbraith retired with the title of Professor Emeritus.

San Diego State College: Mr. E. I. Deaton, University of Texas, Mr. W. E. de Malignon, State University of South Dakota, Dr. Robert DeZur, University of Wyoming, Mr. L. D. Fountain, University of Nebraska, Mr. R. L. Van de Wetering, Stanford University, Mr. Herbert Gindler, and Mr. Raymond Killgrove have been appointed Assistant Professors; Miss Elizabeth Otten has been appointed Lecturer; Mr. Max Bergstrom, Mr. Rolando Peinado, and Miss Shirley Weihe have been appointed Instructors; Associate Professors A. R. Harvey and L. G. Riggs have been promoted to Professors; Assistant Professors Dean Branstetter, Peter Shaw, N. B. Smith, Le Roy Warren, and Margaret Willerding have been promoted to Associate Professors.

University of California, Berkeley: Professors J. L. Koszul and René Thom, University

of Strasbourg, France, Alex Rosenberg and Daniel Zelinsky, Northwestern University, Wolfgang Rothstein, University of Münster, Germany, and Jerome Sachs, Chicago State Teachers College, have been appointed Visiting Professors; Dr. Stephen Smale, Institute for Advanced Study, has been appointed Associate Professor; Drs. G. E. Bredon and M. W. Hirsch, Institute for Advanced Study, Chen-Chung Chang, University of California, Los Angeles, Adam Koranyi, Harvard University, D. S. Scott, University of Chicago, and Hirofumi Uzawa, Stanford University, have been appointed Assistant Professors; Dr. Steven Bryant, Fresno State College, Dr. L. E. Dubins, University of California, Berkeley, Assistant Professor S. H. Dwivedi, N.R.E.C. College, Khurja, India, Dr. Linda Naim, Maitre de Conferences, Institut Fourier, Grenoble, France, Drs. Hugo Rossi and D. F. Wehn, Princeton University, and Assistant Professor J. H. Wells, University of North Carolina, have been appointed Visiting Assistant Professors; Drs. Gertrude I. Heller, RIAS, and F. J. Kosier, Michigan State University, have been appointed Instructors; Drs. R. H. Abraham, University of Michigan, A. T. Lundell, Brown University, and R. R. Phelps, Institute for Advanced Study, have been appointed Lecturer and Assistant Research Mathematicians; Mr. J. M. Cook, Argonne National Laboratory, has been appointed Lecturer; Mr. P. L. Cavailles, Paris, France, has been appointed Acting Instructor; Associate Professor R. J. de Vogelaere has been promoted to Professor; Assistant Professors Jacob Feldman, P. E. Thomas, and R. L. Vaught, have been promoted to Associate Professors.

University of Cincinnati: Professor I. A. Barnett has been appointed Acting Head of the Department of Mathematics; Associate Professor Charles Saltzer, Case Institute of Technology, has been appointed Professor; Dr. Lee Suyemoto has been appointed Acting Assistant Professor; Mr. R. M. Lotspeich, Oklahoma State Uuiversity, Mr. R. H. Rolwing, Christian Brothers College, Mr. M. L. Brown, Mr. J. R. Downing, Mr. A. L. Haste, and Mr. E. V. Martin have been appointed Instructors.

University of Florida: Dr. J. E. Maxfield, Naval Ordnance Test Station, China Lake, California, has been appointed Professor and Head of the Department of Mathematics; Professor Paul Civin, University of Oregon, has been appointed Visiting Research Professor; Assistant Professor Henryk Minc, University of British Columbia, has been appointed Associate Professor; Mr. Del Willard, Baldwin-Wallace College, Mrs. Jane M. Day, and Mr. J. L. Tilley have been appointed Instructors; Mr. N. W. Hill, Radio Corporation of America at Patrick Air Force Base, Florida, and Mrs. Petee S. Jung, University of Massachusetts, have been appointed Interim Instructors; Professor F. W. Kokomoor retired as Head of the Department of Mathematics on June 30, 1960; Professor C. G. Phipps retired June 30, 1960.

University of Illinois: Assistant Professor S. I. Goldberg, Wayne State University, has been appointed Visiting Associate Professor; Acting Assistant Professor A. Feinstein, Stanford University, has been appointed Assistant Professor; Dr. J. L. Britton, Glasgow University, Scotland, and Assistant Professor E. E. Kohlbecker, University of Utah, have been appointed Visiting Assistant Professors; Mr. M. J. Wicks, University of Malaya, Singapore, Malaya, has been appointed Visiting Instructor; Drs. Betty Detwiler, University of Kentucky, Leone Low, Oklahoma State University, David Sachs, Illinois Institute of Technology, P. M. Weichsel, California Institute of Technology, and E. C. Weinberg, Purdue University, have been appointed Instructors; Associate Professors W. W. Boone and Alex Heller have been promoted to Professors; Assistant Professors D. L. Burkholder, R. C. Langebartel, Echo D. Pepper, L. A. Rubel and R. A. Wijsman have been promoted to Associate Professors; Professor J. L. Doob has been made a charter member of the Center of Advanced Study and Professor W. W. Boone is an associate member for this year.

University of Kansas: Professor S. M. Shah, Muslim University, Aligarh, India, has been appointed Visiting Professor; Dr. W. C. Nemitz, Ohio State University, has been appointed Assistant Professor; Dr. Andrew Page, Kings College, has been appointed

Visiting Instructor; Mrs. Carol H. Bassett has been appointed Instructor; Assistant Professor A. H. Kruse has been promoted to Associate Professor; Associate Professor A. H. Kruse is on leave during the academic year of 1960–61 as Research Professor at the New Mexico State University Research Center; Professor R. M. Schatten is on leave for the academic year of 1960–61 as Visiting Professor at the University of California, Los Angeles; Associate Professor Florence Black retired June, 1960, with the title of Associate Professor Emeritus.

University of Massachusetts: Assistant Professor M. D. Barr, Eastern Michigan University, Mr. J. R. Brown and Miss Eleanor Killam, Yale University, Dr. R. A. McHaffey, Rutgers University, Mr. Torsten Norvig, Brown University, and Mr. C. A. Riley, University of Michigan, have been appointed Assistant Professors; Mr. W. J. Halm, University of Kansas, and Mr. R. K. Mento have been appointed Instructors; Assistant Professors R. R. Archer and D. J. Dickinson have been promoted to Associate Professors; Miss Lorraine D. Lavallee has been promoted to Assistant Professor; Associate Professor A. G. Azpeitia is on leave during the academic year of 1960–61 at Brown University.

University of Miami: Professor Emeritus Tomlinson Fort, University of Georgia, Dr. B. E. Howard, University of Chicago, and Professor Andrew Sobczyk, University of Florida, have been appointed Professors; Miss Jacqueline E. H. Elliott and Mr. R. L. Kelley have been appointed Instructors; Dr. E. F. Low, Jr., has returned from a year's leave of absence at the Institute of Mathematical Sciences, New York University, and has been promoted to Associate Professor.

University of Utah: Assistant Professors W. J. Coles, E. A. Davis, E. E. Kohlbecker and D. V. V. Wend have been promoted to Associate Professors; Associate Professor E. E. Kohlbecker is on leave for the academic year of 1960–61 at the University of Illinois; Associate Professor J. H. Barrett has returned from leave with the Mathematics Research Center, University of Wisconsin; Dr. Lida K. Barrett has returned from leave at the University of Wisconsin.

University of Virginia: Assistant Professors Marvin Rosenblum, P. E. Conner, and E. C. Paige, have been promoted to Associate Professors; Dr. N. F. G. Martin has been promoted to Assistant Professor.

Washington State University: Professor T. G. Ostrom, Montana State University, has been appointed Professor; Messrs. L. D. Coffin, J. A. Johnson, E. E. Mayer, and R. McCormmach have been appointed Instructors.

West Virginia University: Professor J. K. Stewart has been appointed Chairman of the Department of Mathematics; Miss Jean Loudin, Otterbein College, and Mrs. Judith Hall, West Virginia University, have been appointed Instructors; Associate Professor A. B. Cunningham has been promoted to Professor; Dr. H. A. Davis, retired in September, 1960, as Chairman of the Department of Mathematics with the rank of Professor.

Mr. C. B. Baytop, Albama Agricultural and Mechanical College, has been appointed Instructor at Howard University.

Mr. J. A. Brown, University of Delaware, has accepted a position as Operations Analyst with the Johns Jopkins Research Office, Bethesda, Maryland.

Professor Lamberto Cesari, Purdue University, has been appointed Professor at the University of Michigan.

Assistant Professor H. L. Crowson, University of Florida, has accepted a position as Staff Mathematician with International Business Machines, Bethesda, Maryland.

Associate Professor John Dyer-Bennet, Purdue University, has been appointed Associate Professor at Carleton College.

Associate Professor Virginia I. Felder, Mississippi Southern College, has been promoted to Professor.

Associate Professor Leonard Gillman, Purdue University, has been appointed Chairman of the Department of Mathematics at the University of Rochester.

Mr. W. R. Hydeman, Touche, Niven, Biley and Smart, Detroit, Michigan, has accepted a position as Manager of the Administration Systems Planning Division of Lockheed Missiles and Space Division, Lockheed Aircraft Corporation, Sunnyvale, California.

Dr. Ronald Jacobowitz, Princeton University, has been appointed Instructor at the Massachusetts Institute of Technology.

Mr. L. A. Kenna, Radio Corporation of America, Tucson, Arizona, has been appointed Instructor at the University of Arizona.

Assistant Professor H. T. La Borde, University of Cincinnati, has been appointed Associate Professor at the University of South Carolina.

Mr. W. R. LaCour, Northwestern State College, has accepted a position as Mathematician with the United States Department of Agriculture, Dallas, Texas.

Professor L. C. Lay, Pasadena City College, has been appointed Professor at Orange County State College.

Mr. Lowell Leake, Jr., University of Wisconsin, has been appointed Instructor at the University of Cincinnati.

Associate Professor R. C. Meacham, University of Florida, has been appointed Head of the Department of Mathematics at Florida Presbyterian College.

Professor H. M. MacNeille, Washington University, has been appointed Professor and Head of the Department of Mathematics at Case Institute of Technology.

Mr. H. H. Manker, Kansas State College, has accepted a position as Programmer with International Business Machines, Endicott, New York.

Associate Professor G. H. Miller, Western Illinois University, has been appointed Professor at State Teachers College at Towson, Maryland.

Mr. N. L. J. Oldani, University of Detroit, has accepted a position as Analyst with A. C. Spark Plug Company, Oak Creek, Wisconsin.

Mr. Wade Petersen, Washington University, has been appointed Instructor at St. Louis University.

Assistant Professor R. J. Pitts, Los Angeles State College, has been promoted to Associate Professor.

Professor Hans Samelson, University of Michigan, has been appointed Professor at Stanford University.

Mr. R. J. Schwabauer, University of Nebraska, has been appointed Instructor at the State University of South Dakota.

Mr. C. E. Seabold, Standard Oil Company of Ohio, Cleveland, Ohio, has accepted a position as Programmer with Applied Data Systems, Princeton, New Jersey.

Professor O. T. Shannon, Agricultural, Mechanical and Normal College, Pine Bluff, Arkansas, has been appointed Chairman of the Department of Mathematics and Physics.

Dr. Oved Shisha, Harvard University, has accepted a position as Mathematician with the Numerical Analysis Section of the National Bureau of Standards, Washington, D. C.

Mr. E. A. Stavinoha, Douglas Aircraft Company, Tulsa, Oklahoma, has accepted a position as Mathematician in the Research Laboratory of the Research and Development Operations, ABMA, Redstone Arsenal, Alabama.

Dr. Beauregard Stubblefield, University of Michigan, has accepted a position as Programming Specialist with International Electric, Paramus, New Jersey.

Mr. G. L. Sward, Cornell University, has been appointed Instructor at Vassar College.

Mr. John Weissman, Rutgers University, has accepted a position as Senior Engineer-Research with Autonetics, Downey, California.

Mr. J. P. Williams, University of Michigan, has accepted a position as Engineer with Sperry Gyroscope, Syosset, Long Island, New York.

Professor E. T. Bell, California Institute of Technology, died December 21, 1960. He was a Charter Member of the Association and former President of the Association 1931–32.

Rev. J. G. Burke, Mount St. Mary College, died September 6, 1960. He was a member of the Association for 36 years.

Professor C. T. Bumer, Clark University, died March 14, 1960. He was a member of the Association for 36 years.

Mr. Morris Dansky, Creighton University, died November 7, 1960. He was a member of the Association for 11 years.

Mr. J. J. Dodd, Wisconsin State College, died April 3, 1960. He was a member of the Association for 11 years.

Professor R. L. Erickson, retired from Lakeland College, died November 21, 1960. He was a member of the Association for 13 years.

Professor Alfred Errera, L'Université Libre de Bruxelles, Belgium, died September 18, 1960. He was a member of the Association for 21 years.

Associate Professor W. L. Fields, Hampton Institute, died May 29, 1960. He was a member of the Association for 19 years.

Professor Emeritus Edward Fleisher, Brooklyn College, died July 8, 1960. He was a member of the Association for 34 years.

Professor Emeritus C. H. Forsyth, Dartmouth College, died November 2, 1960. He was a Charter Member of the Association.

Professor Emeritus Isabel Harris, University of Richmond, died October 21, 1960. She was a member of the Association for 36 years.

Associate Professor B. M. Ingersoll, Arizona State University, died October 3, 1960. He was a member of the Association for 16 years.

Professor H. H. Irwin, Washington State University, died June 9, 1960. He was a member of the Association for 14 years.

Associate Professor G. B. Lang, University of Florida, died July 21, 1960. He was a member of the Association for 25 years.

Mr. A. L. Milner, University of Alabama, died June 30, 1960.

Associate Professor J. W. Popow, United States Naval Academy, died November 28, 1960. He was a member of the Association for 13 years.

Sister Mary Clementia, S.S.F., St. Mary's Academy, New Orleans, Louisiana, died March 7, 1960.

Associate Professor Hidehiko Yamabe, Northwestern University, died November 20, 1960. He was a member of the Association for 5 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The sixth annual meeting of the Northeastern Section of the Mathematical Association of America was held at Wesleyan University, Middletown, Connecticut on November 26, 1960. Professor J. G. Kemeny, Chairman of the Section, presided at the morning session and Professor D. E. Christie presided at the afternoon session. There were 166 people registered, including 113 members of the Association.

Officers chosen for 1960–61 were Professor D. E. Christie, Bowdoin College, Chairman; Professor R. A. Rosenbaum, Wesleyan University, Vice-Chairman; Mr. R. S.

Pieters, Phillips Academy, Secretary-Treasurer. Written notice of the following amendments to the By-Laws of the Section, as recommended by the Executive Committee, was sent to all members of the Section on October 25, 1960. These were passed by unanimous vote of those members of the Association present at the meeting.

Article III, section 2. Add the words, "and the sectional governor of the Mathematical Association of America." This makes the sectional governor a member of the Executive Committee.

Add "Article V. Registration Fees. The Executive Committee may, at its discretion, charge a registration fee of not more than $1.00 at the Annual Meeting. These fees shall be used to help pay the expenses of conducting the business of the Section."

By invitation of the Executive Committee the following talks were presented:

1. *An example in mathematical logic*, by Professor Hartley Rogers, Jr., Massachusetts Institute of Technology.

A decision procedure for the elementary theory of the ordering of the reals is described (Langford 1927). Its formulation is used to suggest some of the main traditional and present concerns of mathematical logic. The distinction between semantical and syntactical concepts is emphasized.

2. *The transfinite diameter*, by Professor Einar Hille, Yale University.

The transfinite diameter (Fekete, 1923) is a function of closed bounded sets in the complex plane, obtained as the limit of a sequence of maximal average distances between points of the set, the average being furnished by the geometric mean. Polya and Szego (1931), using harmonic, arithmetic, and pth root pth power means instead, extended the definition to three dimensions. All these definitions apply for compact sets in arbitrary metric spaces and so do the corresponding definitions of the Chebichev constant of the set. The notions of capacity also extend for Euclidean spaces at least.

3. *The SMSG geometry program*, by Professor Edwin Moise, Harvard University.

4. *The mathematical training of physics and engineering students*, by Professor R. J. Walker, Cornell University.

5. *Recommendations for teacher training*, by Professor J. G. Kemeny, Dartmouth College.

R. S. PIETERS, *Secretary*

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual fall meeting of the Philadelphia Section of the Mathematical Association of America was held on November 26, 1960 at Swarthmore College, Swarthmore, Pennsylvania. Professor W. S. Lawton, Chairman of the Section, presided at the morning and afternoon sessions. There were 70 persons registered in attendance, including 55 members of the Association.

The following officers were elected to serve during the year 1960–61; Chairman, Professor S. S. McNeary, Drexel Institute of Technology; Secretary-Treasurer, Professor F. L. Dennis, Ursinus College; Second member of the Executive Committee, Mr. A. M. Linton, Jr., William Penn Charter School; Third member of the Executive Committee, Professor David Rosen, Swarthmore College. At the business meeting, Professor J. A. Brown reported on the progress of the Committee on Professional Standards.

The following papers were presented:

1. *Some basic concepts in algebraic topology*, by Professor S. L. Gulden, Lehigh University.

Theoretically, a major aim of topology is the identification and classification of all spaces. However, because of the infinite varieties of spaces possible, this goal seems unachievable. A more realistic goal is to restrict one's investigation to a class of spaces having some "relatively nice" properties and to look for some invariants that would give at least a necessary condition for the

identification of these spaces. Such invariants are the homology and cohomology groups of a space. On the class of simplicial complexes, the above invariants do provide an excellent tool for solving many topological problems.

2. *Some nonlinear aspects of differential equations*, by Professor Solomon Lefschetz, Director, Center for Differential Equations RIAS (by invitation of the Executive Committee).

It would, of course, be extremely desirable to be able to solve at least a very general and extensive class of differential equations. However, this is not in the cards and the best thing that one can do is to obtain some idea of the performance of the solutions. Along this line there are two noteworthy topics: (a) the study, in the large following Poincairé, of a system of two first order equations in the plane; (b) the Liapunov stability theory. These two topics are discussed at some length in the present communication.

3. Panel Discussion: *Professional standards for teachers of mathematics in the schools.* Keynoter, Professor Howard Fehr, Teachers College, Columbia University; Moderator, Professor B. H. Bissinger, Lebanon Valley College; Panel, Professor A. E. Filano, West Chester State Teachers College; Mr. K. S. Kalman, Abraham Lincoln High School, Philadelphia; Mr. Joseph Gavin, Olney High School, Philadelphia.

What teachers must know depends upon the subject matter they will teach. What will be taught depends largely on the calibre of scholar that can be attracted to the teaching of high school mathematics. On the assumption that the high school program will eventually be of the standards shown in the SMSG materials, the teachers must have a five-year training period. Entrance to the program should demand four years (9–12) of high school mathematics study as prerequisite. The four year undergraduate program should consist of calculus and analytic geometry (12 s.h.); algebra (polynomial, linear, abstract), 6 s.h.; geometry (affine, euclidean, vector, projective, algebraic), 6 s.h.; probability and statistical inference, 6 s.h.; professionalized subject matter, 6 s.h.; methods of teaching and practice teaching, 6 s.h. The fifth year should include a 3 to $4\frac{1}{2}$ s.h. course in each of the following: (a) higher analysis or function theory; (b) theory of numbers; (c) structures, i.e., theory of sets, topology, or vector spaces; (d) logic or non-euclidean geometries; (e) applications, i.e. mathematical physics, econometrics, game theory, statistical analysis, etc.; and (f) history of mathematics. All this should be accompanied by a seminar in mathematical education.

F. L. DENNIS, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Annual Fall Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland, College Park, Maryland, on Saturday, December 3, 1960. Professor D. B. Lloyd, Chairman of the Section, presided. One hundred and twenty-three persons were present, including 103 members of the Association. The following papers were presented:

1. *Generalized symmetric random walks*, by Mr. Eugene Albert, University of Virginia.

Define a generalized symmetric random walk as a Markov chain for which: a) for each state E_i, the number of states N_i from which E_i can be reached in one step equals the number of states which can be reached from E_i in one step; b) the one-step transition probabilities from E_i to these N_i states is a constant, for fixed i, to be denoted by a_i. The theorem is proved: The limiting probability of E_i is proportional to $1/a_i$, for any generalized symmetric random walk with a finite number of states.

2. *Preliminary remarks on a distillation problem*, by Mr. H. H. Barnett, Automatic Computations Section, The Martin Company, Baltimore, Maryland.

Periodic provisioning of a component during the operational life of a missile system, in order to reduce to a specified low level the risk of running out of spare components, is a two-stage, multiple-channel flow problem. The first stage is the set of applications of the component; the second stage

is the set of repair areas. For given distributions of failure and repair times, a list of periodic replenishment quantities can be obtained which will satisfy the risk requirements. This is simulated by means of an IBM 709 computer program, using random number generation (Monte Carlo method). Operational life histories are generated until an adequate sample is obtained to make the statistical calculations.

3. *Pursuit games of kind*, by Dr. Rufus Isaacs, Department of Defense, Weapons System Evaluation Group, Washington, D. C.

Pursuit and evasion contests with an essentially two-valued payoff are considered. Under what conditions does the Pursuer P possess a strategy that insures capture of the Evader E despite all opposition? When does E have a strategy guaranteeing escape? The general technique—construction of a surface which separates the space of starting positions into regions of the above two types—is applied to several examples. In the Lifeline and Deadline Games, both players move with fixed speed in a half-plane. Its boundary L is forbidden to E in the latter game, but in the former L is a haven for him: E's reaching L counts as escape. The solutions to the two games are strikingly diverse. In other cases, the motions of P and E are subject to bounded curvature or acceleration. Collision avoidance between moving craft is capable of a modified analysis.

4. *Probability models for measurement with a linear scale*, by Dr. Churchill Eisenhart, Chief, Numerical Analysis Section, U. S. Bureau of Standards.

Let a length of $(m+\delta)w$ be measured with a linear scale having sub-divisions w units apart, where m is some nonnegative integer and $0<\delta<1$. If the scale is positioned at random and the nearest scale divisions at each end of the length "read" correctly in each instance, the sum of n such length determinations will have a binomial distribution with mean $n(m+\delta)w$ and variance $n\delta(1-\delta)$, and standard best-estimate and confidence-interval techniques will be applicable with only slight modifications. Correct estimation of "tenths" corresponds to using a finer mesh, but tenths-estimation errors yield quadrinomial distributions and require modifications in technique.

5. *The training of inservice teachers of mathematics*, by Dr. C. R. Phelps, Program Director for the Academic Year Institutes Program, National Science Foundation.

A discussion of the possible impact of teacher training programs on the schools in the area covered by our section, including present data and future possibilities.

6. *Semantic information*, by Dr. C. J. Maloney, Chief, Bio-mathematics Division, U. S. Army, CmlC Biological Laboratories, Fort Detrick, Maryland.

The theory of information as used in communication is concerned with the development of a procedure for coding messages such that maximum transmission under assigned channel capacity and noise interference is possible. The semantic content of the messages is ignored. Certain applications, including that of information retrieval, require a theory competent to handle the message content. For this purpose two requirements are necessary. Freedom of coding must be restricted to analytical rather than enumerative codes. As words are an enumerative code, messages expressed in words must first be transformed into an analytical code, such as a classification system.

7. *The algebra program in the Soviet Union*, by Professor G. H. Miller, State Teachers College at Towson.

The analysis of the algebra instruction in the U.S.S.R. shows the same traditional topics that we employ in our school systems. During the 6th and 7th years of the Russian schools, the students take the equivalent of a year to a year and a half of algebra. Students graduating from their 10 year and newly revised 11 year school complete the equivalent of our college algebra. Inequalities receive much greater emphasis in the Russian texts than in the American. The teaching of algebra extends over a four and one half year period (or five and one half in the 11 year school) compared to two or three years in our current curriculum.

8. *Evolving patterns in mathematical research*, by Dr. F. J. Weyl, Director of Research, Office of Naval Research. (Invited Address)

An account is rendered of how the scope and substance of mathematical research have been influenced by and have, in turn, influenced the entire supporting apparatus on which its conduct has come to depend in our time. Relevant factors to be analyzed include, aside from the dynamic trends proper to mathematics itself, the new environment created by developments in the rest of science, matters of management and administration, and the general social setting. This sets the stage on which the main protagonists, the universities and colleges, the industrial complexes, and the agencies of the Federal Government, play out their parts in determining the research environment of contemporary mathematics. On the basis of the forces and constraints thus laid bare the attempt is made to describe what can be expected to happen to mathematical research in the course of the next few years ahead.

The meeting concluded with the showing of the color film, *Mathematical Induction*, by Professor L. A. Henkin, University of California, Berkeley.

HERTA T. FREITAG, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA, Rose Polytechnic Institute, Terre Haute, May 6, 1961.

IOWA, Simpson College, Indianola, April 14, 1961.

KANSAS, Ottawa University, April 15, 1961.

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Aberdeen, Maryland, April 29, 1961.

METROPOLITAN NEW YORK, Fordham University, New York, April 15, 1961.

MICHIGAN

MINNESOTA, St. Cloud State College, May 13, 1961.

MISSOURI, University of Missouri, Columbia, April 22, 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY

NORTHEASTERN, University of Vermont, Burlington, June 20, 1961.

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA, Oklahoma State University, Stillwater, Spring, 1961.

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, University of Colorado, Boulder, April 28–29, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

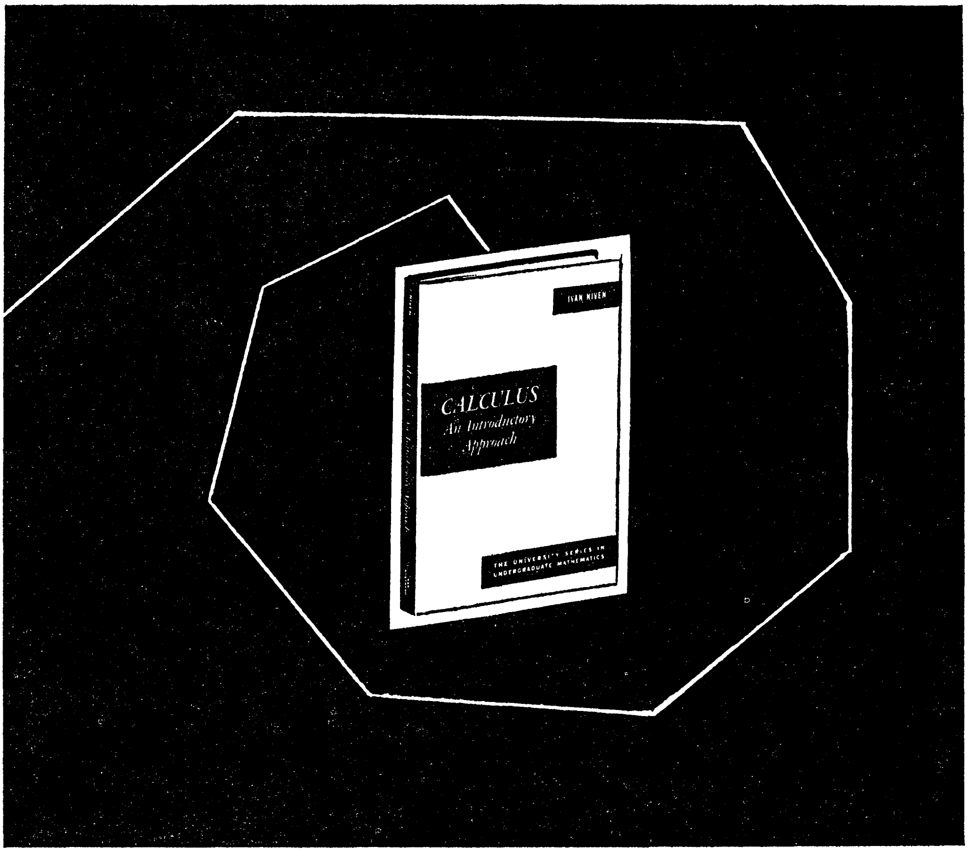
SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.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An Introductory Approach

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THE CIRCLES OF CURVATURE OF THE CURVES OF STEEPEST DESCENT OF GREEN'S FUNCTION*

J. L. WALSH, Harvard University

A number of years ago the present writer published [1] a study of the tangents and circles of curvature of lemniscates and of the level loci of Green's function in the plane. The present note is complementary to that previous study, and considers elementary geometric properties of the curves of steepest descent, namely the lines of force, or orthogonal trajectories of the level loci of Green's function. The method used is a continuation of the previous methods, especially use of an integral representation of Green's function essentially due to Hilbert.

The previous paper pointed out that for an infinite plane region R with finite boundary B, the normal to the level locus of Green's function $G(x, y)$ with pole at infinity, at an arbitrary point P of R, cuts the smallest convex point set K containing B. That normal is, of course, the tangent at P to the curve of steepest descent passing through P, and the tangent thus cuts K. All critical points of $G(x, y)$ in R lie in K, and these are the only multiple points in R of the level loci or of the curves of steepest descent; at every finite point of R exterior to K the tangents and normals to these curves exist and are unique.

The principal result of this note is

THEOREM 1. *Let $G(x, y)$ be Green's function with pole at infinity for an infinite region R whose boundary B is finite. Let a point P of R lie exterior to a circle Γ containing B, and let the tangent at P to the curve of steepest descent of $G(x, y)$ through P pass through the center of Γ. Then the circle of curvature of that curve at P cuts Γ.*

It is sufficient here to prove the conclusion in the case that B consists wholly of a finite number of mutually exterior analytic Jordan curves, for if B is replaced by a locus B_1: $G(x, y) = A$ (const.) in R, Green's function with pole at infinity for the infinite region R_1 bounded by B_1 is $G_1(x, y) \equiv G(x, y) - A$; the curves of steepest descent of $G_1(x, y)$ in R_1 are precisely those of $G(x, y)$. The locus $G(x, y) = A$ in R consists of a finite number of mutually exterior analytic Jordan curves for all but a countable set of values of $A (> 0)$, and this locus can be chosen as near B as desired. Indeed, we consider an infinite sequence of such loci B_k: $G(x, y) = A_k$ in R, where A_k approaches zero monotonically. If the circle of curvature in Theorem 1 cuts each of a sequence of concentric circles Γ_k containing B_k with monotonically decreasing radii, that circle of curvature also cuts the limiting circumference Γ.

Under the assumption that B consists of a finite number of mutually exterior analytic Jordan curves, it can be shown ([1]; [2], Sec. 4.2) that $G(x, y)$ admits a representation for (x, y) in R of the form

* This research was sponsored in part by the U. S. Air Force, Office of Scientific Research of the Air Research and Development Command.

$$(1)\qquad G(x, y) = \int_B \log r d\sigma + g, \qquad r^2 = (x-\alpha)^2 + (y-\beta)^2,$$

where (α, β) are the running coordinates, σ is a suitably chosen distribution on B, and g is a constant. At the point P: $(0, b)$, $b<0$, of R the partial derivatives of $G(x, y)$ are given by the following formulas, where all integrals are taken over B:

$$(2)\qquad \begin{aligned} G_x &= \int -\alpha r^{-2} d\sigma, & G_y &= \int (b-\beta) r^{-2} d\sigma, \\ G_{xx} &= \int \left\{(b-\beta)^2 - \alpha^2\right\} r^{-4} d\sigma, & G_{xy} &= \int 2\alpha(b-\beta) r^{-4} d\sigma. \end{aligned}$$

Green's function $G(x, y)$ is harmonic in R except at infinity, and possesses there a conjugate function $H(x, y)$, also harmonic but not necessarily single valued in R. The level loci of $H(x, y)$ in R are precisely the curves of steepest descent of $G(x, y)$. To determine their circles of curvature we shall need the partial derivatives of $H(x, y)$, of which those of first order are given by the Cauchy-Riemann equations:

$$(3)\qquad G_x = H_y, \quad G_y = -H_x, \quad G_{xx} = H_{xy}, \quad G_{xy} = H_{yy} = -H_{xx}, \quad G_{yy} = -H_{xy}.$$

To prove Theorem 1 we choose Γ as $x^2+y^2=a^2$, so we have

$$(4)\qquad \alpha^2 + \beta^2 \leqq a.$$

Moreover the slope $-H_x/H_y$ of the level locus of $H(x, y)$ through P is infinite at P, whence

$$(5)\qquad H_y = G_x = \int -\alpha r^{-2} d\sigma = 0.$$

It is to be noted that P is not a critical point of $G(x, y)$ and $H(x, y)$, so we have there $H_x = -G_y \neq 0$. The center of curvature C: (X, Y) of the level locus of $H(x, y)$ at P has the coordinates $X = \pm\rho$, $Y=b$, where $\rho(\geqq 0)$ is the radius of curvature (compare [1]) at P:

$$\rho = \pm \frac{\left[H_x^2 + H_y^2\right]^{3/2}}{H_y^2 H_{xx} - 2H_x H_y H_{xy} + H_x^2 H_{yy}} = \pm \frac{H_x}{H_{yy}},$$

$$(6)\qquad \rho = \mp \frac{\int (b-\beta) r^{-2} d\sigma}{\int 2\alpha(b-\beta) r^{-4} d\sigma}.$$

Since the circle of curvature is tangent to OP at P, the condition that that circle should cut Γ can be written

$$X^2 + b^2 \leqq (\rho + a)^2, \tag{7}$$

$$b^2 - a^2 \leqq 2a\rho. \tag{8}$$

We note that $b-\beta<0$, since P is exterior to Γ, so (8) can be written

$$(b^2 - a^2)\int \alpha(b - \beta)r^{-4}d\sigma \leqq -\int a(b - \beta)r^{-2}d\sigma, \tag{9}$$

provided we have

$$\int \alpha(b - \beta)r^{-4}d\sigma > 0. \tag{10}$$

If the first member of (10) is zero, we have $\rho=\infty$, the circle of curvature coincides with the straight line OP, and the conclusion of the theorem is satisfied. If the first member of (10) is negative, reflection of the region R in the y-axis makes no essential change in hypothesis or conclusion, and replaces x by $-x$, α and $-\alpha$, and achieves (10). Thus it remains to study (9) and (10).

We rewrite (9) as

$$0 \leqq \int \frac{(b - \beta)[-\alpha(b^2 - a^2) - ar^2]}{r^4}\, d\sigma. \tag{11}$$

By the Cauchy inequality in the form $|A\alpha+B\beta| \leqq (A^2+B^2)^{1/2}(\alpha^2+\beta^2)^{1/2}$, by $r^2=\alpha^2+(b-\beta)^2$, and by (4), we may write

$$\begin{aligned}
-\alpha(b^2 - a^2) - ar^2 &= - ab^2 - a(\alpha^2 + \beta^2) - (b^2 - a^2)\alpha + 2ab\beta \\
&\leqq - ab^2 - a(\alpha^2 + \beta^2) + [(b^2 - a^2)^2 + (2ab)^2]^{1/2}(\alpha^2 + \beta^2)^{1/2} \\
&= - ab^2 - a(\alpha^2 + \beta^2) + (a^2 + b^2)(\alpha^2 + \beta^2)^{1/2} \\
&= [b^2 - a(\alpha^2 + \beta^2)^{1/2}][-a + (\alpha^2 + \beta^2)^{1/2}] \leqq 0.
\end{aligned}$$

Thus the integrand in (11) is nonnegative, so (8) is established and if B lies interior to Γ is established even with the strong inequality.

A limiting case of Theorem 1 is easily proved:

THEOREM 2. *Let $G(x, y)$ be Green's function with pole at infinity for an infinite region R whose boundary B is finite. Let a point P of R lie exterior to a half-plane (bounded by a line L) containing B, and let the tangent at P to the curve of steepest descent through P be perpendicular to L. Then the circle of curvature of that curve at P cuts L.*

Theorem 2 is readily proved from Theorem 1 by noting that the circle of curvature cuts all circles Γ containing B but not containing P whose centers lie on the perpendicular to L through P. Consequently the circle of curvature

also cuts L. A proof of Theorem 2 can also be given by the methods previously used. Let P be $(0, b)$, $b<0$, and let L be Ox. Then B lies in the half-plane $y\geqq 0$. With the assumption (10) as before, (6) with the upper sign is valid. We are to prove $\rho > -b$, namely,

$$-\int (b-\beta)r^{-2}d\sigma > \int -2\alpha b(b-\beta)r^{-4}d\sigma,$$

$$-\int (b-\beta)[r^2 - 2\alpha b]r^{-4}d\sigma > 0. \tag{12}$$

However, we have $r^2=\alpha^2+(b-\beta)^2\geqq\alpha^2+b^2\geqq 2|\alpha b|$. The inequality $r^2-2\alpha b\geqq 0$ becomes an equality only in the point $\alpha=b$, $\beta=0$; by (5) the set B contains other points, so (12) follows and thereby Theorem 2.

Theorem 1 can be expressed in a form invariant under linear transformation of the complex variable; such a transformation of the extended plane carries every "circle" (where the term is used to include straight line) into a "circle," and also transforms Green's function for a region into Green's function for the transformed region. Thus we have

THEOREM 3. *Let R be a region of the extended plane and $G(x, y)$ Green's function for R with pole in some point Q of R. Let Λ be the circle of curvature at the point P in R to the curve of steepest descent for $G(x, y)$ through P. Let the circle tangent to this curve at P and passing through Q be orthogonal to the boundary Γ of a circular region (closed interior of a circle, closed exterior of a circle, or closed half-plane) not containing P or Q which contains the boundary B of R. Then Λ cuts Γ.*

A limiting case of Theorem 3 is the invariant form of Theorem 2, which can be easily formulated by the reader. If R is simply connected, the curves of steepest descent of $G(x, y)$ are of course the images of the radii (*Radienbilder*) when the unit circle is mapped conformally and one-to-one onto R so that the center of the circle is transformed into Q.

If R is a region which does not possess a Green's function in the classical sense, it may still possess in an extended sense a Green's function $G(x, y)$ $(\not\equiv +\infty)$, found as the limit of the sequence of classical Green's functions $G_k(x, y)$ all with fixed pole Q for a sequence R_k of subregions of R which monotonically increase and exhaust R (compare [1], Sec. 15). Theorem 3 remains valid for $G(x, y)$ in this case. In fact, the functions $G_k(x, y)$ approach $G(x, y)$ uniformly in any closed bounded subset of R, and the partial derivatives of the $G_k(x, y)$ approach uniformly the corresponding partial derivatives of $G(x, y)$ in such a subset. Thus the tangents and circles of curvature of the curves of steepest descent of the $G_k(x, y)$ approach the tangents and circles of curvature of $G(x, y)$. A suitably chosen sequence of circular regions bounded by "circles" Γ_k containing the respective boundaries of the R_k approaches the circular region bounded by Γ containing B, where each Γ_k is normal to the tangent at P to the curve of steepest

descent through P for $G_k(x, y)$. The circles of curvature Λ_k of these curves at P cut the respective circles Γ_k by Theorem 3, so the limit Λ of the Λ_k cuts the limit Γ of the Γ_k.

It is pointed out in [1] that the technique used there applies to the study of curvature of both lemniscates and level loci of Green's function. A lemniscate is defined as a locus

$$|p(z)| = \text{const.}, \qquad p(z) \equiv \prod_1^n (z - z_k), \tag{13}$$

and of course the lemniscate can be defined equivalently as the locus

$$\sum_1^n \log |z - z_k| = A, \qquad A = \text{const.} \tag{14}$$

The first member of (14) is closely analogous to the second member of (1); in (14) there appears a finite sum and in (1) an integral (plus a constant which is here without significance). Results concerning lemniscates can be proved (i) by using a succession of constants A_k in (14), $A_k \to -\infty$, and interpreting $(1/n)[\sum_1^n \log |z-z_k| - A_k]$ as a Green's function or (ii) by using discussions directly involving finite sums rather than integrals. By either method there may be proved

THEOREM 4. *Let $p(z) \equiv \prod_1^n (z-z_k)$ be a polynomial, let a point P lie exterior to a circle Γ containing the points z_k, and let the tangent at P to the curve of steepest descent of $|p(z)|$ through P pass through the center of Γ. Then the circle of curvature of that curve at P cuts Γ.*

Reciprocally, Theorem 1 is readily proved by use of Theorem 4 (compare [2], Ch. 4).

The invariant form of Theorem 4, readily expressed by the reader, is concerned with the curves of steepest descent of $|r(z)|$, where $r(z)$ is an arbitrary rational function having but one pole in the extended plane.

It might be conjectured that Theorem 1 remains valid without the requirement that the tangent at P to the curve of steepest descent through P should pass through the center of Γ. This conjecture is false:

THEOREM 5. *There exists an infinite region R with finite boundary B possessing a Green's function $G(x, y)$ with pole at infinity, and a point P in R exterior to a circle Γ containing B such that Γ is not cut by the circle of curvature at P to the curve of steepest descent of $G(x, y)$ through P.*

It will be recalled that [1] the tangent at P to the curve of steepest descent through P must cut Γ.

By virtue of our discussion of lemniscates, notably method (ii), it is sufficient here to study as defining functions the first members of (13) or (14). We choose $p(z) \equiv z^2 - 1$, and the function conjugate to the harmonic function $\log |p(z)|$ is

$$\arg(z^2 - 1) \equiv \arg(x^2 + 2ixy - y^2 - 1) = \tan^{-1}[(2xy)/(x^2 - y^2 - 1)].$$

The curves of steepest descent of $|p(z)|$ are the curves

$$x^2 - y^2 - 1 + 2cxy = 0, \qquad c = \text{const.}, \tag{15}$$

taken together with the curve $xy=0$, namely the family of equilateral hyperbolas with center O: $(0, 0)$ passing through the points $(\pm 1, 0)$. These curves all have the same shape except for the degenerate hyperbola $xy=0$. The asymptotes of the curve (15) are $x^2-y^2+2cxy=0$ with slopes $c\pm(1+c^2)^{1/2}$, and the axes of the curve bisect the angles between the asymptotes, and are therefore given by $cx^2-2xy-cy^2=0$.

As $c\rightarrow+\infty$ the asymptotes and the curve (15) itself approach the curve $xy=0$ consisting of the coordinate axes. The slope of (15) in the point $(1, 0)$ is $-1/c$, so for large positive c one branch of the curve (15) passes from the fourth quadrant into the first quadrant at $(1, 0)$ as x decreases, and that branch remains in the first quadrant as (x, y) continues to trace the curve monotonically; the vertex lies near the origin. The circle of curvature of an equilateral hyperbola at a vertex lies in the closed interior of the corresponding branch of the curve; that circle subtends a certain angle 2γ at the center of the curve which is independent of the size of the curve, $0<\gamma<45°$. Then when c is sufficiently large (and positive) a vertex P of (15) lies in the first quadrant near the origin, and an asymptote (having the slope $c+(1+c^2)^{1/2}$) makes an angle numerically less than $45°-\gamma$ with Oy; the circle of curvature at P subtends the angle 2γ at O, so that circle of curvature lies wholly in the first quadrant. Then there exists a new circle Γ containing in its interior both the two points $z=\pm 1$ and a suitable lemniscate B: $|z^2-1|=A(>0)$, but to which the circle of curvature is exterior. Theorem 5 is established. Likewise in Theorem 2 there cannot be omitted the requirement that the tangent at P to the curve of steepest descent through P shall be perpendicular to L.

We have seen in Theorem 5 that Theorem 1 is false if there is omitted the requirement that the tangent at P to the curve of steepest descent should pass through the center of Γ. Nevertheless, even if that requirement is omitted, and if Γ subtends an acute angle at P, weaker conclusions can be drawn concerning the circle of curvature, by virtue of the fact that the tangent at P to the curve of steepest descent through P must cut Γ (and indeed is not tangent to Γ). Under these conditions *the circle of curvature must cut at least one of the two lines* L_1 *and* L_2, *each tangent to* Γ, *each perpendicular to a tangent from* P *to* Γ, *each separating* P *from the interior of* Γ. In fact, let us consider all the lines tangent to Γ at points of Γ lying on the shorter arc of Γ bounded by the points of tangency of L_1 and L_2. One of these lines (say L) is perpendicular to the tangent at P to the curve of steepest descent through P, so L satisfies the requirements of Theorem 2; then the circle of curvature cuts L and hence cuts either L_1 or L_2, since L_1 and L_2 separate L from P (L cannot coincide with L_1 or L_2). The conclusion persists if B lies not necessarily in Γ but interior to the infinite convex

region bounded by half-lines of the tangents to Γ through P and the shorter arc of Γ joining their points of tangency.

A more general remark can be made, still with the hypothesis of Theorem 1 except for the requirement that the tangent at P to the curve of steepest descent should pass through the center of Γ. Let $\{\Gamma_\mu\}$ be a family of circles each containing the interior of Γ, let P lie exterior to each Γ_μ, and suppose each half-line from P cutting Γ passes through the center of at least one Γ_μ; then the circle of curvature at P of the curve of least descent through P cuts at least one Γ_μ.

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DEMOSIAN SYSTEMS OF QUASIGROUPS*

A. SADE, Marseille, France

1. Introduction. Definitions. Since the terms which follow have been used with various meanings in the literature, the sense in which they are to be understood in this paper will first of all be made precise.

Anti-automorphism. A groupoid G defined on a set E possesses an anti-automorphism $T \in \mathfrak{S}_E$ if the groupoid G', isomorphic to G under the permutation T, is conjoint to $G(x\cdot y=z \Leftrightarrow yT\cdot xT=zT)$. ([11], p. 212).

Conjoint (*groupoid*, *expression*, *equation*). Two groupoids $G=E(\times)$ and $G'=E(*)$, defined on the same set E, are conjoint if $\forall\ x,\ y \in E,\ x\times y=y*x$ ([6], p. 60, no. 75). The symbols $\times$ and $*$ are similarly conjoint. Two expressions are conjoint if one is obtained from the other by replacing the symbols of operation with their conjoint symbols. Two equations are conjoint if one is obtained from the other by replacing the two members of each equation with their conjoint expressions ([8], p. 78, nos. 4.2, 6.1).

Demosian (*system*, *identity*). Given a set E with laws of composition ϕ_i, $i=1, 2, \cdots$, forming a set Φ, then (E, Φ) is said to be a demosian system. If, in this system, a condition such as (17) below is satisfied, then (17) is said to be a demosian identity [10].

Groupoid. A groupoid is a set E with a law of composition $\times$ such that to each

* Translated from the French.

pair $\langle x, y\rangle \in E^2$ there corresponds a unique element $z\in E$ called the product of x by y and written $x\times y=z$ ([7], p. 157, no. 3).

Left inverse. In the groupoid $G=E(\times)$ with unit u, the element y is a left inverse of x if $y\times x=u$.

Isotopy. Two groupoids $Q=E(\cdot)$ and $R=E(\times)$, defined on the same set E, are isotopic if there exist three permutations of E, (ξ, η, ζ) such that $\forall\, x, y\in E$, $(x\cdot y)\zeta=(x\xi)\times(y\eta)$ ([7], p. 164, no. 15).

Loop. A loop is a quasigroup with a two-sided unit ([7], p. 164, no. 15).

Multigroupoid. A multigroupoid is a set E with an operation $\times$ such that to each pair $\langle x, y\rangle\in E^2$ there corresponds a subset (in general, proper) of E, $\emptyset\subseteq x\times y\subseteq E$ ([9], p. 231).

Quasigroup. A quasigroup is a groupoid satisfying the axiom that each of the equations $ax=b$ and $ya=b$ has a unique solution for x and y, respectively, ([7], p. 15, no. 3; [8], p. 73, no. 1).

Right translation. If $Q=E(\times)$ is a groupoid, the right translation by the element s is the mapping Δ_s: $x\rightarrow x\times s$ of E into itself ([1], p. 509).

Twist. A twist is an isotopy in which the first two components are the identity, *i.e.*, $T=(1, 1, \zeta)$ ([7], p. 171, no. 28).

Left unit. The element u is a left unit of the groupoid $G=E(\cdot)$ if $\forall x\in E$, $u\cdot x=x$ ([13], p. 1).

If the twelve ways of forming the product of three elements a, b, c are equated two at a time, then $\frac{1}{2}\cdot 12\cdot 11=66$ equations are obtained. Of these, 48 have the identity as their group of automorphisms and are isomorphic in sets of 6 to 8 of them. The remaining 18 have a group of automorphisms of the second order and are isomorphic in sets of 3 to 6 of them. The 14 nonisomorphic laws include the usual associative law

$$(ab)c = a(bc) \tag{0}$$

and the 13 which follow, numbered according to the classification of Faragó ([3], p. 133):

(1)	$(ab)c = a(cb)$,	(9)	$a(bc) = c(ab)$,
(2)	$(ab)c = b(ac)$,	(10)	$a(bc) = c(ba)$,
(3)	$(ab)c = b(ca)$,	(11)	$(ab)c = (ac)b$,
(4)	$(ab)c = c(ab)$,	(12)	$(ab)c = (ba)c$,
(5)	$(ab)c = c(ba)$,	(14)	$(ab)c = (ca)b$,
(6)	$a(bc) = a(cb)$,	(15)	$(ab)c = (cb)a$.
(7)	$a(bc) = b(ac)$,		

In this list, equations (8) $a(bc)=b(ca)$ and (13) $(ab)c=(bc)a$ have been omitted since they are isomorphic to (9) and (14), respectively, and are deducible from them through the permutation $(abc)\rightarrow(bca)$.

Some of these laws are already known under different names ([7], p. 154): (1) = Eingewandte Produkt (22); (7) = left permutability or the law of Hosszú (19); (9) = cyclic associativity (18); (10) = the law of Abel-Grassman (21); (11) = right permutability (20). Faragó studied groupoids with a unit and left inverse satisfying the laws (0)–(15).

Hosszú [4] gave the general solution of these equations for continuous, strictly monotone functions over the real field $\mathfrak{R}$, reducing them to four only, namely (0), (7), (9), and (10). He also solved (p. 212), still over the real field, the equation

$$F[x, G(y, z)] = H[K(x, y), z], \tag{h}$$

where F, G, H, K are continuous and differentiable; equation (h) is more general than (0).

If ϕ_1, ϕ_2, ϕ_3, ϕ_4 are the symbols of operation of four quasigroups over any set E, Belousov [2] announced and Hosszú [5] proved that the demosian equation

$$(x\phi_1 y)\phi_2 z = x\phi_3(y\phi_4 z) \tag{0}$$

has the general solution

$$\begin{aligned} x\phi_1 y &= (x\xi\cdot y\theta)\mu^{-1}, & x\phi_2 y &= (x\mu\cdot y\lambda)\zeta^{-1}, \\ x\phi_3 y &= (x\xi\cdot y\eta)\zeta^{-1}, & x\phi_4 y &= (x\theta\cdot y\lambda)\eta^{-1}, \end{aligned} \tag{I}$$

where $E(\cdot)$ is any group and ξ, η, ζ, θ, λ, μ are arbitrary permutations of E, a result which is easily seen to have a counterpart for multigroupoids [9].

In this paper it is proposed to study the demosian systems which satisfy (0)–(15) and, in addition, each of the equations

$$(x\phi_1 y)\phi_2 z = (x\phi_3 y)\phi_4 z, \tag{16}$$

$$z\phi_1(x\phi_2 y) = z\phi_3(x\phi_4 y); \tag{17}$$

these equations are not of interest as long as the study is concerned with a single groupoid (Faragó), but may no longer be neglected in the demosian case. An approach to the problem is to first of all solve (0)–(17) over a demosian system of quasigroups. It will be seen that these equations reduce to only two.

2. Solution of (0)–(17). These equations may be arranged in the following order and in two categories, the first is

(7)	$z\phi_1(x\phi_2 y) = x\phi_3(z\phi_4 y)$,	(3)	$(y\phi_1 x)\phi_2 z = x\phi_3(z\phi_4 y)$,
(10)	$z\phi_1(y\phi_2 x) = x\phi_3(y\phi_4 z)$,	(2)	$y(\phi_1 x)\phi_2 z = x\phi_3(y\phi_4 z)$,
(9)	$z\phi_1(x\phi_2 y) = x\phi_3(y\phi_4 z)$,	(11)	$(y\phi_1 x)\phi_2 z = (y\phi_3 z)\phi_4 x$,
(0)	$(x\phi_1 y)\phi_2 z = x\phi_3(y\phi_4 z)$,	(14)	$(y\phi_1 x)\phi_2 z = (z\phi_3 y)\phi_4 x$,
(1)	$(x\phi_1 y)\phi_2 z = x\phi_3(z\phi_4 y)$,	(15)	$(x\phi_1 y)\phi_2 z = (z\phi_3 y)\phi_4 x$;

and the second,

$$(16) \qquad (x\phi_1 y)\phi_2 z = (x\phi_3 y)\phi_4 z, \qquad\qquad (4) \qquad (x\phi_1 y)\phi_2 z = z\phi_3(x\phi_4 y),$$

$$(12) \qquad (x\phi_1 y)\phi_2 z = (y\phi_3 x)\phi_4 z, \qquad\qquad (17) \qquad z\phi_1(x\phi_2 y) = z\phi_3(x\phi_4 y),$$

$$(5) \qquad (x\phi_1 y)\phi_2 z = z\phi_3(y\phi_4 x), \qquad\qquad (6) \qquad z\phi_1(x\phi_2 y) = z\phi_3(y\phi_4 x).$$

In each of the two series, every equation leads to the one following by replacing an operation ϕ by its conjoint. The solution of (0) is already known and, according to the way in which an equation is derived from the preceding one, its solution follows from that of the preceding equation by a technique of which a single example will suffice. For example, let $x\psi y = y\phi_4 x$; then, if ϕ_1, ϕ_2, ϕ_3, ϕ_4 are solutions of (0), those of (1) will be ϕ_1, ϕ_2, ϕ_3, ψ; *i.e.*, permuting x and y in the expression for ϕ_4,

$$(x\xi\cdot y\theta)\mu^{-1}, \qquad (x\mu\cdot y\lambda)\zeta\lambda^{1}, \qquad (x\xi\cdot y\eta)\zeta^{-1}, \qquad (y\theta\cdot x\lambda)\eta^{-1}.$$

More generally, if a demosian equation (D) goes into another demosian equation (D') by permuting two factors in (D), then the solution of (D') follows from that of (D) by substituting the conjoint groupoid for the groupoid defined by the operation involved in the composition of the two factors. Table I is obtained in

TABLE I

(7)	$(y\mu\cdot x\lambda)\zeta^{-1}$	$(x\xi\cdot y\theta)\mu^{-1}$	$(x\xi\cdot y\eta)\zeta^{-1}$	$(y\theta\cdot x\lambda)\eta^{-1}$
(10)	$(y\mu\cdot x\lambda)\zeta^{-1}$	$(y\xi\cdot x\theta)\mu^{-1}$	$(x\xi\cdot y\eta)\zeta^{-1}$	$(x\theta\cdot y\lambda)\eta^{-1}$
(9)	$(y\mu\cdot x\lambda)\zeta^{-1}$	$(x\xi\cdot y\theta)\mu^{-1}$	$(x\xi\cdot y\eta)\zeta^{-1}$	$(x\theta\cdot y\lambda)\eta^{-1}$
(0)	$(x\xi\cdot y\theta)\mu^{-1}$	$(x\mu\cdot y\lambda)\zeta^{-1}$	$(x\xi\cdot y\eta)\zeta^{-1}$	$(x\theta\cdot y\lambda)\eta^{-1}$
(1)	$(x\xi\cdot y\theta)\mu^{-1}$	$(x\mu\cdot y\lambda)\zeta^{-1}$	$(x\xi\cdot y\eta)\zeta^{-1}$	$(y\theta\cdot x\lambda)\eta^{-1}$
(3)	$(y\xi\cdot x\theta)\mu^{-1}$	$(x\mu\cdot y\lambda)\zeta^{-1}$	$(x\xi\cdot y\eta)\zeta^{-1}$	$(y\theta\cdot x\lambda)\eta^{-1}$
(2)	$(y\xi\cdot x\theta)\mu^{-1}$	$(x\mu\cdot y\lambda)\zeta^{-1}$	$(x\xi\cdot y\eta)\zeta^{-1}$	$(x\theta\cdot y\lambda)\eta^{-1}$
(11)	$(y\xi\cdot x\theta)\mu^{-1}$	$(x\mu\cdot y\lambda)\zeta^{-1}$	$(x\theta\cdot y\lambda)\eta^{-1}$	$(y\xi\cdot x\eta)\zeta^{-1}$
(14)	$(y\xi\cdot x\theta)\mu^{-1}$	$(x\mu\cdot y\lambda)\zeta^{-1}$	$(y\theta\cdot x\lambda)\eta^{-1}$	$(y\xi\cdot x\eta)\zeta^{-1}$
(15)	$(x\xi\cdot y\theta)\mu^{-1}$	$(x\mu\cdot y\lambda)\zeta^{-1}$	$(y\theta\cdot x\lambda)\eta^{-1}$	$(y\xi\cdot x\eta)\zeta^{-1}$

this way. In order to solve (16), suppose that z is a constant a. Then, if Δ_a^i denotes the right translation $(x \rightarrow x\phi_i a)$ by a in the quasigroup $E(\phi_i)$,

$$(x\phi_1 y)\overset{2}{\Delta_a} = (x\phi_3 y)\overset{4}{\Delta_a}, \qquad x\phi_3 y = (x\phi_1 y)\overset{2}{\Delta_a}(\overset{4}{\Delta_a})^{-1}.$$

Thus if $x\phi_1 y = x\times y$ is any quasigroup, $x\phi_3 y$ is isotopic to $x\times y$ under the twist $(1, 1, \eta)$, *i.e.*, $x\phi_3 y = (x\times y)\eta^{-1}$.

Now let $x\phi_3y=t$; then $x\times y=t\eta$ and, if $x\phi_2y=x*y$ is again an arbitrary quasigroup, $\forall t\in E$, $t\eta\phi_2z=t\phi_4z$, $x\phi_4y=x\eta*y$.

The general solution of (16) does not enter into that of (h) and depends on two arbitrary quasigroups and an arbitrary permutation $\eta\in\mathfrak{S}_E$. The same method as that used for Table I yields Table II.

TABLE II

(16)	$x\times y$	$x*y$	$(x\times y)\eta^{-1}$	$x\eta*y$
(12)	$x\times y$	$x*y$	$(y\times x)\eta^{-1}$	$x\eta*y$
(5)	$x\times y$	$x*y$	$y\eta*x$	$(y\times x)\eta^{-1}$
(4)	$x\times y$	$x*y$	$y\eta*x$	$(x\times y)\eta^{-1}$
(17)	$y*x$	$x\times y$	$y\eta*x$	$(x\times y)\eta^{-1}$
(6)	$y*x$	$x\times y$	$y\eta*x$	$(y\times x)\eta^{-1}$

Remarks. (i) *If the solutions of a demosian equation are quasigroups isotopic to the same arbitrary group G, then another solution may be deduced from a given solution by replacing G by a loop isotopic to the conjoint of G.* (It is known ([1], p. 511, Th. 2) that every loop isotopic to a group is again a group.)

(ii) The conjoint of an expression (equation) has been defined as the expression (equation) obtained by replacing the symbols of operation with their conjoint symbols. Thus the equation conjoint to (0) is $(z\psi_4y)\psi_3x=z\psi_2(y\psi_4x)$, where $\psi_1, \psi_2, \psi_3, \psi_4$, are conjoint, respectively, to $\phi_1, \phi_2, \phi_3, \phi_4$. It is clear that, *if two equations are conjoint, then the solutions of the second are conjoint to those of the first.* It is easily verified that (0), (3), (4), (5) are auto-conjoint, that (1) is conjoint to (2), (6) to (12), (7) to (11), (9) to (14), (10) to (15), and (16) to (17). For example, the solutions of (7) are

$$(y\mu\cdot x\lambda)\zeta^{-1},\qquad (x\xi\cdot y\theta)\mu^{-1},\qquad (x\xi\cdot y\eta)\zeta^{-1},\qquad (y\theta\cdot x\lambda)\eta^{-1}.$$

Their conjoint solutions are

$$(x\mu\cdot y\lambda)\zeta^{-1},\qquad (y\xi\cdot x\theta)\mu^{-1},\qquad (y\xi\cdot x\eta)\zeta^{-1},\qquad (x\theta\cdot y\lambda)\eta^{-1},$$

and, putting them in order, they are precisely those of (11). In the same way, taking the conjoint solutions of (1) leads to

$$(y\xi\cdot x\theta)\mu^{-1},\qquad (y\mu\cdot x\lambda)\zeta^{-1},\qquad (y\xi\cdot x\eta)\zeta^{-1},\qquad (x\theta\cdot y\lambda)\eta^{-1}.$$

These four quasigroups, taken in reverse order do indeed satisfy (2). In addition, it may be shown that these solutions are isomorphic to those in the second line of Table I. Let $G(\times)$ be the conjoint of the group $(\cdot)$ so that $x\times y=y\cdot x$. The above solutions become

$$(y\lambda \times x\theta)\eta^{-1}, \qquad (x\eta \times y\xi)\zeta^{-1}, \qquad (x\lambda \times y\mu)\zeta^{-1}, \qquad (x\theta \times y\xi)\mu^{-1}.$$

The permutation $(\lambda\mu\eta\xi)\to(\xi\eta\mu\lambda)$ of the components of the isotopy leads to

$$(y\xi \times x\theta)\mu^{-1}, \qquad (x\mu \times y\lambda)\zeta^{-1}, \qquad (x\xi \times y\eta)\zeta^{-1}, \qquad (x\theta \times y\lambda)\eta^{-1},$$

which is the solution of (2).

It is clear that the way in which the four solutions are rearranged after passing to the conjoint equations depends on the fact that no distinction is made between the equation $A=B$ and the equation $B=A$.

3. Demosian systems. Every quasigroup which is isotopic to a group is isotopic to the conjoint of this group since every group is isomorphic to its conjoint ([6], p. 60, no. 75). If T is the anti-automorphism which transforms $G(\cdot)$ into its conjoint, then, for example, $(y\xi\cdot x\theta)\mu^{-1}=(x\theta T\cdot y\xi T)T^{-1}\mu^{-1}$. Consequently, all the formulas in Table I may be written as isotopes of the same group G.

Consider a demosian system of quasigroups defined on a set E by the laws of composition ϕ_i, $i=1, 2, \cdots$, forming a set Φ, and satisfying any one of the laws (L) of the first category (0)–(15). Let ϕ_1 be fixed and let ϕ_2 range over Φ. For each choice of ϕ_2 there exist $\phi_3, \phi_4 \in \Phi$ satisfying (L) and each time the four quasigroups are isotopic to the same group. Since the isotopy is transitive, all the groups under consideration, being isotopic to $E(\phi_1)$, are isotopic (and hence isomorphic) to each other ([1], p. 511). On the other hand, as ϕ_2 ranges over Φ, $E(\phi_2)$ is always isotopic to $E(\phi_1)$ and hence to a fixed group $G=E(\cdot)$. This gives the following necessary condition:

Every system of quasigroups (E, Φ) satisfying any one of the demosian laws of the first category consists of quasigroups isotopic to the same group $G=E(\cdot)$.

Example. The set of quasigroups defined over any field by the law of composition $x\times y=ax+by+c$; $a, b\neq 0$, satisfies every demosian law of the two categories. The demosian subsets with the same property are obtained by taking $a=1$, or $b=1$, or $c=0$, or two of these conditions simultaneously. Each quasigroup $Q=E(\times)$ is isotopic to the additive group $f(x, y)=x+y$ of the field under the isotopy with components $\xi: x\to ax$, $\eta: x\to bx$, $\zeta: x\to x-c$. Each of the seven demosian sets and subsets described above has the property of being a group with respect to the composition of isotopies. This fact suggests the following two questions:

QUESTION I. *What are the demosian laws which define a set of quasigroups isotopic to the same group?*

QUESTION II. *Among these laws what are the ones for which the set of isotopies which define the quasigroups of a demosian set (E, Φ) is a group with respect to the composition of isotopies?*

It is not easy to state a sufficient condition to impose on the isotopes of the

same group in order that they form a demosian domain; this involves the group of autotopies of a group.

4. Systems of the second category. Since the quasigroups $E(\phi_1)$ and $E(\phi_2)$ may be chosen arbitrarily, there is no one condition which can be imposed on two elements of a demosian system in order that they should appear in one of the laws in Table II. For example, such a system may be generated by two quasigroups $E(\times)$, $E(*)$, and a permutation η of E. Thus, in the case of (16), the general term of $S(E, \Phi)$ would be $(x\eta^m \times y)\eta^{-p}$, $(x\eta^q \times y)\eta^{-r}$. If η is finite of order n, the system contains a finite number, $2n^2$ of quasigroups.

Special case. Putting $\phi_1=\phi_3$, $\phi_2=\phi_4$ in (0) yields the identity,

$$(0_1) \qquad \forall\, x, y, z \in E, \qquad (x\phi_1 y)\phi_2 z = x\phi_1(y\phi_2 z),$$

between $E(\phi_1)$ and $E(\phi_2)$, which has been considered in [11]. The groupoids $E(\phi_1)$ and $E(\phi_2)$, no longer necessarily quasigroups, which satisfy (0_1) are said to be in an associative relation (*en relation associative*, written $\phi_1(\mathbf{RA})\phi_2$). It is easy to see that *this relation is transitive.* For, if $\phi_1(\mathbf{RA})\phi_2$ and $\phi_2(\mathbf{RA})\phi_3$, then, since every element in a groupoid $E(\phi_2)$ may be represented as the product of two elements, say, $u=y\phi_2 z$, it follows that $\forall x, u, t \in E$,

$$\begin{aligned}(x\phi_1 u)\phi_3 t &= [x\phi_1(y\phi_2 z)]\phi_3 t = [(x\phi_1 y)\phi_2 z]\phi_3 t \\ &= (x\phi_1 y)\phi_2(z\phi_3 t) = x\phi_1[y\phi_2(z\phi_3 t)] \\ &= x\phi_1[(y\phi_2 z)\phi_3 t)] = x\phi_1(u\phi_3 t).\end{aligned}$$

Hence $\phi_1(\mathbf{RA})\phi_3$.

Suppose from now on that the ϕ are the laws of quasigroups. If $E(\phi_1)$ and $E(\phi_2)$ satisfy (0_1), the four quasigroups $E(\phi_1)$, $E(\phi_2)$, $E(\phi_1)$, $E(\phi_2)$ are isotopes of the same group $G(\cdot)$ according to the formulas (I). Then, denoting the isotopy with components X, Y, Z by (X, Y, Z),

$$G(\xi, \theta, \mu) = G(\xi, \eta, \zeta), \qquad G(\mu, \lambda, \zeta) = G(\theta, \lambda, \eta),$$

from which it follows that $G(1, \theta\eta^{-1}, \mu\zeta^{-1}) = G = G(\mu\theta^{-1}, 1, \zeta\eta^{-1})$. Moreover, $(1, \theta\eta^{-1}, \mu\zeta^{-1})$ and $(\mu\theta^{-1}, 1, \zeta\eta^{-1})$ are two autotopies of G. The solutions ϕ_1, ϕ_2 are found by choosing two autotopies of $G(\cdot)$ of the form $(1, Y, Z)$ and $(X', 1, Z')$. The components $\xi, \theta, \mu, \cdots$, of (I) will then be determined by the equations

$$\theta\eta^{-1} = Y, \qquad \mu\zeta^{-1} = Z, \qquad \mu\theta^{-1} = X', \qquad \zeta\eta^{-1} = Z',$$

from which it follows, taking ξ, η, λ arbitrarily, that $\theta = Y\eta$, $\zeta = Z'\eta$, $\mu = ZZ'\eta$, with the condition $\mu\theta^{-1} = ZZ'\eta\eta^{-1}Y^{-1} = X'$ or $ZZ' = X'Y$. Hence:

If two quasigroups $E(\phi_1)$ and $E(\phi_2)$ satisfy the demosian equation (0_1), they are isotopic to the same group G under $(\xi, \eta, Z'\eta)$ and $(Y\eta, \lambda, \eta)$, respectively, where ξ, η, λ are any permutations of E, and $(1, Y, Z)$, $(X', 1, Z')$ are two autotopies of G satisfying the condition $ZZ' = X'Y$.

A demosian domain (E, Φ) of quasigroups satisfying the identity,

$$\forall x, y, z \in E, \qquad \forall \phi_1 \in \Phi, \qquad \exists \phi_2 \in \Phi, \qquad \phi_1(\mathbf{RA})\phi_2, \tag{18}$$

may be constructed in the following way. Let Y and Z' be fixed. Then, starting with two quasigroups derived from $G(\cdot)$ under the isotopies $(\xi, \eta, Z'\eta)$ and $(Y\eta, \lambda, \eta)$, a third quasigroup $E(\phi_3)$ is constructed such that $\phi_2(\mathbf{RA})\phi_3$ by taking $Y\eta = \xi_1$, $\lambda = \eta_1$, $\eta = Z'\eta_1$. Then $E(\phi_3)$ will be isotopic to G under $(Y\eta_1, \lambda_1, \eta_1) = (Y\lambda, \lambda_1, Z'^{-1}\eta)$ with the condition $Z'^{-1}\eta = \eta_1 = \lambda$. This requires that the second isotopy ϕ_2 be chosen in the form $(Y\eta, Z'^{-1}\eta, \eta)$; the third isotopy is then $(YZ'^{-1}\eta, Z'^{-2}\eta, Z'^{-1}\eta)$ since $\lambda_1 = Z'^{-1}\eta_1$; the fourth is $(YZ'^{-2}\eta, Z'^{-3}\eta, Z'^{-2}\eta)$, etc. If Z' is finite of order k, the demosian domain will contain k other quasigroups besides $E(\phi_1)$. The first, $E(\phi_1)$ will be (**RA**) to all the others; the k remaining ones forming a circular chain, will each be (**RA**) to all the others and, for these k quasigroups, the associative relation (**RA**) will then be reflexive and symmetric. Furthermore, each of them will be mutually associative to the $k-1$ other quasigroups and will itself be a semigroup [11] and hence a group isomorphic to G. The argument is no longer valid if Z' is not of finite order since then the associative relation (**RA**) is only transitive. In this case a direct proof is necessary. It is known ([1], p. 510, Th. I) that the necessary and sufficient condition for the isotope $G'(*)$ of a loop $G(\cdot)$ to be again a loop under (ξ, η, ζ), $x\xi * y\eta = (x\cdot y)\zeta$, is that the permutations $\xi\zeta^{-1}$ and $\eta\zeta^{-1}$ be respectively a right and a left translation of $G(\cdot)$.

Since $(1, Y, Z)$ is an autotopy of $G(\cdot)$, the permutations Z^{-1} and YZ^{-1} are the translations $Z^{-1} = \Delta_b$, $YZ^{-1} = \Gamma_a$, so that $Z = \Delta_b^{-1}$, $Y = \Gamma_a\Delta_b^{-1}$. Similarly, since $(X', 1, Z')$ is also an autotopy, $Z' = \Gamma_c^{-1}$, $X' = \Delta_d\Gamma_c^{-1}$. The condition $ZZ' = X'Y$ may then be expressed in the form

$$\Delta_b^{-1}\Gamma_c^{-1} = \Delta_d\Gamma_c^{-1}\Gamma_a\Delta_b^{-1}, \qquad \Delta_b = \Gamma_c\Delta_b\Delta_d\Gamma_c^{-1}\Gamma_a.$$

Then $\forall x \in G$,

$$x\cdot b = (c\cdot x\cdot b\cdot d)\Gamma_c^{-1}\Gamma_a = a\cdot x\cdot b\cdot d, \quad x\cdot b\cdot d^{-1}\cdot b^{-1} = a\cdot x = x\Gamma_a.$$

Finally, $xY = x\Gamma_a\Delta_b^{-1}$ or $xY\Delta_b = x\Gamma_a = x\cdot b\cdot d^{-1}\cdot b^{-1}$, $xY = x\cdot b\cdot d^{-1}\cdot b^{-2}$, $Y = \Delta_{b\cdot d^{-1}\cdot b^{-2}}$. Thus Y is a right translation of $G(\cdot)$. Similarly, $Z'^{-1} = \Gamma_c$ is a left translation. By the lemma cited above, the isotope of $G(\cdot)$ under $(Y, Z'^{-1}, 1)$ is a loop. But a loop which is isotopic to a group is necessarily a group isomorphic to that group 1. It has therefore been shown that, *if* $(1, Y, Z)$ *and* $(X', 1, Z')$ *are two autotopies of a group G satisfying the condition $ZZ' = X'Y$, then the isotope of G under* $(Y, Z'^{-1}, 1)$ *is a group isomorphic to G.*

If this is applied to the demosian system constructed as above by starting with a group $G(\cdot)$ and two isotopes $E(\phi_1)$ and $E(\phi_2)$, it is seen that *the system* (E, Φ), *satisfying* (18) *and generated by two isotopes* $E(\phi_1)$, $E(\phi_2)$ *of G, is always composed, in addition to* $E(\phi_1)$, *of groups isomorphic to G, each of which is* (**RA**) *to all the rest.*

Example I. Let $G=\mathfrak{S}_3=012345$, $G\cdot 1=120534$, $G\cdot 2=201453$, $G\cdot 3=345012$; $Y=(102)(345)=Z$, $X'=Z'=(102)(354)$. Taking $\eta=1$ it is found that ϕ_1: $(\xi, 1, Z')$, ϕ_2: $(Y, Z'^{-1}, 1)$, ϕ_3: $(YZ'^{-1}, Z'^{-2}, Z'^{-1})$, ϕ_4: $(YZ'^{-2}, 1, Z'^{-2})$. The latter three are isomorphic to $\mathfrak{S}_3$ under YZ', Y, and YZ'^{-1}, respectively.

Example II. Consider the additive group $G(\cdot)$, $x\cdot y=x+y$ of the ring $\mathfrak{Z}/n$. It is found that every pair of autotopies of the cyclic group satisfying $ZZ'=X'Y$ is of the form $X=1$, $Y=Z=(x\rightarrow x+a)=A$, $X'=(x\rightarrow x+b)=Z'=B$, $Y'=1$. Then ϕ_1: $(\xi, \eta, B^{-1}\eta)$,

$$\phi_{i+1}\colon (AB^{i-1}\eta,\ B^i\eta,\ B^{i-1}\eta) \qquad (i = 1, 2, \cdots),$$

where ϕ_i is isomorphic to G under the permutation $AB^{i-1}\eta$. The demosian domain contains $E(\phi_1)$ and k groups isomorphic to G, where k is the order of B.

If it is no longer assumed that the demosian system is made up only of quasigroups, such a domain may be constructed by application of the general methods for the formation of groupoids (**RA**) [11].

A study analogous to this one may be made for each of the 16 equations of Section 2.

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THE ROLLING OF ONE CURVE OR SURFACE UPON ANOTHER

WILLIAM CLIFFORD AND JAMES J. McMAHON, Fordham University

It is well known in dynamics [1] that any general motion of a lamina in its own plane may be generated by the rolling of one curve upon another. In his book on continuous groups, Kowalewski [2] gives a mathematical treatment of this result, and here, by using matrix methods, we shall generalize his theorem to spaces of higher dimensions.

The motion of a body in Euclidean space of n dimensions is generated by the transformation

$$X = AX_0 + C, \tag{1}$$

where A is a proper orthogonal $n\times n$ matrix, and X_0 and C are $n\times 1$ vectors. The elements of A and C are continuously differentiable functions of t, and the elements of X_0 are the coordinates of a point in the body at the initial time, $t=t_0$. To avoid the cases of pure translation and pure rotation we assume that $\dot{A}\neq 0$, $\dot{C}\neq 0$, where $\dot{A}=dA/dt$. We shall discuss whether such motions may be considered as generated by the rolling of one curve or surface upon another curve or surface fixed in space.

In a motion generated by the rolling of one curve upon another, the point of contact is an instantaneous center of rotation for the rolling curve. For those points which are instantaneous centers of rotation at time t, we must have $\dot{X}=0$. Hence, in order to find the position at time t_0 of those points which will become instantaneous centers at time t, we must solve the equations

$$0 = \dot{X} = \dot{A}X_0 + \dot{C} \tag{2}$$

for the vector X_0. The equations

$$\dot{A}X_0 = -\dot{C} \tag{2a}$$

are uniquely solved by

$$X_0 = -(\dot{A})^{-1}\dot{C} \tag{3}$$

if $\dot{A}$ is nonsingular. However, since $A^TA=I$, we have

$$0 = \dot{A}^TA + A^T\dot{A} = (A^T\dot{A})^T + (A^T\dot{A}),$$

and so $(A^T\dot{A})$ is an antisymmetric $n\times n$ matrix with zero determinant if n is odd, and except when its pfaffian vanishes, with nonzero determinant when n is even. A motion in Euclidean space of even dimensions for which the pfaffian of $(A^T\dot{A})$ vanishes shall be called a *singular* motion. In E_n, if n is even, and if the motion is not singular, the initial positions of those points which at time t become instantaneous centers of rotation, are given by (3). At time t, these are transformed into the points

$$X = -A(A)^{-1}C + C \tag{4}$$

by (1). Thus the *rolling curve* is given by (3), and the *curve rolled upon* by (4), when the motion (1) is not singular and takes place in Euclidean space of even dimensions.

The cases in which n is odd, or in which n is even and the motion is singular, need further consideration. Here, $\det(A^T\dot{A}) = \det(\dot{A}) = 0$, and so equations (2a) are solvable if and only if

$$(A^T)Y = 0 \Rightarrow (C^T)Y = 0. \tag{5}$$

In general, these conditions will not be satisfied for an arbitrary motion. We shall now develop those physical conditions necessary to insure solvability of (2a), or equivalently, to interpret the motion as the rolling of one surface upon another.

We consider an infinitesimal motion of a body, given by (1), with the consistency conditions (5) imposed. For an infinitesimal displacement of the body,

$$X + \dot{X}\delta t = AX_0 + C + (AX_0 + \dot{C})\delta t. \tag{6}$$

Since $\det(A) \neq 0$, A^{-1} exists, and we may solve (1) for X_0: $X_0 = A^{-1}(X - C)$. Substituting this expression in (6), we obtain

$$X + X\delta t = C + [(I + AA^T\delta t)(X - C)] + C\delta t. \tag{7}$$

Since the matrix $(I + \dot{A}A^T\delta t)$ is the matrix of an infinitesimal orthogonal transformation [3], equations (7) represent an infinitesimal rotation on the vector $(X - C)$, followed by an infinitesimal translation, $\dot{C}\delta t$.

Now, in any rotation, the points on the axis of rotation remain fixed. Hence, if Y is a vector along the axis of the rotation represented by the matrix $(I + \dot{A}A^T\delta t)$ then

$$(I + AA^T\delta t)Y = Y,$$

which implies that $\dot{A}A^TY = 0$. Since $AA^T = I$ yields $\dot{A}A^T = -A\dot{A}^T$, we have $A\dot{A}^TY = 0$ or $\dot{A}^TY = 0$.

Thus we see that the condition (5) states that the infinitesimal translation $\dot{C}\delta t$ must be perpendicular to any vector Y which lies along the axis of infinitesimal rotation. *Therefore, for any motion in Euclidean space of odd dimensions, and for singular motions in space of even dimensions, a necessary and sufficient condition for finding X_0, or the center of instantaneous rotation, is that the infinitesimal translation at any moment be perpendicular to the axis of infinitesimal rotation.*

If we suppose that condition (5) is fulfilled and that the matrix A in (2) has rank r, then the solution of (2) is

$$X_0 = Q_0 + \sum_{i=1}^{s} \lambda_i P_i, \tag{8}$$

where $r+s=n$. Here Q_0 is a particular solution of (2); the P_i are the s linearly independent solutions of the homogeneous equation, and the λ_i are parameters. Hence we obtain, not a curve, but an $(s+1)$-dimensional manifold generated by s-dimensional flats, as the rolling manifold. The manifold rolled upon is

$$X = AX_0 + C, \tag{9}$$

where X_0 is given by (8). Thus the situation is similar to that in three-dimensional space.

It seemed interesting to investigate if any particular curve upon the manifold in (9) deserved to be considered as that curve upon which the rolling takes place, so that the picture would again be that of one curve rolling upon another as in a space of even dimensions. As the motion is not one of pure rotation, there is no fixed center of rotation. However, we can demand that the movement of the center of rotation should be minimized, and this is achieved if we choose on the manifold that curve whose velocity at each instant has minimum absolute value. A curve on the manifold is obtained by making the λ_i functions of t in (8) and (9), and we now wish to minimize

$$|\dot{X}|^2 = |AX_0 + \dot{C} + A\dot{X}_0|^2 = |A\dot{X}_0|^2 = |\dot{X}_0|^2$$

for variable λ_i and $\dot{\lambda}_i$. Thus we find the equations

$$\left.\begin{aligned} &(10\text{a}) \qquad P_i^T\left\{\dot{Q}_0 + \sum_{j=1}^{s}(\dot{\lambda}_j P_j + \lambda_j P_j)\right\} = 0, \\ &(10\text{b}) \qquad P_i^T\left\{\dot{Q}_0 + \sum_{j=1}^{s}(\dot{\lambda}_j P_j + \lambda_j \dot{P}_j)\right\} = 0, \end{aligned}\right\} \qquad (i = 1, \cdots, s)$$

and these yield values for the λ_i and $\dot{\lambda}_i$. It is not true, in general, that when we solve (10a) and (10b) for λ_i and $\dot{\lambda}_i$ the value obtained for $\dot{\lambda}_i$ is the derivative of the value obtained for λ_i. However, the values found for the λ_i are those which determine the *line of striction* on the manifold, and it has long been recognized [4] that the line of striction is not in general perpendicular to the generating flats, or equivalently, that the λ_i and their derivatives need not satisfy (10a). By using these values for the λ_i in (8) and (9), the lines of striction are obtained as corresponding curves on the two manifolds, and we consider these two curves to have intrinsic interest for the motion.

We shall illustrate the above theory with two simple examples. Consider the motion in the Euclidean plane generated by the transformation of the type (1), when

$$A = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \qquad C = \begin{bmatrix} t \\ 0 \end{bmatrix},$$

where in the matrix A, c and s denote $\cos t$ and $\sin t$. Here $\det(\dot{A}) = 1$, and so the rolling curve X_0 is, by (3),

$$X_0 = \begin{bmatrix} s & c \\ -c & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ -c \end{bmatrix},$$

which is a circle. The curve rolled upon is, by (4),

$$X = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} s \\ -c \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ -1 \end{bmatrix},$$

which is a straight line.

For the second example we consider a motion in Euclidean space of three dimensions generated by

$$A = (1/a) \begin{bmatrix} ac & 0 & as \\ -s^2 & -1 & sc \\ -s & s & c \end{bmatrix}, \qquad C = (1/a) \begin{bmatrix} at \\ s \\ 1 - a \end{bmatrix},$$

where $a = \sqrt{(1+s^2)}$. In this case

$$A = -(1/a^3) \begin{bmatrix} a^3 s & 0 & -a^3 c \\ 2sc + s^3 c & -sc & -c^2 + s^2 + s^4 \\ c & -c & s + s^3 + sc^2 \end{bmatrix}, \qquad \dot{C} = (1/a^3) \begin{bmatrix} a^3 \\ c \\ -sc \end{bmatrix},$$

and $\det(\dot{A}) = 0$. However, this is not a general motion as the conditions (5) are satisfied. The general solution to (2) is, as in (8),

$$X_0^T = [s + \lambda c^2, \lambda(1 + s^2), -c + \lambda sc],$$

which is the rolling surface, and the surface rolled upon is, by (9)

$$X^T = [t + \lambda c, -\lambda a, -1 + \lambda as].$$

In both of the cases we obtain the line of striction by giving λ the value $(3s+2s^3-s^5)/(2+8s^2-6s^4)$ in accordance with (10a) and (10b).

The theory presented in this paper may be slightly generalized by taking as the fundamental form preserved under the transformations, not $\sum_{i=1}^n x_i^2$, but a nondegenerate quadratic form, $\sum_{i=1}^n \sum_{j=1}^n c_{ij} x_i x_j$.

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REMARKS ON A MULTIVARIATE GAMMA DISTRIBUTION

P. R. KRISHNAIAH,* University of North Carolina, AND M. M. RAO,
Carnegie Institute of Technology

1. Summary. In this note, the uniqueness problem of a "natural" formulation of a multivariate gamma distribution (Th. 2) is considered. Also the usefulness of the knowledge of the arithmetical character of distributions (in the sense of P. Lévy [6]), concerning extensions of domains of certain parameters (Sec. 4) is discussed.

2. A multivariate gamma distribution. Let $\mathbf{Z}=(Z_1, \cdots, Z_p)$ be a vector of p correlated random variables, the density of the marginal distribution function (d.f.) of each of which is

$$f(x) = \frac{x^{\beta-1}e^{-x/\alpha}}{\alpha^{\beta}\Gamma(\beta)} \quad \text{if } x > 0, \text{ and zero otherwise,} \tag{1}$$

where $\alpha, \beta > 0$ are real numbers. Now let us consider a vector $\mathbf{X}_u=(X_{1u}, \cdots, X_{pu})$ distributed as a p-variate normal vector $(u=1, \cdots, n)$ with mean vector zero and covariance matrix M. Writing for each i, $Z_i=\sum_{u=1}^{n} X_{iu}^2$, where $M=(\sigma_{ij})$, we see that Z_i is distributed as (1) with $\alpha_i=2\sigma_{ii}$ and $\beta=\frac{1}{2}n$. Then a "natural" formulation of the distribution of $\mathbf{Z}$ is this. Since for all "i" the X_i's (hence Z_i's), $(i=1, \cdots, p)$, are correlated, a possible distribution of $(Z_1, \cdots, Z_p)$ is obtained from a multivariate normal d.f. ([5], [3]). It is no restriction to take M as $\sigma^2 P$, where P is the correlation matrix and $\sigma^2>0$ is a real constant, and one obtains the characteristic function (ch.f.) of the d.f. of $\mathbf{Z}$ from that of $\mathbf{X}$ as (cf. [1] p. 159, for the derivation of the ch.f. of the Wishart distribution and then see [2], p. 297),

$$\phi(t_1, \cdots, t_p) = E[\exp\{i(t_1Z_1 + \cdots + t_pZ_p)\}] = |I - 2i\sigma^2 D_t P|^{-n/2}, \tag{2}$$

where D_t is a diagonal matrix whose elements are $(t_1, \cdots, t_p)$.

We observe immediately that (2) becomes the ch.f. of the Wishart distribution if D_t is replaced by D, where D is a full (symmetric) matrix of t's. Hence setting all the off-diagonal elements of D in the ch.f. of the latter d.f. equal to zero, we get (2), *i.e.*, (2) is precisely the ch.f. of a marginal distribution of the Wishart distribution.

Thus, we may state this observation in terms of the following

THEOREM 1. *Let $\alpha=2\sigma^2$, and $\beta>0$ be real members. For $\beta=\frac{1}{2}n$, n an integer, the multivariate gamma distribution is a marginal distribution of the diagonal elements of a Wishart matrix and its ch.f. is given by* (2).

Note. This statement does not assert that (2) holds for any real $\beta>0$, nor does it imply that the choice of the Wishart distribution is determined by $\alpha, \beta>0$, alone. These points are examined in some detail in the following sections.

* Now at Remington Rand Univac, Philadelphia, Pa.

3. Relations between the parameters of the gamma and the "accompanying" Gaussian distributions. Consider the relations between the correlations in the two distributions. For simplicity we assume in the following, with the p-variate normal, the p-variate gamma distribution is nonsingular.

LEMMA 1. *Let $K=(k_{ij})$ be the correlation matrix of a gamma distribution and $P=(\rho_{ij})$ be that of the "accompanying" normal distribution. Then for all α, $\beta>0$, consistent with the conditions on these distributions, the relation*

$$k_{ij} = \rho_{ij}^2 \qquad (i, j = 1, \cdots, p)$$

holds. $(\rho_{ii}=1=k_{ii})$.

Thus the correlations between any two variates of a gamma distribution defined by (2) are nonnegative. (*cf.* [2], p. 317.)

Proof. Writing out (2) in which we set $\beta=\frac{1}{2}n$ and $\alpha=2\sigma^2$, we have the determinant

$$\Delta = \begin{vmatrix} 1 - i\alpha t_1\rho_{11} & -i\alpha t_1\rho_{12} \cdots & -i\alpha t_1\rho_{1p} \\ -i\alpha t_2\rho_{12} & 1 - i\alpha t_2\rho_{22} \cdots & -i\alpha t_2\rho_{2p} \\ \vdots & \vdots & \vdots \\ -i\alpha t_p\rho_{1p} & -i\alpha t_p\rho_{2p} \cdots & 1 - i\alpha t_p\rho_{pp} \end{vmatrix}$$

$=1-y$, say. Taking $t_1, \cdots, t_p$ sufficiently small, we can make $|y|<1$ so that

$$\begin{aligned} \phi(t_1, \cdots, t_p) = \Delta^{-\beta} &= 1 + i\alpha\beta \sum t_i\rho_{ii} + \tfrac{1}{2}i^2\alpha^2\beta(\beta+1) \sum t_i^2\rho_{ii}^2 \\ &\quad + i^2\beta\alpha^2 \sum\sum t_it_j[(\rho_{ij}^2 - \rho_{ii}\rho_{jj}) + (\beta+1)\rho_{ii}\rho_{jj}] + \cdots, \end{aligned} \tag{3}$$

where the summation ranges are $i, j=1, \cdots, p$, $i\neq j$.

From (3) by a routine computation which involves taking partial derivatives with respect to t_i, t_j, and setting them equal to zero in the resulting expressions, it follows that $k_{ij}=\rho_{ij}^2$ $(i, j=1, \cdots, p)$ as stated in the lemma.

Remark. Since α, β do not appear in the final result (even in the proof we did not use the fact that $\beta=\frac{1}{2}n$), the result is valid for all real α, $\beta>0$.

One may, at this point, wish to find the corresponding relations between the correlations of the Wishart and normal distributions. All the correlations associated with the "square terms" will be the same as above, but the remaining $\frac{1}{8}p(p-1)(p^2+3p-2)$ correlations have complicated expressions and hence are omitted.

THEOREM 2. *Let $K=(k_{ij})$ be the covariance matrix of a p-variate gamma distribution and α, β be the other two parameters as before. If h_i $(i=1, \cdots, p-1)$ are the number of nonzero elements (i.e., correlations) in the ith row above the main diagonal of K and $q=h_1+ \cdots +h_{p-1}$, then, for given α, $\beta>0$, and K (symmetric*

positive definite (p.d.)) *with* k_{ij} *small* ($<(p-1)^{-2}$, *say*), *there exist* 2^q *distinct* p-*variate gamma distributions, all of whose ch.f.'s are of the form* (2), *and have the same* α, β *and* K.

Proof. Let $P=(\rho_{ij})$, where $\rho_{ij}=\pm\sqrt{k_{ij}}$ $(i, j=1, \cdots, p)$. Of course k_{ij} are all positive (or zero) so that ρ_{ij} are real, and have *all* possible combinations of signs. Then under the conditions given, for all the choices, it is seen that P is a (symmetric) p.d. matrix. With each such P as covariance matrix with mean vector zero, define a p-variate normal d.f. (which is unique). Corresponding to α, 2β (integer) and each P so obtained, define a $\phi(t_1, \cdots, t_p)$ by (2). Clearly there are 2^q such possible distinct ch.f.'s, since there are that many distinct P's, giving different distributions, all of which by Lemma 1 have the *same* covariance matrix K and, of course, also the same α, β. That is, corresponding to each normal d.f. obtained above we could define a separate multivariate gamma d.f. using (2). This completes the proof.

Remark. A gamma distribution unlike the Gaussian is not completely characterized by the covariance matrix K and α, β. Obviously if $k_{ij}=0$ $(i\neq j)$, $q=0$ and the distribution is unique.

4. Some further considerations. We need the following

DEFINITION. *A d.f. is said to be infinitely divisible* (*or to have an arithmetical character*) *if it can be written as an n-fold convolution of some d.f. for every n* (*or equivalently, if its ch.f. is the nth power of some ch.f. for every n*).

It is easily seen that such a ch.f. never vanishes, and we use this fact in the sequel. First we have

LEMMA 2. *Suppose* $\mathbf{Z}=(Z_1, \cdots, Z_p)$ *is a correlated sample from the population whose ch.f. is given by* (2). *If* $\mathbf{a}$ *is any nonnull vector* $(1\times p)$, *then* $\mathbf{aZ}'$ *has the d.f. whose ch.f. is* $|I-2it\sigma^2D_aP|^{-\beta}$.

Proof. Set $t_j=a_jt$ $(j=1, \cdots, p)$ in (2), where $\mathbf{a}=(a_1, \cdots, a_p)$.

LEMMA 3. *In the above lemma, the d.f. of* $\mathbf{aZ}'$ *holds true if* $\beta>0$ *is any real number, provided that the matrix* D_aP *is at least positive semidefinite* (*p.s.d.*) *and is similar to a real diagonal matrix.*

Proof. If $\beta=\frac{1}{2}n$, n an integer, from Lemma 2, the ch.f. of $\mathbf{aZ}'$ is given by

$$\psi(t) = |I - 2it\sigma^2D_aP|^{-\beta}. \tag{4}$$

For convenience let $\alpha=2\sigma^2$ and $B=D_aP$. Let $(c_1, \cdots, c_p)$ be the characteristic roots of B. From the hypothesis on B, it is seen that all c_i $(i=1, \cdots, p)$ are nonnegative. Then

$$\psi(t) = |I - it\alpha B|^{-\beta} = [\prod (1 - it\alpha c_j)]^{-\beta} = [\prod (1 - itd_j)]^{-\beta},$$

where the product ranges are $j=1, \cdots, p$, and $d_j=\alpha c_j\geqq 0$.

Since $\beta=\frac{1}{2}n$ in Lemma 2, $\psi(t)=[\prod(1-itd_j)]^{-n/2}=[\eta(t)]^n$, say. But $\eta(t)=\prod(1-itd_j)^{-\frac{1}{2}}$ is a ch.f. of the convolution of p gamma distributions, and $\psi(t)$, being the product of n such ch.f.'s, each of which has the arithmetical character [7], is infinitely divisible and hence never vanishes. Therefore, the following operations are meaningful.

From $\eta(t)=\exp\{(1/n)\log\psi(t)\}$, one obtains

$$[\eta(t)]^m = \exp\{(m/n)\log\psi(t)\} \tag{5}$$

is a ch.f., for all integers $m, n>0$. Thus if β is rational, (4) is a ch.f. If β is real, then there exist sequences of rationals n/m such that limit $n/m=\beta$. Thus letting $n, m\to\infty$, so that their ratio tends to β, we observe from (5) that this is a sequence of ch.f.'s tending to a limit which is continuous at $t=0$. Hence by the continuity theorem for ch.f.'s [2], the limit is a ch.f. so that $\exp\{\beta^{-1}\log\psi(t)\}$ (hence (4)) is a ch.f., if β is any real (positive) number, as was to be proved.

We remark here that in (4) one may be tempted to replace the half-integer by any real positive number. However, that is not correct in general. More precisely, let $\phi(t_1, \cdots, t_p; n_1, \cdots, n_r)$ be a ch.f. for every set of positive integers $n_1, \cdots, n_r$. It does not follow from this that $\phi(t_1, \cdots, t_p; \beta_1, \cdots, \beta_r)$ is a ch.f. for every set of positive real numbers $\beta_1, \cdots, \beta_r$. A simple example* of this is the ch.f. of the binomial distribution

$$\phi(t; n) = (q + pe^{it})^n \qquad (p, q > 0, p + q = 1). \tag{6}$$

Another example, which is a multivariate distribution, is the following (given in another context in [4]). This is a slightly restrictive example which nevertheless is of interest. Consider the ch.f. given by ($m, n>0$ integers)

$$(1 - it_1)^{-mn/2}(i - it_2)^{-m/2}\left[1 + \frac{\rho_{12}^2 t_1t_2}{(1 - it_1)(1 - it_2)}\right]^{-m/2}. \tag{7}$$

Then a "generalization" is to write this form as

$$(1 - it_1)^{-M}(1 - it_2)^{-N}\left[1 - \frac{\rho_{12}^2 t_1t_2}{(1 - it_1)(1 - it_2)}\right]^{-Q} \tag{8}$$

for a joint ch.f. of two positive random variables X and Y, where $M, N, Q>0$ are any real numbers. However, on expansion and inversion, one finds that the "frequency function" of (7) being convergent for $|\rho_{12}|<1$ is negative for certain positive values of x and y if $Q\rho_{12}^2>\min(M, N)$, [4], which shows that the convergence condition alone does not ensure the desired conclusion, and consequently for general Q, M, N, (7) cannot be a ch.f. But because of Lemma 3, infinite divisibility is a sufficient condition for this "replacement."

* This was pointed out by the referee. We wish to thank the referee for this and various other useful remarks.

If X_1 and X_2 are independent normal random variables with zero means and unit variances, then Lévy showed, in [6], that the joint distribution of any two of X_1^2, X_1X_2, and X_2^2 is infinitely divisible, but that the joint distribution of the three, which is a Wishart distribution, is not infinitely divisible. In general, for multivariate distributions, infinite divisibility is a nontrivial property to establish ([6], [8]).

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A SINGLE POSTULATE FOR GROUPS

MICHAEL SLATER, Magdalen College, Oxford, England

Marlow Sholander [1] shows that if we make certain natural presuppositions (*e.g.*, about the structure of English grammar and the definition of an operator) we may characterize an abelian group by a single postulate. This note provides a similar result for an arbitrary group.

Let G be a nonempty set on which two operations are defined: a binary *multiplication* and a singularly *inversion*. For each a, $b\in G$ we write (ab) for the product in that order of a and b, and a' for the inverse of a.

THEOREM. *If $(ab)c=(ad)f$ implies $b=d(fc')$ for every a, b, c, d, $f\in G$, then G is a group relative to the operations described above.*

Proof. 1. Since $(ab)c=(ab)c$ we have $b=b(cc')$. Let us write a^* for aa'. Then $(ab^*)c=(ad^*)c$ so that

$$b^* = d^*c^* = d^*.$$

Thus for all $a\in G$, $aa'=e_r$ (a right identity).

2. If $ab=ad$ then $(ab)c=(ad)c$, so that

$$b = d(cc') = d.$$

We thus have left cancellation.

3. We have $(ae_r)b=(ab)e_r$, so that $e_r=b(e_rb')$. But $e_r=bb'$, so that, by left cancellation, $b'=e_rb'$. In particular,

$$e_r' = e_re_r' = e_r.$$

Next, $(ab)e_r=(ae_r)b$, so that $b=e_r(be_r')=e_rb$.

4. We have $(e_rc)c'=(e_rd)d'$ so that $c=d(d'c'')$. If we take $d=e_r$, we find, in particular, that $c=c''$. So

$$c = d(d'c),$$

and the equation $a=bx$ has a solution $x=b'a$.

5.† Given a, b, c let us choose d so that $[a(bc)]d=(ab)c$. Then $bc=b(cd')$, $ce_r=c=cd'$, $e_r=d'$, and $d=e_r$. We thus have associativity, and the rest is easy.

Added in proof. Since writing the above, I have come across an article‡ in which a single axiom for groups is given. The axiom is written in terms of the operator δ, where $ab\delta=ab'$.

More generally, let $W(x_1, \cdots, x_n, \delta)$ be a ("meaingful") word in the symbols $x_1, \cdots, x_n, \delta$. Suppose a class $\mathfrak{A}$ of groups is such that a given group G with unit element e is a member of $\mathfrak{A}$ if and only if $W(a_1, \cdots, a_n, \delta)=e$ for all $a_1, \cdots, a_n \in G$. For example, if $\mathfrak{A}$ is the class of *all* groups, we may take $W=xx\delta$; if $\mathfrak{A}$ is the class of abelian groups, we may take $W=xyyx\delta\delta\delta$.

It is then shown that the class $\mathfrak{A}$ is characterized by the single axiom

$$(\alpha) \qquad xxx\delta W\delta y\delta z\delta xx\delta x\delta z\delta\delta\delta = y,$$

where $W=W(x_1, \cdots, x_n, \delta)$. That is, if G is a set on which a binary operation δ is defined, then G is a member of $\mathfrak{A}$ if and only if (α) is satisfied for all $x, y, z, x_1, \cdots, x_n \in G$.

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1. Marlow Sholander, Postulates for commutative groups, this MONTHLY, vol. 66, 1959, pp. 93–95.

† I am indebted to the referee for pointing out a simplification in my treatment of 5.

‡ Graham Higman and B. H. Neumann, Groups as groupoids with one law, Publ. Math. Debrecen, vol. 2, 1952, pp. 215–221.

A THEOREM IN THE FAREY SERIES

DAVID S. ROBERTS, University of Colorado

In this note we shall prove the following

THEOREM. *Let F_n be the Farey series of order n with the end fractions* 0 *and* 1 *deleted. Then, for $n \geqq 4$, $f_1 = f_2 f_3$, where $f_i \in F_n$ $(i = 1, 2, 3)$ and $f_2 \neq f_3$, if and only if f_1 does not lie among the largest $[\frac{1}{2}(n+1)]$ elements of F_n.*

Let S_n be the set of all $f \in F_n$ which are expressible as $f_2 f_3$, where $f_2, f_3 \in F_n$ and $f_2 \neq f_3$. Let F_n be divided into two exhaustive mutually exclusive subsets R_{n_1}, R_{n_2} as follows:

$$f_a \in R_{n_1} \text{ if and only if } 0 < f_a \leqq (n-2)/n,$$

$$f_a \in R_{n_2} \text{ if and only if } (n-2)/n < f_a < 1.$$

LEMMA 1. *$f_a \in R_{n_2}$ if and only if f_a is of the form $(x-1)/x$ with $x > \frac{1}{2}n$.*

Proof. If $a/b > (n-2)/n$, then $n(b-a) < 2b$ and $b - a < (2b)/n \leqq 2$. Hence $a = b-1$ and $a/b = (b-1)/b$, which is of the form $(x-1)/x$. Also, $(x-1)/x > (n-2)/n$ implies $x > \frac{1}{2}n$.

Conversely, if $x > \frac{1}{2}n$, then $(x-1)/x = 1 - (1/x) > 1 - (2/n) = (n-2)/n$.

LEMMA 2. *The number of elements in R_{n_2} is $[\frac{1}{2}(n+1)]$.*

Proof. By Lemma 1, all the elements of R_{n_2} are of the form $(x-1)/x$ with $x > \frac{1}{2}n$, and any element of that form is in R_{n_2}. If n is odd, there are $\frac{1}{2}(n+1)$ integers x such that $\frac{1}{2}n < x \leqq n$. Consequently, $\frac{1}{2}(n+1)$ fractions of the form $(x-1)/x$ with $x > \frac{1}{2}n$ can be formed. If n is even, then $\frac{1}{2}n$ such fractions can be formed. In either case the number of such fractions is $[\frac{1}{2}(n+1)]$.

LEMMA 3. *If $f_a \in R_{n_2}$ then $f_a \notin S_n$.*

Proof. From Lemma 1 we conclude that the two largest elements of F_n are $(n-1)/n$ and $(n-2)/(n-1)$. Their product, consequently, will be the greatest possible product of two distinct elements of F_n, that is, the greatest element of S_n. Since the product is equal to $(n-2)/n$ and since $f_a \in R_{n_2} \Rightarrow f_a > (n-2)/n$, no element of R_{n_2} is in S_n.

LEMMA 4. *If $f_a \in R_{n_1}$ then $f_a \in S_n$.*

Proof. If $f_a \in R_{n_1}$, then f_a cannot be of the form $(x-1)/x$ with $x > \frac{1}{2}n$ (by Lemma 1). Therefore, if $a/b \in R_{n_1}$, either $a < b-1$ or $a = b-1$ and $b \leqq \frac{1}{2}n$.

In the first case, $1 \leqq a < b-1 < b \leqq n$. Therefore $1 \leqq a < a+1 < n$ and $1 < a+1 < b \leqq n$; hence $a/(a+1) \in F_n$ and $(a+1)/b \in F_n$. Since

$$\frac{a}{b} = \left(\frac{a}{a+1}\right)\left(\frac{a+1}{b}\right),$$

$a/b \in S_n$ in the first case.

In the second case, $1 < 2b-1 < 2b \leqq n$ and $1 < 2b-2 < 2b-1 < n$; hence $(2b-1)/(2b) \in F_n$ and $(2b-2)/(2b-1) \in F_n$. Since

$$\frac{a}{b} = \frac{b-1}{b} = \left(\frac{2b-1}{2b}\right)\left(\frac{2b-2}{2b-1}\right),$$

$a/b \in S_n$ in the second case also.

Proof of the theorem. In Lemmas 1–4 we have proved that all elements of R_{n_1} are in S_n, that no element of R_{n_2} is in S_n, and that the number of elements of R_{n_2} is $[\frac{1}{2}(n+1)]$. Therefore the proof of the theorem is complete.

THE ERDÖS INEQUALITY AND OTHER INEQUALITIES FOR A TRIANGLE

A. OPPENHEIM, University of Malaya

Note added in proof (February, 1961).* From a copy of the paper by A. Florian kindly sent to me by the author, I learn that the reviewer in *Mathematical Reviews* has misquoted the results for $k<0$. Florian's results agree with mine.

Through the kindness of Professor H. S. M. Coxeter and the erudition of Professor E. H. Neville, I learn that the triangle theorem of Section 9 is not new. It is stated in the last edition of Casey's *A Sequel to Euclid*, 1892, page 253, and is apparently due to Neuberg. I find it remarkable that an elementary theorem of such charm should disappear from view.

My extension to plane polygons appears to be new.

* A delay in transit prevented this note from being added to the original article, this MONTHLY, vol. 68, 1961, pp. 226–230.

CORRECTION

It has been pointed out by Donald W. Western that Theorem 1 of his paper *Inequalities of the Markoff and Bernstein type for integral norms*, Duke Math. J., vol. 15, 1948, pp. 839–869, contains as a special case Theorem 1 of the paper by Q. I. Rahman, *Some inequalities for polynomials*, this MONTHLY, vol. 67, 1960, pp. 847–851. This fact was not known to Mr. Rahman nor to the referee.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

RATIONAL ORTHOGONAL MATRICES

MARLOW SHOLANDER, Carnegie Institute of Technology

The construction of a 3×3 rotation matrix R over the rationals is equivalent to the construction of a similarity matrix S whose rows are mutually orthogonal vectors α, β, and γ which have integral components and share an integral length D. Some textbooks give the impression the construction is essentially unique with $\alpha=[1, 2, 2]$, $\beta=[2, 1, -2]$, and $\gamma=[2, -2, 1]$.

It is well known that a rotation may be obtained as a product of rotations about the z, y, and z axes, respectively, and that R may be expressed as the Eulerian matrix

$$\begin{pmatrix} c_1c_2c_3 - s_1s_3 & c_1c_2s_3 + s_1c_3 & -c_1s_2 \\ -s_1c_2c_3 - c_1s_3 & -s_1c_2s_3 + c_1c_3 & s_1s_2 \\ s_2c_3 & s_2s_3 & c_2 \end{pmatrix}$$

where $c_i=\cos\theta_i$ and $s_i=\sin\theta_i$ for $i=1, 2, 3$. We note that to have these elements rational it is not necessary that all c_i and s_i be rational. Suppose that we have found integers satisfying

$$(*) \qquad \alpha_3^2+\beta_3^2+\gamma_3^2 = D^2 = \gamma_1^2+\gamma_2^2+\gamma_3^2.$$

We construct R by choosing $\cos\theta_2=\gamma_3/D$, $\tan\theta_1=\beta_3/\alpha_3$, and $\tan\theta_3=\gamma_2/\gamma_1$.

The construction of S now is reduced to finding integral solutions of (*). This Diophantine problem is discussed in L. E. Dickson's *History of the Theory of Numbers*, vol. II, Washington, 1919, pp. 261–269. Elegant parametric representations for classes of matrices S, based on identities due to Lebesgue, Catalan, and Dainelli (loc. cit.) are given below.

First, let

$$\begin{aligned} \alpha &= [a^2+b^2-c^2-d^2,\ 2(ac+bd),\ 2(ad-bc)], \\ \beta &= [2(ac-bd),\ b^2+c^2-a^2-d^2,\ 2(ab+cd)], \\ \gamma &= [2(ad+bc),\ 2(cd-ab),\ b^2+d^2-a^2-c^2]. \end{aligned}$$

Here $D=a^2+b^2+c^2+d^2$. When d is chosen as shown below and a factor 2 is removed, we get the following special cases. Where $d=a+b+c$, let $\alpha=[-ab-cd,\ ac+bd,\ ad-bc]$, $\beta=[ac-bd,\ -bc-ad,\ ab+cd]$, and $\gamma=[bc+ad,\ cd-ab,\ ac+bd]$. Here $D=a^2+b^2+c^2+ab+bc+ca$. If we take $c=0$, we obtain the next case. Where $d=a+b$, let $\alpha=[-ab, bd, ad]$, $\beta=[-bd, -ad, ab]$, and $\gamma=[ad, -ab, bd]$. Here $D=a^2+ab+b^2$.

REMARK ON THE CONSTRAINT SETS IN LINEAR PROGRAMMING

FRANK EUGENE CLARK, Rutgers, The State University

In the standard linear programming problem, one is given a real $m\times n$ matrix A and vectors $b\in\mathfrak{R}^m$ and $c\in\mathfrak{R}^n$ where $\mathfrak{R}^k$ is euclidean k-space. One forms the *primal constraint set*, here designated by $\mathfrak{F}(A, b)=\{x\mid Ax\leqq b,\ 0\leqq x\}$, and the *dual constraint set* $\mathfrak{F}(c, A)=\{u\mid c^T\leqq u^TA,\ 0\leqq u\}$. The primal problem is to maximize c^Tx over all $x\in\mathfrak{F}(A, b)$ and the dual problem is to minimize u^Tb over all $u\in\mathfrak{F}(c, A)$. The "existence theorem" [**3**; p. 61] states that either problem has a solution (and hence both do) if and only if neither constraint set is empty. It appears not to have been noticed that it is impossible for both constraint sets to be bounded.

THEOREM. *If a linear programming problem has a solution, then either the primal constraint set or the dual constraint set is unbounded.*

Proof. By [**2**; p. 49], if $\mathfrak{F}(A, b)$ is not empty, then it is bounded if and only if $\mathfrak{F}(A, 0)=\{0\}$. Similarly if $\mathfrak{F}(c, A)$ is not empty, then it is bounded if and only if $\mathfrak{F}(0, A)=\{0\}$. The assumption that the linear programming problem has a solution implies that b and c have been so chosen that neither constraint set is empty. Beyond this condition no further use is made of b and c; we prove that it is impossible to have both $\mathfrak{F}(A, 0)=\{0\}$ and $\mathfrak{F}(0, A)=\{0\}$.

Now $\mathfrak{F}(A, 0)=\{x\mid Ax\leqq 0,\ 0\leqq x\}$, a pointed, convex polyhedral cone. Its polar cone is $\mathfrak{F}^*(A, 0)=\{y\mid x\in\mathfrak{F}(A, 0)\Rightarrow y^Tx\leqq 0\}$, the set of all vectors in $\mathfrak{R}^n$ which do not form an acute angle with any vector of $\mathfrak{F}(A, 0)$. Thus $\mathfrak{F}(A, 0)=\{0\}$ if and only if $\mathfrak{F}^*(A, 0)=\{\text{all } y\in\mathfrak{R}^n\}$. But by the theorem of Farkas [**1**; p. 31], $\mathfrak{F}^*(A, 0)=\{y\mid y=A^Tu-v,\ 0\leqq u,\ 0\leqq v\}=\{y\mid y^T\leqq u^TA,\ 0\leqq u\}$. Select any $y\in\mathfrak{R}^n$ with $0<y$. Since $y\in\mathfrak{F}^*(A, 0)$, there exists a nonzero $u\in\mathfrak{R}^m$ with $y^T\leqq u^TA$, $0\leqq u$.

One may reason in a similar fashion for $\mathfrak{F}(0, A)$. Instead one may simply observe that $\mathfrak{F}(0, A)=\mathfrak{F}(-A^T, 0)$. Hence $\mathfrak{F}^*(0, A)=\{v\mid v^T\leqq x^T(-A^T),\ 0\leqq x\}$ and this set must consist of all $v\in\mathfrak{R}^m$ if $\mathfrak{F}(0, A)=\{0\}$. Select any $v\in\mathfrak{R}^m$ with $0<v$; then there exists a nonzero $x\in\mathfrak{R}^n$ with $Ax\leqq -v$, $0\leqq x$. Now if both $\mathfrak{F}(A, 0)=\{0\}$ and $\mathfrak{F}(0, A)=\{0\}$ it would follow that $0<y^Tx\leqq u^TAx\leqq -u^Tv<0$, a contradiction.

One may also prove the theorem more briefly but less directly by showing that if the theorem were false, this would deny the property of complementary slackness as given in [**1**; Theorem 6]. With the appropriate changes in notation, the theorem referred to states (in part) that there must be some solution of the inequalities $0\leqq u$, $0\leqq A^Tu$, $Ax\leqq 0$, $0\leqq x$ for which $Ax<u$ and $-A^Tu<x$; thus $x=0$, $u=0$ cannot be the only solution of this system of inequalities.

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IRREDUCIBILITY OF CERTAIN BERNOULLI POLYNOMIALS*

P. J. McCarthy, Florida State University

The Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \cdot$$

$B_m(x)$ is a polynomial of degree m having rational coefficients. Its constant term, $B_m = B_m(0)$, is the mth Bernoulli number in Nörlund's notation [2, Chapter 2]. We have [2, p. 18]

$$B_m(x) = \sum_{r=0}^{m} \binom{m}{r} B_r x^{m-r}.$$

We are interested in determining values of m for which $B_m(x)$ is irreducible (over the field of rational numbers). When m is odd, $B_m(x)$ is divisible by $x(x-1)(x-1/2)$, so we shall confine our attention to even values of m. The only known results are those of Carlitz [1] that $B_m(x)$ is irreducible whenever $m = k(p-1)p^t$, p a prime, $t \geqq 0$, and $1 \leqq k \leqq p$.

Carlitz obtained his results by showing that for the values of m mentioned above, $pB_m(x)$ is the Eisenstein polynomial, *i.e.*, it satisfies the hypotheses of the Eisenstein irreducibility criterion, with p as the determining prime. There are other cases when this is true and we shall determine all of them here. The leading coefficient of $pB_m(x)$ is p while its constant term is pB_m. Hence it follows from the Staudt-Clausen theorem [2, p. 33] that if $pB_m(x)$ is to be an Eisenstein polynomial, with p as the determining prime, we must have $m = k(p-1)$. The coefficient of x^{m-r} in $pB_m(x)$ is then

$$\binom{k(p-1)}{r} pB_r. \tag{1}$$

Suppose that $0 < r < k(p-1)$. If $p-1$ does not divide r, it follows from the Staudt-Clausen theorem that $pB_r \equiv 0 \pmod{p}$, so that (1) is divisible by p. If $p-1$ does divide r, then $pB_r \equiv -1 \pmod{p}$, again from the Staudt-Clausen theorem. Hence $pB_m(x)$ is an Eisenstein polynomial, with p as the determining prime, if and only if the binomial coefficient in (1) is divisible by p for all r such that $0 < r < k(p-1)$ and $p-1$ divides r. A necessary and sufficient condition for

* This work was supported by NSF-G11293.

this to be true has been given by Schäffer [3, Lemma 3]. If $n=a_0+a_1p+a_2p^2+\cdots$, where $0\leqq a_i\leqq p-1$ for $i=0, 1, \cdots$, we set $A(n, p)=a_0+a_1+\cdots$. Schäffer's lemma is then the following:

If p is a prime and n is an arbitrary positive integer, then

$$\binom{n}{s(p-1)} \equiv 0 \pmod p$$

for all integers s such that $0<s(p-1)<n$, if and only if $A(n, p)\leqq p-1$.

We therefore have the following result.

THEOREM. *$pB_m(x)$ is an Eisenstein polynomial, with p as the determining prime, if and only if $p-1$ divides m and $A(m, p)\leqq p-1$.*

Using this we can state several special results. We give only one example: *if $k+1\leqq p$ then $B_{(kp+k+1)(p-1)}(x)$ is irreducible.*

If m is even and $0\leqq m\leqq 200$, the result of Carlitz tells us that $B_m(x)$ is irreducible except for 25 values of m. Of these 25 cases application of the above theorem shows that $B_{76}(x)$, $B_{114}(x)$, $B_{152}(x)$, and $B_{170}(x)$ are irreducible. Of the remaining 21 cases, m is twice a prime in 15 cases. This points up the desirability of showing that $B_{2p}(x)$ is irreducible when p is a prime.

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THE CONVERGENCE OF SEQUENCES WITH LINEAR FRACTIONAL RECURRENCE RELATION

HANS LIEBECK, University of Cape Town

For the sake of brevity, and by analogy with [1], we make the

DEFINITION. *A (complex-valued) sequence with initial z_1 and recurrence relation*

$$z_{n+1} = \frac{az_n + b}{z_n + d}, \qquad ad - b \neq 0 \tag{1}$$

is called the Moebius sequence $M(z_1; a, b, d)$.

In this note we shall give a formula for the nth term of a Moebius sequence. From this, convergence can readily be discussed. Finally, we obtain a necessary and sufficient condition for a Moebius sequence to take only a finite number of values.

To include the case where $z_n=-d$ for some n, we add the point $z=\infty$ to the

complex plane. $M(z_1; a, b, d)$ has a member $z_r = \infty$ (*i.e.*, $z_{r-1} = -d$) if and only if z_1 is a member of $M(-d; -d, b, -a)$.

As usual, the sequence $\{z_n\}$ is said to be convergent if and only if $\lim_{n\to\infty} z_n$ exists and is not infinite.

Rewriting (1) in the form

$$z_{n+1}z_n + dz_{n+1} - az_n - b = 0, \tag{1*}$$

we form the "auxiliary equation"

$$z^2 + (d - a)z - b = 0 \tag{2}$$

whose roots, α and β, are the only values to which z_n can converge.

$$\alpha + \beta = a - d; \qquad \alpha\beta = -b. \tag{3}$$

Remarks.

(i) If α is a root of (2) and if $z_n = \alpha$ for some value of n, then $z_n = \alpha$ for all n, *i.e.* the sequence is single valued.

(ii) If the sequence is not single valued, then $z_{n+1} \neq z_n$ for $n = 1, 2, \cdots$.

In general, two cases arise:

(a) $\alpha \neq \beta$. From (1*) and (3) and the remarks above we obtain, for any n,

$$\frac{z_{n+1} - \alpha}{z_{n+1} - \beta} = \lambda \frac{z_n - \alpha}{z_n - \beta}, \quad \lambda = \frac{d + \beta}{d + \alpha} \cdot \tag{4}$$

$\lambda \neq 0$ or ∞, since if $d = -\beta$ or $-\alpha$ then $ad - b = 0$, and the recurrence relation (1) collapses.

(b) $\alpha = \beta$. In this case

$$\frac{1}{z_{n+1} - \alpha} = \frac{1}{z_n - \alpha} + \mu, \quad \mu = \frac{1}{d + \alpha}. \tag{5}$$

Repeated application of (4) and (5) gives

THEOREM 1. *The Moebius sequence $M(z_1; a, b, d)$ has nth term*

$$\begin{aligned} z_n &= \frac{\alpha(z_1 - \beta) - \lambda^{n-1}\beta(z_1 - \alpha)}{(z_1 - \beta) - \lambda^{n-1}(z_1 - \alpha)}, \qquad \alpha \neq \beta, \qquad \lambda = \frac{d + \beta}{d + \alpha}; \\ z_n &= \frac{(z_1 - \alpha)(d + \alpha)}{(n - 1)(z_1 - \alpha) + (d + \alpha)} + \alpha, \quad \alpha = \beta, \end{aligned} \tag{6}$$

where α and β are the roots of the auxiliary equation (2).

Concerning the convergence of Moebius sequences:

THEOREM 2. *$M(z_1; a, b, d)$ is single valued if $z_1 = \alpha$ or β. For other initial values, the sequence*

(*a*) *converges to α, if $\alpha = \beta$ or $|d+\alpha| > |d+\beta|$,*

(*b*) *diverges if $|d+\alpha| = |d+\beta|$, $\alpha \neq \beta$.*

Proof. (*a*) follows from (6). As for (*b*), let $\lambda=e^{i\theta}$, $\theta\not\equiv 0 \pmod{2\pi}$. Then, from (4),

$$\frac{z_n-\alpha}{z_n-\beta}=e^{i(n-1)\theta}\frac{z_1-\alpha}{z_1-\beta}. \tag{7}$$

Thus z_n cannot converge to α or β. We note that by suitable selection of the root α, one of the cases (*a*), (*b*) must occur.

DEFINITION. *A sequence* $\{z_n\}$ *is cyclic of period m if m is the least integer such that, for all n,* $z_{n+m}=z_n$.

THEOREM 3. *The Moebius sequence* $M(z_1; a, b, d)$, $z\neq\alpha$ *or* β, *is cyclic of period* $m>1$ *if and only if there is an integer k,* $(k, m)=1$, *for which*

$$\frac{(a+d)^2}{(a-d)^2+4b}=-\cot^2\frac{k\pi}{m}. \tag{8}$$

Proof. From (7), $z_{m+1}=z_1$ and $z_n\neq z_1$ for $1<n\leqq m$ if and only if $m\theta=2k\pi$, for some integer k which is relatively prime to m. But

$$e^{2k\pi i/m}=e^{i\theta}=\lambda=\frac{d+\beta}{d+\alpha}=\frac{(a+d)\pm\sqrt{[(a-d)^2+4b]}}{(a+d)\mp\sqrt{[(a-d)^2+4b]}}$$

from which the result follows.

Because of its frequent application, we give the following special case of Theorem 2 as

THEOREM 2R. *The real Moebius sequence* $M(x_1; a, b, d)$, x_1, a, b, d *real, whose auxiliary equation* (2) *has real roots* α *and* β, $\alpha\geqq\beta$, *is single valued if* $x_1=\alpha$ *or* β. *For other initial values the sequence*

(*a*) *converges to* α, *if* $\alpha=\beta$ *or* $a+d>0$, *and converges to* β *if* $a+d<0$

(*b*) *is cyclic of period* 2 *if* $a+d=0$.

Proof. Since $a\geqq\beta$, $2(d+\alpha)=(a+d)+\sqrt{[(a-d)^2+4b]}$ and $2(d+\beta)=(a+d)-\sqrt{[(a-d)^2+4b]}$. Thus $|d+\alpha|>|d+\beta|$ implies that $a+d>0$. The case $a+d=0$, $\alpha=\beta$ cannot arise, for this would imply that $ad=b$. If $a+d=0$, $\alpha\neq\beta$, then the denominator of (8) is not zero, and Theorem 3 applies.

The only real Moebius sequences not covered in Theorems 3 and 2*R* are those which are not cyclic and for which α and β are not real. Clearly such sequences cannot converge, and, as required by Theorem 2, $|d+\alpha|=|d+\beta|$.

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THE RECIPROCAL ITERATED LIMIT THEOREM

OSCAR VARSAVSKY, Universidad Central de Venezuela

Let I and J_i, $i \in I$, be directed sets (for nomenclature see Kelley, *General Topology*, Ch. II), and let $S = \{s_{ij}: i \in I, j \in J_i\}$ be a double net in a topological space T, which is simply convergent: $r_i = \lim_{j \in J_i} s_{ij}$. Let P be the product directed set $I \times \prod_{i \in I} J_i$, and if $p = \{i, \{j_k: k \in I\}\} \in P$, let $s_p = s_{ij_i}$. The net $\{s_p: p \in P\}$ is called the diagonal net of S, and an important elementary theorem of general topology states:

ITERATED LIMIT THEOREM. *If the net $\{r_i: i \in I\}$ (obtained from the limits of a simply convergent double net) is convergent, then its diagonal net converges to the same limit.*

Here we wish to prove the following elementary reciprocal theorem:

THEOREM. *A topological space T is regular if and only if, for every simply convergent double net S it is true that* $\lim_{p \in P} s_p = r$ *implies* $\lim_{i \in I} r_i = r$ *(notation as above).*

Proof. 1) Let T be regular and $\lim s_p = r$, and suppose that r_i does not converge to r. Then there are a closed neighborhood V of r and a subnet of r_i which do not meet.

Since $\lim_{p \in P} s_p = r$, there is a $p^0 = \{i^0, \{j_k^0: k \in I\}\}$ such that $p \geqq p^0$ implies $s_p \in V$. Then for each $i \geqq i^0$ and $j \geqq j_i^0$, put $p' = \{i, \{j_k': k \in I\}\}$, with $j_i' = j$ and $j_k' = j_k^0$ for $k \neq i$. Then $p' \geqq p^0$, and $s_{ij} = s_{p'} \in V$. Let $i \geqq i^0$ and such that $r_i \notin V$, and let W be a neighborhood of r_i disjoint from V (V is closed). Since $s_{ij} \to r_i$, there is a $j^0 \in J_i$ such that $j \geqq j^0$ implies $s_{ij} \in W$. Take $j \geqq j^0, j_i^0$ and we have a contradiction.

2) Suppose that T is not regular, that is, there is a point r in T and a neighborhood V of r such that for each neighborhood W of r the set difference $\overline{W} \sim V$ is not empty. Take $r_W \in \overline{W} \sim V$, and, for each W and each neighborhood U of r_W take $s_{WU} \in U \cap W$. Call $I(J_W)$ the directed set of all neighborhoods of $r(r_W)$. Then obviously $\lim_{U \in J_W} s_{WU} = r_W$ and the diagonal net $s_p = s_{WU_W}$, with $p = \{W, \{U_S: S \in I\}\}$, converges to r, since for each $W^0 \in I$, $W \subset W^0$ implies $s_{WU_W} \in W^0$. Yet of course r_W does not converge to r.

NOTE ON THE DIVISORS OF A NUMBER

R. L. DUNCAN, Pennsylvania State University

Let $\nu(n)$ and $\tau(n)$ denote the number of distinct prime divisors and the number of divisors of n respectively. Define $\beta(n) = \sum_{d/n} \nu(d)$ and $\alpha(n) = \beta(n)/\tau(n)$. Thus $\alpha(n)$ is the average number of distinct prime divisors of the divisors of n. The object of this note is to derive a simple expression for $\alpha(n)$ and to calculate the average order of $\alpha(n)$. In particular, it is shown that $\alpha(n)$ has average order $\frac{1}{2}\nu(n)$.

THEOREM 1. *$\beta(n)=\sum_{p/n}\tau(n/p)$, where the sum extends over the distinct prime divisors of n.*

Proof. For $s>1$ we have

$$\sum_{n=1}^{\infty}\frac{\beta(n)}{n^s}=\zeta(s)\sum_{n=1}^{\infty}\frac{\nu(n)}{n^s}=\zeta^2(s)\sum_{p}\frac{1}{p^s}$$
$$=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}\sum_{n=1}^{\infty}\frac{x(n)}{n^s}=\sum_{n=1}^{\infty}\frac{\gamma_n}{n^s},$$

where $x(n)=1$ if n is a prime and zero otherwise and

$$\gamma_n=\sum_{d|n}x(d)\tau\left(\frac{n}{d}\right)=\sum_{p|n}\tau\left(\frac{n}{p}\right).$$

The result now follows from the uniqueness theorem for Dirichlet series.

THEOREM 2. *$\alpha(n)=\sum_{k=1}^{r}\alpha_k/\alpha_k+1$, where $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ is the representation of n as a product of powers of distinct primes.*

Proof. Since $\tau(n)=(\alpha_1+1)\cdots(\alpha_r+1)$, we have

$$\tau\left(\frac{n}{p_k}\right)=\frac{\alpha_k}{\alpha_k+1}\tau(n).\quad\text{Hence}\quad\beta(n)=\tau(n)\sum_{k=1}^{r}\frac{\alpha_k}{\alpha_k+1}$$

by Theorem 1 and the desired result follows from the definition of $\alpha(n)$. The following corollary is now obvious.

COROLLARY. *$\frac{1}{2}\nu(n)\leqq\alpha(n)<\nu(n)$ and these inequalities are the best possible. In particular, $\alpha(n)=\frac{1}{2}\nu(n)$ if and only if n is square-free.*

THEOREM 3. *$\sum_{n\leqq x}\alpha(n)=\frac{1}{2}\sum_{n\leqq x}\nu(n)+cx+O(x/\log x)$.*

Proof. By Theorem 1 we have

$$\sum_{n\leqq x}\alpha(n)=\sum_{n\leqq x}\sum_{p|n}\frac{\tau(n/p)}{\tau(n)}=\sum_{p\leqq x}\sum_{\substack{n\leqq x\\ p|n}}\frac{\tau(n/p)}{\tau(n)}$$
$$=\sum_{p\leqq x}\sum_{k=1}^{\infty}\frac{k}{k+1}\left(\left[\frac{x}{p^k}\right]-\left[\frac{x}{p^{k+1}}\right]\right)$$
$$=\frac{1}{2}\sum_{p\leqq x}\left[\frac{x}{p}\right]+\sum_{p\leqq x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)}\left[\frac{x}{p^k}\right]$$
$$=\frac{1}{2}\sum_{n\leqq x}\nu(n)+x\sum_{p\leqq x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}+O(\pi(x)).$$

But $O(\pi(x))=O(x/\log x)$ by Tchebycheff's inequality. Also,

$$\sum_{p\leq x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}=\sum_{p}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}-\sum_{p>x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}$$

$=c+O(1/x)$ and this proves the theorem.

The following corollaries are a consequence of the well-known asymptotic formula

$$\sum_{n\leq x}\nu(n)=x\log\log x+c_1x+O\left(\frac{x}{\log x}\right).$$

COROLLARY. $\sum_{n\leq x}\alpha(n)=\frac{1}{2}x\log\log x+c_2x+O\,(x/\log x)$.

COROLLARY. *The average order of* $\alpha(n)$ *is* $\frac{1}{2}\nu(n)$.

The constant c of the preceding theorem can be put in a more suitable form by observing that

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}&=\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)\frac{1}{p^k}\\&=\sum_{k=2}^{\infty}\frac{1}{kp^k}-p\sum_{k=3}^{\infty}\frac{1}{kp^k}\\&=\frac{1}{2p^2}+(p-1)\left(\frac{1}{p}+\frac{1}{2p^2}\right)-(p-1)\sum_{k=1}^{\infty}\frac{1}{kp^k}\\&=1-\frac{1}{2p}+(p-1)\log\left(1-\frac{1}{p}\right).\end{aligned}$$

Thus

$$c=\sum_{p}\left\{1-\frac{1}{2p}+(p-1)\log\left(1-\frac{1}{p}\right)\right\}.$$

We recall that the normal order of $f(n)$ is $g(n)$ if $(1-\epsilon)g(n)<f(n)<(1+\epsilon)g(n)$ for every $\epsilon>0$ and almost all n, *i.e.*, if the number of $n\leqq x$ which do not satisfy these inequalities is $O(x)$.

THEOREM 4. *The normal order of* $\alpha(n)$ *is* $\frac{1}{2}\nu(n)$.

Proof. Suppose that $\alpha(n)\geqq(1+\epsilon)\frac{1}{2}\nu(n)$, where $\epsilon>0$. Then, by Theorem 2,

$$\begin{aligned}2\alpha(n)-\nu(n)&=\sum_{k=1}^{r}\left(\frac{2\alpha_k}{\alpha_k+1}-1\right)=\sum_{k=1}^{r}\frac{\alpha_k-1}{\alpha_k+1}\\&=\frac{1}{\tau(n)}\sum_{p^2|n}\tau\left(\frac{n}{p^2}\right)\geqq\epsilon\nu(n).\end{aligned}$$

Thus

$$\sum_{p^2|n} \tau\left(\frac{n}{p^2}\right) \geqq \epsilon\nu(n)\tau(n),$$

$$\tau(n) \sum_{p^2|n} 1 \geqq \sum_{p^2|n} \left(\frac{n}{p^2}\right) \geqq \epsilon\nu(n)\tau(n), \qquad i.e., \sum_{p^2|n} 1 \geqq \epsilon\nu(n).$$

But $\Omega(n)-\nu(n)\geqq \sum_{p^2|n} 1$, so that $\Omega(n)\geqq(1+\epsilon)\nu(n)$, where $\Omega(n)=\sum_{k=1}^{r}\alpha_k$ is the total number of prime divisors of n.

It is known that $\Omega(n)$ and $\nu(n)$ both have normal order log log n and it follows that the number of $n\leqq x$ for which $\Omega(n)\geqq(1+\epsilon)\nu(n)$ is $O(x)$. Also, $\alpha(n) > (1-\epsilon)\frac{1}{2}\nu(n)$ for all n. Hence,

$$(1-\epsilon)\tfrac{1}{2}\nu(n) < \alpha(n) < (1+\epsilon)\tfrac{1}{2}\nu(n)$$

for almost all n and this proves the theorem.

Reference

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SOME REMARKS ABOUT THE CURL OF A VECTOR FIELD

J. D. WESTON, University of Durham, England

If one defines the curl of a differentiable vector field in the usual manner of elementary vector analysis (in terms of partial derivatives with respect to Cartesian coordinates), it is necessary to show that the defining expression is invariant under right-handed orthogonal transformations. It is not difficult to do this, without using Stoke's integration theorem or the machinery of tensor calculus; but if one starts in a straightforward way, the ensuing manipulations are apt to seem tediously complicated. The desire for a simple demonstration led to the following considerations, in which the notion of curl is generalized in an elementary way.

Let $\mathbf{R}^n$ be Euclidean n-space, consisting of all sequences of n real numbers, with the usual vector operations and with inner multiplication defined by

$$\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{n} x_i y_i,$$

where $\mathbf{x}=(x_1, \cdots, x_n)$ and $\mathbf{y}=(y_1, \cdots, y_n)$. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be the fundamental unit vectors, so that $x_i=\mathbf{x}\cdot\mathbf{e}_i$ and $\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}$. Suppose that $\mathbf{R}^n$ is made into an algebra by introducing a binary operation $\wedge$ which is linear but not necessarily associative (this can be done by setting up a "multiplication table" specifying the n^2 vectors $\mathbf{e}_i\wedge\mathbf{e}_j$).

A differentiable vector field, defined on an open set G in $\mathbf{R}^n$, is a function $\mathbf{v}$ which maps G into $\mathbf{R}^n$ and has continuous partial derivatives $\partial\mathbf{v}/\partial x_i$. Given

such a function, we can define a vector field $\nabla \wedge \mathbf{v}$ by the formula

$$\nabla \wedge \mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i \wedge \frac{\partial \mathbf{v}}{\partial x_i} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_i} \mathbf{e}_i \wedge \mathbf{e}_j, \tag{1}$$

where $v_j = \mathbf{v} \cdot \mathbf{e}_j$ (so that $\mathbf{v} = \sum_{j=1}^{n} v_j \mathbf{e}_j$). Now an orthogonal transformation is characterized by the vectors, say $\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n$, to which $\mathbf{e}_1, \cdots, \mathbf{e}_n$ are respectively transformed; these are subject only to the conditions $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$, and if we write $\hat{x}_j = \mathbf{x} \cdot \hat{\mathbf{e}}_j$ we can define the transformation by the identity

$$\sum_{i=1}^{n} x_i \mathbf{e}_i \equiv \sum_{j=1}^{n} \hat{x}_j \hat{\mathbf{e}}_j. \tag{2}$$

Thus

$$\frac{\partial \hat{x}_j}{\partial x_i} = \hat{\mathbf{e}}_j \cdot \mathbf{e}_i,$$

so that, by the chain rule for partial derivatives, and the linearity of $\wedge$,

$$\nabla \wedge \mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i \wedge \sum_{j=1}^{n} \hat{\mathbf{e}}_j \cdot \mathbf{e}_i \frac{\partial \mathbf{v}}{\partial \hat{x}_j} = \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} \hat{\mathbf{e}}_j \cdot \mathbf{e}_i \mathbf{e}_i \right\} \wedge \frac{\partial \mathbf{v}}{\partial \hat{x}_j} \cdot$$

But $\sum_{i=1}^{n} \hat{\mathbf{e}}_j \cdot \mathbf{e}_i \mathbf{e}_i = \hat{\mathbf{e}}_j$, so that

$$\nabla \wedge \mathbf{v} = \sum_{j=1}^{n} \hat{\mathbf{e}}_j \wedge \frac{\partial \mathbf{v}}{\partial \hat{x}_j} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial \hat{v}_k}{\partial \hat{x}_j} \hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k, \tag{3}$$

where $\hat{v}_k = \mathbf{v} \cdot \hat{\mathbf{e}}_k$. Comparing (1) and (3), we see that the vector field $\nabla \wedge \mathbf{v}$ is an orthogonal invariant of $\mathbf{v}$. Let us say that $\mathbf{v}$ is "irrotational," in a given domain, if $\nabla \wedge \mathbf{v} = \mathbf{0}$ throughout the domain.

When $n=1$ and $\wedge$ means ordinary multiplication of real numbers, $\nabla \wedge \mathbf{v}$ is the ordinary derivative of $\mathbf{v}$ and the irrotational fields are the constants. When $n=2$, $\mathbf{R}^n$ consists of the complex numbers, and we can take $\wedge$ to mean the usual multiplication of these ($\mathbf{e}_1 \wedge \mathbf{e}_1 = \mathbf{e}_1$, $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_2 \wedge \mathbf{e}_1 = \mathbf{e}_2$, $\mathbf{e}_2 \wedge \mathbf{e}_2 = -\mathbf{e}_1$); then we find from (1) that

$$\nabla \wedge \mathbf{v} = \left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} \right) \mathbf{e}_1 + \left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_2.$$

In this case, therefore, $\mathbf{v}$ is irrotational if and only if it satisfies the Cauchy-Riemann equations,

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} \quad \text{and} \quad \frac{\partial v_2}{\partial x_1} = -\frac{\partial v_1}{\partial x_2} \cdot$$

Thus the irrotational 2-vector fields are precisely the holomorphic functions.

When $n=3$ and $\wedge$ means the usual vector multiplication (defined by the

skewness property $\mathbf{x}\wedge\mathbf{y}=-\mathbf{y}\wedge\mathbf{x}$ together with the cyclic rule $\mathbf{e}_1\wedge\mathbf{e}_2=\mathbf{e}_3$ etc.), we see from (1) that

$$\nabla\wedge\mathbf{v}=\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1+\cdots=\operatorname{curl}\mathbf{v}.$$

But, since $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j=\delta_{ij}$, we have $\hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_2=\epsilon\hat{\mathbf{e}}_3$ etc., where $\epsilon=\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_3=\pm 1$; hence, by (3),

$$\epsilon\nabla\wedge\mathbf{v}=\left(\frac{\partial\hat{v}_3}{\partial\hat{x}_2}-\frac{\partial\hat{v}_2}{\partial\hat{x}_3}\right)\hat{\mathbf{e}}_1+\cdots.$$

Now the transformation defined by (2) is right-handed if and only if $\epsilon=1$; thus we have established the invariance of the Cartesian formula for the curl under right-handed orthogonal transformations.

For a general value of n there are, of course, many possible meanings for $\wedge$, and hence for $\nabla\wedge\mathbf{v}$. Skewness of $\wedge$ will ensure that gradients (of scalar fields having continuous second-order derivatives) are irrotational; for,

$$\nabla\wedge\operatorname{grad}\phi=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\phi}{\partial x_i\partial x_j}\mathbf{e}_i\wedge\mathbf{e}_j=\mathbf{0}$$

if $\mathbf{e}_i\wedge\mathbf{e}_j=-\mathbf{e}_j\wedge\mathbf{e}_i$, since $\partial^2\phi/\partial x_i\partial x_j=\partial^2\phi/\partial x_j\partial x_i$. Similarly, the divergence of $\nabla\wedge\mathbf{v}$ will be zero if the multiplication table has the property $\mathbf{e}_i\wedge\mathbf{e}_j\cdot\mathbf{e}_k=-\mathbf{e}_k\wedge\mathbf{e}_j\cdot\mathbf{e}_i$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

ON THE CONVERGENCE OF THE SERIES $\sum n^{-\alpha}$

H. SCHWERDTFEGER, McGill University

1. Here is a simple proof of the convergence of $\sum n^{-\alpha}$ for $\alpha>1$. The idea originates from a recent note by V. P. Palamodov [1] where he begins with putting

$$1^{-\alpha}+2^{-\alpha}+3^{-\alpha}+\cdots=A,$$

admitting A to be "finite or infinite!—that we don't know yet." He then observes that

$$2^{-\alpha} + 4^{-\alpha} + 6^{-\alpha} + \cdots = 2^{-\alpha}A$$
$$3^{-\alpha} + 5^{-\alpha} + 7^{-\alpha} + \cdots \leqq 2^{-\alpha}A,$$

and therefore

$$\begin{aligned} A &= 1 + (2^{-\alpha} + 4^{-\alpha} + \cdots) + (3^{-\alpha} + 5^{-\alpha} + \cdots) \\ &\leqq 1 + 2^{-\alpha}A + 2^{-\alpha}A \quad \text{whence} \quad A \leqq 2^{\alpha-1}(2^{\alpha-1} - 1)^{-1}. \end{aligned}$$

At the cost of a few more lines this proof is readily given in a more conventional form avoiding inequalities involving the symbol A of which it is to be shown that it represents a positive number, and which *a priori* could also be ∞.

Let

$$A_n = 1 + 2^{-\alpha} + 3^{-\alpha} + \cdots + n^{-\alpha} \qquad (n = 2, 3, \cdots).$$

Then for all $n \geqq 2$

$$2^{-\alpha} + 4^{-\alpha} + \cdots + (2n-2)^{-\alpha} + (2n)^{-\alpha} = 2^{-\alpha}A_n$$
$$3^{-\alpha} + 5^{-\alpha} + \cdots + (2n-1)^{-\alpha} \qquad\qquad < 2^{-\alpha}A_n.$$

Consequently

$$A_{2n} < 1 + 2^{-\alpha}A_n + 2^{-\alpha}A_n = 1 + 2^{1-\alpha}A_n \qquad (2^{1-\alpha} < 1).$$

In particular for $m = 2, 3, \cdots$

$$\begin{aligned} A_{2^m} &< 1 + 2^{1-\alpha}A_{2^{m-1}} < 1 + 2^{1-\alpha} + (2^{1-\alpha})^2 A_{2^{m-2}} < \cdots \\ &< 1 + 2^{1-\alpha} + (2^{1-\alpha})^2 + \cdots + (2^{1-\alpha})^m < 2^{\alpha-1}(2^{\alpha-1} - 1)^{-1}. \end{aligned}$$

Thus we have a bound, independent of m for all A_{2^m}. In the usual way we conclude that this is also a bound for all A_n. Hence $\lim_{n\to\infty} A_n$ exists. The bound is the same as that obtained by Palamodov for his sum value A.

In this form the proof is, in its idea, similar to the one based on Cauchy's convergence theorem (cf. [2], pp. 118–119).

In a similar way one can decide about divergence of $\sum n^{-\alpha}$ if $0 < \alpha < 1$. Indeed, still

$$2^{-\alpha} + 4^{-\alpha} + \cdots + (2n)^{-\alpha} \qquad = 2^{-\alpha}A_n$$

and therefore

$$1^{-\alpha} + 3^{-\alpha} + \cdots + (2n-1)^{-\alpha} > 2^{-\alpha}A_n.$$

Thus

$$A_{2n} > 2^{1-\alpha}A_n \qquad (2^{1-\alpha} > 1),$$

and in particular

$$A_{2^m} > 2^{1-\alpha}A_{2^{m-1}} > (2^{1-\alpha})^2 A_{2^{m-2}} > \cdots > (2^{1-\alpha})^m,$$

which proves the divergence. The case $\alpha=1$ escapes, whereas it is covered by the test in Cauchy's theorem.

2. The same principle can be applied in more general cases. We prove the following

THEOREM. *Let $A=a_1+a_2+\cdots$ be an infinite series with monotonically decreasing positive terms, p a natural number >1, and γ a positive real number such that $p\gamma<1$. Also let $A_n=a_1+\cdots+a_n$. The series A is convergent if for all $n=1, 2, \cdots$ the sum*

$$(*)\qquad a_p+a_{2p}+\cdots+a_{(n-1)p}+a_{np}\leqq \gamma A_n.$$

Moreover then $A\leqq A_{p-1}(1-p\gamma)^{-1}$.

Proof. Because by supposition $a_p>a_{p+1}>\cdots$ we conclude from (*) that

$$\begin{array}{ll} a_{p+1}+a_{2p+1}+\cdots+a_{(n-1)p+1} & <\gamma A_n \\ a_{p+2}+a_{2p+2}+\cdots+a_{(n-1)p+2} & <\gamma A_n \\ \cdots\cdots\cdots\cdots\cdots & \\ a_{2p-1}+a_{3p-1}+\cdots+a_{np-1} & <\gamma A_n \end{array}$$

and therefore

$$A_{np}<A_{p-1}+p\gamma A_n.$$

In particular

$$\begin{aligned} A_{p^m}&=A_{p-1}+p\gamma A_{p^{m-1}}<A_{p-1}+p\gamma A_{p-1}+(p\gamma)^2A_{p^{m-2}}<\cdots \\ &<A_{p-1}(1+p\gamma+(p\gamma)^2+\cdots+(p\gamma)^{m-1})+(p\gamma)^mA_1. \end{aligned}$$

Thus if $p\gamma<1$ we have for all m

$$A_{p^m}<A_{p-1}(1-p\gamma)^{-1},$$

that is, a bound independent of m. This is also an upper bound for all A_n.

Conversely if there is for a fixed $p>1$ a positive δ such that for all n

$$a_p+a_{2p}+\cdots+a_{np}>\delta A_n$$

for all positive integers n, then it is readily shown that

$$A_{np}>p\delta A_n$$

and we conclude that the series $\sum a_n$ is divergent if $p\delta>1$.

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AN ABSTRACTION OF A COMBINATORIAL CONCEPT

A. A. MULLIN, University of Illinois

This note is concerned with giving a set-theoretical abstraction of the usual definition of the concept of combinations that appear in combinatorial analysis [1] and to show an anomaly that occurs in the transfinite case when compared to the finite case. We adopt the notation of Cantor [2] in representing cardinal numbers (*i.e.*, the double-bar notation).

DEFINITION. *Let A and B be sets with $B \subseteq A$. Let ϕ be a mapping. The cardinal-combinations of $\overline{\overline{A}}$ taken $\overline{\overline{B}}$ at a time, denoted $(\overline{\overline{A}}/\overline{\overline{B}})$, is $\overline{\overline{\Lambda}}$, where $\Lambda = \{\phi : \phi(C) = D$, for some C for which $\overline{\overline{A}} = \overline{\overline{C}}$ and for all D for which $\overline{\overline{B}} = \overline{\overline{D}}\}$.*

From an interpretation of the binomial theorem it is trivial to show that for every A for which $\overline{\overline{A}}$ is finite,

$$(\overline{\overline{A}}/\overline{\overline{B}}) < 2^{\overline{\overline{A}}}, \qquad \text{for all } B \subseteq A.$$

THEOREM. *For every A for which $\overline{\overline{A}} \geqq \aleph_0$, $\exists B \subseteq A$ such that $(\overline{\overline{A}}/\overline{\overline{B}}) = 2^{\overline{\overline{A}}}$.*

Proof. Put $\alpha = \{C : C \subseteq A\}$ and put $\beta = \sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}})$.† It can be shown [3] that $\overline{\overline{\alpha}} = 2^{\overline{\overline{A}}}$. But by the algebra of cardinal numbers [4] and our definition, $\overline{\overline{\alpha}} = \sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}})$. No term of β can be greater than $2^{\overline{\overline{A}}}$, for otherwise, by the algebra of cardinal numbers $\sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}}) > 2^{\overline{\overline{A}}}$. Not all terms of β can be strictly less than $2^{\overline{\overline{A}}}$, for otherwise, by the generalized continuum hypothesis and the algebra of cardinal numbers $\sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}}) \leqq \overline{\overline{A}} < 2^{\overline{\overline{A}}}$.

It is well known [5] that there exists a relation R for which every collection of cardinal numbers can be well-ordered with respect to R and, *a fortiori*, every pair of cardinal numbers is comparable. (In this respect, Kelley's last sentence in the first paragraph on page 28 seems to have a contradiction in it, as per his theorem 162 of the appendix, although this example is probably the " . . . single nontrivial exception . . . " that he refers to in the footnote on his page 1.)

Hence by comparability and the law of excluded-middle the existence of the required B is established.

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† At all its appearances this summation is for distinct $\overline{\overline{B}}$'s.

SEQUENCES GENERATED BY USE OF THE MEAN VALUE THEOREM

JACQUELINE P. EVANS,* Wellesley College

Classroom presentations of applications of the mean value theorem often include obtaining first approximations to given functional values $f(a)$. It is interesting to note that under certain conditions the mean value theorem may yield a good deal more. The theorem can be used to generate a sequence $\{f(b_n)\}$ which actually converges to $f(a)$. The result is stated in the following theorem.

THEOREM. *Let f' be continuous and decreasing in the closed interval $[a, b]$ and assume that $f'(b)>0$. Then the numbers $b_n=f^{-1}[f(b_{n-1})-f'(b_{n-1})(b_{n-1}-a)]$ exist, with $a<b_n<b_{n-1}$, for $n=1, 2, \cdots$, and with $b_0=b$. The sequence $\{f(b_n)\}$ decreases to $f(a)$.*

The first part of the theorem is established by induction. By the mean value theorem applied to f in $[a, b]$, there exists an X, $a<X<b$, such that $f(a)=f(b)-f'(X)(b-a)$. Then, since f' is decreasing and $f'(b)$ is positive, $f(a)<f(b)-f'(b)(b-a)<f(b)$. It follows from the hypothesis of the theorem that f is increasing and continuous in $[a, b]$. Hence, f^{-1} exists, is increasing and continuous in $[f(a), f(b)]$, and thus $b_1=f^{-1}[f(b)-f'(b)(b-a)]$ exists, with $a<b_1<b$. Now assume b_n exists with $a<b_n<b_{n-1}$. As before, application of the mean value theorem to f in $[a, b_n]$ yields $f(a)<f(b_n)-f'(b_n)(b_n-a)<f(b_n)$, so the number $b_{n+1}=f^{-1}[f(b_n)-f'(b_n)(b_n-a)]$ exists with $a<b_{n+1}<b_n$ and our induction is complete.

The sequence $\{b_n\}$ is decreasing and bounded below by a, so $\lim_{n\to\infty} b_n$ exists. Denoting this limit by x_0, we have $a \leqq x_0 < b$. Since $f(b_n) = f(b_{n-1})-f'(b_{n-1})(b_{n-1}-a)$, it follows from the continuity of f and f' at x_0 that $f(x_0)=f(x_0)-f'(x_0)(x_0-a)$. Thus, $x_0=a$ and $\lim_{n\to\infty} f(b_n)=f(a)$. The sequence $\{f(b_n)\}$ is decreasing because $\{b_n\}$ is.

The problem of approximating roots affords a simple application of the theorem. For example, to approximate $\sqrt{3}$ we take $f(x)=\sqrt{x}$, $a=3$, $b=4$. Since the values of f^{-1} are easily determined and $f'(b_n)$ can be expressed in terms of $f(b_n)$, the successive values of $f(b_n)=\sqrt{b_n}$ are readily found: $\sqrt{b_1}=\sqrt{b}-(b-a)/2\sqrt{b}=7/4=1.75$, $\sqrt{b_2}=\sqrt{b_1}-(b_1-a)/2\sqrt{b_1}=97/56=1.73214 \cdots$ and so forth.

If f' is continuous and increasing in $[a, b]$ with $f'(a)>0$, the mean value theorem can be used as above to generate a sequence $\{f(a_n)\}$ increasing to $f(b)$; here $a_n=f^{-1}[f(a_{n-1})+f'(a_{n-1})(b-a_{n-1})]$. In the case where f is decreasing in $[a, b]$ we need merely replace $f(x)$ by $-f(x)$.

* The author wishes to express her appreciation of suggestions made by the referee.

IRREDUCIBILITY OF POLYNOMIALS

J. H. WAHAB, Louisiana State University in New Orleans

Van der Waerden [1] states and proves a criterion for reducibility of polynomials which is attributed to G. Dumas [2]. It is the purpose of this note to call attention to the stronger and more useful formulation of this result as stated and proved in [3]. It will be assumed that $f(x)$, $g(x)$ and $h(x)$ are polynomials in x with integral coefficients, p is a prime, and a_i is an integer relatively prime to p. Let

$$f(x) = \sum_{i=0}^{n} a_i p^{b_i} x^i$$

and let the points (i, b_i) be plotted in the usual way. From this set of points a subset $\{P_j\}_{j=0}^{r}$ is generated by choosing P_0 as the point $(0, b_0)$ and P_j $(j=1, \cdots, r)$ as (k_j, b_{k_j}), where k_j is the greatest integer such that no point (i, b_i) lies below the line through P_{j-1} and P_j. P_r, of course, is (n, b_n). The figure composed of the line segments $P_{j-1}P_j$ for $j=1, \cdots, r$ is called the Newton polygon for $f(x)$ corresponding to the prime p. If for example $f(x)=27+6x^3+4x^4+6x^5+12x^6$, the Newton polygons corresponding to 3 and 2 appear in Figure 1.

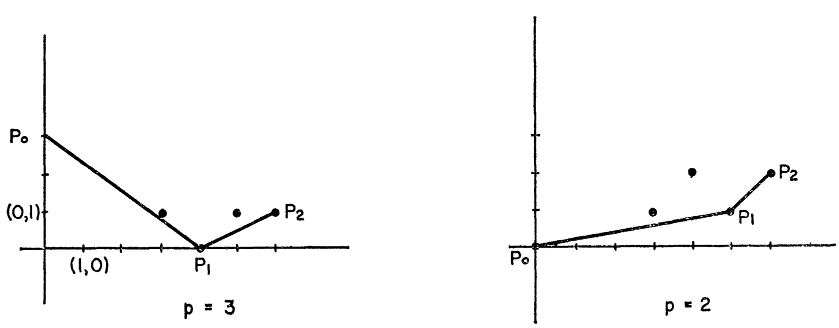

FIG. 1

THEOREM OF DUMAS. *Let the segments of the Newton polygon for $f(x)$ corresponding to p, be subdivided by the lattice points occurring on them and let the resulting segments connecting adjacent points of division be called the elements of the polygon. If $f(x)=g(x)h(x)$, then the Newton polygon for $g(x)$ corresponding to p can be formed by joining some of the elements of the polygon for $f(x)$ without changing their lengths or slopes. Moreover, the Newton polygon for $h(x)$ corresponding to p can be formed in the same manner by precisely those elements not used for the polygon of $g(x)$.*

With this theorem it is easy to show that the Newton polygons for the factors of $27+6x^3+4x^4+6x^5+12x^6$ must appear as in Figure 2.

From the polygons for $p=3$ it is necessary that one factor be of degree 2 and the other of degree 4, but for $p=2$ it is indicated that one degree must be 5 and

the other 1. Consequently, $27+6x^3+4x^4+6x^5+12x^6$ is irreducible over the field of rational numbers, whereas van der Waerden's statement of the theorem does not rule out a factorization with polynomials of degrees 5 and 1.

Eisenstein's criterion is an immediate corollary of this theorem and, to the best of the author's knowledge, so also are all of the criteria concerning divisibility of the coefficients.

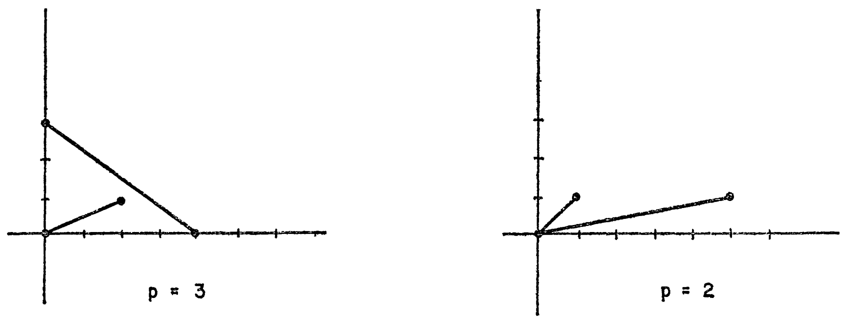

FIG. 2

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A MATRIX APPLICATION OF NEWTON'S IDENTITIES

D. W. ROBINSON, Brigham Young University

It is well known that the characteristic polynomial of a square matrix can be obtained by computing the traces of the powers of the matrix. Conversely, the trace of any power of a matrix can be obtained from the coefficients of the characteristic polynomial of the matrix. Although these computational techniques follow as an immediate application of Newton's identities on symmetric polynomials, the author is not aware of any mention in the matrix algebra literature of a simple proposition that combines these results. The purpose of this note is to formulate such a theorem for use in the classroom.

We begin with a review of some elementary facts about symmetric polynomials. First, it is well known that any symmetric polynomial in the elements $\alpha_1, \cdots, \alpha_n$ is expressible as a polynomial in the elementary symmetric polynomials $p_1, \cdots, p_n$ in $\alpha_1, \cdots, \alpha_n$. In particular, this applies to the sums of like powers: $s_k=\alpha_1^k+\cdots+\alpha_n^k$. A specific algorithm for successively determining $s_1, s_2, \cdots$ in terms of $p_1, \cdots, p_n$ is provided by Newton's identities:

$$s_1 - p_1 = 0, \qquad s_2 - p_1 s_1 + 2p_2 = 0, \cdots,$$
$$s_k - p_1 s_{k-1} + \cdots + (-1)^{k-1} p_{k-1} s_1 + (-1)^k k p_k = 0,$$

where $p_k = 0$ for $k > n$. Furthermore, it is clear that these identities also provide a means of determining $p_1, 2p_2, \cdots, n!p_n$ in terms of $s_1, \cdots, s_n$.

As a consequence of the preceding paragraph, we have the following:

LEMMA. *Let $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_m$ be elements of a field of characteristic zero or prime $p > n \geqq m$. Let p_k and $s_k (q_k$ and $t_k)$ denote, respectively, the elementary symmetric polynomials and the sums of like powers of $\alpha_1, \cdots, \alpha_n (\beta_1, \cdots, \beta_m)$. Then $p_k = q_k$, $k = 1, \cdots, m$ and $p_k = 0$, $k = m+1, \cdots, n$ if and only if $s_k = t_k$, $k = 1, \cdots, n$.*

As a direct application of this lemma, we now have the following:

THEOREM. *Let A be an n-by-n matrix and B an m-by-m matrix over a field of characteristic zero or prime $p > n \geqq m$. Then the characteristic polynomial of A is x^{n-m} times the characteristic polynomial of B if and only if* tr $A^k =$ tr B^k for $k = 1, \cdots, n$.

Proof. Using the notation of the preceding lemma, let

$$(x - \alpha_1) \cdots (x - \alpha_n) = x^n - p_1 x^{n-1} + \cdots + (-1)^n p_n,$$
$$(x - \beta_1) \cdots (x - \beta_m) = x^m - q_1 x^{m-1} + \cdots + (-1)^m q_m,$$

be the characteristic polynomials of A and B, respectively, over some extension of the given field. Since the trace of any matrix is equal to the sum of the characteristic values of the matrix, tr $A^k = \alpha_1^k + \cdots + \alpha_n^k = s_k$ and tr $B^k = \beta_1^k + \cdots + \beta_m^k = t_k$. The theorem now follows immediately from the lemma above.

It is of interest to note that for the proof of sufficiency in the theorem the restriction on the characteristic of the field is indeed required. This proposition is demonstrated by taking A to be the direct sum of the p-by-p identity matrix and the $(n-p)$-by-$(n-p)$ zero matrix and B to be the m-by-m zero matrix over a field of characteristic prime $p \leqq n$.

Finally, two corollaries are given. The first gives the known trace criterion for nilpotence of a matrix. (See, for example, [1], p. 10.) The second is very well known over fields of any characteristic, but the simple proof given here is valid only under the restriction of the theorem above. (Compare for example [2], p. 106.)

COROLLARY 1. *Let A be an n-by-n matrix over a field of characteristic zero or prime $p > n$. Then A is nilpotent if and only if* tr $A^k = 0$, $k = 1, \cdots, n$.

Proof. This result follows directly from the theorem by taking $B = 0$ and using the fact that A is nilpotent if and only if its characteristic polynomial is x^n.

COROLLARY 2. *Let A be an n-by-m matrix and B an m-by-n matrix over a field*

of characteristic zero or prime $p>n\geqq m$. Then the characteristic polynomial of AB is x^{n-m} times the characteristic polynomial of BA.

Proof. Since the trace of a product of two matrices is the trace of the product in reverse order, $\operatorname{tr} AB=\operatorname{tr} BA$ and $\operatorname{tr}(AB)^k=\operatorname{tr}((AB)^{k-1}A)B=\operatorname{tr} B((AB)^{k-1}A)$ $=\operatorname{tr}(BA)^k$ for $k>1$. The conclusion is now a consequence of the theorem above.

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A DERIVATION OF THE GENERAL SOLUTION FOR HOMOGENEOUS, LINEAR, DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

DAVID ZEITLIN, Remington Rand Univac, St. Paul, Minnesota

In obtaining the general solution for homogeneous, linear, difference equations, with constant coefficients, one discusses two cases for the characteristic equation, *i.e.*, (*i*) distinct roots and (*ii*) repeated roots, real or complex. The proofs of (*i*) and (*ii*) use techniques analogous to those used in obtaining the general solution for homogeneous, linear, differential equations with constant coefficients. A typical proof of (*ii*) usually considers only a double root, while roots with higher multiplicities are not treated effectively, since the general result is often stated without proof. In this note, we will establish a theorem which treats cases (*i*) and (*ii*) *simultaneously*, and which reveals a relationship, not generally known, between the solutions of linear difference and linear differential equations.

THEOREM. *Let*

$$y(x)=\sum_{i=1}^{m}\left\{\sum_{j=0}^{n_i-1}C_{i(j+1)}x^j\right\}e^{r_ix}+\sum_{j=(n_1+\cdots+n_m)+1}^{n}C_je^{r_jx} \tag{1}$$

be the general solution of the following homogeneous, linear, differential equation of nth order:

$$D^ny(x)+a_1D^{n-1}y(x)+a_2D^{n-2}y(x)+\cdots+a_ny(x)=0, \qquad a_n\neq 0, \tag{2}$$

where $C_{i(j+1)}$ and C_j are arbitrary constants, a_i, $i=1,\cdots,n$, are real constants, $n_i\geqq 1$, $i=1,\cdots,m$, with $(n_1+\cdots+n_m)\leqq n$, and where the characteristic equation, $r^n+a_1r^{n-1}+\cdots+a_n=0$, has the roots, r_i, with multiplicity, n_i, $i=1,\cdots,m$, and the simple roots, r_j.

Let u_k be the general solution of the following homogeneous, linear, difference equation of nth order:

$$u_{n+k}+a_1u_{n+k-1}+a_2u_{n+k-2}+\cdots+a_nu_k=0. \tag{3}$$

Then $u_k=D^ky(0)$ and

$$u_k = \sum_{i=1}^{m} \left\{ C_{i1} + \sum_{\nu=1}^{n_i-1} \gamma_{i\nu} k^\nu \right\} r_i^k + \sum_{j=(n_1+\cdots+n_m)+1}^{n} C_j r_j^k, \tag{4}$$

where the $\gamma_{i\nu}$ are arbitrary constants.

Proof. If we differentiate (2) k times with respect to x and then set $x=0$, we obtain (3), where $u_k = D^k y(0)$. Let $A_k(x) = D^k[\sum_{j=0}^{n_i-1} C_{i(j+1)} x^j e^{r_i x}]$. Using Leibnitz's rule, we obtain

$$\begin{aligned} A_k(x) &= C_{i1} r_i^k e^{r_i x} + \sum_{j=1}^{n_i-1} C_{i(j+1)} \left\{ \sum_{p=0}^{k} \binom{k}{p} r_i^{k-p} e^{r_i x} D^p x^j \right\} \\ &= C_{i1} r_i^k e^{r_i x} + \sum_{j=1}^{n_i-1} C_{i(j+1)} \sum_{p=0}^{j} \binom{k}{p} r_i^{k-p} e^{r_i x} j^{(p)} x^{j-p}, \end{aligned} \tag{5}$$

where $k^{(j)} = k(k-1)\cdots(k-j+1) = \sum_{\nu=1}^{j} S_\nu^j k^\nu$, ([1], p. 142), and S_ν^j is a Stirling number of the first kind. Setting $x=0$ in (5), we obtain

$$\begin{aligned} A_k(0) &= C_{i1} r_i^k + \sum_{j=1}^{n_i-1} C_{i(j+1)} \binom{k}{j} r_i^{k-j} (j!) = C_{i1} r_i^k + \sum_{j=1}^{n_i-1} C_{i(j+1)} r_i^{k-j} k^{(j)} \\ &= C_{i1} r_i^k + \sum_{j=1}^{n_i-1} C_{i(j+1)} r_i^{k-j} \sum_{\nu=1}^{j} S_\nu^j k^\nu = C_{i1} r_i^k + \sum_{\nu=1}^{n_i-1} \left(\sum_{j=\nu}^{n_i-1} C_{i(j+1)} r_i^{k-j} S_\nu^j \right) k^\nu . \end{aligned}$$

For each i, $i=1, \cdots, m$, set $C_{i(j+1)} = \beta_{ij} r_i^j$, $j=1, \cdots, n_i-1$, and $\gamma_{i\nu} = \sum_{j=\nu}^{n_i-1} \beta_{ij} S_\nu^j$, $\nu = 1, \cdots, n_i - 1$. Thus

$$A_k(0) = C_{i1} r_i^k + \sum_{\nu=1}^{n_i-1} \left(\sum_{j=\nu}^{n_i-1} \beta_{ij} S_\nu^j \right) r_i^k k^\nu = \left[C_{i1} + \sum_{\nu=1}^{n_i-1} \gamma_{i\nu} k^\nu \right] r_i^k .$$

Remarks. The example, $y''(x) = y'(x) + y(x)$, is interesting, since the associated difference equation, $u_{k+2} = u_{k+1} + u_k$, generates a sequence of Fibonacci numbers. The choice of $x=0$ in the theorem is convenient, since it minimizes the number of notational changes for the arbitrary constants.

If the solution of (2), given by (1), is written as a power series, $y(x) = \sum_{k=0}^{\infty} \sigma_k x^k$, where $\sigma_k = D^k y(0)/k!$, then $u_k = k!\sigma_k$. Thus (3) yields, after substituting for u_k and then dividing through by $k!$, the recursion formula satisfied by the series coefficients, σ_k.

It should be noted that the theorem can be extended to nonhomogeneous, linear, difference equations of nth order. For example, if the right-hand side of (3) is given by $\sum_{i=1}^{N} \mu_i (\eta_i)^k$, where μ_i and η_i are specified, real constants, and if the right-hand side of (2) is given by $\sum_{i=1}^{N} \mu_i e^{\eta_i x}$, then the formula $u_k = D^k y(0)$, is still valid, where $y(x)$, given by (1), now includes the particular integral for (2), as modified.

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SPOOF OF THE FUNDAMENTAL THEOREM OF CALCULUS

R. L. EISENMAN, U. S. Air Force Academy

A "spoof" is hereby defined as a short proof with much missing. This is not inconsistent with Webster's definition as "hoax."

The integral undoes differentiation since

$$\text{Derivative} = \text{Limit of } [\text{Quotient of Differences}]$$

$$\text{ANTIderivative} = \text{Limit of } [\text{Sum of Products}]$$

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

SUPPORT OF HIGHER EDUCATION BY THE FEDERAL GOVERNMENT*

MINA REES, Hunter College of the City of New York

In discussing support of higher education by the Federal Government we should recognize that although there is still much discussion of the pros and cons of such support, and although there has been no overall policy decision by the Congress or by the executive branch of the government or by any group of representative leaders of American higher education that such support should be given, it is true nonetheless that a substantial amount of support is currently provided to institutions of higher education by various agencies of the national government. This support is estimated as between 1.5 and 2.0 billion dollars per year. To quote from an article by Homer D. Babbidge, Jr., Assistant Commissioner and Director, Division of Higher Education, Office of Education:

> "A half-dozen different graduate fellowship programs are now in operation, each administered by a different Federal agency. Some 6,000 students are being financed through these programs in their graduate work, at an annual Federal cost of $35 million. Approximately 75 percent of these awards are in mathematics and the sciences, and 17 percent are in the humanities and social sciences, which account for slightly more than a fourth of the Ph.D.'s. The average Federal fellowship in the sciences pays the student $700 more per year than the fellowship in a nonscience field.
>
> "Five Federal agencies conduct major research programs, using U.S. universities as the principal resource. This year they will spend at least $750 million for such research, $450 million directly in universities and the remaining $300 million in research centers associated

* Keynote speech at a meeting on Federal Support of Higher Education called by the Conference Board of the Mathematical Sciences, November 12–13, 1960, Washington, D. C.

with universities. As a result, more than 70 percent of all research conducted by our universities is federally financed: 86 percent of all university research in the physical sciences and 25 percent in the social sciences. C. V. Kidd, whose studies of this area of Federal activity are highly illuminating, reports that 14 universities received 55 percent of Federal research funds in 1953–54 and that 10 universities received about one-third in 1959.

"A recent comparison of 10 large universities with 10 small colleges indicates that the universities received about 33 percent of their general educational income from Federal sources, while the small institutions received about 3 percent of their income from the Federal Government.

"Legislation pending before the Congress would extend Federal participation in higher education to include physical plant construction. Leaders of both political parties and both the executive and legislative branches of Government are agreed to such an extension, and disagree only on methods and amounts. During the last 3 years, $1 out of every $4 spent on higher education construction throughout the nation came from Federal sources—the college housing loan program.

"More than 135,000 young men and women are receiving Federal financial assistance amounting to more than $75 million this year under the National Defense Student Loan Program. Borrowing for the financing of colleges has become an established fact as a direct result of this legislation.*

"The teacher-training function of the nation's colleges and universities has been considerably expanded through the conduct of federally financed institutes for secondary school personnel in mathematics, science, modern foreign languages, and counseling and guidance. No Federal funds have been made available for such institutes in the humanities or social sciences."

In this article I shall discuss the nature and needs for Federal support as these relate to mathematics. It may be worthwhile to consider briefly certain aspects of the history of these questions. At the end of World War II the urgent need of the Government for new scientific results, sometimes in narrowly delimited fields, and the recognition that the universities of the United States were not planning and were financially unable to embark on a program of basic research in the sciences adequate to provide for these needs, led to the decision to initiate a widespread program of Federal support of university research in the sciences.

The first steps in this new program (and we should not forget already existing efforts in the universities like those of the Department of Agriculture) were taken by the Navy through the Office of Naval Research. With the passage of time many other agencies entered the field. The establishment of the National Science Foundation in 1950 gave an added impetus to this development.

Mathematics was early recognized as an essential element in any vigorous program for the development of leadership in the sciences, and this recognition included the purest of mathematical disciplines as well as applied mathematics and the development of computers. As more extensive programs of support have emerged the position of mathematics has been maintained, not however without the determined and selfless effort of many leaders in the profession. It is worth noting that the nature of the support needed for mathematicians is different from that needed for experimental scientists.

* Higher Education, vol. 17, 1960, p. 4.

There were two main reasons for federal contributions toward the support of research in the experimental sciences: (1) the need for expensive equipment which in many fields like nuclear physics grew more and more expensive with the passage of time; and (2) the need for results in particular areas of research and for basic advances across a broad front of scientific inquiry. The first of these reasons was not present in mathematics except in computer development, in which mathematicians participated with engineers, physicists and other scientists.

Though the criteria for the support of research programs that have been emphasized in the past are primarily the two I have listed, a third criterion, the need for trained personnel, was constantly in the picture. As we assess our situation in mathematics today, this need for trained personnel becomes the critical issue. To the extent that this personnel need is acute in other fields, like the social sciences and certain of the humanities, it may be that programs comparable to those suggested for mathematics should be considered.

In recent years the nature of the involvement of higher education in federal programs has changed. Federally supported research programs continue, but the extent to which research people in the arts and sciences are now committed to participation in course content improvement programs at the secondary and even at the elementary level has made new demands on the time of scholars. And in mathematics the Mathematical Association, through the work of the Committee on the Undergraduate Program, has emerged as a significant force in the modernization of collegiate mathematics courses as well as in the design of radical new plans for the training of teachers to handle all the new curricula. All this takes people. Federal government money for research programs tends to go to a relatively few of the large universities with the effect of strengthening these institutions and increasing their power to attract the ablest scholars. Too often they are attracted away from the smaller liberal arts colleges. Moreover, the teachers at the liberal arts colleges often lack the stimulation and vitality gained from actual participation in research. The National Science Foundation, through its education projects, has come to grips with the problem of providing better subject matter competence to teachers in liberal arts colleges and in secondary schools. One need only mention some of the programs of the Foundation—

> Science Faculty Fellowships, Summer Fellowships for Secondary School Teachers, Summer Institutes for High School Teachers of Science and Mathematics, Academic Year Institutes for High School Teachers of Science and Mathematics, In-Service Institutes for High School Teachers of Science and Mathematics, Summer Conferences for College Teachers, Visiting Scientists Programs for Colleges and for Secondary Schools

to recognize the thoroughness with which the Foundation has identified the needs, and the imagination with which it has embarked on programs to try to meet those needs.

For mathematics the outstanding need is for more trained mathematicians.

If we give our attention for a moment to the need for teachers in universities, colleges, and two-year colleges, we see how critical is our situation with regard to the production of Ph.D.'s. In 1953–54 34.2% of all new mathematics teachers in 4-year colleges had Ph.D.'s; in 1958–59 only 19.9% of such teachers had Ph.D.'s.* In 1930 about 15% of the Ph.D.'s in mathematics were in non-academic employment.† This percentage has been rising steadily, except immediately after World War II. Preliminary results indicate that in 1960, for the first time, a little over half of the approximately 300 new mathematics Ph.D.'s entered industry or government (final results are not available at this writing).

What is the prospect for providing the teachers to handle the crowds of new students? Except during the war the number of mathematics Ph.D.'s has constituted an almost constant percentage of all science and mathematics Ph.D.'s (a little under 5%) and also of all Ph.D.'s in all fields (around 2.8%).‡ Thus if the total number of Ph.D.'s in all fields does double within the next decade along with the doubling of the college population, as has been predicted, we may hope to produce around 600 Ph.D.'s in mathematics in 1970. But a little arithmetic will show a terrifying imbalance even if, as seems most unlikely, the drift to industry and government were stemmed.

The junior year enrollments of mathematics majors increased by more than 30% between the fall of 1957 and the fall of 1958, while the increase was only a little over 6% for the biological sciences and about 4% for the physical sciences. During this period total junior year enrollments in institutions of higher education increased by less than 4%.§ Thus enrollment of mathematics majors increased about eight times as fast as the total enrollment. If the enrollments in mathematics continue to climb at this rate while the production of Ph.D.'s merely keeps pace with the total enrollment, the picture seems black. The situation calls for the most imaginative combined efforts of the mathematical community if we are to produce more people qualified to carry on the work that relies on mathematicians: more research in mathematics, pure and applied; more use in applications, not only in the traditional fields like physics and engineering but also in new fields like economics, and in even newer fields like biology and medicine (and here I refer not only to the applications of statistics, but to the efforts to evolve mathematical models to provide insights that are only now suggesting themselves, into the unsolved problems of biological research). More teachers are needed for the increasing number of students preparing to enter these varied fields. This need for teachers is the most acute need of all.

Why is it that the yield of mathematical doctorates is so small a proportion of all doctorates in science and mathematics together? Why is it that so few

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* Statistical Handbook of Science Education, National Science Foundation, NSF 60-13, p. 82.

† A Survey of Research Potential and Training in the Mathematical Sciences, Part 1, The University of Chicago, March 15, 1957, p. 109.

‡ Statistical Handbook of Science Education, National Science Foundation, NSF 60-13, p. 79.

§ Statistical Handbook of Science Education, National Science Foundation, NSF 60-13, p. 69.

mathematics majors go on to the Ph.D.? Must a student be a genius to receive a Ph.D. in mathematics? Some of our students seriously think the answer to this question is "Yes." In physics, a B student at college can do a very good job in his Ph.D. research; but a B student in mathematics will rarely be accepted as a candidate for a doctorate in mathematics. We shall certainly need some of our B students as teachers, particularly in our two-year colleges if these continue to spring into being as they have been doing recently. In several states, the master plan for the development of higher educational facilities to take care of the doubling of collegiate enrollment within the next decade calls for the establishment of many new community and junior colleges within a few years.*

How shall we staff their mathematics faculties? What can the mathematical community do to attract more able young people into graduate work in mathematics? Are the graduate departments of mathematics getting their share of the mathematically gifted students?

In this connection there is one program of the National Science Foundation which seems to me threatening. This is, in itself, an excellent undertaking, but it was designed without the inclusion of an adequate provision for mathematics. It provides financial assistance to undergraduates who participate with faculty members in scientific research programs in liberal arts colleges. Because of the nature of mathematical research, there has been only a handful of proposals for support in genuinely mathematical projects. These were submitted to the Foundation with a mass of proposals in the experimental sciences. The result has been that many liberal arts colleges were given funds with which to pay honor students in the experimental sciences for their undergraduate research while no comparable awards are available for mathematics honor students. The possibility that this lure may prove sufficient to drain off some of the few promising prospective mathematicians into experimental sciences is worrisome. It seems important that mathematicians develop programs that the Foundation will be willing to consider as comparable to the experimental science research programs so that mathematics undergraduates will also receive financial assistance and have special opportunities, while they are still undergraduates, to taste the flavor of creative work in mathematics. The importance of the financial motivation should not be underestimated, particularly for some of our gifted but financially pressed young people. Paid summer work may, in some cases, determine the choice of a career.

How are we to take account of the vastly expanded need for new mathematicians, particularly new teachers of mathematics? The situation is so desperate that very serious consideration should be given to the proposal from the Committee on the Undergraduate Program to the Mathematical Association of America and the American Mathematical Society that a new doctoral degree be created. We should note that there are now some schools of education that

* See, for example, Meeting the Increasing Demand for Higher Education in New York State, A Report to the Governor and the Board of Regents, November, 1960, p. 30.

award the Ph.D. degree in mathematics on the basis of course work devoted half to mathematics and half to education courses, and followed by a thesis that is expository or historical. If we keep this fact in mind, it may not be quite so shocking to contemplate a doctor's degree based on course work probably more extensive and less intensive than the course work now used as the basis for a Ph.D. (but given at the same level as is now customary at the best graduate schools) followed by a thesis that is critical or expository. Such a degree could be expected to provide some of the teachers for liberal arts colleges and many of the mathematicians who enter industry. These men would be equipped with the kind of background they most need. Such a degree could be the basis for a steady push to prevent the widespread staffing of our institutions of higher education with faculties whose education in mathematics is meager. If such preparation were combined with constant efforts for continuing education like that provided in the National Science Foundation Summer Conference Program, we might look forward to liberal arts mathematics faculties that are mathematically mature and alert, in spite of the frightening pressure that is developing. Even in 1953–54, only about one-fourth of mathematics faculty members spent any time on research; so that the programs of the National Science Foundation designed to keep liberal arts faculty members in touch with new results seem to be our best hope for maintaining and improving the quality of mathematical education. The introduction of the proposed degree would have no significant effect on the Ph.D. awarded for original research. The real test of a creative mathematician is not, of course, his possession of a degree, but his contributions toward the advancement of mathematics; and there seems to be nothing in the proposed program that would threaten the development of genuine mathematical talent. Moreover, the trend toward a postdoctoral year or more for the really creative mathematician is so clearly established that it seems natural to look to this kind of experience to develop the gifted mathematical leaders of the next generation.

Our first and outstanding requirement seems to be that we make whatever adjustments are possible in our educational system to encourage the production of more persons equipped to perform the mathematical tasks that must be performed in the years immediately ahead. Where will we look for the additional people to perform these tasks? One immediate answer here, as it has been in many comparable situations, must be, "What of women?" We know, historically, that women have characteristically not been productive mathematicians. The few efforts I know of that have attempted to determine reasons for this fact have not produced results of great importance. But whether or not we understand the reason, there seems to be no question of the fact. In teaching, however, and in many phases of applied mathematics that are now claiming large numbers of our trained youth, women have performed most effectively. It would seem likely, that, with a degree such as that suggested by the Committee on the Undergraduate Program, and with serious efforts to persuade guidance counselors in the nation's high schools that able and interested girls

should be encouraged rather than discouraged from thinking of mathematics as a career, and, of course, with a serious effort to acquaint girls with the increasing trend of women back to the labor market after they have raised their children, we might hope to draw additional numbers of young women into careers in mathematics. Though my feminine loyalties would tend to urge that extensive and impressive improvement in the quality of the whole mathematical effort would occur if we could achieve this end, my honest assessment of the situation leads me to the conclusion that we would benefit primarily on the quantitative side. However, the effort seems to me critically important.

I have spoken only of the need for people with the mathematical sophistication and dedication necessary to carry on the emerging work of our world. Though this is certainly the outstanding demand that the times make upon us, there are clearly other programs that might move us forward in achieving better uses of the men we have. These I shall not attempt to delineate here for I want to emphasize my belief that our biggest problem is to produce more mathematicians at *all* levels. We should not lose a single qualified student because of race, sex, or the multitudes of accidents that determine the choice of an undergraduate college. We need all the imagination we can bring to bear to secure more nearly our share of able youth to carry on the research, the teaching, and the applications functions that this generation of mathematicians must handle. We must use every device in our power to insure sound and stimulating undergraduate opportunities for all the mathematically gifted youth in the land, and to attract these young people into the careers in mathematics for which their talents and interests equip them, and for which the needs of our society make so urgent an appeal.

The Advanced Studies Program at St. Paul's School

The Advanced Studies Program of St. Paul's School, Concord, New Hampshire, will for the first time this coming summer admit girls to its six weeks session. With the approval of principals, superintendents, and the State Department of Education, the Program supplements and enlarges upon New Hampshire public high school courses. For each of the past three summers the Program has enrolled more than one hundred academically talented students from New Hampshire public and parochial high schools. The Reverend Matthew M. Warren, Rector of St. Paul's School, in making the announcement of this change, pointed out that the Program has always been anxious to include girls as well as boys. He said that the limitations of financial aid to the Program and the fact that students have been able to pay an average of only half the cost, have prevented the Program from increasing its contribution to education in the State of New Hampshire.

Noble Memorial Lecture on the Teaching of Mathematics

The first Miss Charlie M. Noble Memorial Lecture on the Teaching of Mathematics was given, December 3, 1960, by Dr. H. J. Ettlinger, Professor of Mathematics at the University of Texas, at Texas Christian University. The lecture was sponsored by the Fort Worth Public Schools, Texas Christian University and the Texas Academy of Science. Miss Noble taught mathematics sixty-two years, most of them in Central High School and R. L. Paschal Senior High School of Fort Worth and at Texas Christian

University. She also for many years taught junior astronomy groups at the Children's Science Museum of Fort Worth. Professor Ettlinger has taught mathematics at the University of Texas for forty-eight years.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1461. *Proposed by Underwood Dudley, University of Michigan*

Solve the cryptic addition

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FIFTY
 FOUR
 FOUR
  TWO
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SIXTY
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remembering that FOUR$+12$ is a perfect square.

E 1462. *Proposed by Michael Skalsky, Southern Illinois University*

In how many ways can 4 white, 3 black, and 3 red balls be arranged in a row so that no two adjacent balls are of the same color?

E 1463. *Proposed by V. F. Ivanoff, San Carlos, California*

Given an imaginary point P: $(a+pi, b+qi, c+ri)$ in 3-space. Find the locus of real points (x, y, z) whose distance from P is real.

E 1464. *Proposed by Freddy Storey, Princeton University*

Show that for $n \geqq 2$, $\prod_{i=0}^{n} \binom{n}{i} \leqq \{(2^n-2)/(n-1)\}^{n-1}$.

E 1465. *Proposed by U. R. Kodres, IBM Corp., Poughkeepsie, New York*

It is well known that the graph which represents the classical problem of connecting three houses to three utilities is not planar. Prove that a generalization of this graph, namely the graph which represents connecting seven houses to seven utilities is not biplanar, *i.e.*, the graph cannot be factored into two planar factors.

SOLUTIONS

A Non-Linear Difference Equation

E 1431 [1960, 802]. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Advanced Development*

If $a_{n+1}=(1+a_na_{n-1})/a_{n-2}$ and $a_1=a_2=a_3=1$, show that a_n is an integer.

I. *Solution by J. L. Pietenpol, Columbia University.* Define a sequence $\{b_n\}$ of integers by

$$b_1 = b_2 = b_3 = 1, \quad b_4 = 2, \quad b_n = 4b_{n-2} - b_{n-4} \qquad (n > 4).$$

Then

$$\begin{aligned} b_{n+1}b_{n-2} - b_nb_{n-2} &= (4b_{n-1} - b_{n-3})b_{n-2} - (4b_{n-2} - b_{n-4})b_{n-1} \\ &= b_{n-1}b_{n-4} - b_{n-2}b_{n-3}, \end{aligned}$$

so that, by induction, $b_{n+1}b_{n-2}-b_nb_{n-1}=1$, or $b_{n+1}=(1+b_nb_{n-1})/b_{n-2}$, and hence $\{a_n\}=\{b_n\}$.

II. *Solution by H. E. Bray, Rice University.* The solution of the problem is implicit in the following

THEOREM. *If $a_{n+1}=(k+a_na_{n-1})/a_{n-2}$ and $a_1=a_2=1$, $a_3=p$, where k, p are positive integers such that $(k, p)=1$, a necessary and sufficient condition that a_n be an integer is that $k=rp-1$, where r is an integer.*

Sufficiency. Since $a_{n+1}a_{n-2}=k+a_na_{n-1}$ and $a_na_{n-3}=k+a_{n-1}a_{n-2}$, it follows that $(a_{n+1}+a_{n-1})/a_n=(a_{n-1}+a_{n-3})/a_{n-2}$, and this ratio is equal to $(a_3+a_1)/a_2$ $=p+1$ if n is even and to $(a_4+a_2)/a_3=(k+p+1)/p=r+1$ if n is odd and $k=rp-1$. Thus $(a_{n+1}+a_{n-1})/a_n$ is an integer, and by recurrence a_{n+1} is an integer if $a_1, a_2, \cdots, a_n$ are integers, whether n is even or odd.

Necessity. We have $a_5=k+p(k+p)$, $a_6=[k+\{k+p\}\{k+p(k+p)\}]/p$, and since a_6 is of the form $(k+k^2+mp)/p$, where m is an integer, it follows that p divides $k(k+1)$. But if $p=1$, k is of the form $rp-1$; and if $p>1$ then, since $(p, k)=1$, p must divide $k+1$. That is, $k=rp-1$. This completes the proof.

Also solved by W. E. Barnes and P. A. Clement (jointly), W. J. Blundon, D. A. Breault, Brother Joseph Heisler, M. D. Burrow, Leonard Carlitz, W. J. Carpenter, Underwood Dudley, B. E. Fristedt and P. D. Rosenbaum (jointly), L. D. Goldstone, Virginia Hanly, Edward Harris, J. E. Homer, Jr. and David Zeitlin (jointly), A. R. Hyde, Erwin Just, William Kantor, Betty Levine, Y. L. Luke, Paul Manos, D. C. B. Marsh, Otto Mond, D. A. Moran, D. R. Morrison, R. F. Norris, H. O. Pollak, John Rainwater, David Rothman, William Ruckle, David Sachs, Mahmoud Sayrafiezadeh, Paul Schillo, Donna J. Seaman, George Senge, Arnold Singer, James Singer, Sister

Rita Jean Tauer, F. C. Smith, R. A. Spinelli, D. C. Stevens, Guy Torchinelli, Alan Wayne, Clement Winston, Dale Woods, and the proposers. Late solutions by A. C. Aitken, Vern Hoggatt, R. H. C. Newton, I. D. Ruggles, Norman Schaumberger, and Dmitri Thoro.

Pollak showed that the recurrence relation gives integers whenever a_1, a_2, $(a_3+a_1)/a_2$, $(a_4+a_2)/a_3$ are integers. Spinelli showed that, apart from translations, there are only two different positive integer sequences satisfying the recurrence relation, namely $a_1=a_2=a_3=1$ and $a_1=a_3=1$, $a_2=2$.

Distinct Products in the Multiplication Table

E 1432 [1960, 802]. *Proposed by O. Lowenschuss and A. Rosenfeld, Budd Lewyt Electronics, Inc., Long Island City, N. Y.*

How many distinct products appear in the multiplication table through $N \times N$?

Remarks by Paul Erdös, Israel Institute of Technology, Haifa, Israel. The problem amounts to determining the number of integers not exceeding N^2 which can be written as the product of two integers not exceeding N. In a paper in the *Riveon Lematematika* in 1955 (in Hebrew) I proved that the number of these integers is $o(N^2/(\log N)^\alpha)$ for a certain $\alpha > 0$. In a recent paper which appeared in the publications of the University of Leningrad (in Russian) I proved that if $A(N)$ denotes the number of these integers then for $N > N_0(\epsilon)$, $\epsilon > 0$ arbitrary,

$$(N^2/\log N)(e \log 2)^{\log \log N/\log 2}(\log N)^{-\epsilon} < A(N)$$
$$< (N^2/\log N)(e \log 2)^{\log \log N/\log 2}(\log N)^{\epsilon}.$$

I do not see any way to obtain an asymptotic formula for $A(N)$.

An Inequality for a Triangle

E 1433 [1960, 802]. *Proposed by Alexander Oppenheim, University of Malaya*

Let P be a point in the interior of a triangle and let the distances of P from the vertices of the triangle be x, y, z and from the sides of the triangle be p, q, r. Show that $xyz \geqq (q+r)(r+p)(p+q)$.

Solution by Leon Bankoff, Los Angeles, California. In a triangle ABC let p, q, r denote the distances of P from the sides BC, CA, AB and let x, y, z be its distances from the vertices A, B, C. In the circles described on the diameters x, y, z we have

$$(q + r)/x \leqq 2 \sin (A/2), \quad (r + p)/y \leqq 2 \sin (B/2), \quad (p + q)/z \leqq 2 \sin (C/2),$$

or $(q+r)(r+p)(p+q)/xyz \leqq 8 \sin (A/2) \sin (B/2) \sin (C/2)$, with equality only when $p=q=r$. Since $8 \sin (A/2) \sin (B/2) \sin (C/2) \leqq 1$, with equality only when $A=B=C$, it follows that $(q+r)(r+p)(p+q) \leqq xyz$, with equality only when P is the incenter of an equilateral triangle.

Also solved by A. N. Aheart, Samuel Beatty, Leonard Carlitz, Ragnar Dybvik, L. D. Goldstone, Erwin Just, D. C. B. Marsh, L. J. Mordell, William Ruckle, Norman Schaumberger, D. C. Stevens, Dale Woods, and the proposer. Late solutions by A. C. Aitken, J. Basile, D. A. Breault, Robert Carlos, and M. V. Mielke.

Editorial Note. What is the analogue in 3-space of Oppenheim's inequality?

Structure of Open Sets

E 1434 [1960, 802]. *Proposed by Anatole Beck, University of Wisconsin*

Show that every open set in the plane can be represented as a disjoint union of closed straight line segments.

I. *Solution by D. A. Moran, University of Illinois.* Every open set in the plane is easily seen to be the disjoint union of open linear sets, *e.g.*, the intersections of the given open set with the set of all horizontal lines in the plane. But it is known that every open linear set is the disjoint union of open straight line segments. Thus the problem is reduced to showing that every open straight line segment can be represented as a disjoint union of closed straight line segments. But this is easily accomplished if we note, *e.g.*, that $(0, 1)$ is the disjoint union of the closed sets $[1/3, 2/3]$, $[1/9, 2/9]$, $[7/9, 8/9]$, $\cdots$ (the union of all closed "middle thirds").

II. *Solution by J. C. Mathews, University of Oklahoma.* It is well known that every open set in the plane is a disjoint union of "half-open" rectangles of the form $\{(x, y): a \leqq x < b \text{ and } c \leqq y < d\}$. Thus if each of these rectangles can be shown to be a disjoint union of closed straight line segments, the problem is finished. To illustrate a general scheme that accomplishes this, consider the rectangle $R = \{(x, y): 0 \leqq x < 1 \text{ and } 0 \leqq y < 1\}$. First fill in the bottom half of R as follows: (1) Let the vertical segment (abbreviated *VS*) $[0, 1/2]$ move on the horizontal segment (abbreviated *HS*) $[0, 1/2)$. (2) Let *HS* $[1/2, 3/4]$ move on *VS* $[0, 1/2]$. (3) Let *VS* $[0, 1/2]$ move on *HS* $(3/4, 7/8)$. (4) Let *HS* $[7/8, 15/16]$ move on *VS* $[0, 1/2]$. Etc. Next we note that the unfilled part of R is again a half-open rectangle and a procedure like that above will fill it in. Etc.

III. *Solution by the Proposer.* A triangle with the entire base missing can be constructed as indicated in Figure 1. A triangle with two whole sides omitted is now made from this as indicated in Figure 2. Next we build a rectangle with two whole sides omitted as indicated in Figure 3. With such rectangles we can build any planar open set.

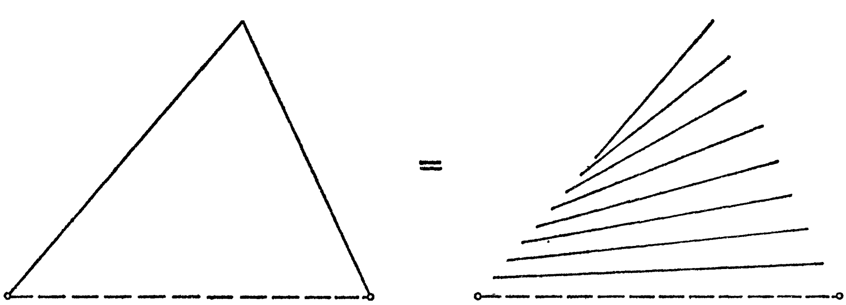

FIG. 1

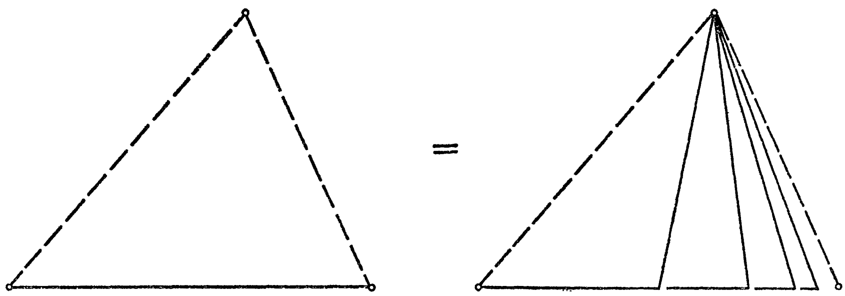

FIG. 2

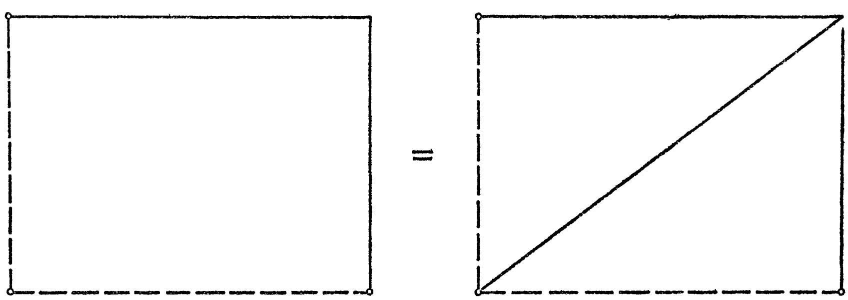

FIG. 3

Also solved, similarly to Solution II, by L. A. Ringenberg.

Editorial Note. Solution I is easily generalized to show that any open set in euclidean n-space can be represented as a disjoint union of closed straight line segments.

Superior and Inferior Limits of Sequences of Sets

E 1435 [1960, 802]. *Proposed by J. F. Leetch, Ohio State University*

Find the limit superior and the limit inferior for each of the following sequences of sets of real numbers:

(a) $\{A_n\}_{n=1}^{\infty}$ where A_n is the set of integers mod n,

(b) $\{A_n\}_{n=1}^{\infty}$ where $A_n=\{a/n \mid a\in J(\text{integers})\}$.

Solution by E. L. Cohen and Gerald Leibowitz, Massachusetts Institute of Technology. (a) $A_n=\{0, 1, \cdots, n-1\}$, whence $A_1\subset A_2\subset A_3\subset \cdots$ and $\limsup A_n =\liminf A_n=\bigcup_{n=1}^{\infty} A_n=\{0, 1, 2, \cdots\}$.

(b) Since $m/n=2m/2n=3m/3n= \cdots$, every rational belongs to infinitely many A_n. It follows that $\limsup A_n=$ set of all rationals.

Since $m=mn/n$, $J\subset A_n$. Let m/n be a nonintegral rational, $n>1$, $(m, n)=1$. Then $m/n\notin A_{nk+1}$ for $k=1, 2, 3, \cdots$ (since $m(nk+1)/n$ is not an integer). Thus there are infinitely many A_i not containing m/n, and $m/n\notin\liminf A_n$. It follows that $\liminf A_n=J$.

Also solved by D. C. B. Marsh, D. A. Moran, D. C. Stevens, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications eoncerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4959. *Proposed by H. S. Shapiro and A. L. Shields, New York University*

(a) Exhibit an analytic Jordan curve Γ in the complex plane (not a circle about the origin) such that $\int_\Gamma z^n ds=0$ for $n=2, 3, \cdots$, where s is arc length.

(b) Exhibit an analytic arc Γ (not a circle) such that $\int_\Gamma z^n ds=0$ for all $n\geqq 1$.

4960. *Proposed by Donald L. Shell, Cincinnati, Ohio*

Prove for every positive integer n,

$$\sum_{p=1}^{n}(-1)^p\binom{n}{p}(p+1)^{p-1}p^{n-p} = (-1)^n.$$

4961. *Proposed by I. S. Gál, Yale University*

For σ, t real, $\alpha(s)=\sum_{n=0}^{\infty} a_n n^{-s}$ and $a(u)=u^{-\sigma}\sum_{n\leq u} a_n$. Let

$$\hat{a}(t) = \int_0^\infty a(u)u^{-it}\frac{du}{u},$$

and define $\beta(s)$ and $b(v)$ similarly. Let

$$(a*b)(u) = \int_0^\infty a\left(\frac{u}{v}\right)b(v)\frac{dv}{v}\cdot$$

Determine $\hat{a}$ and $a*b$ in terms of α and β.

4962. *Proposed by D. J. Newman, Yeshiva University*

Let $P(x, y)$ and $Q(x, y)$ be polynomials with real coefficients. Suppose $xP+yQ=1$ for all points on the unit circle. Prove that $P=0$, $Q=0$ are satisfiable simultaneously.

4963. *Proposed by E. J. Burr, University of New England, N.S.W., Australia*

Let k, r, n be three given integers such that $0<k\leqq r\leqq n$. From a set of n objects arranged in a line, r objects may be selected in $\binom{n}{r}$ ways. How many of these selections have the property that at least one set of k or more consecutive objects is included in the selection?

4964. *Proposed by Ernst Trost, Zürich, Switzerland, and Anders Bager, Hjørring, Denmark*

Consider a triangle *abc* divided into four smaller triangles, a central one *def* inscribed in *abc* and three others on the three sides of *def*. Show that *def* cannot have the smallest perimeter of the four unless all four perimeters are equal with *d*, *e*, and *f*, the midpoints of the sides of *abc*. (See also No. 4908 below.)

SOLUTIONS

Linear Combinations of Continuous Functions

4904 [1960, 382]. *Proposed by D. J. Newman, Yeshiva University*

Suppose a set, S, of functions continuous on $[0, 1]$ has the property that every linear combination of them has a zero on $[0, 1]$. Prove that there exists a nondecreasing function $\alpha(x)$ such that, for all $f \in S$, $\int_0^1 f(x)d\alpha(x) = 0$.

Solution by I. J. Schoenberg, University of Pennsylvania. For the case when the set S is finite, the existence of $\alpha(x)$ is known (I. J. Schoenberg, *Convex domains and linear combinations of continuous functions*, Bull. Amer. Math. Soc., 1933, Thm. 2, p. 274. References to work of L. L. Dines are also given.) Let now the set S be infinite. We associate to every $f \in S$ a set $M(f)$ of monotone functions defined by

$$M(f) = \left\{ \alpha(x) \,\middle|\, \int_0^1 f(x)d\alpha(x) = 0, \quad \alpha(0) = 0, \quad \alpha(1) = 1 \right\}.$$

This set is visibly nonvoid because $f(x)$ vanishes somewhere by assumption. The problem before us is to show that $\bigcap_{f \in S} M(f) \neq \emptyset$. By our remark concerning the case of finite S a solution is directly implied by the following general

THEOREM. *Let $\{M\}$ be a collection of sets M where each M is a set of nondecreasing functions $\alpha(x)$, $0 \leqq x \leqq 1$, having the following properties*:

(i) *If $\alpha(x) \in M$ and the monotone $\beta(x)$ agrees with $\alpha(x)$ at $x=0$, $x=1$ and at all continuity points of $\alpha(x)$, then also $\beta(x) \in M$.*

(ii) *The elements of each M are equi-bounded.*

(iii) *Each M is closed with respect to point-wise convergence.*

If for every finite subcollection of $\{M\}$ there is a common element, then there exists an element $\alpha(x)$ common to all the M.

This will be seen to be a corollary of the following

THEOREM OF F. RIESZ. *If a collection $\{F\}$ of bounded and closed sets in R^m has the property that the elements of every finite subcollection have a common point, then all the F's have a common point.*

Riesz's theorem follows by a straightforward application of the Heine-Borel theorem. It was first published and proved by D. König [*Über konvexe Körper*, Math. Zeit., vol. 14 (1922), pp. 208–210]. To derive our theorem, let n be a fixed

natural integer and let us denote by $\alpha^{(n)}$ the point of the (2^n+1)-dimensional space defined by

$$\alpha^{(n)} = (\alpha(0), \alpha(1/2^n), \alpha(2/2^n), \cdots, \alpha(1)).$$

The euclidean sets $F_m^{(n)} = \{\alpha^{(n)} | \alpha \in M\}$ are evidently bounded and closed by (ii) and (iii). Moreover, every finite number of them have a common point. By Riesz' theorem they all have a common point. This means: From each M we can select a $\alpha_{n,M}(x)$ such that all these functions agree in value for $x=0, 1/2^n, \cdots, 1$.

Let M_0 be an arbitrary but fixed set of the collection $\{M\}$. Out of $\{\alpha_{n,M_0}(x)\}$ we select a subsequence such that for each x in $[0, 1]$, $\lim_{\nu\to\infty} \alpha_{n_\nu,M_0}(x) = \beta(x)$. By (iii) we have $\beta(x) \in M_0$. Let now M be an element of $\{M\}$. By construction we see that $\lim_{\nu\to\infty} \alpha_{n_\nu,M}(x) = \beta(x)$ for all binary $x = m/2^n$. By selection of an everywhere convergent subsequence of $\{\alpha_{n_\nu,M}(x)\}$ and by (i) we see that $\beta(x) \in M$. Thus $\beta(x)$ is common to all the M and the theorem is established.

Also solved, using standard methods concerning linear operations in Banach space, by N. J. Fine, G. Lorentz, B. J. Pettis, Albert Wilansky, and the proposer.

A Corollary of a Theorem of Schwartz

4906 [1960, 479]. *Proposed by P. L. Butzer, Technical University, Aachen, Germany*

If the function $f(x)$ is continuous on an interval (a, b) and, as $h \to 0$, $h^{-3}\int_0^h [f(x+u) + f(x-u) - 2f(x)]du \to 0$ for all x in (a, b), then $f(x)$ is a linear function.

Solution by Stephen Andrea, Olean, N. Y. The numerator and denominator of

$$(1/h^3) \int_0^h [f(x+u) + f(x-u) - 2f(x)]du$$

are both continuous differentiable functions of h. Hence we can apply l'Hospital's rule with the result

$$\lim_{h\to\infty} (1/h^2)[f(x+h) + f(x-h) - 2f(x)] = 0.$$

Schwartz's theorem now applies (Titchmarsh, *Theory of Functions*, 2nd ed., 13.84, p. 431) and the desired conclusion follows.

Also solved by Robert Breusch, C. Kassimatis, Y. Matsuoka, David Zeitlin, and the proposer.

Subgroup of a Finite Abelian Group

4907 [1960, 479]. *Proposed by P. T. Bateman, University of Illinois*

Suppose $\lambda_1, \cdots, \lambda_r$ are among the elements of a finite abelian group G (written additively) and let H be the subgroup of G generated by $\lambda_1, \cdots, \lambda_r$. Show that there exist positive integers $k_1, \cdots, k_r$ such that every element of

H is uniquely expressible in the form

$$n_1\lambda_1 + \cdots + n_r\lambda_r,$$

where n_i is a nonnegative integer less than k_i $(i=1, \cdots, r)$.

Solution by N. J. Fine, Institute for Advanced Study. Let $G_0=\{0\}$, $G_i=$the subgroup generated by $\lambda_1, \cdots, \lambda_i$ $(i=1, \cdots, r)$. Define k_i as the least positive integer for which $k_i\lambda_i \in G_{i-1}$. Clearly every element of $H=G_r$ has a representation $\sum n_i\lambda_i$ with $0 \leqq n_i < k_i$. This yields a mapping from the set N of r-tuples $(n_1, \cdots, n_r)$, $0\leqq n_i \leqq k_i$, onto G_r. The number of elements in N is $k_1 \cdots k_r$, and the number of elements in G_r is

$$(G_r:G_{r-1})(G_{r-1}:G_{r-2}) \cdots (G_1:G_0) = k_r \cdots k_1.$$

Hence the mapping is one-to-one.

Also solved by Stephen Andrea, Robert Breusch, Joe Lipman, D. C. B. Marsh, J. G. Mauldon, D. A. Moran, D. T. Sandberg, Wu Ta-Sun, and the proposer.

Partition of a Triangle

4908 [1960, 479]. *Proposed by John Rainwater, University of Washington, Seattle*

Consider a triangle *abc* divided into four smaller triangles, a central one *def* inscribed in *abc* and three others on the three sides of *def*. Show that *def* cannot have the smallest area of the four unless all four are equal with d, e, and f the midpoints of the sides of *abc*.

I. *Solution by P. H. Diananda, University of Malaya, Singapore.* Let α, β, $\gamma(0<\alpha\leqq\beta\leqq\gamma)$ be the areas of the corner triangles and δ the area of the central triangle. Then we will prove that $\delta \geqq \surd(\alpha\beta)$ with equality if and only if d, e, f are the midpoints of the sides of triangle *abc*. This result is slightly stronger than that proposed.

Let bc, ca, ab be divided at d, e, f, respectively in the ratios $x:x'$, $y:y'$, $z:z'$ with $x+x'=y+y'=z+z'=1$. Also let *abc* be of unit area. Then the corner triangles are of areas $y'z$, $z'x$, $x'y$. Also $\delta=1-\sum y'z=xyz+x'y'z'$.

If $\gamma<\frac{1}{4}$ then $\alpha\leqq\beta\leqq\gamma$ and $\alpha+\beta+\gamma+\delta=1$ imply $\delta\geqq\surd(\alpha\beta)$. If $\gamma\geqq\frac{1}{4}$ then $\delta=xyz+x'y'z'\geqq 2\surd(xyzx'y'z')=2\surd(\alpha\beta\gamma)\geqq\surd(\alpha\beta)$.

Equality is obtained if and only if $xyz=x'y'z'$ and $\gamma=\frac{1}{4}$, which are true if and only if $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ and $x=y=z=\frac{1}{2}$.

II. *Solution by Anders Bager, Hjørring, Denmark.* Because the proposition is affinely invariant we may suppose *def* equilateral. Let the notation be such that $\sphericalangle a \geqq \sphericalangle b \geqq \sphericalangle c$. Then $\sphericalangle a \geqq 60° = \sphericalangle dfe$, and the circular arc *fde* reaches in the least as much "above" the common chord *fe* as the arc *fae* reaches "below." But d is the "highest" point of the first mentioned arc. Hence we conclude that $\triangle def \geqq \triangle fea$, and the solution is complete.

The problem was proposed by H. Debrunner in *Elemente der Mathematik*, January 1956, and the solution here given appeared in the same journal in March 1957. The problem becomes much more difficult when each area is replaced by the corresponding perimeter. It is to this latter problem that P. Erdös gave some attention. It seems to have been Ernst Trost who called Erdös' attention to both problems.

Also solved by A. C. Aitken, Leon Bankhoff, W. J. Blundon, Robert Breusch, N. J. Fine, Michael Goldberg, Peter Yff, G. Laman, J. G. Mauldon, E. J. Mickle, Edmundo Morgantini (who published his solution as a note in *Rendiconti del Seminario Matematico dell' Universita di Padova*, XXX, 246–7), D. J. Newman, K. A. Post, G. E. Raynor, R. C. Read, G. B. Robison, I. J. Schoenberg, P. J. van Albada, J. H. van Lint, and the proposer.

Zeros of a Special Polynomial

4909 [1960, 479]. *Proposed by D. J. Newman, Yeshiva University*

Prove that the product of the zeros of $z^n \pm z^{n-1} \pm \cdots \pm z \pm 1$ which lie outside the unit circle is less than $\sqrt{(n+1)}$.

Solution by Robert Breusch, Amherst College. More generally:

If $f(x) \equiv z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ (a_i real, $a_n = \pm 1$), then the product of the zeros of f which lie outside the unit circle is less than $\sqrt{(1+a_1^2+\cdots+a_n^2)}$.

Proof. Let $f(z) = \prod_{k=1}^{n}(z-z_k)$, not $|z_k|=1$ for all k, $|z_k|>1$ for $k \leqq r$, $|z_k| \leqq 1$ for $k > r$. Let

$$f_1(z) \equiv \prod_{k=1}^{r}(z-z_k) = z^r + b_1 z^{r-1} + \cdots + b_r.$$

The b_i are real because with any z_k its conjugate is also a zero of f_1; and $b_r = (-1)^r \prod_{k=1}^{r} z_k$. Define further functions as follows:

$$f_2(z) = \prod_{k=r+1}^{n}(z-z_k) = z^{n-r} + \cdots \pm 1/b_r;$$

$$g_1(z) \equiv \prod_{k=1}^{r}(z - 1/z_k) = (-1)^r z^r \prod_{k=1}^{r}\left(\frac{1}{z} - z_k\right) \Big/ \prod_{k=1}^{r} z_k$$

$$= \frac{z^r}{b_r} f_1(1/z) = z^r + \cdots + 1/b_r;$$

$$g_2(z) = \prod_{k=r+1}^{n}(z - 1/z_k) = z^{n-r} + \cdots \pm b_r;$$

$$g(z) \equiv g_1(z)g_2(z) = z^n + \frac{a_{n-1}}{a_n}z^{n-1} + \cdots + \frac{a_1}{a_n}z + \frac{1}{a_n};$$

$$A(z) \equiv f_1(z)g_2(z) = z^n + c_1 z^{n-1} + \cdots + c_n, \quad (c_n = \pm b_r^2);$$

$$B(z) \equiv f_2(z)g_1(z) = z^n + \frac{c_{n-1}}{c_n}z^{n-1} + \cdots + \frac{c_1}{c_n}z + \frac{1}{c_n}\cdot$$

Thus $A(z)\cdot B(z) = f(z)\cdot g(z)$. The coefficients of z^n in the two products give

$$(1 + c_1^2 + \cdots + c_n^2)/c_n = (1 + a_1^2 + \cdots + a_n^2)/a_n.$$

Thus $|c_n| = b_r^2 < (1 + \cdots + a_n^2)$, $|b_r| = \prod_{k=1}^r |z_k| < (1 + \cdots + a_n^2)^{1/2}$.

Also solved by T. V. Lakshminarasimhan, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Differential Equations. By Tomlinson Fort. Holt, Rinehart and Winston, New York, 1960. viii+184 pp. $4.75.

This is one of the better books available for an undergraduate course in differential equations. The level of mathematical rigor is satisfactory and the selection of topics is excellent.

The author's use of loosely defined differentials at times is at odds with the requirements of rigor; and the "symmetry in x and y" which he achieves is illusory since he must frequently revert to derivatives in his discussion. Yet, common usage of the differential notation makes another completely consistent choice both difficult and awkward to maintain.

The careful selection of topics has both advantages and disadvantages. For the instructor who agrees with the author's choice it offers a consistent development of the material with little or no cutting either desirable or possible. However, for the instructor who disagrees, a compendium of methods and solutions gives him an opportunity to make some selection of topics as he cannot in such a book as Fort's. Furthermore, the author's pruning does result in a somewhat thin coverage, particularly in the latter part of the volume.

The book is clearly and carefully written and is most readable for a student. The approach is theoretical in nature; existence theorems and general solutions are both discussed; proofs are offered for most of the theorems. The physical applications have been sorted out into two chapters spotted through the book; one on "certain applications" and the other on "vibrations."

Throughout the book there is a nice balance between the desire for complete rigor and the teachability of an intuitive approach. The number of editorial and proofing errors is pleasantly small.

DONALD A. NORTON
University of California, Davis

Combinatory Logic, Volume I. By Haskell B. Curry and Robert Feys, with two sections by William Craig. North-Holland, Amsterdam, 1958. xvi+417 pp. $11.75.

Combinatory logic (here understood as including the intimately related theory of lambda-conversion) is a branch of mathematical logic concerned with the precise formal analysis of certain "prelogical" processes which are considered only informally in the usual current treatments; its primary aim is to provide for the "complete formalization" of logical systems. [Quoted phrases are from the book under review.]

This first volume of a projected two-volume treatise breaks naturally into two parts. Chapters 1–7 form a discussion of the fundamentals of the subject; in addition to bringing together a large number of known technical results from the theory of combinators (mostly due to Curry), this includes a treatment of lambda conversion and its relationship to the theory of combinators. With some interesting exceptions, the material in Chapters 1–7 is not new. In Chapters 8–10, the authors bring together much material from "illative" combinatory logic, which concerns the organization of objects into categories, especially as this is achieved using Curry's "functionality primitive". In large part, the material in Chapters 8–10 was previously unpublished.

This book is a valuable contribution to research in its field; as such it is not surprising that it is not entirely suitable as an introductory treatment. For the latter purpose, Chapter III, Section 4, of Rosenbloom's *Elements of Mathematical Logic* (1950) is probably the best source in English yet available. (The first part of Rosser's *Deux Esquisses de Logique* (1955) is even better.)

Many interesting questions are raised by this book. As presented, the theory aims (roughly speaking) at supplying underpinnings for the whole of mathematics (insofar as that is possible) in the same sense as does formal set theory (but more constructively). As developed, this involves formulation of an inherently combinatory predicate calculus; what are the details of such a formulation? Again: as developed, the underlying intuitive concept is taken to be function rather than set; precisely what system of set theory corresponds to the function theory realized in this book? Again: it is possible to formulate formal set theory (including the machinery of the classical predicate calculus) exactly as a combinatory logic. What are the most natural ways to achieve this formulation? And from the other side of the fence: what are the properties of combinatory logic as formulated within the framework of set theory? It is hoped that answers to some of these questions will be found in Volume II, along with material (promised by the authors) connecting combinatory logic with the theory of recursive functions.

David E. Schroer
University of Rochester

Introduction to Linear Programming. By Walter W. Garvin. McGraw-Hill, New York, 1960. xiv+281 pp. $8.75.

This is a particularly well-written account of the major topics in linear programming. Part I, entitled The General Linear Programming Problem, consists of five chapters which include a detailed description of the simplex method, sensitivity analysis (*i.e.*, the effect of changes in the coefficients on the optimal solution of a problem), and the gasoline-blending problem. Part II (five chapters), The Transportation Problem and Its Variants, includes chapters on unbalance and transshipment, assignment problems, a tanker-routing problem, and the generalized transportation problem. Part III on Special Methods (eight chapters) covers the following: upper bounds, statistical linear programming, revised simplex method, parametric linear programming with an application to a simple economic model, duality theory with application to the warehouse problem, and the resolution of degeneracy.

A clear and unhurried presentation, together with an abundance of numerical illustrations and flow diagrams, makes the book eminently suitable for self-study. The author's considerable industrial experience is evident in the well-chosen applications. The practitioner, teacher, or prospective student can all profit from a serious study of this volume.

H. KAUFMAN
McGill University

Combinatorial Topology, Volume 3. By P. S. Aleksandrov. Graylock Press, Albany, 1960. viii+144 pp. $6.50.

This volume, containing Parts Four and Five, apparently completes a translation of the first (1947) Russian edition of Aleksandrov's book. Part Four commences with (1) the duality theorems of Poincaré and of Alexander and (2) Pontryagin's theorems on linked systems of cycles. These results pertaining to homological manifolds are followed by an exposition of concepts needed for the topology of compacta and locally bicompact Hausdorff spaces. Part Four culminates in the Alexander-Pontryagin Duality and its consequences.

The final part is devoted to (1) the theory of continuous mappings of polyhedra, based ultimately on the work of Brouwer, and (2) the fixed-point theorem for topological polyhedra, as extended by Hopf from Lefschetz' fixed-point theorem for manifolds.

Aleksandrov's point of view is primarily geometric. His exposition is careful and detailed. The three volumes are a valuable compendium of fundamental concepts and results.

S. S. CAIRNS
University of Illinois

Differential and Integral Calculus. By J. R. F. Kent. Houghton Mifflin, Boston, 1960. xv+511 pp. $6.75.

A principal aim of the author was to write a text that a motivated student could understand without the aid of an interpreter and that is rigorous enough to answer some of his more searching questions. The reviewer feels that the author has constructed an interesting approximate solution to this difficult problem and that the text is worthy of critical perusal by those seeking a new calculus text.

Material is presented in the classical manner, that is, differential calculus followed by integral calculus. Exercises are carefully conceived and contain numerous thought-provoking questions as well as applications to the social and physical sciences. Definitions are given in the body of the text and are called out by italics. For the most part good heuristic explanations are given in lieu of proofs. Mean value theorems are given proper emphasis. The author's writing style and the publisher's printing format are both pleasing.

E. H. CRISLER
Hughes Aircraft Company

Axiomatic Set Theory. By Patrick Suppes. Van Nostrand, Princeton, N. J., 1960. xii+265 pp. $6.00.

Professor Suppes' book is an excellent, rigorous, clear text on axiomatic set theory for use in courses where students may not have any previous acquaintance with mathematical logic or set theory. Such a book has been needed for a long time. Students who work through this book and its many exercises should have no trouble with more advanced books on set theory, as for instance the *Axiomatic Set Theory* by P. Bernays and A. A. Fraenkel (North Holland Publishing Co., Amsterdam, 1958).

The Zermelo-Fraenkel system is developed here in detail. For didactic reasons, Professor Suppes adds an additional axiom (originally due to Tarski) when developing the theory of cardinals, but later he shows that the theory could have been developed without it. The book also covers the highlights of the theory of rational and real numbers, transfinite induction, ordinal arithmetic, well-ordered sets, the axiom of choice and its equivalents. There is a brief mention of the relative consistency of the axiom of choice on page 250, but some mention should be made there of the known results on the independence of the axiom (e.g., E. Mendelson, *The Axiom of Fundierung and the Axiom of Choice*, Arch. Math. Logik Grundlagenforsch., 1958, pp. 65–70, and E. Specker, *Zur Axiomatik der Mengenlehre* (*Fundierungs- und Auswahlaxiom*), Z. Math. Logik Grundlagen Math., 1957, pp. 173–210.)

Now that such a lucid textbook is available it is hoped that its existence will encourage more colleges and universities to give courses on axiomatic set-theory.

L. N. GÁL
Yale University

Naive Set Theory. By Paul R. Halmos. Van Nostrand, Princeton, N. J., 1960. vii+104 pp. $3.50.

From the author's preface: "Every mathematician agrees that every mathematician must know some set theory; the disagreement begins in trying to decide how much is some. This book contains my answer to that question. The purpose of the book is to tell the beginning students of advanced mathematics the basic set-theoretic facts of life, and to do so with the minimum of philosophical discourse and logical formalism. The point of view throughout is that of a prospective mathematician anxious to study groups, or integrals, or manifolds. From this point of view the concepts and methods of this book are merely some of the standard mathematical tools; the expert specialist will find nothing new here. . . . Instead of *Naive Set Theory* a more honest title for the book would have been *An Outline of the Elements of Naive Set Theory*. "Elements" would warn the reader that not everything is here; "outline" would warn him that even what is here needs filling in . . . . The student's task in learning set theory is to steep himself in unfamiliar but essentially shallow generalities till they become so familiar that they can be used with almost no conscious effort. In other words, general set theory is pretty trivial stuff really, but, if you want to be a mathematician, you need some, and here it is; read it, absorb it, and forget it."

Having worked both in formal logic and in plain mathematics, Halmos is admirably qualified to write this book. It is possible that he was first prompted to set pen to paper when (as a Van Nostrand editor) he saw the manuscript of Suppes's *Axiomatic Set Theory*. In any event the two books form a natural contrasting pair, Suppes's for the careful logician investigating (say) the independence of the axiom of choice, Halmos's for the mathematician-in-the-street who just wants to stay out of trouble when he does (say) measure theory.

One small cavil. It seems to the reviewer that Halmos should have relaxed his "naive" principles occasionally, at least in his examples. A mathematician indifferent to most logical subtleties might nonetheless be delighted by the weird un-integer-like objects that happen to satisfy Peano's postulates.

H. MIRKIL
Hanover, N. H.

Ordinary Differential Equations and Their Solutions. By George M. Murphy Van Nostrand, New York, 1960. ix+451 pp. $8.50.

This book is designed as a relatively exhaustive table of ordinary differential equations for which a solution, or a method of solution, is available. Considerable effort has been applied to provide a systematic classification of these differential equations, so that the locating of a particular equation is quite straightforward.

The text is composed of two parts. The first part contains a brief discussion of the methods available for solutions, and follows the classification scheme. The second part consists of over two thousand equations together with their solu-

tions, or appropriate references. Again, the cross-indexing makes it an easy matter to track down any particular differential equation.

The excellent organization of the material, and the ease with which it can be used, makes this work very valuable as a reference.

G. E. LATTA
Stanford University

Elementary Statistics. By Paul G. Hoel, Wiley, New York, 1960. vii+261 pp. $5.50.

Professor Hoel has written a text that will prove extremely useful for a one-semester service course in statistics for the student whose mathematical background has been limited to high-school algebra.

The topics considered include probability, theoretical frequency distributions, sampling, estimation, testing hypotheses, correlation, regression, chi-square distribution, nonparametric tests, analysis of variance, and time series and index numbers. The exercise lists are excellent and the tables necessary to solve them are included in the appendix.

Professor Hoel has a clarity of style and aptness of illustration that make each topic considered easy to understand and apply.

JOHN C. BRIXEY
The University of Oklahoma

BRIEF MENTION

High School Mathematics, Unit V, Relations and Functions. University of Illinois Committee on School Mathematics. University of Illinois Press, Urbana, Illinois, 1960. 278 pp. Student's Edition $1.50, Teacher's Edition $3.00.

The fifth unit of UICSM Mathematics deals with relations and functions. The emphasis, of course, is on set theory. This admirable presentation and its many interesting exercises can well be considered by mathematicians everywhere.

Digital Computer Fundamentals. By Thomas C. Bartee. McGraw-Hill, New York, 1960. 334 pp., $6.50.

Some mathematics and some electrical circuit theory and a good bit of valuable information.

Probability Theory. (2nd ed.) By Michel Loéve. Van Nostrand, Princeton, N. J., 1960. 685+xvi pp., $14.75.

Two new chapters concerning random analysis have been added to Loéve's well-known text. A scholarly volume.

Time Series Analysis. By E. J. Hannan. Wiley, New York, 1960, 142+bib. pp., $3.50.

This book also assumes a reasonable statistical knowledge as a prerequisite, as well as some knowledge of infinite-dimensional vector spaces.

Modern Factor Analysis. By Harry H. Harman. University of Chicago Press, 1960 486 pp., $10.00.

If your library doesn't have a book on factor analysis available, this is one to be considered. With modern psychologists and sociologists becoming more and more interested in factor analysis, mathematicians will do well to have a reference book available. This one should help bridge the gap in vocabulary usage between mathematicians and psychologists.

Mathematical Snapshots, by H. Steinhaus. Oxford University Press, 1960. 318 pp., $6.75.

This revised and enlarged edition of Steinhaus's well-known pictorial work is a delightful book which should be called to the attention of high school teachers seeking suggestions for their libraries, as well as being welcome on college shelves.

Introduction to Modern Algebra (official textbook for Continental Classroom). By John L. Kelley. Van Nostrand, New York, 1960. ix+338 pp., $2.75.

By all means get up early and watch the program. One may wonder why modern algebra is introduced without an example of a postulational system such as a group, field, ring, etc., but upon reading the text, one finds that the basic heart of postulational algebra has not been slighted in the text, in spite of this. This reviewer's congratulations to the author and the committee for this courageous undertaking.

Foundations of Geometry. By Borsuk and Szmielew. North-Holland Pub. Co., 1960. 439+index pp., $12.00.

This is no gentle rehash of geometry designed for consumption by high school teachers at summer institutes, but a scholarly work on Euclidean and Bolyai-Lobachevskian geometry, as well as an introduction to projective geometry. Some excellent mathematical logic is displayed. This is a source book for serious geometers everywhere.

The Number Story. By H. Freitag and A. Freitag. National Council of Teachers of Mathematics, 1960. 76 pp., $0.85.

Special Relativity, by W. Rindler. Interscience, New York, 1960. 196+index pp., $2.25.

Relativity, the General Theory. By J. L. Synge. North-Holland Pub. Co., 1960. 414+app. and bib., $16.50.

This is a beautiful, mathematical presentation, which gets quickly to the heart of things without wasting the time of the specialist by attempting a lay presentation. Congratulations to Professor Synge.

From An Ivory Tower. By Bernard A. Hausmann, S. J. Bruce Pub. Co., Milwaukee, Wisconsin, 1960. 122 pp., $3.50.

A Jesuit scholar discusses certain philosophical problems of mathematical origin.

Markov Chains with Stationary Transition Probabilities. By Kai Lai Chung. Springer-Verlag, Berlin, 1960. ix+278 pp., DM 65.60. (About $16.25)

A well-written advanced volume in spite of the distracting translucency of the paper on which it is printed. A number of research problems still to be solved are mentioned at various points.

Statistical Theory of Communication. By Y. W. Lee. Wiley, New York, 1960. xviii+509 pp., $16.75.

Don't let the title mislead you. When Lee discusses the statistical theory of communication it is indeed a branch of higher mathematics, and rightly so. Probability theory and generalized harmonic analysis, as well as statistics and statistical mechanics, are used freely in this development of the Wiener-Lee theories.

Stationary Processes and Prediction Theory. By Harry Furstenberg, Princeton University Press, 1960, 283 pp., $5.00. Princeton Annals of Mathematics Studies, No. 44.

An advanced and carefully prepared volume on statistical predictability using Markoff and Weiner-Kolmogoroff methods.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor D. E. Christie, Bowdoin College, represented the Association at the inauguration of Dr. Kenneth Brooks as President of Gorham State Teachers College on December 7, 1960.

Professor George Dubay, University of St. Thomas, has been named "Piper Professor, 1960" and has received the Minnie Stevens Piper Foundation $1,000 award for outstanding scholarly and academic achievement.

Professor K. O. May, Carleton College, represented the Association at the inauguration of Dr. O. M. Wilson as President of the University of Minnesota on February 23, 1961.

Kansas State Teachers College: Mr. T. E. Bonner and Mr. D. L. Bruyr have been appointed Instructors; Assistant Professor L. E. Laird has been promoted to Associate Professor.

Oregon State College: Dr. C. S. Ballantine, University of California, Berkeley, has been appointed Instructor; Miss Florence A. Bakkum has been promoted to Assistant Professor; Professor I. M. Hostetter retired July 1, 1960.

Mr. S. O. Albert, Fort Bliss, Texas, has accepted a position as Retail Industry Representative in the Data Processing Division of International Business Machines, New York, New York.

Mr. C. D. Alders, Mankato State College, has been promoted to Assistant Professor.

Professor Howard Alexander, Earlham College, will be on leave during the year 1961–62 at Bowdoin College to lecture at a National Science Foundation Institute.

Mr. A. D. Brock, University of Oklahoma, has been appointed Assistant Professor at Radford College.

Miss Emalou Brumfield, Kent State University, has been appointed Teacher at Byron Junior High School, Shaker Heights, Ohio.

Mr. J. W. Calvert, University of Kentucky, has accepted a position as a member of the Technical Staff of Hughes Aircraft Corp., Fullerton, California.

Mr. C. J. Cillay, Texas State Board of Insurance, Austin, Texas, has accepted a position as Mathematician at the United States Naval Radiological Laboratories, San Francisco, California.

Mr. W. P. Durbin, Conviar, Fort Worth, Texas, has accepted a position as Engineer with the Radio Corporation of America, Moorestown, New Jersey.

Professor Wade Ellis, Oberlin College, has been awarded a Science Faculty Fellowship for 1961–62 by the National Science Foundation and will study at the University of Michigan.

Mr. L. C. Fletcher, Eastern Kentucky State College, has been appointed Head of the Department of Mathematics at Jackson Township High School, Circleville, Ohio.

Mr. H. S. Hall, Pennsylvania State University, has been appointed Assistant Professor at Rhode Island College.

Mr. G. G. Harrington, Jr., Boeing Airplane Company, Seattle, Washington, has accepted a position as Senior Engineer with the Martin Company, Denver, Colorado.

Mr. P. J. Hawkins, University of Connecticut, has been appointed Instructor in Electrical Engineering at Ohio State University.

Mr. M. O. Holoien, North Dakota State University, has accepted a position as Associate Research Engineer with Boeing Airplane Company, Seattle, Washington.

Mr. J. P. Lamb, University of Notre Dame, has accepted a position as Computer Programmer with International Business Machines, Washington, D. C.

Dr. E. O. Nelson, University of North Dakota, has been appointed Assistant Professor at the University of Utah.

Dr. P. B. Norman, Long Island University, has accepted a position as a member of the Technical Staff of Aerospace Corporation, El Segundo, California.

Mr. L. G. Salvin, Clark University, has accepted a position as Mathematical Statistician at the Cancer Chemotherapy National Service Center, National Institutes of Health, Silver Spring, Maryland.

Dr. E. T. Welmers, Bell Aircraft Corporation, Niagara Falls, New York, has accepted a position as a member of the Technical Staff of Aerospace Corporation, El Segundo, California.

Professor Emeritus W. E. Anderson, Miami University, died December 3, 1960. He was a Charter Member of the Association.

Professor L. S. Hill, Hunter College, died January 10, 1961. He was a member of the Association for 31 years.

Professor G. E. Moore, Eastern Michigan College, died June 20, 1960. He was a member of the Association for 36 years.

Mr. W. A. Riley, Jr., University of Kentucky, died July 15, 1960.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1961:

Cornell University, June 29 to August 11: Mrs. Hertzig, survey of mathematics; Professor Walker, higher geometry.

DePaul University, Day, June 26 to August 4: Dr. De Cicco, lattice theory, infinite series; Mr. Czarnecki, introduction to Fourier series. Evening, June 12 to August 4: Dr. Yao, mathematical statistics.

Indiana University, June 14 to August 11: introduction to modern mathematics I; number theory; introduction to analysis I; introduction to topology I; non-euclidean geometry; mathematical reading and research.

Kent State University, June 19 to July 22: Professor Bush, selected topics for classroom teachers; Professor Dressler, differential equations I; Professor Jenkins, theory of numbers, history of mathematics; Professor Brooks, Boolean algebra. July 24 to August

26: Professor Johnson, college geometry; Professor Kaiser, differential equations II, functions of a complex variable.

Northwestern University, June 24 to August 5 and June 24 to August 19: advanced calculus; engineering mathematics I; numerical methods; statistics for teachers; statistics for experimenters; vectors, matrices, and quadratic forms; introduction to the theory of numbers; algebra for teachers; geometry for teachers; the history of mathematics II; advanced geometry for teachers; foundations of calculus for teachers; complex variables for applications; topics in modern mathematics for teachers; introduction to topology.

Syracuse University, July 5 to August 11: Professor Exner, mathematics of statistics, introduction to logic; Professor Baum, modern algebra; Professor Davis, axiomatic algebra in the "modern" school programs; Professor Hemmingsen, linear algebra; Professor Reid, real variables; Staff, digital computers and numerical analysis.

University of Chicago, June 19 to September 1: Dr. Fong, introduction to the theory of groups and rings; Professor Calderon, singular integrals and differential equations; Professor Stone, Hilbert space II; Professor Stinespring, introduction to functional analysis; Professor Baily, algebraic geometry. In addition to the above, special seminars in homological algebra and algebraic topology will be organized by Professors MacLane and Lashof and visiting mathematicians. Resident members of the Department of Mathematics will also make available to qualified students on an individual basis the customary courses entitled, "reading and research in mathematics."

University of Michigan, June 26 to August 19: Dr. Brumfiel, introduction to matrices; Dr. Clarke, theory of statistics I; Dr. Coburn, operational mathematics, Fourier series and applications; Mr. Cohn, differential equations, introduction to matrices; Dr. Craig, statistical analysis II; Dr. Dickson, advanced mathematics for engineers; Dr. Dushnik, operational methods for systems analysis; Dr. Dwyer, statistical analysis II, mathematical theory of probability I; Dr. Griffen, advanced calculus; Dr. Donald Jones, differential equations, mathematical theory of probability I; Dr. P. S. Jones, history of geometry and trigonometry; Dr. Kincaid, introduction to differential equations, operational mathematics; Dr. Lee, introduction to differential equations; Dr. Livingstone, topics in algebra, algebra; Dr. Nesbitt, mathematics of life insurance, calculus of finite differences; Dr. Reade, introduction to differential geometry, introduction to functions of a complex variable with applications; Dr. Rosen, theory of equations and determinants, set-theoretic topology; Dr. Schaefer, advanced mathematics for engineers; Dr. Ullman, introduction to functions of a complex variable with applications, real analysis I; Dr. Mrowka, introduction to the foundations of mathematics; Dr. Shimrat, Fourier series and applications, intermediate course in differential equations.

University of Minnesota, College of Science, Literature and the Arts, June 13 to July 15: Professor Harper, advanced algebraic theory; Dr. Joichi, advanced analytic geometry, theory of numbers; Dr. Miracle, differential equations, critical reasoning in mathematical analysis. July 18 to August 19: Professor Guggenheimer, non-euclidean geometry; Professor Harper, advanced algebraic theory; Professor Turner, critical reasoning in mathematical analysis, calculus of variations; Dr. Govindarajulu, probability.

University of Oklahoma, June 8 to August 5: Professor Bernhart, theory of games, college geometry; Professor Brixey, principles of mathematical statistics, theory of groups; Professor Huneke, theory of equations; Professor Springer, ordinary and partial differential equations.

University of Pittsburgh, June 5 to August 4: Professor Blumberg, advanced calculus; Professor Bowers, functions of a complex variable; Professor Bryson, partial differential equations and Fourier series; Professor Taylor, geometry of the complex domain; Professor Laush, integral equations. June 26 to August 4: Professor Knipp, differential equations; Professor Kovacs, mathematical theory of statistics; Professor Teats, history of mathematics; Professor Myers, introduction to modern algebra.

University of South Carolina, June 12 to August 12: Professor Lee, introduction to modern algebra; Professor Weber, college geometry.

University of Tennessee, June 12 to August 24: Professor Albert, Laplace and Fourier transform; Professors Harrold and Mahavier, seminar in topology.

University of Utah, June 26 to August 19: Dr. Nelson, integral equations; Dr. Barrett, topics in topology.

University of Washington, linear algebra; introduction to modern algebra; fundamental concepts of analysis; topics in applied analysis; foundations of geometry; non-euclidean geometry; special topics in mathematics; foundations of mathematics; Visiting Professor Mackey, special topics in analysis.

BOOKS FOR ASIAN STUDENTS

The Books for Asian Students program of The Asia Foundation is now in its sixth year. Under this program over two million books and journals have been shipped to Asian educational institutions. The Asia Foundation will welcome donations of books suitable for the program. Suitable books are:

1. University, college, and secondary school books in good condition, published after 1945.
2. Scholarly, scientific, and technical journals in runs of five years or more.
3. Works by standard authors (e.g., Dickens, Hawthorne, Hemingway, Plato, W. James, T. Huxley, etc.)

The Asia Foundation will pay transportation costs from the donor to San Francisco and thence to Asia. Before shipping books to the Foundation, further information should be requested from: Books for Asian Students, 21 Drumm Street, San Francisco 11, California.

EXPERIMENTS IN MATHEMATICS

The Board of Directors of the National Council of Teachers of Mathematics appointed a committee to gather information and to analyze experimental mathematics programs. This committee would like your help in identifying such programs. It has limited its analysis to programs that (1) have printed instructional materials other than courses of study, (2) are of at least one semester in duration, and (3) are not sponsored by a commercial publishing firm. The committee will examine all material, ask the experimenter to make a short personal report, and publish as many of the reports as possible.

If you know of experimental programs or have one of your own, we would appreciate receiving your material or having you contact the chairman, whose address is: Philip Peak, Chairman, Committee on the Analysis of Experimental Programs, School of Education, Indiana University, Bloomington, Indiana.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH APPLIED MATHEMATICS PROGRAM

The Air Force Office of Scientific Research has announced a broad expansion of its support for research projects in the area of applied mathematics. This program which will emphasize both the traditional and modern aspects of applied mathematics will be a part of the program of the Directorate of Mathematical Sciences of AFOSR. It is the hope of the Air Force that this new program will help fill the increasing need for more mathematical methods that are required to solve today's complex problems in the physical and engineering sciences. Present plans call for the program to start in the fall of 1961. Interested applied mathematicians are encouraged to submit proposals for support under this program to the Director of Mathematical Sciences, Air Force Office of Scientific Research, Washington 25, D. C.

MATHEMATICS AND THE WASHINGTON SCENE

An article under the above title by G. Baley Price, Executive Secretary, Conference Board of the Mathematical Sciences, appeared in the February, 1961, *Notices of the American Mathematical Society*. It is called to the attention of the readers of the MONTHLY since it was not possible to publish it in the MONTHLY.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NEW EDITOR-IN-CHIEF

The Board of Governors of the Association has elected Professor Frederick A. Ficken of New York University as Editor-in-Chief of the AMERICAN MATHEMATICAL MONTHLY for a five-year period beginning January 1, 1962.

Professor Ficken has served as Associate Editor of the MONTHLY from 1951 to 1956 in charge of *Mathematical Notes*. He has also served on the Committee on Slaught Memorial Papers from 1958 to 1960, and on the current Committee on Publications. He has been an MAA Secondary School Lecturer and a member of the Committee on Secondary School Lecturers from 1958 to the present time. He was Chairman of the Nominating Committee in 1960. He has wide interests and experience in both pure and applied mathematics.

After July 1, 1961, articles intended for publication in the MONTHLY should be sent to Professor Ficken at this address: Department of Mathematics, New York University, University Heights, New York 53, N. Y.

HENRY L. ALDER, *Secretary*

THE FORTY-FOURTH ANNUAL MEETING OF THE ASSOCIATION

The forty-fourth Annual Meeting of the Mathematical Association of America was held at the Willard Hotel, Washington, D. C., from Wednesday to Friday, January 25 to 27, 1961, in conjunction with meetings of the American Mathematical Society, the Association for Symbolic Logic, and the Society for Industrial and Applied Mathematics. There were registered 1,527 persons, including 921 members of the Association.

Sessions of the Association were held on Wednesday and Thursday morning and on Friday morning and afternoon in the Grand Ballroom of the Willard Hotel. Presiding officers were Professor Rothwell Stephens on Wednesday morning, Second Vice-President Harley Flanders for the first lecture on Thursday morning, President C. B. Allendoerfer for the remainder of Thursday morning, First Vice-President A. S. Householder on Friday morning, and Professor B. J. Pettis on Friday afternoon. The Program Committee for the meeting consisted of E. E. Moise, Chairman; W. H. Durfee, Samuel Goldberg, S. A. Jennings, and B. J. Pettis.

FIRST SESSION OF THE ASSOCIATION

Probability and Statistics

Finite random walks, by Professor Hale F. Trotter, Princeton University.

Suppose a particle takes successive unit steps in the lattice of points with integral coordinates in k-dimensional space, with each of the $2k$ possible directions equally likely at each step. Let $P(x, N)$ be the probability that a particle starting at x visits the origin within the first N steps. Then $\lim_{N\to\infty} P(x, N)=1$ when $k=2$ but not when $k=3$. (This was first proved by Pólya in 1921.) An elementary proof was given, showing that the different behavior in 2 and 3 dimensions reflects

the fact that the 2-dimensional (logarithmic) potential is unbounded at infinity while the corresponding 3-dimensional potential is not.

Hitting probabilities, by Professor Frank L. Spitzer, Princeton University.

The interplay between potential theory and Markov processes is used to study random walk on the lattice points of Euclidean space. It is shown how to calculate hitting probabilities (*viz.*, the probability that a finite set A is first visited by the process x_n at $a \in A$, given that $x_0 = b \notin A$). Such probabilities are represented in terms of the kernel $a(x, y) = \sum_{n=0}^{\infty} [P(x_n = x) - P(x_n = y)]$. The crucial theorem asserts the existence of the kernel for every pair of points x, y, with a positive probability of being visited, even in the recurrent case, when each series separately diverges.

Applied statistical decision theory, by Professor Howard Raiffa, Harvard University.

The problem is to choose an experiment e from E and after observing the experimental outcome z in Z, to choose an act a from A to maximize the expected utility $E[u(e, z, a, \theta)]$, where $\theta \in \Theta$ is the unknown state parameter. It is assumed that to each e in E there is a given probability measure on $Z \times \Theta$. Special techniques are developed when (1) $u(e, z, a, \theta)$ is expressible in the form $u_s(e, z) + k_a + K_a w(\theta)$, (2) the conditional sampling measures on Z given e and θ are in the exponential family and (3) the conditional measures on Θ given e and z are conjugate to the sampling measures of (2).

SECOND SESSION OF THE ASSOCIATION

Metric entropy and approximation, by Professor George G. Lorentz, Syracuse University.

Let F be a compact subset of a linear metric space X. We wish to characterize the "massiveness" of F. For $\epsilon > 0$, let $n_\epsilon(F)$ be the minimal number of points in an ϵ-net in F. Then $H_\epsilon(F) = \log n_\epsilon(F)$ is the entropy of F. An essential contribution to the 13th problem of Hilbert is possible with this tool. Let $d(F, L_n)$ be the supremum of the distances of points $x \epsilon F$ to some n-dimensional subspace $L_n \subset X$. Then $d_n(F) = \inf_{L_n} d(F, L_n)$ is the n-dimensional width of F. For sets of functions, the exact values of $H_\epsilon(F)$ and $d_n(F)$ are usually unknown, but methods to determine their asymptotic behavior for $\epsilon \to 0$, $n \to \infty$ can be given.

Annual Business Meeting of the Association

Undergraduate preparation for graduate work, by Professor Andrew H. Gleason, Harvard University.

Analysis of even a strong program in mathematics at what is customarily regarded as undergraduate level reveals that only a small amount of mathematics is covered from a theoretical point of view. This is because it is impossible to present abstract mathematics to students until they have developed a certain *savoir faire*. Some students almost seem to be born with it; they can, and often do, start a graduate program in mathematics early in their undergraduate careers. Others acquire savoir faire slowly; still others never do. The speaker submits that the one thing that is truly important as preparation for graduate training in mathematics is the ability to think like a mathematician.

THIRD SESSION OF THE ASSOCIATION

Applied Mathematics

Quasi-conformal mappings, by Professor Lipman Bers, New York University.

This talk reviews the concept of quasi-conformal mappings and indicates its applications to applied mathematics, partial differential equations, and theory of functions, with emphasis on moduli of Riemann surfaces.

Some problems in applied mathematics, by Professor J. B. Keller, New York University.

Of all cylindrical rods with convex cross sections of unit area, which is the stiffest? This

optimum design problem in elasticity leads to the following mathematical problem in the calculus of variations. Which convex plane region of unit area yields the largest value for the minimum of the moment of inertia about any line through the centroid? The solution of this variational problem yields an isoperimetric inequality which gives an upper bound on the stiffness of any cylindrical rod with a convex cross section. Other optimum design problems in elasticity, heat conduction and other fields of application of mathematics lead to similar variational problems and isoperimetric inequalities.

Applied mathematics as a science, by Professor H. P. Greenspan, Massachusetts Institute of Technology.

There is, as yet, no general agreement concerning the nature of study and research in applied mathematics. As a direct consequence, this important area is badly neglected and does not appear in most university curricula as an independent entity or body of knowledge. Recognizing the need, a few major institutions, including the Massachusetts Institute of Technology, have developed comprehensive programs based on the philosophy that applied mathematics is primarily a science which seeks knowledge and understanding of physical phenomena through the use of mathematical methods. The emergent concept of the applied mathematician is that of a versatile scientist—a specialist in mathematics—with broad and active interests in many scientific areas, whose ultimate efforts are directed to the creation of ideas, concepts, and methods that are of basic and general applicability. The nature of applied mathematics is best explained by discussing its relationship to pure mathematics, theoretical physics, and engineering science. The distinctive attitude, approach and way of thinking is illustrated by several examples taken from current research in the fields of magneto-hydrodynamics and compressible gas dynamics.

FOURTH SESSION OF THE ASSOCIATION

Topology

The topology of group-like spaces, by Professor Eldon Dyer, University of Chicago.

As an example of the method of Algebraic Topology, a discussion is given of certain homology structures. The homology groups of cell-complexes are described. If the spaces in question have a continuous multiplication, their homology groups, with suitable coefficients, take on a richer structure, that of a Hopf algebra. The structure of such algebras is discussed and geometric applications are given.

Some applications of homotopy theory to geometric problems, by Professor Raoul Bott, Harvard University.

The notion of a smooth manifold is discussed with special emphasis on the parallelizability question.

Some geometric applications of algebraic topology, by Professor John W. Milnor, Princeton University.

An example is given of two finite simplicial complexes which are homeomorphic, but do not have isomorphic subdivisions. The proof depends on a recent theorem of B. Mazur. Let L_i denote a 3-dimensional Lens manifold of type $(7, i)$; and let K_i denote the space obtained from the Cartesian product of L_i with a 5-cell by collapsing the boundary to a point. Then K_i can be triangulated in a natural way. Using methods due to Reidemeister, Franz, and J. H. C. Whitehead, it is shown that no cell-subdivision of K_1 is isomorphic to a cell-subdivision of K_2. On the other hand, following Mazur, K_1 is homeomorphic to K_2.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Wednesday afternoon in room 220 of the Willard Hotel in Washington with twenty-nine members present. Among the

items of business transacted were the following:

The Board approved the appointment by President Allendoerfer of the following Nominating Committee for 1961: R. P. Boas, Chairman; J. A. Clarkson, and C. R. Wylie, Jr.

The Board elected Professor Robert A. Rosenbaum of Wesleyan University as Second Vice-President for the two-year term 1961–1962.

The Board expressed its regret at the resignation of Professor Walter B. Carver as a member of the Finance Committee and approved the following resolution:

> "Resolved that
>
> The Board of Governors greatly regrets the resignation of Walter B. Carver as a member of the Finance Committee.
>
> In accepting this resignation we wish to take note of his more than 40 years of service to the Association including terms as President, Secretary-Treasurer, and Editor of the MONTHLY, and to thank him deeply for all that he has contributed to the Association.
>
> We wish him rapid recovery from his recent illness, and hope to see him frequently at future meetings of the Association."

The Board then elected Professor Carl B. Allendoerfer to fill the unexpired part of the term of Professor Carver ending in 1963 as a member of the Finance Committee.

The Board elected Professor Frederick A. Ficken of New York University as Editor-in-Chief of the MONTHLY for a five year period beginning January 1, 1962.

The Board acting upon the recommendations of its Committee on Publications under the Chairmanship of Professor R. P. Dilworth approved the following:

1. As soon as feasible, the volume year for the *Mathematics Magazine* should be changed to coincide with the calendar year.

2. The first editor of the *Mathematics Magazine* shall be appointed for a term extending through December 1963. Succeeding editors shall be appointed for five-year terms with the proviso that no editor may serve two consecutive terms.

3. Appropriate steps should be taken to insure that the transfer of the *Mathematics Magazine* to the Association be legally valid.

4. The Board of Governors, upon recommendation by the Editor, shall appoint the Editorial Board of the *Mathematics Magazine* for terms to coincide with the term of the Editor.

The Board then elected Professor Robert E. Horton of Los Angeles City College as Editor of the *Mathematics Magazine* for the three-year term January 1, 1961, to December 31, 1963.

The Board authorized continuation of the present subscription rates for the *Mathematics Magazine* with appropriate discounts to dealers and with a special subscription rate of $5.00 for 2 years available to members of the Association for subscriptions placed directly through the office of the Association.

The Board acting upon the recommendations of the Joint Committee on the Doctor of Arts Degree under the Chairmanship of Professor E. E. Moise voted that the degree of Doctor of Arts be established, in mathematics, at most of the universities which are qualified to grant the Ph.D. While the name of the degree was not considered part of the substance of the Committee's proposal, the Board approved a motion that the preferred name for the new degree be the Doctor of Arts. A more detailed report on the recommendations of this Committee will appear in the May issue.

The Board approved the following schedule of future meetings: Oklahoma State University, August 28–30, 1961; Sheraton-Gibson Hotel, Cincinnati, Ohio, January 1962; University of British Columbia, August 1962; University of California, Berkeley, January 1963; University of Colorado, August 1963; University of Michigan, August 1964;

Cornell University, August 1965; Rutgers-The State University, New Brunswick, New Jersey, August 1966.

The Board acting upon the recommendations of the Joint Committee on the Setting of Winter Meetings under the Chairmanship of Professor G. A. Hedlund approved:

1. that the practice of holding the Annual (Winter) Meeting near the end of January be continued,

2. that the representatives of the Society and Association appointed with the task of determining the place of the Annual Meeting should not preclude the possibility of holding the Annual Meeting in conjunction with a university,

3. that the determination of the time and place of such meetings be entrusted to a committee consisting of the Secretaries and Executive Directors of the stated organizations.

The Board authorized the Executive Committee to proceed with plans for celebration of the fiftieth anniversary of the Association which will occur in 1965.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday, January 26, 1961 in the Grand Ballroom of the Willard Hotel, Washington, D. C. with President Allendoerfer presiding. The Secretary reported that the membership of the Association was 10,197, an increase of 1,084 since the corresponding date last year.

The balloting for officers in which 2,032 votes were cast resulted in the election of Professor A. W. Tucker of Princeton University as President for the two-year term 1961–1962, and of Professors P. R. Halmos of the University of Chicago and John L. Kelley of the University of California, Berkeley, as Governors for the three-year term 1961–1963.

The Secretary then reported on some of the actions taken by the Board of Governors the previous day. He expressed, on behalf of the Association, the highest appreciation to all those responsible for the excellent arrangements for the meeting and singled out for special commendation Professor Everett Pitcher, the Chairman of the Arrangements Committee, Dr. Gordon L. Walker, Executive Director of the Society, Mrs. Robert Drew-Bear, for their most helpful efforts in coordinating the activities of the Society and the Association, and Professor John W. Brace for the latter's unusually effective efforts as Publicity Director for the meeting.

The two amendments to the By-laws which were printed on page 951 of the November 1960 issue of the MONTHLY were unanimously adopted.

President Allendoerfer then presented a summary of the status and activities of the Association.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Monday, January 23, through Thursday, January 26. The thirty-fourth Josiah Willard Gibbs Lecture was delivered by Professor J. J. Stoker of New York University on Tuesday evening at 8:00 p.m. on "Problems in Non-linear Elasticity." At other sessions, Professor Lars Hörmander delivered an invited address entitled "On the Range of Differential Operators" and Professor Helmut Wielandt an address entitled "On the Structure of Finite Groups."

The Association for Symbolic Logic met on Tuesday. An invited address was given given at 2:00 p.m. by Professor S. C. Kleene on "Foundations of Intuitionistic Mathematics."

The Society for Industrial and Applied Mathematics met on Wednesday evening at 7:30 p.m. when Dr. Brockway McMillan delivered the retiring presidential address entitled "An Elementary Approach to the Theory of Information."

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of Everett Pitcher, Chairman; H. L. Alder, J. W. Brace, M. W. Oliphant, G. L. Walker.

Registration headquarters were located in the Caucus Room on the first floor of the Willard Hotel. The book exhibits were located in the Jackson Room on the first floor and the employment register in room 220. The Willard Hotel was the official hotel for the meeting, but the Washington Hotel (next door to the Willard) and the Raleigh Hotel (two blocks away) cooperated in reserving blocks of rooms.

HENRY L. ALDER, *Secretary*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The nineteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at the City College of New York on April 2, 1960. Dr. B. G. Gallagher, President of the City College, gave the address of welcome. Professor J. P. Russell, Collegiate Vice-Chairman of the Section, presided at the morning session and Dr. George Grossman, High School Vice-Chairman, presided at the afternoon session. One hundred eighty-three persons, including 103 members of the Association, attended the meeting.

Professor Azelle B. Waltcher, Chairman of the Section, presided at the business meeting. The following proposal to amend the By-laws of the Section was approved: "The Executive Committee of the Section will include, in addition to the other members provided by the By-laws, one or more representatives of science and industry to be selected by the officers of the Section." Previously, the By-laws provided for only one such representative on the Executive Committee. Reports were presented by the Treasurer, Mr. Aaron Shapiro, by Professor J. N. Eastham for the Speaker's Bureau, and by Professor C. T. Salkind for the Committee on Contests and Awards.

The following papers were presented at the meeting:

1. *Nonstandard models of axiomatic theories*, by Professor Elliot Mendelson, Columbia University.

From the completeness theorem for first-order logic, it follows that axiomatic set theory has a model of every infinite power, and, in particular, a denumerable model, in apparent (but not real) contradiction of the fact that the existence of nondenumerable sets is provable in the theory. There are also nonstandard models in which the collection of ordinals of the model is not well-ordered by ϵ. For any infinite power, there are at least $2^{\aleph_0}$ models of that power for formalized elementary number theory, and, even stronger, models having all elementary properties (including Goldbach's conjecture and Fermat's theorem or their negations) in common with the nonnegative integers.

2. *Some aspects of numerical analysis*, by Dr. H. H. Goldstine, International Business Machines, Yorktown Heights, New York.

The author first discussed the number system and arithmetical processes of digital calculation and showed the circumstances under which the familiar associative, distributive and commutative laws hold. Secondly, he discussed the topic of numerical stability and illustrated the concept with an analysis of the Bessel recurrence relations.

3. *The graph of a group*, by Professor Wilhelm Magnus, New York University.

The graph of a group was introduced by A. Cayley 100 years ago and utilized as an instrument of research by M. Dehn in 1910. It offers an access to group theory which has the advantage of giving a geometric (*i.e.*, visible) interpretation to groups. Also, the idea that a group element may be interpreted as a path provides an easy transition from group theory to some of its applications in topology, at least in an intuitive manner.

MARY P. DOLCIANI, *Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The fifth annual meeting of the New Jersey Section of the Mathematical Association of America was held at Rutgers, The State University on November 5, 1960. Dr. H. O. Pollak, Chairman of the Section, presided at the morning session and Professor E. P. Starke presided at the afternoon session. There were 80 persons in attendance, including 59 members of the Section.

At the business meeting, an amendment to the By-Laws creating the office of Associate Secretary-Treasurer was approved. It was felt that this change would make it possible to divide the work of the office, and also provide a desirable continuity in the work. The Associate Secretary-Treasurer, elected for a three-year term, may succeed himself only once, but may subsequently serve as Secretary-Treasurer of the Section.

The following officers were elected: Chairman, Professor E. P. Starke, Rutgers, The State University; Senior Member of the Executive Committee, 1960–61, Dr. George Cherlin, Mutual Benefit Life Insurance Company; Member of the Executive Committee, 1960–62, Dr. Sheldon Meyers, Educational Testing Service; Member of the Executive Committee, 1960–63, Professor John Schumaker, Montclair State College; Associate Secretary-Treasurer, 1960–62, Mr. F. A. Varrichio, Saint Peter's College.

The report of Mr. R. S. Lockhart, Chairman of the Contest Committee, was presented by Dr. Cherlin, the Committee Secretary. 5,272 students from 139 schools participated in the 1960 contest, as contrasted with 4,462 students from 133 schools in 1959.

At the morning session the following papers were presented:

1. *Strategy in antimissile defense*, by Dr. W. T. Read, Jr., Bell Telephone Laboratories, Murray Hill, New Jersey (by invitation).

The application of mathematical analysis to problems lying in the broad, ill defined area between weapons system engineering on the one hand and military policy planning on the other was discussed. The approach was illustrated by using a simple model of anti-missile defense to derive various results relating to firing doctrine, deployment, offensive and defense strategies, and overall evaluation of defensive effectiveness.

2. *The rise of analysis through mechanics*, by Professor Solomon Bochner, Princeton University (by invitation).

Analysis rather than algebra constitutes the difference between Greek mathematics and "modern" mathematics which came into being during and after the Renaissance. Mathematics became the language of science mainly through analysis. Only after the great needs of classical mechanics and electrodynamics were satisfied could the present day algebraization of mathematics (and analysis) begin in earnest.

At the afternoon session, Professor B. E. Meserve, Montclair State College, reported on the work of the Panel on Teacher Training of the CUPM. There then followed a panel discussion:

3. *Careers for the mathematically gifted*, by Dean E. C. Easton, Rutgers, The State University (engineering), Dr. George Cherlin, Mutual Benefit Life Insurance Company (actuarial work), Professor E. R. Ott, Rutgers, The State University (statistics), Dr. H. O. Pollak, Bell Telephone Company Laboratories, Murray Hill, New Jersey (industry), Dr. Donald Thomson, International Business Machines Corporation, New York City (computing), and Professor B. E. Meserve, Montclair State College (teaching).

I. L. BATTIN, *Secretary*

REPORT OF THE TREASURER FOR THE YEAR 1960

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1960. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of

the Association who wishes the complete report of the Treasurer may obtain it by writing to the Buffalo office of the Association.

The Current Fund ended the year with a surplus of $6,790 of which $5,000 has been transferred to the General Fund. Again this year contributions amounting to $204 have been added to the General Fund.

Among the temporary funds held by the Treasurer are certain items being held for the account of the Conference Board of the Mathematical Sciences. As soon as that Board becomes fully incorporated, these funds will be transferred to the Treasurer of the Conference Board.

Assets of the Association		January 1, 1960		December 31, 1960
M. & T. Trust Co., Buffalo		$ 25,081		$119,699
Savings Accounts		111,699		89,550
Securities		275,542		388,041
		$412,322		$597,290
Funds of the Association				
Current Fund		$ 2,526		$ 4,316
Carus Fund		28,798		35,066
Chace Fund		9,546		9,062
Houck Fund		12,028		11,012
Chauvenet Fund		1,642		1,671
Dunkel Fund		19,745		21,410
General Fund		64,266		77,316
		$138,551		$159,853
Visiting Lecturers		$ 78,423		$ 79,580
CUPM (Ford Grant)		54,923		41,340
CUPM (NSF Grant)		—		164,809
High School Contests	Deficit	77	Deficit	5,828
Survey Non-Teaching Math. Employment		10,384		3,768
Secondary School Lecturers		73,883		52,091
Production of Films		15,437		8,711
Washington Office		34,154		48,732
Survey European Math. Education		6,644		3,980
Institutes		—		10,254
Conference Board Survey		—		30,000
		$412,322		$597,290

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CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.
ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.
INDIANA, Rose Polytechnic Institute, Terre Haute, May 6, 1961.
IOWA
KANSAS
KENTUCKY, Western Kentucky State College, Bowling Green, April 29, 1961.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, St. Cloud State College, May 13, 1961.
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, University of Vermont, Burlington, June 20, 1961.
NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.
OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.
OKLAHOMA, Oklahoma State University, Stillwater, May, 12, 1961.
PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.
PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, University of Wisconsin, Madison, May 13, 1961.

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CONTENTS

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EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

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SOME ELEMENTARY CRYPTANALYSIS OF ALGEBRAIC CRYPTOGRAPHY

JACK LEVINE, North Carolina State College

1. Introduction. By algebraic cryptography we mean the process of encipherment which converts a plain message into a cipher message by means of n simultaneous linear congruences, where n is an arbitrary integer ([**1**], [**2**], [**3**], [**4**]). The plain letters are written in blocks of n, and if $P_{i\beta}$ denotes the letter in position β of block i, then

$$C_{i\beta} \equiv a_{\beta 1}P_{i1} + \cdots + a_{\beta n}P_{in} \pmod{26}, \tag{1.1}$$

where $i=1, 2, \cdots, \beta=1, \cdots, n$, so that the encipherment of plain block $P_{i1} \cdots P_{in}$ is cipher block $C_{i1} \cdots C_{in}$.

In (1.1) the matrix $A = [a_{\alpha\beta}]$ of the coefficients is such that $|a_{\alpha\beta}|$ is prime to 26. It is convenient for cryptographic purposes to take A such that $A = A^{-1}$, and we assume A is so chosen. Hence we can also write

$$P_{i\beta} \equiv a_{\beta 1}C_{i1} + \cdots + a_{\beta n}C_{in} \pmod{26}. \tag{1.2}$$

The 26 letters of the alphabet are given numerical values according to some permutation of the normal sequence 0, 1, $\cdots$, 25; and it is these numerical values which are actually used for the $P_{i\beta}$ and $C_{i\beta}$ above. For the purposes of this article there is no loss of generality in using the normal sequence itself, giving

(1.3)	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

For illustrations in the actual use of (1.1), (1.3), [3] may be consulted.

In its most general form the cryptanalytic problem involved here may be stated in the following way. Given one (or more) cryptograms obtained by use of (1.1), but with no other information, to find the corresponding plain messages, and the matrix A. Even for such small values of n as $n=5$ this would ordinarily be a problem of very great difficulty.

Our purpose here is to treat what may be considered as one of the simplest special cases of the general problem. We assume as known the numerical values of the alphabet letters (see (1.3)), and also as known some portion of the plain message of a cryptogram. However, the exact location of this portion in the message is not known. This is an instance of the classical "method of the probable word," well known in the cryptanalytic art. The solution of this case involves two steps:

Problem (A): Determining the location of the known plain-text.

Problem (B): Determining matrix A.

The final test is of course deciphering the cryptogram based on the recovered matrix.

2. Some basic relations. Consider any $n+1$ blocks of plain-text $P_{i1} \cdots P_{in}$ for $i=i_1, \cdots, i_{n+1}$. By (1.2) we can write

$$P_{i_t\beta} \equiv \sum_{\sigma=1}^{n} a_{\beta\sigma} C_{i_t\sigma} \qquad (t = 1, \cdots, n+1), \tag{2.1}$$

where in (2.1), and hereafter unless otherwise stated, all congruences are taken mod 26. In (2.1) consider β as fixed (a value from $1, \cdots, n$). Regarding (2.1) as a system of $n+1$ congruences in n unknowns $a_{\beta 1}, \cdots, a_{\beta n}$, we may eliminate these to obtain the condition

$$\Delta_\beta = \begin{vmatrix} P_{i_1\beta} & C_{i_1 1} & \cdots & C_{i_1 n} \\ \vdots & \vdots & & \vdots \\ P_{i_{n+1}\beta} & C_{i_{n+1}\beta} & & C_{i_{n+1}n} \end{vmatrix} \equiv 0 \qquad (\beta = 1, \cdots, n). \tag{2.2}$$

In a like manner we derive from (1.1),

$$\Delta'_\beta = \begin{vmatrix} C_{i_1\beta} & P_{i_1 1} & \cdots & P_{i_1 n} \\ \vdots & \vdots & & \vdots \\ C_{i_{n+1}\beta} & P_{i_{n+1}1} & \cdots & P_{i_{n+1}n} \end{vmatrix} \equiv 0 \qquad (\beta = 1, \cdots, n). \tag{2.3}$$

Relations $\Delta_\beta \equiv 0$, $\Delta'_\beta \equiv 0$ will be found useful in our problem since they do not involve the (unknown) matrix elements $a_{\alpha\beta}$. Their use is illustrated below.

From (2.1) select any n values of i_t, say, $j_1, \cdots, j_n$, and form the determinant based on the resulting n^2 congruences (using $\beta = 1, \cdots, n$). This gives

$$\left| P_{j_t\beta} \right| \equiv \left| \sum a_{\beta\sigma} C_{i_t\sigma} \right| \equiv \left| a_{\beta\sigma} \right| \left| C_{j_t\sigma} \right|,$$

or

$$\begin{vmatrix} P_{j_1 1} & \cdots & P_{j_1 n} \\ \vdots & & \vdots \\ P_{j_n 1} & \cdots & P_{j_n n} \end{vmatrix} \equiv \pm \begin{vmatrix} C_{j_1 1} & \cdots & P_{j_1 n} \\ \vdots & & \vdots \\ C_{j_n 1} & \cdots & C_{j_n n} \end{vmatrix} \tag{2.4}$$

since $|a_{\alpha\beta}| \equiv \pm 1$, as follows from $A = A^{-1}$.

In (2.2), (2.3), (2.4), the cipher values are known. The plain values will be selected from the known probable text. Briefly, the correct location of this known text will be found by fitting it in all positions until one is obtained for which (2.2), (2.3), (2.4) are satisfied. The exact method for doing this with a minimum of trials is explained in the sections to follow.

3. Problem (A) **and the use of modulo 2.** Consider the system of n congruences (1.1) for a fixed i and for modulo 2. We write these as

$$C_\beta \equiv a_{\beta 1} P_1 + \cdots + a_{\beta n} P_n \pmod 2 \qquad (\beta = 1, \cdots n) \tag{3.1}$$

so that $P_1 \cdots P_n$, $C_1 \cdots C_n$ represent any block of n plain and corresponding

n cipher values. As $P_1 \cdots P_n$ assumes all possible sequences of n letters, (3.1) establishes a unique 1-1 reciprocal correspondence between the 2^n length n binary sequences of 0's and 1's. These 2^n sequences represent the 2^n integers (in base 2) from 0 through 2^n-1. Hence by (3.1) there is associated with each block of n plain letters and its corresponding block of n cipher letters a pair of numbers in the range 0 through 2^n-1. The number thus associated with a block of n letters is called its binary value.

The following example illustrates these pairings and their use in Problem (A).

(a_1)	CRY	PTO	GRA	PHY	BYA	LGE	BRA	ICE	QUA	TIO	NSX
(b_1)	SUI	RIM	AYG	DIK	VFG	LTE	RUK	KRC	QHA	JLY	JOB

(a_2)	3 18 25	16 20 15	7 18 1	16 8 25	2 25 1	12 7 5	2 18 1
(b_2)	19 21 9	18 9 13	1 25 7	4 9 11	22 6 7	12 20 5	18 21 11

(a_2)	9 3 5	17 21 1	20 9 15	14 19 24
(b_2)	11 18 3	17 8 1	10 12 25	10 15 2

(a_3)	101	001	101	001	011	011	001	111	111	011	010
(b_3)	111	011	111	011	001	001	011	101	101	001	010

(a_4)	5	1	5	1	3	3	1	7	7	3	2
(b_4)	7	3	7	3	1	1	3	5	5	1	2

Here, (a_1), (b_1) represent a plain-text with its cipher text; (a_2), (b_2) are the numerical values of the letters by (1.3); (a_3), (b_3) give these values (mod 2); and (a_4), (b_4) are the binary values according to the scheme

$$000 = 0,\ 000 = 1,\ 010 = 2,\ 011 = 3,\ 100 = 4,\ 101 = 5,\ 110 = 6,\ 111 = 7.$$

Thus, the binary value of CRY $=5$, of SUI $=7$, etc. Because of $A=A^{-1}$ the binary value pairing as in (a_4), (b_4), will always be such that $a \leftrightarrow b$, $0 \leftrightarrow 0$ where a may equal b. (In the above example it is seen that $1\leftrightarrow 3$, $2\leftrightarrow 2$, $5\leftrightarrow 7$). It follows that the two binary-value sequences derived from any plain-text and its corresponding cipher text must always be of the same pattern. This pattern in the present illustration could be represented by the sequence *ababccbddce* (see (a_4), (b_4)). Furthermore, any portion of the plain-text must produce a common pattern with its associated cipher portion.

It is this last property which is used to obtain preliminary locations (apparent settings) of the probable text. The cipher text is converted to its binary values (as in (b_4) above), and the probable text is similarly converted, assuming it starts in each of the n positions of a block. There will thus be n such binary conversions, $B_1, \cdots, B_n$, each of which is then matched against the cipher text conversion for like patterns. Each such matching is an apparent setting of the probable text, and these are next tested against conditions (2.2), (2.3), (2.4), depending on the material available. This feature is taken up in the next section.

To demonstrate the pattern matching, consider the example,

(3.2)

MIU	GNJ	WWU	YHZ	DNS	WVK	RFV	LLK	AMP	IGS	MIU
7	4	7	4	1	5	0	1	6	7	7
WKN	OEM	IEK	ORW	WAE	KZB	APL	KYP	MEU	ZMO	ZIX
6	7	7	5	7	4	4	6	7	3	6
FHS	SJI	DDJ	KFY	BWW	HQP	KLI	NKG	TMJ	ROB	TZE
1	5	0	5	3	2	5	3	2	2	1

The number under each group is its binary value. We assume the probable text THREE CONGRUENCES, but use the first 14 letters only. Let S_α indicate the first letter starts in position α of a block ($\alpha = 1, 2, 3$). We then obtain

S_1:	THR	EEC	ONG	RUE	NCx
B_1:	0	7	5	3	

S_2:	xTH	REE	CON	GRU	ENC
B_2:		3	6	5	5

S_3:	xxT	HRE	ECO	NGR	UEN	Cxx
B_3:		1	7	2	6	

From (3.2) the cipher text binary sequence is

(3.3) 7 4 7 4 1 5 0 1 6 7 7 6 7 7 5 7 4 4 6 7 3 6 1 5 0 5 3 2 5 3 2 2 1

The B_1 pattern, 0 7 5 3, must obviously match at 0 1 6 7 or 0 5 3 2 of (3.3), giving

(3.3)	0 1 6 7	0 5 3 2
B_1:	0 7 5 3	0 7 5 3

Each of these matchings can be eliminated by inspection, the first by 7 paired with both 1 and 3, and the second by 3 with 2 and 5.

The B_2 pattern, 3 6 5 5, results in four apparent settings:

(3.3)	1 6 7 7	7 6 7 7	5 7 4 4	5 3 2 2
B_2:	3 6 5 5	3 6 5 5	3 6 5 5	3 6 5 5

of which all but the first can be eliminated at once. The first setting is kept for further testing.

The B_3 pattern is matched at

(3.3)	7 4 1 5	4 6 7 3	7 3 6 1	3 6 1 5
B_3:	1 7 2 6	1 7 2 6	1 7 2 6	1 7 2 6

all of which are inconsistent. If the probable text is actually present, then it must start in block 7 of (3.2):

(3.4)

	1	6	7	7
RFV	LLK	AMP	IGS	MIU
xTH	REE	CON	GRU	ENC
	3	6	5	5

4. Problem (A) **continued. Testing of apparent settings.** In the above example only one apparent setting, (3.4), survived the binary pattern test. Ordinarily with longer messages, shorter probable texts, or larger values of n, a large number of apparent settings will remain, and these must be further eliminated.

Let λ equal the length of the probable text. Then the actual procedure used will largely depend on the value of λ.

Suppose first $\lambda \geqq n(n+1)+(n-1)$. For any starting point S_α write the probable text corresponding to an apparent setting beginning in block i and position α in an array of consecutive blocks vertically as given below (corresponding cipher blocks are also shown).

$$\begin{array}{ccccc|ccc} x & \cdots & \cdots x\, P_{i\alpha} & \cdots & P_{in} & C_{i1} & \cdots & C_{in} \\ P_{i+1,1} & \cdots & P_{i+1,\alpha} & \cdots & P_{i+1,n} & C_{i+1,1} & \cdots & C_{i+1,n} \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot \\ P_{i+n+1,1} & \cdots & P_{i+n+1,\alpha} & \cdots & P_{i+n+1,n} & C_{i+n+1,1} & \cdots & C_{i+n+1,n} \end{array} \tag{4.1}$$

Then regardless of the values of i and α there will always be at least $n+1$ complete blocks of (presumably) known plain-text as indicated in (4.1). Now if there are a large number of apparent settings with a common value of α, it will be found that (2.3), or $\Delta_\beta' \equiv 0$, is best to use to test these settings. For n columns of (2.3) can be kept fixed, these being selected from the array of plain-text columns in (4.1). The remaining column is chosen from the cipher-text array at the right in (4.1), and (2.3) then expanded using cofactors of this C column,

$$P_1 C_{i\beta} + \cdots + P_{n+1} C_{i+n+1,\beta} \equiv 0. \tag{4.2}$$

The cofactors $P_1, \cdots, P_{n+1}$ being independent of i need be calculated only once as i varies through values corresponding to the apparent settings.

If (4.2) be first tested (mod 2), many apparent settings will be eliminated very easily. The remaining are then tested (mod 26).

If we apply these ideas to the apparent setting of (3.4) we have corresponding to (4.1),

$$\begin{array}{ccc|ccc} x & \mathrm{T} & \mathrm{H} & \mathrm{R} & \mathrm{F} & \mathrm{V} \\ \mathrm{R} & \mathrm{E} & \mathrm{E} & \mathrm{L} & \mathrm{L} & \mathrm{K} \\ \mathrm{C} & \mathrm{O} & \mathrm{N} & \mathrm{A} & \mathrm{M} & \mathrm{P} \\ \mathrm{G} & \mathrm{R} & \mathrm{U} & \mathrm{I} & \mathrm{G} & \mathrm{S} \\ \mathrm{E} & \mathrm{N} & \mathrm{C} & \mathrm{M} & \mathrm{I} & \mathrm{U} \end{array} \tag{4.3}$$

Forming Δ_1' gives

$$\begin{vmatrix} 18 & 5 & 5 & 12 \\ 3 & 15 & 14 & 1 \\ 7 & 18 & 21 & 9 \\ 5 & 14 & 3 & 13 \end{vmatrix},$$

which is $\equiv 0 \pmod 2$ and (mod 26). It is found that Δ_2', Δ_3' are also both $\equiv 0 \pmod 2$ and (mod 26). In addition, (2.4) is satisfied for any choice of 3 rows from (4.3). This setting would next be used to obtain matrix A, (Problem (B)).

Suppose next $n^2 < \lambda < n(n+1)+n-1$. In this case certain starting points will not produce $n+1$ complete plain-text blocks in array (4.1), and the method described above will consequently fail. For such cases we can use (2.2), (and (2.4) when available), since, if $\lambda > n^2$, there must always be at least one column in the plain-text array of (4.1) containing $n+1$ letters, this column then being used in (2.2). Thus, for certain settings (values of α), (2.2) is used, and for the remaining (2.3) is used. In addition (2.4) is used when necessary.

We discuss again the previous example (3.2), this time using only 10 letters of the probable text (this being the minimum for $n=3$), or THREE CONGR. The three starting points give

S_1		S_2		S_3	
T H R	0	x T H	0/4	x x T	
E E C	7	R E E	3	H R E	1
O N G	5	C O N	6	E C O	7
R x x		G R x	4/5	N G R	2

The binary patterns are written at the right of each array. In S_2 we indicate the two possibilities 0/4, 4/5 as shown. These patterns are now matched against the cipher pattern (3.3), with the following preliminary matchings:

B_1:

0	7	5		0	7	5
0	1	6		0	5	2

B_2:

0/4	3	6	4/5
7	4	1	5
0	1	6	7
4	6	7	3
6	7	3	6
7	3	6	1
3	6	1	5
0	5	3	2
5	3	2	5
3	2	5	3
2	5	3	2

B_3:

1	7	2
7	4	1
4	1	5
1	6	7
5	7	4
4	6	7
6	7	3
7	3	6
3	6	1
6	1	5
5	3	2
2	5	3
5	3	2

All but the following seven are eliminated by inspection,

0 7 5	0 3 6 4/5	4 3 6 5	4 3 6 4	1 7 2	1 7 2	1 7 2
0 1 6	0 1 6 7	7 3 6 1	5 3 2 5	5 7 4	6 7 3	5 3 2
$S_1(a)$	$S_2(b)$	$S_2(c)$	$S_2(d)$	$S_3(e)$	$S_3(f)$	$S_3(g)$

The corresponding arrays (4.1) are

T H R	R F V	x T H	R F V	M E U	K F Y	x x T	I E K	A P L	D D J
E E C	L L K	R E E	L L K	Z M O	B W W	H R E	O R W	K Y P	K F Y
O N G	A M P	C O N	A M P	Z I X	H Q P	E C O	W A E	M E U	B W W
R x x	I G S	G R x	I G S	F H S	K L I	N G R	K Z B	Z M O	H Q P
	$S_1(a)$		$S_2(b)$	$S_2(c)$	$S_2(d)$		$S_3(e)$	$S_3(f)$	$S_3(g)$

Since none of the S_α (plain-text) settings contain four complete blocks we must use text (2.2), first (mod 2). The column TEOR of length 4, from the probable text, is of course used as the plain-text column in (2.2) in all cases. Cases (c), (e), (f) are eliminated using modulo 2 and (2.2). The remaining four are tested (mod 26). This eliminates (d) and (g), leaving $S_1(a)$ and $S_2(b)$. We now use (2.4) on $S_1(a)$ as three complete blocks are present in the probable text, and this eliminates $S_1(a)$.

$S_2(b)$ is the only case left ((2.4) cannot be used on it) and the next step would be to recover matrix A. This is taken up in the next section.

5. Problem (B). **Determination of matrix** A. Assuming a probable text has been located (as in $S_2(b)$ above), (1.2) give a series of n^2 congruences for the determination of the elements $a_{\alpha\beta}$ of matrix A. For a fixed α, (1.2) determine at least n congruences in the n unknowns $a_{\alpha 1}, \cdots, a_{\alpha n}$. If their coefficient matrix contains an $n\times n$ determinant prime to 26, all these unknowns are determined uniquely. Otherwise, there will be several solutions possible. Finally the use of condition $A^2=I$ will pick out the correct matrix which should decipher the cryptogram.

We carry out this procedure by completing the solution of above example (3.2). The setting $S_2(b)$ is to be used with (1.2). Convert all letters to numerical values, and write the corresponding (1.2) in matrix form:

$$(5.1)\qquad \begin{bmatrix} x & 18 & 3 & 7 \\ 20 & 5 & 15 & 18 \\ 8 & 5 & 14 & x \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 18 & 12 & 1 & 9 \\ 6 & 12 & 13 & 7 \\ 22 & 11 & 16 & 19 \end{bmatrix}$$

or, $P\equiv AC$, the various rows of $S_2(b)$ now appearing as columns. The C matrix contains no 3×3 determinant prime to 26. The 10 congruences of (5.1) split into 3 sets of 3, 4, 3 respectively for the 9 unknowns $a_{\alpha 1}, a_{\alpha 2}, a_{\alpha 3}$ ($\alpha=1, 2, 3$).

Solving the congruences of (5.1) using mod 2 and mod 13 gives

$$(5.2)\qquad \begin{aligned} a_{11} &= 7, & a_{12} &= 8, & a_{13} &= 16, \text{ or} \\ a_{11} &= 20, & a_{12} &= 21, & a_{13} &= 16; \end{aligned}$$

$$\begin{aligned} &a_{21} = 6, \quad && a_{22} = 9, \quad && a_{23} = 3, \text{ or} \\ &a_{21} = 19, \quad && a_{22} = 22, \quad && a_{23} = 3. \end{aligned} \tag{5.3}$$

The 3 congruences for a_{31}, a_{32}, a_{33} have a zero determinant, so it will be simpler to use condition $A^2 = I$, taking each of the four pairs of solutions from (5.2), (5.3) for the first two rows of A. The choice

$$\begin{bmatrix} 7 & 8 & 16 \\ 6 & 9 & 3 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 7 & 8 & 16 \\ 6 & 9 & 3 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is consistent, with $a_{31} = 20$, $a_{32} = 18$, $a_{33} = 11$. The other three cases give contradictions. We finally have then for matrix A,

$$A = \begin{bmatrix} 7 & 8 & 16 \\ 6 & 9 & 3 \\ 20 & 18 & 11 \end{bmatrix},$$

giving $P_1 = 7C_1 + 8C_2 + 16C_3$, $P_2 = 6C_1 + 9C_2 + 3C_3$, $P_3 = 20C_1 + 18C_2 + 11C_3$ to be used in deciphering the cryptogram. This is left for the reader.

6. Concluding remarks. In case the length of the probable text is $\lambda \leqq n^2$, the above methods are in general unavailable. Even assuming a correct location would involve a considerable amount of work, too lengthy to be discussed here. If two or more short probable texts are known, the simultaneous testing of assumed locations may enable the present methods to be used.

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THE BEHAVIOR OF ENTIRE FUNCTIONS AND A CONJECTURE OF ERDÖS

S. M. SHAH,* University of Kansas

1. Introduction. Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function. The maximum term of $f(z)$ is defined to be $\max_n |a_n| r^n$ and is denoted by $\mu(r, f)$, or for brevity, by $\mu(r)$. The largest value of n such that $\mu(r) = |a_n| r^n$ is denoted by $\nu(r, f)$ or $\nu(r)$. It is known ([4], p. 32) that

$$\mu(r) < M(r) < \mu(r)\left\{2\nu\left(r + \frac{r}{\nu(r)}\right) + 1\right\}, \tag{1.1}$$

where $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$. The following theorem was proved by J. Clunie [1].

THEOREM A. *Let $f(z) = \sum_1^\infty a_n z^n$ be an entire function and β be a positive number. If*

$$\left(\frac{n+1}{n}\right)^\beta \left|\frac{a_n}{a_{n+1}}\right| > \left(\frac{n}{n-1}\right)^\beta \left|\frac{a_{n-1}}{a_n}\right| \tag{1.2}$$

for all sufficiently large n, then

$$M(r, f) < \{1 + o(1)\}\beta^{-\beta-1}\Gamma(1+\beta)e^\beta \nu(r, f)\mu(r, f). \tag{1.3}$$

In the following theorem, to be proved in Section 3, we suppose that the coefficients (a_n) satisfy somewhat less restrictive conditions and arrive at an extension of Clunie's theorem.

THEOREM 1. *Let $f(z) = \sum_1^\infty a_n z^n$ be an entire function satisfying the following conditions:*

$$\max_{0 \leqq \theta \leqq 2\pi} \left| \sum_1^N \exp(in\theta + i \arg a_n) \right| \leqq CN^\alpha, \tag{i}$$

where α and C are constants, $\frac{1}{2} \leqq \alpha \leqq 1$, $C > 0$, for $N = 1, 2, \cdots$; (ii) $|a_{n-1}/a_n|$ *is strictly increasing for $n > n_0$;* (iii) $|a_{n-1}/a_n| \geqq |a_{p-1}/a_p| \{n/(n-1)\}^\alpha$, $p = [nC_1]$, *where C_1 is a constant such that $0 < C_1 < 1$, for $n > n_0$. Then*

$$M(r, f) < C\{1 + 2C_1^{-\alpha} + o(1)\}\mu(r, f)(\nu(r, f))^\alpha. \tag{1.4}$$

COROLLARY. *If $f(z) = \sum_1^\infty a_n z^n$ is an entire function satisfying conditions* (ii) *and* (iii) *with $\alpha = 1$, then*

$$M(r, f) < \{1 + 2C_1^{-1} + o(1)\}\mu(r, f)\nu(r, f). \tag{1.5}$$

We note that our hypothesis on $|a_n/a_{n+1}|$ is less restrictive than (1.2), since (1.2) implies (ii) and (iii) with $0 < C_1 < \beta/(\beta + \alpha)$, whereas (1.2) does not follow from

* I am indebted to the referee for helpful comments.

conditions (ii) and (iii). Furthermore, (1.4) gives a sharper result than (1.3) when $\alpha<1$; our proof will not depend upon the Wiman-Valiron theory ([4], [6]). Since (1.3), (1.4) and (1.5) are asymptotic inequalities, they hold also when $f(0)\neq 0$.

2. Conjecture of Erdös. Paul Erdös ([2], problem 25) conjectured that if for an entire function $f(z)$, $\lim_{r\to\infty}\mu(r,f)/M(r,f)$ exists, then this limit must be equal to zero. We prove this conjecture when the coefficients (a_n) satisfy a certain condition (Th. 2 (i)) or when $f(z)$ can be written as a product of two entire functions $F(z)$ and $G(z)$ satisfying certain conditions (Th. 3). We also obtain inequalities for $U(f)=\limsup_{r\to\infty}\mu(r,f)/M(r,f)$ and $u(f)=\liminf_{r\to\infty}\mu(r,f)/M(r,f)$. Let us define

$$L = L(f) = \limsup_{n\to\infty} \left| a_n^2/(a_{n-1}a_{n+1}) \right| ; \qquad l = l(f) = \liminf_{n\to\infty} \left| a_n^2/(a_{n-1}a_{n+1}) \right| .$$

THEOREM 2. *Let $f(z)=\sum_0^\infty a_nz^n$ be an entire function satisfying* (ii) *of Theorem* 1. (i) *If $L=1$, then $U(f)=0$;* (ii) *if $L>1$, then $U(f)\geqq(\sqrt{L}-1)/(\sqrt{L}+1)$;* (iii) *if $l>1$, then $u(f)\geqq\{2+3l^{-1}+(2/(l^3-1))\}^{-1}$;* (iv) *if $l=\infty$, then $u(f)=\frac{1}{2}$;* (v) *if $L=\infty$, then $U(f)=1$.*

We note that the condition $l>1$ implies condition (ii) of Theorem 1.

THEOREM 3. *Let $f(z)=F(z)G(z)$, where $F(z)$ and $G(z)$ are entire functions satisfying the following conditions:*

(i) $M(r, F)M(r, G)=O(M(r,f))$,

(ii) $\mu(r, F)/M(r, F)=O(r^{-\alpha})$, $\alpha>0$,

(iii) $M(ar, G)/M(r, G)=o(r^{\alpha})$, $a>1$,

as r tends to infinity. Then $U(f)=0$. If any one of the three conditions holds for a sequence of values of $r\uparrow\infty$, and the remaining two conditions for $r\to\infty$, then $u(f)=0$.

THEOREM 4. *Let $f(z)$ be an entire function* of mean type of integer order ρ such that the canonical product formed with zeros of $f(z)$ is at most zero type of order ρ. Let $\phi(z)$ be any entire function such that $M(r,\phi)=o(M(r,f))$. Then $U(f+\phi)\leqq\frac{1}{2}\sqrt{2}$, $u(f+\phi)\leqq\frac{1}{2}$.*

The proofs of these theorems will be given in Sections 5–8.

3. Proof of Theorem 1. If $f_1(z)=P(z)+Af(z)$, where $P(z)$ is any polynomial and A any nonzero constant, then for any number $c>0$,

$$M(r,f)/\mu(r,f)(\nu(r,f))^c \sim M(r,f_1)/\mu(r,f_1)(\nu(r,f_1))^c \tag{3.1}$$

and so we may suppose without loss of generality that

* For the definition *mean type of order ρ*, see, for example, R. P. Boas, Jr., Entire Functions, New York, 1954.

$$f(z) = \sum_1^\infty a_n z^n = \sum_1^\infty z^n \exp(i\alpha_n)/(R_1 \cdots R_n),$$

where $0 < R_1 < R_2 \cdots$. Let

$$S_n(\theta) = \sum_1^n \exp(ij\theta + i\alpha_j), \qquad S_0(\theta) = 0;$$

$$\begin{aligned} f_M(z) = f_M(re^{i\theta}) &= \sum_1^M r^n \exp(in\theta + i\alpha_n)/(R_1 \cdots R_n) \\ &= \sum_1^M S_n(\theta) \frac{r^n}{R_1 \cdots R_n}\left(1 - \frac{r}{R_{n+1}}\right) + \frac{r^{M+1} S_M}{R_1 \cdots R_{M+1}} \cdot \end{aligned}$$

Let $R_k \leqq r < R_{k+1}$, $n_0 < k < M-1$ and write

$$\sum\nolimits_1 = \sum_1^{k-1} \frac{n^\alpha r^n}{R_1 \cdots R_n}\left(\frac{r - R_{n+1}}{R_{n+1}}\right), \qquad \sum\nolimits_2 = \sum_k^M \frac{n^\alpha r^n}{R_1 \cdots R_n}\left(\frac{R_{n+1} - r}{R_{n+1}}\right).$$

Then $|f_M(z)| \leqq C(\sum_1 + \sum_2 + O(1/M))$, $\sum_1 < \mu(r)(\nu(r))^\alpha$,

$$\begin{aligned} \sum\nolimits_2 &\leqq \alpha \sum_{k+1}^M \frac{r^n n^{\alpha-1}}{R_1 \cdots R_n} + C_2 \sum_{k+1}^M \frac{r^n n^{\alpha-2}}{R_1 \cdots R_n} + \mu(r)\nu(r))^\alpha \\ &< \{\alpha + C_2/\nu(r)\} \sum\nolimits_3 + \mu(r)(\nu(r))^\alpha, \end{aligned}$$

where C_2 is a constant and

$$\sum\nolimits_3 < \left(\sum_{k+1}^P + \sum_{P+1}^\infty\right)\left(\frac{r^n n^{\alpha-1}}{R_1 \cdots R_n}\right) = \sum\nolimits_4 + \sum\nolimits_5, \text{ say.}$$

Take $P = [(k+1)/C_1]$. Then

$$\sum\nolimits_4 < \mu(r)\{(C_1^{-\alpha} - 1)(k+1)^\alpha/\alpha + (k+1)^{\alpha-1}\},$$

$$\sum\nolimits_5 \leqq (P+1)^{\alpha-1} \sum_{P+1}^\infty \mu(r)(r/R_{P+1})^{n-P}.$$

Now $R_{p+1} \geqq R_{k+1}\{(P+1)/P\}^\alpha$, $R_{k+1} > r$. Hence

$$\sum\nolimits_5 < \{\mu(r) P^\alpha\}/\{(P+1) - (P+1)^{1-\alpha} P^\alpha\}.$$

Letting $M \to \infty$ we have

$$M(r,f) < C\mu(r,f)(\nu(r,f))^\alpha \left\{2 + (C_1^{-\alpha} - 1)\left(\frac{k+1}{k}\right)^\alpha + C_1^{-\alpha} + o(1)\right\}$$

and the theorem follows.

4. Corollary and example. If we take $C = 1 = \alpha$, condition (i) is satisfied and (1.5) follows from (1.4).

Example. Let

$$f(z) = \sum_{1}^{\infty} z^n \exp(in \log n)/(R_1 \cdots R_n),$$

where $R_n = (n+1)^a(\log(n+1))^{a_1}$, $a > 0$; or $a = 0$, $a_1 > 0$. The conditions (i)–(iii) are satisfied with $\alpha = \frac{1}{2}$ ([5], pp. 116–118), C an absolute constant, and C_1 any number < 1. From (1.4) we have

$$\limsup_{r \to \infty} M(r, f)/\{\mu(r, f)(\nu(r, f))^{1/2}\} \leqq 3C.$$

5. Proof of Theorem 2. (i–a). Consider first

$$\phi(z) = \sum_{1}^{\infty} z^n/(R_1 \cdots R_n), \qquad 0 < R_1 < R_2 \cdots, R_n \sim R_{n+1}.$$

For $R_k \leqq r < R_{k+1}$ and p any fixed number,

$$\frac{M(r, \phi)}{\mu(r, \phi)} > \frac{r}{R_{k+1}} + \cdots + \frac{r^p}{R_{k+1} \cdots R_{k+p}} > \frac{1}{2} p$$

for $r > r_0(p)$. Hence $\liminf_{r \to \infty} M(r, \phi)/\mu(r, \phi) \geqq \frac{1}{2}p$. Since p can be chosen arbitrary large

$$\lim_{r \to \infty} M(r, \phi)/\mu(r, \phi) = \infty. \tag{5.1}$$

(i-b). Consider now

$$\phi_1(z) = \sum_{1}^{\infty} z^n \exp(i\theta_n)/(R_1 \cdots R_n),$$

where $0 < R_1 < R_2, \cdots, R_{n+1} \sim R_n$. Let

$$f_1(z) = \sum_{1}^{\infty} z^n/(R_1 \cdots R_n)^2, \qquad f_2(z) = \sum_{1}^{\infty} z^{2n}/(R_1 \cdots R_n)^2.$$

We have

$$\mu(r, f_2) = \{\mu(r, \phi_1)\}^2 = \mu(r^2, f_1),$$
$$\{M(r, \phi_1)\}^2 \geqq M(r, f_2) = M(r^2, f_1).$$

Hence by (i-a)

$$M(r, \phi_1)/\mu(r, \phi_1) \geqq \sqrt{\{M(r^2, f_1)/\mu(r^2, f_1)\}} \tag{5.2}$$

which tends to ∞ as $r \to \infty$.

(i-c). Any entire function $f(z)$ satisfying condition (ii) of Theorem 1 and (i) of Theorem 2 can be written as $f(z) = P(z) + A\phi_1(z)$, where P is a polynomial and A a constant. Hence from (3.1) and (5.2), (i) follows.

(ii-a). Consider first

$$\phi(z) = \sum_1^\infty z^n/(R_1 \cdots R_n), \quad 0 < R_1 < R_2 \cdots ; \quad \limsup_{n\to\infty} R_{n+1}/R_n = L > 1.$$

Then $R_{n+1} > L_1 R_n$ where $1 < L_1 < L$ for a sequence of values of $n = n_p \uparrow \infty$. Write $n_p = N$ and consider $M(r, \phi)/\mu(r, \phi)$ when $r = \sqrt{L_1 R_N} = aR_N$ (say). Then $R_N < r < R_{N+1}$ and

$$\frac{M(r, \phi)}{\mu(r,\phi)} < \left\{1 + a^{-1} + a^{-2} + \cdots \right\} + \left\{aL_1^{-1} + a^2 L_1^{-2} + \cdots \right\} + o(1).$$

Hence

$$\liminf_{r\to\infty} M(r, \phi)/\mu(r, \phi) \leqq (\sqrt{L} + 1)/(\sqrt{L} - 1). \tag{5.3}$$

(ii-b). If

$$\phi_1(z) = \sum_1^\infty z^n \exp(i\theta_n)/(R_1 \cdots R_n), \qquad 0 < R_1 < R_2 \cdots ; \quad \limsup R_{n+1}/R_n = L,$$

then $M(r, \phi_1) \leqq M(r, \phi)$, $\mu(r, \phi_1) = \mu(r, \phi)$; and we can complete the argument as in (i-c).

(iii) We consider

$$\phi(z) = \sum_1^\infty z^n(R_1 \cdots R_n), \qquad 0 < R_1 < R_2 \cdots ; \qquad \liminf_{n\to\infty} R_{n+1}/R_n = l > 1;$$

and then complete the argument as in (ii-b). We have $R_{n+1} > l_1 R_n$, where $1 < l_1 < l$ for $n > n_0$. Further, for $R_n \leqq r < R_{n+1}$,

$$\begin{aligned}\frac{M(r, \phi)}{\mu(r, \phi)} &= 1 + \left(\frac{R_n}{r} + \frac{R_n R_{n+1}}{r^2} + \cdots \right) + \left(\frac{r}{R_{n+1}} + \frac{r^2}{R_{n+1}R_{n+2}} + \cdots \right) \\ &\leqq 1 + \left(\frac{R_n}{r} + \frac{r}{R_{n+1}}\right) + 2(l_1^{-1} + l_1^{-1-2} + l_1^{-1-2-3} + \cdots) + o(1) \\ &< 2 + 3l_1^{-1} + \left\{2/(l_1^3 - 1)\right\} + o(1).\end{aligned}$$

Hence

$$\limsup_{r\to\infty} M(r, \phi)/\mu(r, \phi) \leqq 2 + 3l^{-1} + \left\{2/(l^3 - 1)\right\}.$$

(iv) From the argument for (iii), we have $u(f) \geqq \frac{1}{2}$. If

$$\phi_1(z) = \sum_1^\infty z^n \exp(i\theta_n)/(R_1 \cdots R_n), \quad 0 < R_1 < R_2 \cdots ; \quad \lim_{n\to\infty} R_{n+1}/R_n = \infty,$$

then

$$\phi_1\{R_n \exp(i(\theta_{n-1}-\theta_n))\} = 2\mu(R_n)\exp\{in\theta_{n-1} - i(n-1)\theta_n\} + o(\mu(R_n)).$$

Hence $u(\phi_1) \leqq \frac{1}{2}$ and (iv) follows by an argument similar to (i-c).

(v) By the argument of (ii) we have $U(f) \geqq 1$, and hence (v) follows from (1.1).

6. Proof of Theorem 3. Write

$$f(z) = \sum_0^\infty a_n z^n, \qquad G(z) = \sum_0^\infty b_n z^n, \qquad F(z) = \sum_0^\infty c_n z^n.$$

Then

$$\begin{aligned} a_n &= c_n b_0 + c_{n-1} b_1 + \cdots + c_0 b_n \\ &= \frac{1}{2\pi i}\int_\Gamma G(z)\left(\frac{c_n}{z} + \cdots + \frac{c_0}{z^{n+1}}\right) dz, \end{aligned} \tag{6.1}$$

where Γ is $|z| = R$. Hence

$$|a_n| r^n \leqq M(R, G)\left\{r^n|c_n| + r^{n-1}|c_{n-1}|\left(\frac{r}{R}\right) + \cdots + |c_0|\left(\frac{r}{R}\right)^n\right\}.$$

Let $R > r$. Then

$$|a_n| r^n \leqq M(R, G)\mu(r, F)\left\{\sum_0^\infty \left(\frac{r}{R}\right)^n\right\} = M(R, G)\mu(r, F)\{R/(R-r)\}.$$

Hence

$$\mu(r, f) \leqq M(R, G)\mu(r, F)\{R/(R-r)\}. \tag{6.2}$$

Taking $R = ar$, $a > 1$, and assuming (i)–(iii), we have

$$\frac{\mu(r,f)}{M(r,f)} \leqq O\left(\frac{M(ar, G)\mu(r, F)}{M(r, G)M(r, F)}\right) = o(1), \tag{6.3}$$

and so $U(f) = 0$. If one of the three conditions, say (iii), holds for a sequence of values of $r \uparrow \infty$, and the remaining two, (i) and (ii), for r tending to infinity, then from (6.3) we have $u(f) = 0$.

7. Remarks. (i). If we take $F(z) = \exp(Az^p)$, A any constant, p a positive integer, then condition (ii) of Theorem 3 is satisfied with $\alpha = \frac{1}{2}p$ [**3**].

(ii) If $f(z) = \exp(Az^p)P(z)$, P any polynomial, p any positive integer, then $U(f) = 0$.

(iii) If

$$F(z) = \sum_0^\infty c_n z^n, \qquad G(z) = \sum_0^\infty b_n z^n, \qquad f(z) = F(z)G(z)$$

satisfy the following:

(a) $$\sum_0^\infty |b_n| r^n = O(M(r, G)), \qquad \text{as } r \to \infty;$$

(b) $$M(r, F)M(r, G) = O(M(r, f)), \qquad \text{as } r \to \infty;$$

(c) $$U(F) = 0,$$

then $U(f) = 0$. For from (6.1) we have

$$\begin{aligned} r^n |a_n| &\leqq |c_n| r^n |b_0| + \cdots + |c_0| r^n |b_n| \\ &\leqq \mu(r, F)\left\{\sum_0^\infty |b_n| r^n\right\} \leqq O(\mu(r, F)M(r, G)). \end{aligned}$$

Hence $\mu(r, f) \leqq O(\mu(r, F)M(r, G))$, and so from (b), (c), we have $U(f) = 0$. In particular, if b_n and c_n be nonnegative for $n > n_0$, and $U(F) = 0$, then $U(f) = 0$.

Example. Let

$$f(z) = F(z)G(z), \qquad F(z) = e^{-z}, \qquad G(z) = \sum_0^\infty (-1)^n |b_n| z^n,$$

where b_n are any numbers such that $G(z)$ is an entire function. Then $M(r, f) = M(r, F)M(r, G)$, $M(r, G) = \sum_0^\infty |b_n| r^n$, $U(F) = 0$. Hence $U(f) = 0$.

8. **Proof of Theorem** 4. We have

$$\mu(r, f + \phi)/M(r, f + \phi) \leqq \{1 + o(1)\}\mu(r, f)/M(r, f).$$

For $r > r_0(\delta)$,

$$\{\mu(r, f)\}^2 < I(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \frac{\pi + 2\rho\delta}{2\pi}\{M(r, f)\}^2 + O(1),$$

and, for a sequence of values of $r \uparrow \infty$, $2\{\mu(r, f)\}^2 < I(r)$, and the theorem follows.

Remark. If $f(z) = \sum_0^\infty a_n z^n$ be any entire function such that $M(r, f) \sim \sum_0^\infty |a_n| r^n$, then $u(f) \leqq \frac{1}{2}$. *Added in proof:* If $P(z)$ is any polynomial of degree n, then it can be proved, by an argument similar to that of W. K. Hayman (J. Reine Angew. Math., vol. 196, 1956, pp. 67–95), that $\mu(r, e^P)/M(r, e^P) = O(r^{-n/2})$.

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A PROPOSAL OF MARRIAGE*

T. E. HULL, University of British Columbia

1. Introduction. A great deal of effort is currently being devoted to revising the school and college mathematics curricula. New material and new points of view are being introduced through the organizing of special courses, the development of new textbooks, the writing of articles, and so on.

The present period is an opportune one for this sort of activity. There has for some time been a need for change, and, moreover, most of us in this post-sputnik era are ready to change or even anxious to change.

There need be no quarrel about the desirability of the goals being sought. Most of us would agree, for example, that it is desirable for students to understand the concept of a set or the notion of a function as early as possible, and that it is desirable to appreciate abstractness and axiomatics. One could perhaps argue about exactly *when*, or exactly *how*, these ideas should be introduced. But the question being raised here concerns the possibility that, in our enthusiasm for the new approach, we are neglecting to develop still another equally important aspect of the needed change.

My answer to this question is that we are in fact neglecting one very important point of view, and one which could easily be wedded to what is presently being adopted. In the next section I describe what I believe is being neglected. Then in the following section I indicate a number of ways in which I think the situation could be improved. Finally in the remaining sections of the paper I give a precise treatment of one example, to show explicitly how the idea I am proposing would work out in practice. The example will be concerned with ordinary differential equations. Our treatment will contain some novel features, but we shall be primarily concerned with the *way* in which they are treated.

2. What is being neglected? The neglected point of view to which I refer is that of the applied mathematician. And it is my contention that his point of view is not only compatible with that of the pure mathematician, but that the two cannot live without each other. I am therefore proposing a marriage of these two points of view.

That the principals need each other follows from the way in which mathematics has developed in the past. They certainly appear together in the work of such very great mathematicians as Archimedes, Newton, Euler, Gauss, Poincaré, and von Neumann. Much of the initial motivation and stimulation for past research in pure mathematics has come from the problems of applied mathematics. On the other hand the applied mathematician looks to the pure mathematician for generalizations and refinements of his techniques, and sometimes even for whole new techniques.

When the two mathematicians are not speaking to each other the cause of

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each will suffer. Consider the long delay in developing the Laplace transform, which was caused by Heaviside and what he called "those Cambridge mathematicians" being so disdainful of each other.

Mathematical activity is now increasing at a very rapid rate. This means it is most important that we understand the various aspects of this growth. But it also means that we are in great danger of specializing and thereby losing just the understanding that we need. This danger is intensified for the student who is under so much pressure to complete his thesis, or to pass a particular course. It can also be intensified later by university administrations if they insist on the appearance of a large number of publications in the journals.

Such specializing leads, on the one extreme, to the pure mathematician who is concentrating on just one small point in some abstract point set. He might even publish several papers on this one point. On the other extreme, we have the applied mathematician who is concentrating on the intricacies of a large, inelegant, inefficient, and probably unnecessary computer program. He scorns, and is scorned by, his pure colleague.

It is my claim that the points of view being caricatured by these two extremes belong together. Both the rigor and the realism of mathematics need each other. Alone they can only be sterile.

3. What can be done? To bring these points of view together quickly and effectively, we must do so in the classroom. What we are here concerned about is the manner in which material is presented. The two points of view are rarely presented together in a proper way. Pure mathematicians seem to be in a majority in university mathematics departments, and so it turns out that it is the point of view of the applied mathematician that is most often neglected. This is especially unfortunate for the students, since most of them are primarily interested in applied mathematics.

I am not advocating that we set up separate pure and applied courses, so that the student will have to choose between them. I am claiming instead that if a course in mathematics is taught properly it will thereby necessarily contain the essential ingredients of both points of view. In keeping with our allusions to the possibility of marriage, this ideal is perhaps a case of "two living as cheaply as one."

There are many ways in which this ideal can be approached. I would like to illustrate what I mean with a few examples, appropriate to courses usually taught at the intermediate college level. Because I feel that it is the applied aspect which is most often neglected, my examples will mostly take the form of suggestions for modifying presently existing courses so that they will include enough of the motivation and understanding that comes from a proper consideration of the applications.

Let us begin by considering the introduction of the definite integral. The definition and the fundamental theorem are basic to any theoretical considerations. Numerical quadrature is on the other hand one of the basic problems in

practice. These ideas are conceptually very close together, and they can be taught together quite naturally. The student can in this way be led to a more thorough understanding of both the meaning of a definite integral, and of the problem of integration in practice.

The situation referred to in the preceding paragraph is typical. The theory is concerned with a particular limit, while the practice is concerned with a finite approximation, and hence with the consequences of stopping just short of this limit. The connection between these two points of view is perhaps even more dramatically illustrated in the case of ordinary differential equations. Beginning with the next section I shall show how easy and how profitable it is to consider such equations from both points of view at the same time.

As for differentiation, an early application could be to the use of Newton's method for finding the zeros of a polynomial. For a complete understanding of the method one could write a computer program, for which some simple interpretive language should be available. A useful application of partial differentiation which is usually omitted is in finding the normal equations which result when a linear expression is fitted to data by "least-squares." And this brings up the subject of matrices.

In courses concerned with matrices there are almost unlimited opportunities to at least refer to the interesting, practical, and nontrivial problems related to matrix inversion and finding eigenvalues.

Applications can be introduced in many other situations as well. There are of course the standard applications of calculus and differential equations in physics, of complex variables in circuit theory, and of mathematical statistics in the biological and social sciences. Linear programming and the simplex method can easily be introduced in a course on finite-dimensional spaces.

In the examples referred to above I have emphasized especially the numerical aspects. This is partly because the basic numerical problem happens to be very close to the basic theoretical problem in so many branches of mathematics (particularly in analysis, of both the hard and the soft varieties). It is also partly because I believe that an interplay between pure and applied is most urgently needed in this area. Moreover numerical analysis is the one area of applied mathematics which is of common interest to all users of mathematics.

We can blame the computers for opening up vast new problem possibilities in this area. (But we can also thank them for taking out of numerical analysis most of the drudgery of making calculations.) Just one such possibility comes from looking upon mathematics as an experimental science. In the past, pencil and paper were the most common pieces of apparatus. Now with a computer it is possible to perform experiments on a scale and of a kind which were quite unthinkable to most of us only ten years ago. A great deal of thought must be put into the design and interpretation of these experiments. And to be profitable, much of this thought will have to be motivated and encouraged by a joining together of the point of view of the mathematician who has everyday problems

to solve, with that of the mathematician who can afford to remain relatively detached and uncommitted.

4. An example concerning numerical methods and existence theorems. I shall now present an example of the sort of outcome we should expect from the proposed marriage. The example is an outline of the basic steps to be used in a general treatment of ordinary differential equations. It is in my opinion a good example of how one can teach both the applied and the pure aspects of a subject at the same time, and to thereby gain a deeper understanding of each one separately. In this case the applied aspect is represented by numerical methods of solution, while the pure aspect is represented by existence theorems.

I also claim that the numerical methods are introduced in a new and very efficient way, and that the existence proof has been improved somewhat.

The problem to be considered concerns the solution (if any) of the following:

$$x' = f(t, x), \qquad x(t_0) = x_0,$$

where for the moment we assume only continuity of $f(t, x)$ in some region of the tx-plane which contains the point (t_0, x_0).

A treatment of this first-order problem is basic from the point of view of both the numerical methods and the existence theorems. This is so because a system of first-order problems can be handled in almost exactly the same way, from both points of view, by considering x and f to be vectors and considering absolute values to be norms. And of course any nth-order problem can be interpreted as a system of n first-order problems.

Again from both points of view, it is natural to begin consideration of the problem with some approximation to the hoped-for solution. We can define such an approximation by first introducing the numbers t_i with $t_i - t_{i-1} = h$ for $i = 1, 2, \cdots$. We then by some means or other obtain some numbers y_i which are meant to approximate the values $x_i = x(t_i)$, if such exist. Our approximate solution is then obtained by linear interpolation between succeeding points (t_i, y_i) and will be denoted by $y(t)$.

What is crucial is, of course, the way in which the y_i are defined. Let us suppose that $y_0, y_1, \cdots, y_{n-1}$ have already been defined and that we require a formula to define y_n. To motivate the development of suitable formulas we first point out that, if there is a solution $x(t)$ of the original problem, it will, by the mean-value theorem, have to satisfy

$$x_n = x_{n-1} + hx'(\xi) = x_{n-1} + hf(\xi, x(\xi)),$$

where $(\xi, x(\xi))$ is some point in the strip of the tx-plane between the abscissas t_{n-1} and t_n. We of course do not know ξ, but for our approximation we are led in a natural way to take $y_0 = x_0$, and to try

$$y_n = y_{n-1} + h\bar{f}_n, \tag{1}$$

where $\bar{f}_n$ is the average of some values of f in the neighborhood of the last com-

puted point (t_{n-1}, y_{n-1}).

Here and in what follows it is useful to have in mind the standard geometrical interpretation in terms of the direction field. It also justifies the use of terms like "neighborhood" and "point."

The simplest possible choice for $\bar{f}_n$ is $f(t_{n-1}, y_{n-1})$, and this choice defines Euler's method. It is good enough for the proof of the basic existence theorem, but too crude for numerical work. With the more sophisticated formulas of this section, we shall see that we can obtain well-known and useful numerical methods without in any way increasing the complexity of the existence proof. We thereby also establish the convergence of these methods.

There are two principal ways in which $\bar{f}_n$ is defined in numerical work. One way leads to Runge-Kutta methods. These methods are obtained by expressing $\bar{f}_n$ as a linear combination of k values of f,

$$\bar{f}_n = c_1 f_{n,1} + \cdots + c_k f_{n,k},$$

where $f_{n,1} = f(t_{n-1}, y_{n-1})$, and where the other values of f are obtained in turn from formulas of the form

$$\begin{aligned} f_{n,2} &= f(t_{n-1} + \alpha h, y_{n-1} + \alpha h f_{n,1}), \\ f_{n,3} &= f(t_{n-1} + \beta h + \gamma h, y_{n-1} + \beta h f_{n,1} + \gamma h f_{n,2}), \end{aligned}$$

and so on. (These formulas may appear at first to be awkward, but they have a completely natural and simple geometrical interpretation. For example, try the case with $k=3$, $c_1=\frac{1}{6}$, $c_2=\frac{4}{6}$, $c_3=\frac{1}{6}$, $\alpha=\frac{1}{2}$, $\beta=-1$, $\gamma=2$.) In practice k is usually taken to be 4. The remaining parameters are determined primarily by equating coefficients of like terms on either side of (1), when the two sides of (1) are replaced by expansions in powers of h about the point (t_{n-1}, y_{n-1}). (The expansions are those obtained by substituting a solution $x(t)$ for $y(t)$ into each side of (1) and assuming sufficient differentiability of x and f.) For our present purposes it is enough to note that the first equation so obtained simply requires that $\sum_{i=1}^{k} c_i = 1$. This condition was to be expected since $\bar{f}_n$ was to be an "average" value of f. The condition is the special case here of what we later call "consistency." It is enough for our present purposes because we can prove existence theorems with the approximate solutions generated by *any* consistent Runge-Kutta method.

The other principal way of defining $\bar{f}_n$ leads to Adams's methods. They are obtained by expressing $\bar{f}_n$ as a linear combination of values of f at the points (t_i, y_i). Putting $y_i' = f(t_i, y_i)$ we write

$$\bar{f}_n = b_k y_{n-k}' + b_{k-1} y_{n-k+1}' + \cdots + b_0 y_n'.$$

The b's are found as before by equating coefficients in (1) after expanding in powers of h, but the calculations are now quite a bit simpler. We note that the first equation again gives us the consistency condition $\sum_{i=0}^{k} b_i = 1$.

Before proceeding to the existence theorem we should point out that some

new difficulties have arisen. We notice that some other procedure will be needed to obtain the starting values $y_1, \cdots, y_{k-1}$. And we also notice that (1) now defines y_n only implicitly if $b_0 \neq 0$. Another formula will then be needed for predicting a trial value of y_n, before we can iterate on (1). On the other hand fewer evaluations of f will be needed at each step with Adams's methods, and this could make them worth the extra trouble compared to Runge-Kutta methods.

Let us turn now to the question of existence. We want to show that any consistent Runge-Kutta or Adams's method can be used as easily as Euler's method in the proof of existence. At the same time we want to show that the usual existence proof can be improved somewhat. If we were to assume a Lipschitz condition at this stage we would be able to avoid the use of equicontinuity and Ascoli's lemma. We prefer however to present the more general result that continuity *alone* implies existence, a Lipschitz condition being needed only for uniqueness.

One introduces a region in the tx-plane, and an appropriate interval of the t-axis, in the usual way. Then for t, t' in this interval, one obtains (directly from the definition of $y(t)$, that is, from (1) followed by linear interpolation) the following inequality

$$|y(t) - y(t')| \leqq M|t - t'|,$$

where M is the usual bound for $|f(t, x)|$ times $\sum|b_i|$ or $\sum|c_i|$.

Since this result is independent of h we conclude that the approximations for different values of h are equicontinuous as well as being uniformly bounded. Ascoli's lemma therefore guarantees that some subsequence of these approximations will converge, as $h \to 0$, uniformly to a limit function which we denote by $Y(t)$. The limit is continuous because the approximations are continuous.

To show that this limit is a solution of the original problem we choose a typical member $y(t)$ of the convergent subsequence and write

$$y(t) - x_0 = (y(t) - y_n) + (y_n - y_0),$$

where $n = [t/h]$. Then

$$y(t) = x_0 + (t - t_n)f_n + \sum_{i=0}^{n-1} h\bar{f}_i.$$

The uniform convergence of $y(t)$ to $Y(t)$, and the continuity of $f(t, x)$, guarantee that the sum on the right approaches a limit as $h \to 0$ so that we end up with

$$Y(t) = x_0 + \int_{t_0}^{t} f(u, Y(u))du,$$

and we have thus completed the existence proof.

Besides having our subsequence of approximate solutions approaching the limit $Y(t)$, we also have their slopes, where they exist, approaching the limit $Y'(t)$. The existence proof is improved to the extent that this result is not con-

sidered to be an integral part of the proof, as it usually is, but only as a corollary. As a matter of fact this result does not even hold for some of the more general approximate solutions to be considered in the next section.

Finally we point out that uniqueness can be established in the usual way, once some sort of Lipschitz condition has been assumed. It is here that one might introduce the method of successive approximations and refer to Picard's form of the existence and uniqueness theorem. But for our purposes this form of the theorem is of relatively little interest.

5. The general case. Let us now consider very briefly the general multistep methods. Adams's methods are special cases of these general methods, but the latter will have one important feature not found in either Runge-Kutta or Adams's methods. This feature makes possible the phenomenon of instability, which in turn leads to questions which are important both for numerical calculations and for existence theorems.

The general multistep methods can be introduced in a natural way as follows. We recall that in Adams's methods the use of earlier values of y_i' caused trouble with getting the procedure started. If we are however willing to put up with this extra trouble, we might as well also make use of the information we have about the earlier values of y_i. This leads us to consider

$$(2) \qquad y_n = a_k y_{n-k} + \cdots + a_1 y_{n-1} + h(b_k y'_{n-k} + \cdots + b_0 y'_n)$$

in place of (1). Having almost twice as many parameters as (1), we might expect that it will be possible to make such a formula more accurate than (1). This will turn out to be the case, but not, as we shall see, in the way we might at first expect.

Of course we will have to have $\sum_{i=1}^{k} a_i = 1$, and this turns out to be the first equation we get from (2) on expanding in powers of h and equating coefficients. But we do not this time expect the sum of the b's to be unity, because we are no longer extrapolating from the value y_{n-1} at t_{n-1}. We are instead extrapolating from an average value of earlier y's, at the value of t at which this average is effective. Taking moments about t_n we find that this value of t, say t^*, must satisfy

$$t_n - t^* = ha_1 + 2ha_2 + \cdots + kha_k.$$

We are thus led to the equation $\sum_{i=0}^{k} b_i = \sum_{i=1}^{k} ia_i$, and this turns out to be the second equation obtained from (2) by expanding and equating coefficients. We now have two equations to be satisfied by the a's and b's. These two equations together guarantee the "consistency" of the methods.

The new feature that appears with (2) can be described in the following way. We notice that both of the difference equations (1) and (2) will in general have more than one solution. Since only one of their solutions will be close to the solution of the differential equation for small h, there is the danger that it will be dominated by one of the extraneous solutions of the difference equation.

Then both the numerical calculation and the proof of existence would be invalidated. Such unwanted behavior of the extraneous solutions is called instability. The new feature is that with (2) it is possible to have such instability, whereas with (1) the extraneous solutions all approach zero as $h \to 0$.

In trying to make (2) as accurate as possible we would naturally try to equate as many coefficients as possible in the expansions of the two sides of (2). It turns out that we cannot in general match up any more coefficients in (2) than we can for Adams's formula, with the same k, if we at the same time insist on stability. Matching more coefficients would give us formulas which represent the differential equation more accurately, but such formulas are too sensitive to be stable.

I shall not go further into details but I should like to indicate two parallel directions in which we could go from here. One direction we could go is towards the development of efficient numerical methods. It is natural of course to begin with the consistent and stable methods, and to ask which from all these methods is best in practice. The additional degrees of freedom afforded by the multistep methods can be used to minimize the error which accumulates during a long sequence of calculations. This point of view has been exploited by Hull and Newbery [7] who are in this way led to a number of new methods which are in a certain sense best possible.

The other direction in which we might go concerns the establishing of existence theorems and so also the convergence of the multistep methods. Such questions turn out to be much more difficult when one begins with the approximate solutions which are generated by multistep methods. It turns out however that existence and uniqueness can be established if and only if the multistep method is both consistent and stable. The meaning of "stable" must of course be made more precise, and there are some surprises such as the fact that the derivatives do not in general converge. The details are given by Dahlquist [2, 3], and by Hull and Luxemburg [6].

6. Concluding remarks. In the last two sections I have summarized very briefly a treatment of ordinary differential equations which combines the point of view of the pure mathematician interested in existence theorems with that of the applied mathematician interested in numerical calculations. A great deal has of course been omitted. Coddington and Levinson ([1], Chs. 1, 2) give a complete treatment of questions regarding the weakening of hypotheses, the continuing of solutions, the dependence on parameters, and so on. Hildebrand ([5], Ch. 6) and Henrici [4] give a complete treatment of questions regarding the rate of convergence, convenience in practice, round-off, and so on.

One of the chief advantages of the presentation given in this paper is that the numerical methods have been introduced in a way which is both completely natural and completely general. All too often only very special cases are treated, and they are usually derived from a bewildering variety of formulas for numerical differentiation and numerical integration, or from a juggling around with

finite difference formulas. Such formulas have their uses of course, and so even do the unstable methods they sometimes yield, but they ought not to be mixed up in a discussion of differential equations.

An advantage of using realistic numerical procedures in the proof of an existence theorem is that we thereby at the same time establish convergence of the numerical procedures. In the case of multistep methods it is particularly important to know which methods converge and which do not.

The early sections of this paper were devoted to advocating a joining together of the pure and applied points of view in mathematics. Several areas of mathematics were referred to in which this joining together ought to be considered. In the last two sections I have shown, by way of illustration, just how advantageous such a union can be in one particular situation.

Of course the advantages will vary tremendously in kind and degree between different areas of mathematics, and between different applications. But I believe there are worthwhile advantages in each of the situations already referred to.

I therefore wish to return to and repeat my original proposal of a marriage between the pure and applied points of view. I hope they will learn to live together, happily ever after.

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ON THE AREA OF CURVED SURFACES

JAMES SERRIN, University of Minnesota

It is a common impression that if one is to say anything rigorously correct about the area of a surface he must first spend several months of his life with some of the most difficult books in the mathematical library. Be that as it may, certainly there are no textbooks of calculus, or of advanced calculus, which present more than a heuristic account of the subject even for nonparametric surfaces. It would seem worthwhile, therefore, to present a simple account of the basic theory of surface area, and it is to this end that the paper is devoted.

Starting from Lebesgue's definition of area, we shall show that the fundamental results of the theory (for surfaces of the form $z=f(x, y)$) follow at once from a single inequality whose geometric intent is roughly that integral smoothing is an area shrinking operation. The inequality itself is due to Radó, but our proof is a considerable simplification of his, and at the same time gives the result in a somewhat more general form. It is hoped that the treatment will be within the ability of an honors class in advanced calculus. If so, then the theory of surface area can take a place alongside arc length as a possible part of the curriculum.

1. Definition of surface area. In this section we shall briefly outline Lebesgue's concept of surface area. As motivation for this, it is convenient to begin by considering the class of polyhedral surfaces. Here the notion of area is entirely elementary; if $z=f(x, y)$ is such a surface, then we have

$$\text{Area} = \sum \text{areas of the plane faces} = \int_R \sqrt{(1 + f_x^2 + f_y^2)}\,dA,$$

the integral being evaluated over the set R where the surface is defined. We assume throughout that R is a closed region whose boundary is a rectifiable curve of finite length.

To define the area of a *curved* surface, one might simply decide to use the same integral expression. This procedure, however, tends to blur the geometrical idea of area, and in any case is analytically vague: To what class of functions should the formula be applied? Should one allow Lebesgue integration, or must the function $\sqrt{(1+f_x^2+f_y^2)}$ be strictly Riemann integrable? etc. That these questions are relevant is clear if we consider a surface $z=g(x)$ for which g is a monotonically increasing function whose derivative is zero almost everywhere. If R is a unit square, we arrive at the absurd conclusion that the area of the surface is one.

A more appealing definition parallels that of arc length; that is, the area of the surface is to be the limit of the areas of approximating polyhedra. Unfortunately, we have only to recall Schwarz's famous example [1] to see the incorrectness of this method.

Lebesgue's happy idea was to modify the preceding definition by using the

lower limit of the areas of approximating polyhedra. To state this precisely, let us introduce for piecewise smooth functions $u(x, y)$ the notation

$$I[u] = \int_R \sqrt{(1 + u_x^2 + u_y^2)}dA$$

(the area of a polyhedral surface $z=f(x, y)$ over R is thus simply the integral $I[f]$). Now let $f(x, y)$ be a real continuous function defined on R, and let $\{f_n(x, y)\}$ be a sequence of quasilinear functions such that $f_n(x, y) \to f(x, y)$ uniformly on R. We set $A = \lim\inf I[f_n]$, observing that A depends on the particular sequence of functions f_n.

The Lebesgue area $L[f]$ of the surface $z=f(x, y)$ is then defined to be the infimum of the numbers A corresponding to all possible sequences $\{f_n\}$ converging to f as above.

This definition may alternatively be expressed in the convenient and concise form

$$L[f] = \underset{f_n \to f}{\text{minimum}} \text{ limit } I[f_n].$$

It should be emphasized that this definition gives a value (possibly infinite) for the area of *any* continuous surface over R. Moreover, the area thus defined is clearly a lower semicontinuous functional with respect to uniform convergence, that is, if $f_n \to f$ uniformly on R, then $L[f] \leqq \lim\inf L[f_n]$. The property of lower semicontinuity is of considerable importance in the general theory of area, though we shall make no particular use of it here.

In order that the Lebesgue's definition of surface area be useful, it must still be verified that the Lebesgue area of a polyhedron agrees with its elementary area. This follows, however, from our main proposition, which we now state.

THEOREM 1. *If f is continuously differentiable, then $L[f] = I[f]$.*

Theorem 1 obviously justifies the well-known formula for surface area. The proof of Theorem 1 will be given in the next three sections of the paper.

2. Preliminary results. We shall be interested in the operation of integral averaging or smoothing. Let us associate with each positive number h a corresponding function $K(\xi, \eta) = K(\xi, \eta; h)$ with the following properties:

(i) K is nonnegative and continuously differentiable for all values of ξ and η,
(ii) $K \equiv 0$ outside the circle $\xi^2 + \eta^2 = h^2$,
(iii) $\int K(\xi, \eta) dA_{\xi\eta} = 1$.

In (iii) and in subsequent formulas it is tactitly assumed that the integration is carried out over all values of ξ and η.

The *integral average* ϕ_h of a function $\phi(x, y)$ is now defined by the formula

$$\phi_h = \phi_h(x, y) = \int K(\xi, \eta)\phi(x + \xi, y + \eta)dA_{\xi\eta}, \tag{1}$$

or, alternatively, by the equivalent expression

$$\phi_h = \int K(\xi - x, \eta - y)\phi(\xi, \eta)dA_{\xi\eta}. \tag{2}$$

Certain properties of the integral average will be needed later on. Assuming to begin with that ϕ is continuous, then by virtue of the properties of $K(\xi, \eta)$ one has

$$\phi_h(x, y) \to \phi(x, y) \text{ as } h \to 0. \tag{3}$$

Moreover, from (2) it follows that ϕ_h is continuously differentiable; indeed, we have

$$\phi_{hx} = -\int_R K_\xi(\xi - x, \eta - y)\phi(\xi, \eta)dA_{\xi\eta}, \tag{4}$$

and a similar formula holds for ϕ_{hy}. Finally, if ϕ is piecewise smooth, differentiation of (1) yields

$$\phi_{hx} = \phi_{xh}, \qquad \phi_{hy} = \phi_{yh}. \tag{5}$$

One more elementary fact will be required later:

If $F(p, q)$ is a convex function of the variables p and q, and if u and v are functions of (x, y), then $F(u_h, v_h) \leqq F(u, v)_h$.

This is nothing more than Jensen's inequality. (See, e.g., G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, p. 53.)

3. The fundamental inequality. Let $f(x, y)$ be a continuous function defined in R. Its integral average f_h is defined in the subregion R_h of R consisting of all points of R which are farther than h from the boundary. We put

$$I[f_h] = \int_{R_h} \sqrt{(1 + f_{hx}^2 + f_{hy}^2)}dA,$$

where the fact that the integration is only over the subregion R_h should cause no confusion. Then for all $h>0$ we have

$$I[f_h] < L[f]. \tag{6}$$

This is the fundamental inequality (*cf.* also the last paragraph of Sec. 4).

In order to prove (6), let $\{f_n\}$ be a sequence of quasilinear functions with the property that

$$\lim f_n = f, \qquad \lim I[f_n] = L[f]; \tag{7}$$

such a sequence surely exists in view of the definition of area. Now (5) and (4) imply

$$|f_{hx} - (f_n)_{xh}| = |f_{hx} - (f_n)_{hx}| \leqq C(h) \max |f - f_n|,$$

and a similar estimate holds with x replaced by y. Therefore, for any convex

function $F(p, q)$,

$$F(f_{hx}, f_{hy}) \leqq F((f_n)_{xh}, (f_n)_{yh}) + \epsilon \leqq F(f_{nx}, f_{ny})_h + \epsilon,$$

where $\epsilon = \epsilon(n, h) \to 0$ as $n \to \infty$. Supposing in particular that $F = \sqrt{(1+p^2+q^2)}$, integration of the preceding inequality over the region R_h yields

$$I[f_h] \leqq \int_{R_h} F(f_{nx}, f_{ny})_h dA + \epsilon \text{ meas } R_h.$$

Now the operation of averaging is in fact an integration, so that the first term on the right can be treated as an iterated integral. By reversing the order of integration we can write this term in the form

$$\int K(\xi, \eta) \int_{R'_h} F(f_{nx}, f_{ny}) dA_{xy} dA_{\xi\eta},$$

where R_h' denotes the parallel translation of the set R_h by the vector (ξ, η). The inner integral is clearly dominated by $I[f_n] - \text{meas } (R - R_h)$; whence by property (iii) of the function K we find

$$I[f_h] \leqq I[f_n] - \text{meas } (R - R_h) + \epsilon \text{ meas } R_h$$

Letting $n \to \infty$, and taking account of (7), leads at once to the required inequality.

4. Proof of Theorem 1. Since by hypothesis the partial derivatives of f are continuous, we have by (3), (5), and (6),

$$I[f] = \lim_{h \to 0} I[f_h] \leqq L[f].$$

On the other hand, there certainly exists a sequence of quasilinear functions tending to f such that $\lim I[f_n] = I[f]$, (the reader may verify that this can be attained even for inscribed polyhedra). Therefore from the definition of $L[f]$ it follows that

$$L[f] \leqq \lim I[f_n] = I[f].$$

Combining the two preceding inequalities yields $L[f] \leqq I[f] \leqq L[f]$, and the proof of Theorem 1 is complete.

Remark. By virtue of Theorem 1 it is evident that $I[f_h] = L[f_h, R_h]$, the notation being obvious. Hence the fundamental inequality can be written in the form

$$L[f_h, R_h] < L[f, R], \tag{6'}$$

and in this sense averaging is indeed an area shrinking operation.

5. Further results. In this section we shall consider some further properties of the Lebesgue area. Though these results do not have the fundamental im-

portance of Theorem 1, they nevertheless serve to round out the theory.

To begin with, we note from the introductory discussion that there are functions for which the integral $I[f]$ does not represent surface area. This fact leads to two main problems: first, to characterize precisely the class of functions for which $L[f]=I[f]$ (so far we know only that this class includes all continuously differentiable functions), and second, to give a constructive method for calculating the area of an arbitrary surface. The first problem is unfortunately beyond the scope of this paper, for a satisfactory answer turns out to require rather deep tools of the theory of integration [4]. On the other hand, the second problem has an elegant and simple solution.

Let us say that an averaging procedure is *regular* if there is a constant M, independent of h, such that

$$\int |\nabla K|\, dA_{\xi\eta} \leqq Mh^{-1}.$$

This condition is satisfied, for example, by any averaging procedure whose kernel is of the form $K(\xi, \eta)=h^{-2}\tilde{K}(\xi h^{-1}, \eta h^{-1})$, where the function $\tilde{K}(u, v)$ is independent of h. We now have the following

THEOREM 2. *For an arbitrary continuous function $f(x, y)$ defined on R, and for any regular averaging procedure, we have* $L[f]=\lim_{h\to 0} I[f_h]$.

Proof. Because of (6), it is obvious that $\lim I[f_h]\leqq L[f]$. To prove the opposite inequality, let us first extend f so that it is defined and uniformly continuous in some open region containing R. Denoting the resulting function by $g(x, y)$, we have $g\equiv f$ in R, while (for sufficiently small h) the integral average g_h exists and is continuously differentiable in all of R. Now for each $h>0$, let g_h' be a quasilinear function in R such that

$$|g_h' - g_h| \leqq h^{-1}, \qquad |I[g_h'] - I[g_h]| \leqq h^{-1}.$$

Obviously $g_h'\to f$ as $h\to 0$. Therefore by the definition of area,

$$L[f] \leqq \lim I[g_h'] = \lim I[g_h].$$

To complete the proof it is sufficient to show that the difference $I[g_h] - I[f_h]\to 0$ as $h\to 0$. By writing out the integrals involved, one easily verifies that

$$0 \leqq I[g_h] - I[f_h] \leqq \max \{1 + |g_{hx}| + |g_{hy}|\} \text{ meas } (R - R_h).$$

Now the boundary of R is rectifiable, so that the set $R-R_h$ can be covered by at most Lh^{-1} circles of radius h, where L depends only on the length of the boundary. Thus meas $(R-R_h)\leqq$const. h. On the other hand, it is clear from (4) that

$$g_{hx} = -\int K_\xi(\xi - x, \eta - y)\{g(\xi, \eta) - g(x, y)\}dA_{\xi\eta} \leqq Mh^{-1}\omega(h),$$

where $\omega(t)$ is a modulus of continuity for $g(x, y)$. It follows easily from the pre-

ceding estimates that

$$I[g_h] - I[f_h] \leqq \text{const. } (h + \omega(h)),$$

and the proof is complete.

We conclude by showing that Lebesgue area is an additive function of sets (for continuously differentiable functions this is an obvious consequence of Th. 1).

THEOREM 3. *Let S_1 and S_2 be disjoint open regions with rectifiable boundaries. Let R_1 and R_2 be the closures of S_1 and S_2, and R the union of R_1 and R_2. Then*

$$L[f, R] = L[f, R_1] + L[f, R_2].$$

Proof. This theorem can be demonstrated directly from the definition of area, but an even more immediate proof can be given on the basis of Theorem 2. Indeed, by that result we have

$$L[f, R] - L[f, R_1] - L[f, R_2] = \lim_{h \to 0} \int_S \sqrt{(1 + f_{hx}^2 + f_{hy}^2)}\, dA,$$

where $S = R_h - R_{1h} - R_{2h}$. But, exactly as in the proof of Theorem 2, the last limit is zero.

Note. Since this paper is to some extent expository in character, we have not attempted to discuss refinements and extensions of the theory, nor have we paid particular attention to questions of attribution and priority. The reader interested in these matters may consult [3], [4], and [5], and the bibliographies listed there. The last named reference, in particular, includes an alternative definition of surface area which applies even to "surfaces" represented by locally integrable functions $f(x, y)$ defined over an arbitrary open region.

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SUCCESSOR AXIOMS FOR THE INTEGERS

ANGELO MARGARIS, The Ohio State University

1. Introduction. There are two standard approaches to the integers. One is a set of axioms for an ordered integral domain whose set of positive elements is well-ordered. The other is the construction of the integers as equivalence classes of ordered pairs of natural numbers. In this paper the Peano axioms for the natural numbers are modified to provide an axiom set for the integers. In showing the adequacy of the axioms, we shall omit many details by assuming that the reader is familiar with the first few pages of Landau [1].

For the natural numbers based on the Peano axioms, the chief method of proof is induction. For the integers based on the modified Peano axioms, the chief method of proof is *symmetric induction.* In symmetric induction, the basis step may be given for any integer, and there are two induction steps: *forward,* from x to its successor x', and *backward,* from x' to x.

2. The axioms. The axioms for the set Z of integers are

1. Z is not empty.
2. To each integer x there is associated a unique integer x' (called the *successor* of x).
3. For all integers x and y, if $x'=y'$, then $x=y$.
4. For each integer y there is an integer x such that $x'=y$. (This x, which is unique by Axiom 3, is called the *predecessor* of y, and is denoted $'y$.)
5. If M is a set of integers such that (i) M is not empty; (ii) for every integer x, x is in M if and only if x' is in M; then $M=Z$.
6. There is a subset Q of Z such that (i) Q is not empty; (ii) for every integer x, if x is in Q, then x' is in Q; (iii) $Q\neq Z$.

For comparison and reference we now give the Peano axioms.

1*–3*. These are the same as 1–3, with "Z" replaced by "N" and "integer" replaced by "natural number".

4*. There is a natural number 1 such that for every natural number x, $x'\neq 1$.

5*. If M is a set of natural numbers such that (i) 1 is in M; (ii) for every natural number x, if x is in M, then x' is in M; then $M=N$.

Axiom 4 is the negation of 4*. From 3 and 4 follows

For every integer x, $'(x')=('x)'=x$.

Axiom 5 is symmetric induction. In the presence of the other axioms,

Axiom 5 *is equivalent to the statement obtained by replacing clause* (ii) *by*: *For every integer* x, *if* x *is in* M, *then* x' *and* $'x$ *are in* M. This equivalent of Axiom 5 we call the *second form of symmetric induction.*

Axiom 6 or some equivalent is necessary to rule out finite models, such as a model consisting of a, b and c, with $a'=b$, $b'=c$, $c'=a$. For the natural numbers, finite models are ruled out by 4* (in conjunction with 2* and 3*). Without Axiom 6 one cannot prove that $x \neq x'$, for a one-element model exists.

The axiom set for the integers differs from the Peano set in one respect, important from a logical standpoint. Axiom 6 cannot be translated into the symbolism of the first-order predicate calculus, in which quantification (for all x; there exists an x) is permitted only on elements. Axioms 5 and 5* are translated by a trick, in which the subset M is replaced by a property $P(x)$, and the universal quantifier "for all M" which is implicit, is omitted (as it must be), resulting in an infinite bundle of axioms, one for each $P(x)$. This trick does not work for Axiom 6 because the quantifier "there exists a Q" is an existential quantifier.

3. Addition and multiplication. In this section addition and multiplication are defined for Z, and Z is shown to be a ring. No use is made of Axiom 6. 0 is an arbitrary but fixed integer, and $1=0'$. To save repetition, we make the conventions that $a, b, \cdots, z$ always stand for integers, and every set is a set of integers.

Addition in Z is defined by the identities

$$x + 0 = x, \tag{1}$$

$$x + y' = (x + y)'. \tag{2}$$

A proof of existence and uniqueness of a binary composition satisfying (1) and (2) can be obtained by straightforward modification of the proof of Theorem 4 in Landau [1]. That the induction in [1] is based on 1 instead of 0 is inessential. The additional details for the backward induction steps can be supplied without difficulty.

Then, imitating Landau, the associative and commutative laws can be proved. There is no counterpart in [1] of the following theorem.

For every x there is a y such that $y+x=0$.

Proof. We use the second form of symmetric induction. Let M be the set of integers for which the theorem holds. 0 is in M since $0+0=0$. Assume x is in M and $y+x=0$. Then

$$'y + x' = ('y + x)' = (x + 'y)' = x + ('y)' = x + y = 0,$$
$$y' + 'x = 'x + y' = ('x + y)' = (y + 'x)' = y + ('x)' = y + x = 0.$$

Hence $M=Z$.

Thus Z is an abelian group under addition. As usual, the negative of x is denoted $-x$, and $x-y$ is an abbreviation for $x+(-y)$. From (2) it follows that for every x, $x'=x+1$ and $'x=x-1$. Z is a cyclic group, because by Axiom 5, the integers

$$\cdots, -1-1, -1, 0, 1, 1+1, \cdots,$$

exhaust Z.

Multiplication is defined by the identities

$$x0 = 0, \tag{3}$$

$$x(y+1) = xy + x. \tag{4}$$

The proof of the theorem justifying addition is easily modified to prove the corresponding theorem for multiplication. Then Landau can be imitated to prove the commutative, distributive and associative laws.

As matters stand, Z is recognizable as the ring of integers modulo an unspecified n. To show $n=0$, Axiom 6 is necessary.

4. Order. In this section we show that Z is an ordered ring, and its set of positive elements satisfies the Peano axioms.

For the set Q of Axiom 6, there is an x such that x' is in Q and x is not in Q. Otherwise, $Q=Z$ by Axiom 5, contradicting Axiom 6. Let 0 be such an x and let $1=0'$. The 0 and 1 of Section 3 are now fixed to be this 0 and 1.

Let P be the intersection of all sets M such that

I. 1 is in M;

II. For every x, if x is in M, then x' is in M;

III. 0 is not in M.

P exists because the set Q of Axiom 6 is such an M. Then P has properties I–III, and is a subset of every set satisfying I–III.

P satisfies the Peano axioms, with the successor function of Z restricted to P as domain.

Proof. 1*–4* are easily verified. For 5*, let M be a subset of P satisfying the hypotheses of 5*. Then M satisfies I–III, making P a subset of M. Hence $M=P$.

We now show that P is a set of positive elements for Z. That P is closed under addition and multiplication is an easy exercise in ordinary (not symmetric) induction, which holds in P. In proving the trichotomy law, we shall use the fact that if x is in P, then $x=1$ or $x-1$ is in P. A proof of the trichotomy law follows and closes this paper.

For every integer x, exactly one of the following holds:
(1) $x=0$; (2) x is in P; (3) $-x$ is in P.

Proof. Neither the pair (1), (2) nor the pair (1), (3) can hold simultaneously because $0=-0$ is not in P. The pair (2), (3) is also impossible, because otherwise $x+(-x)=0$ is in P. Now let M be the set of all x for which at least one of (1)–(3) holds. We show $M=Z$ by symmetric induction. M is not empty because (1) holds for 0. Assume x is in M. If $x=0$, then $x+1=1$ is in P. If x is in P, then $x+1$ is in P. If $-x$ is in P, then either $-(x+1)=-x-1$ is in P, or $-x=1$, and

$x+1=0$. Assume $x+1$ is in M. If $x+1=0$, then $-x=1$ is in P. If $x+1$ is in P, then either x is in P, or $x=0$. If $-(x+1)$ is in P, then $-x$ is in P since $-x=-(x+1)+1$.

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PROPERTIES OF THE CANTOR SET AND SETS OF SIMILAR TYPE

N. C. BOSE MAJUMDER, Calcutta University, Calcutta, India

1. Introduction. Kestelman [2] proved the following theorem:

Let C be a closed bounded set in R_N (N-dimensional Euclidean space) having positive measure and let p be any positive integer. Then there exists a positive number δ (depending only on p and the measure of C) with the following property: if $\lambda_1, \cdots, \lambda_p$ are any vectors in R_N whose lengths are less than δ, then the set of all ξ such that $\xi \in C$, $\xi+\lambda_r \in C$ $(r=1, \cdots, p)$ is closed and has positive measure.

In the present note we propose to show that there exists a set E of zero measure with an almost similar property. We shall consider two special cases ((i) $N=2$, $p=1$; (ii) $N=N$, $p=1$) of the above theorem.

THEOREM 1. *Let C be the Cantor middle-third set in $[0, 1]$ and let E be the product $C \times C (=C^2)$ in R_2. (Obviously E is a nondense perfect set of Lebesgue plane measure zero.) Then if λ is a vector in R_2, whose length d is less than $\sqrt{2}$, the set of ξ such that $\xi \in E$, $\xi+\lambda \in E$, is nonempty for each of an infinite number (of power c) of directions of λ and the set of such ξ is closed for each particular direction of λ.*

Proof. Let λ be a vector such that $|\lambda|=d$, $0<d<\sqrt{2}$. We can find an infinite set of pairs (of power c), (d', d''), such that $d'^2+d''^2=d^2$, $0<d'<1$, $0<d''<1$. Since d' may take any value in some interval contained in $[0, 1]$, we can choose d' (the number of choices being infinite of power c) such that the set of pairs (x_1, x_2), where $x_1 \in C$, $x_2 \in C$, $x_2-x_1=d'$, has power c ([4], Ths. I, II). Having chosen d' in this way, we have now, corresponding to d'', at least one pair of points (y_1, y_2), $y_1 \in C$, $y_2 \in C$, $y_2-y_1=d''$ by Randolph's theorem [1].

Consider any pair (x_1, x_2) corresponding to the chosen d'. The points $F(x_1, y_1)$, $G(x_2, y_1)$, $H(x_2, y_2)$, $T(x_1, y_2)$ are points in E and form a rectangle in the unit square (Fig. 1). Now consider the vector $\lambda=\overrightarrow{FH}$, where $FH=d$ is a diagonal of the rectangle $FGHT$. Then $|\lambda|=d$ and λ makes an angle $\theta=\arctan(d''/d')$ with the x-axis. If F is the point ξ, then $\xi+\lambda$ is the point H, where $\xi \in E$, $\xi+\lambda \in E$. Since the set of pairs (x_1, x_2) has the power c, so does the set of rectangles like $FGHT$. It therefore follows that the vectors like $\overrightarrow{FH}(=\lambda)$ form an infinite set

of power c, each such vector being of length d, having the constant slope $\tan\theta$, and its endpoints in E.

Thus, since the number of choices for d' is infinite (of power c), with d fixed, we get an infinite number (of power c) of directions θ of λ. It should be noted that the vector $\overrightarrow{GT}$ also satisfies the conditions of the theorem, the direction in this case being $\pi-\theta$.

We now show that for each direction of λ for which the set of ξ such that $\xi\in E$, $\xi+\lambda\in E$, is nonempty, the set of all such ξ is closed. Let E_1 be obtained by translating E by an admissible vector $-\lambda$ (*cf.* [**2**], where $E_1\equiv T(E;\ -\lambda)$). Since each of E and E_1 is closed so is $E\cap E_1$, and this is precisely the set of ξ such that $\xi\in E$, $\xi+\lambda\in E$.

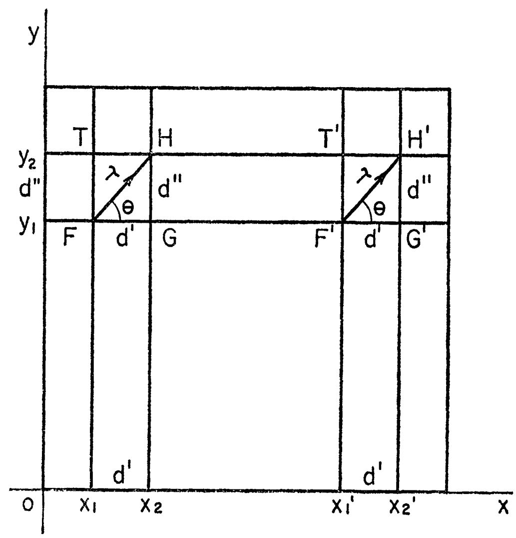

FIG. 1

Theorem 1 may easily be generalized as follows:

THEOREM 2. *Let C be the Cantor middle-third set in* $[0, 1]$ *and let E be the product $C\times\cdots\times C(=C^N)$ in R_N. Let $n\leqq N$ be any positive integer. Then if λ is a vector in R_N whose length is less than $\sqrt{n}$, the set of ξ, such that $\xi\in E$, $\xi+\lambda\in E$, is nonempty for each of an infinite number (of power c) of directions of λ, and the set of such ξ is closed for each particular direction of λ.*

Proof. After taking $|\lambda|=d$, we write $d_1^2+\cdots+d_n^2=d^2$, where at least one d_r has the same property as d' in Theorem 1 and $0<d_r<1$, $r=1,\cdots,n$. The proof is then completed exactly as above.

2. Consider the straight lines $y=3^{-n}x$, $n=0, 1,\cdots$, which cut the side KM of the unit square (Fig. 2) at $P(1, 3^{-n})$. Let $OP=\sqrt{\{(3^{2n}+1)/3^{2n}\}}$ be de-

noted by L and let θ be the angle that OP makes with the x-axis. Then we have the following:

THEOREM 3. *There is a subset of $E = C \times C (= C^2)$ lying on OP alone such that the distance set between points of this subset completely fills the closed interval* $[0, L]$.

Proof. Consider any l satisfying $0 < l \leqq L$. Then $0 < l \cos\theta \leqq L \cos\theta = 1$. Therefore, by Randolph's theorem [1], we can find at least one pair of points (x_1, x_2), $x_1 \in C$, $x_2 \in C$ such that $l \cos\theta = x_2 - x_1$. We draw ordinates through x_1 and x_2 to meet OP at $A(x_1, y_1)$ and $B(x_2, y_2)$, respectively. Now if

$$x = c_1 3^{-1} + c_2 3^{-2} + \cdots$$

is in C, then

$$y = 3^{-n}x = c_1 3^{-n-1} + c_2 3^{-n-2} + \cdots$$

is also in C. Therefore each of A and B lies on E and obviously $AB = l$.

There is, of course, a similar theorem for the straight lines $y = 3^n x$ or $x = 3^{-n} y$.

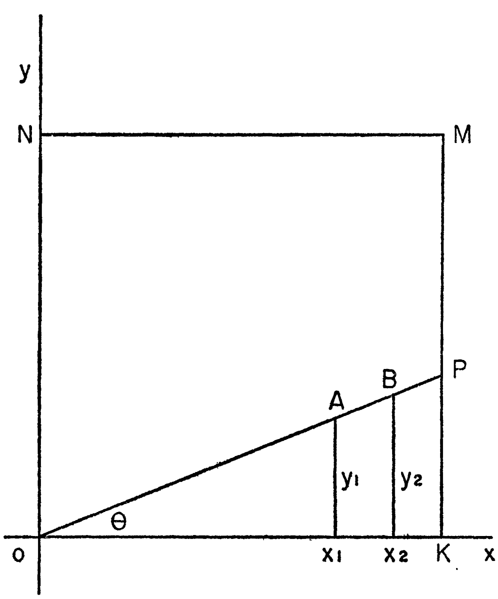

FIG. 2

3. In [3] two sets E_n' and E_n'', each of Cantor's type, were constructed on the x- and y-axes, respectively, in $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$. The only difference was that the intervals were subdivided at each stage into $2n+1$ equal parts and the middle $(n+1)$th interval was suppressed. Each of E_n' and E_n'' is a nondense perfect set of measure zero. It was shown that for the product set $E_n = E_n' \times E_n''$, the distance set between its points completely fills the closed interval $[0, \sqrt{2}]$.

In this section we prove

THEOREM 4. *There are subsets of E_n, one lying on the diagonal OM and the other on the diagonal KN of the unit square* (Fig. 3) *such that the distance set between points of each subset completely fills the closed interval* $[0, \sqrt{2}]$.

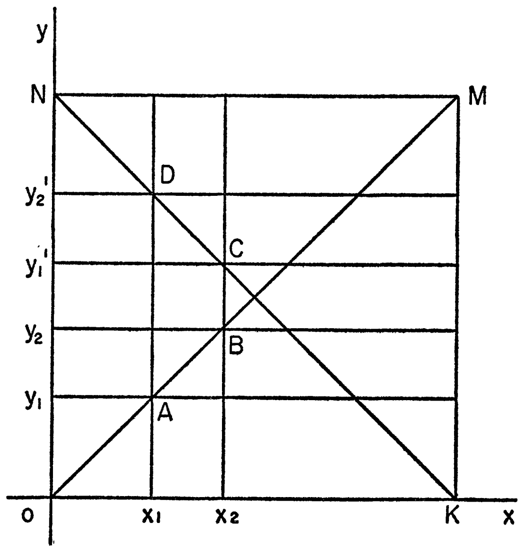

FIG. 3

Proof. Take the diagonal of the unit square joining $O(0, 0)$ and $M(1, 1)$ and consider any d satisfying $0 \leqq d \leqq \sqrt{2}$. Since $0 \leqq d/\sqrt{2} \leqq 1$ we can choose [3] at least one pair of points (x_1, x_2) on E_n' such that $x_2 - x_1 = d/\sqrt{2}$. Now $y_1 = x_1$ and $y_2 = x_2$ on E_n'' are such that $y_2 - y_1 = d/\sqrt{2}$. These points give rise to a square, two of whose corners A and B (obviously points on E_n) lie on the diagonal OM. It is evident that $AB = d$ so that the distance set between the points of E_n lying on OM completely fills $[0, \sqrt{2}]$.

It is, of course, easy to see that the distance set between the points of E_n lying on the other diagonal KN also fills $[0, \sqrt{2}]$.

I am grateful to Dr. H. M. Sen Gupta for his kind help and guidance in the preparation of this paper.

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ON THE DUAL OF A TRIVALENT MAP

C. R. MARATHE, Indian Institute of Technology, Bombay

1. Introduction. In the literature on the four-color problem use is made of a "regular" map introduced by Heawood [1]. However, since "regular" now has quite another meaning, it has been suggested that maps with vertices of order three be called *trivalent*. The dual map of a trivalent map has the property that every country in such a map has three sides. Indeed, we may study the dual map as a triangular network. In this paper we shall study certain properties of the dual map of a trivalent map. This study leads very easily to certain known results, one of them a theorem due to Petersen [3].

2. Definitions and notation. In the literature on the four-color problem a term may be used with different meanings. To avoid misunderstanding, we shall explain the meaning of the terms we use.

(i) *Map.* Unless otherwise stated, all maps will be on a closed, simply connected surface and, without loss of generality, we may consider the simply connected surface as a sphere. A simple construction will transform any map on a sphere into a topologically equivalent map which may be drawn on a plane surface. The construction is as follows:

Suppose that the sphere is made up of a thin rubber sheet. Let the map over this surface contain $n+1$ countries C_k $(k=0, 1, \cdots, n; n>0)$ with the boundary b_k. Cut out the country C_0 along the boundary b_0 and deform the map until all the countries C_k $(k\neq 0)$ lie flat over a plane. This resulting "flat" map has n countries C_k' with the corresponding boundaries b_k'. Here C_k' and C_k and also b_k' and b_k are topologically equivalent. The country C_0' also exists in the flat map. The boundary contour of the flat map is precisely b_0', only one must imagine that C_0' now lies on the *reverse* side of the flat map. This construction of representing a spherical map on a plane surface is useful.

In general, a map M is divided into countries C_k by the corresponding boundaries b_k. These boundaries are closed curves. It may happen that b_s and b_t have some portion in common. Then C_s and C_t are called *adjacent*, otherwise they are *disjoint*. If b_s and b_t have only one point in common then C_s and C_t are also called disjoint. Since C_k is on a closed surface, the boundary b_k is divided into m mutually exclusive and exhaustive segments $b_{k_1}, \cdots, b_{k_m}$. When three or more sides meet, they can do so only at a point. This point is called a *vertex* of M. The *order* of a vertex V is the number of sides which meet at V.

(ii) *Trivalent map.* This is a map with each vertex of order three.

(iii) *Dual map.* Consider a map M with n countries C_k $(k=1, \cdots, n)$ and with corresponding boundaries b_k. Take a point P_i in C_i. If C_i and C_j are adjacent then join P_i and P_j by an arc P_iP_j which cuts only once the common boundary of C_i and C_j but does not touch or intersect any other b_i in M. This construction will give a map M' with P_k $(k=1, \cdots, n)$ as the vertices. M' is called a dual

map of M. It is easily seen that all dual maps of M are topologically equivalent. Hence we say that M' is the dual map (or simply dual) of M. We note that $(M')'$ is equivalent to M.

(iv) *Network.* Consider any map M with countries C_k having boundaries b_k. Let b_k be decomposed into b_{k_i} $(i=1, \cdots, m)$, where b_{k_i} is the ith side of C_k. Then the configuration $\{b_{k_i}\}$ for all permissible values of k and i, constitutes a *network.* This network is called *closed* if M is a map on a closed surface. Otherwise the network is *open.* A *path* in a network is a succession of sides in the network. The path is thus continuous and connects two or more vertices of M. A path is *closed* if the sides may be named cyclically so that only successive sides meet at a vertex. A closed path is called a *cycle.* Each b_k is a cycle. Cycles like the b_k are called *prime* cycles; we can not further decompose them into other cycles. Two cycles are *adjacent* when they have one or more common sides. A cycle is *even* when it has an even number of sides, otherwise it is *odd.* We may *add* or *unite* two or more adjacent cycles and get another cycle. In Figure 1 are shown

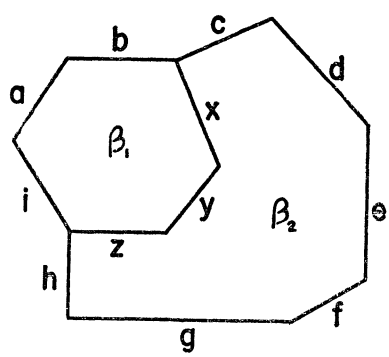

FIG. 1

two cycles β_1 and β_2. Here β_1 has sides a, b, x, y, z, i, while β_2 has sides c, d, e, f, g, h, z, y, x. We define $\beta_1+\beta_2$ as the cycle of sides a, b, c, d, e, f, g, h, i. By the product $\beta_1\times\beta_2$ we denote the path of the sides x, y, z. The product is not a cycle unless β_1 and β_2 are the same cycles. If we denote an odd cycle by 1 and an even cycle by 0 then for adjacent cycles the following relations hold:

$$0+0=0, \qquad 0+1=1, \qquad 1+0=1, \qquad 1+1=0.$$

In what follows, we shall mostly study *triangular* networks. A triangular network corresponds to a map which has all three-sided countries. The dual of a trivalent map contains all three-sided countries (see Lemma 1). Hence the network which corresponds to the dual of a trivalent map is triangular.

Every cycle in any network (whether closed or open) must be adjacent to at least one other cycle of the network; further, each side of a closed network belongs to at least two cycles.

3. LEMMA 1. *Let R' be the dual of a trivalent map R on a sphere. Then*

I. *All the countries in R' are three-sided.*
II. *The number of countries in R' is even.*

Proof. I. All the vertices of R are of order three. Consider a vertex V_1 in R. Let countries C_1, C_2, C_3, of R surround V_1. Thus, in R' there is a triangle, say T_1, with sides P_1P_2, P_2P_3, P_3P_1, and vertices P_1, P_2, P_3. Thus, to each vertex V_i there corresponds a triangle T_i.

II. For every closed network we have by Euler's formula $F-E+V=2$, where F, E, V are, respectively, the number of faces, sides and vertices of the network. But in the triangular network of R' we have $3F=2E$, since each triangular face has three sides and each side is common to two faces. Hence we get $F=2(V-2)$.

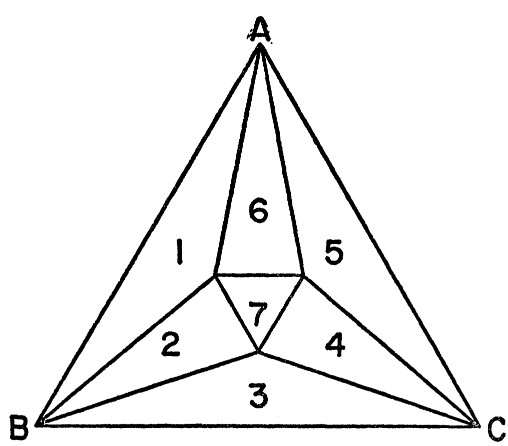

FIG. 2

The general appearance of the dual of a trivalent map is shown in Figure 2 for the dual of a trivalent map with six countries each of four sides. The number of triangles in the dual is $7+1=8$. The seven faces are numbered while the 8th face is the triangle on the *reverse* and is bounded by the cycle ABC. Figures 2 and 3 show the corresponding vertices of the closed trivalent map which may be taken as made up of the four-sided faces of a cube.

LEMMA 2. *Any open triangular network which contains an even number of triangles can be colored with two colors—red and green—in such a way that each triangle has one red side and two green sides while the boundary of the network has sides of only green color.*

Proof. Assume the result true for an open triangular network N_{2m} with $2m$ triangles. Consider an open triangular network N_{2m+2} with $2m+2$ triangles. Let β_{2m+2} be the boundary of N_{2m+2}. In N_{2m+2} select a triangle, say T_1, with boundary

b_1 so that β_{2m+2} and b_1 are adjacent. (Such a selection of T_1 is always possible). There must exist a triangle, say T_2, with boundary b_2 such that T_1 and T_2 are adjacent. Consider the cycle β_{2m} obtained from β_{2m+2} after deleting from it the sides $b_1 \times \beta_{2m+2}$, $b_2 \times \beta_{2m+2}$, $b_1 \times b_2$. Let β_{2m} be the boundary of the open triangular network N_{2m}. By the induction hypothesis, the network N_{2m} satisfies Lemma 2. Hence β_{2m} may be colored so that all sides are green. Color the sides in b_1+b_2 green and color $b_1 \times b_2$ red. Since $\beta_{2m+2}=(b_1+b_2)+\beta_{2m}$, the cycle β_{2m+2} gets all green sides and the network is colored as required. This establishes the lemma as the result is certainly true for $m=1, 2$.

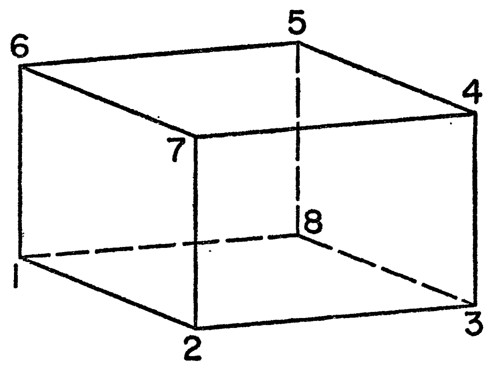

FIG. 3

THEOREM 1. *Any triangular network (whether closed or open) that contains an even number of triangles can be colored with two colors—red and green—so that each triangle has one red side and two green sides.*

Proof. The theorem is already proved above for an open triangular network with even number of triangles. Now consider a closed triangular network N_{2m+2} with $2m+2$ triangles. Take two adjacent triangles T_1 and T_2 with boundaries b_1 and b_2, respectively, in N_{2m+2}. Transform N_{2m+2} into a planar network by renoving $T_1 \cup T_2$ from the surface and flattening out the remainder of the surface and its network N_{2m}. The boundary of N_{2m} is now b_1+b_2, and T_1, T_2, $b_1 \times b_2$ must be taken exterior to b_1+b_2. This *flat* network contains $2m$ triangles and is open. Hence, by Lemma 2, this network can be colored so that the boundary is all green and the triangles get one red side and two green sides each. But the boundary of N_{2m} is precisely the cycle b_1+b_2. Now color $b_1 \times b_2$ red and the theorem holds for the closed network N_{2m+2}.

COROLLARY 1. *Every closed triangular network can be colored with two colors—red and green—so that each triangle has one red side and two green sides.*

Proof. By Lemma 1, every closed triangular network has an even number of triangles. Hence by Theorem 1 we get the result.

COROLLARY 2 (PETERSEN'S THEOREM). *The network which corresponds to a trivalent map can be colored with two colors—red and green— so that at each vertex there meet one red side and two green sides.*

Proof. Let R be the trivalent map and R' its dual. At any vertex V in R there meet three sides, say s_1, s_2, s_3. Let s_j' in R' intersect s_j in R ($j=1, 2, 3$). Color s_j and s_j' with the same color. Now a necessary and sufficient condition that Corollary 2 should hold is that Theorem 1 holds.

COROLLARY 3. *Every green cycle in a closed triangular network, colored according to Theorem* 1, *is even.*

Proof. The triangular network will be reduced to a quadrilateral network if we delete all the sides colored red. Then all prime cycles in this quadrilateral network are even since they have four sides each. Any cycle in this network is green and is obtained as the sum of two or more prime cycles. So, every cycle in the network is even.

The above corollary shows that it is possible to color any cycle of the quadrilateral network in two colors, say yellow and blue, so that the sides of the cycles are alternately yellow and blue. Unfortunately, this does not imply that *every* triangle of the triangular network has one red, one yellow, and one blue side.

4. LEMMA 3. *Four colors suffice to color a trivalent map R (and hence any map), so that no two adjacent countries of R have the same color, if the triangular network of R' can be colored with three colors so that each triangle has one side of each color.*

(This result is due to Tait [2]. He obtains it for the network of a trivalent map R. With suitable modification this result may be applied to the network of R'.)

Proof. Let V_i and V_j be two vertices of R'. There are cycles in R' which pass through V_i and V_j. There is at least one path between any two vertices of the network of R'. Consider the network of R' and let this be colored by three colors α, β, γ, so that each triangle of this network gets one side of each color. Now define a commutative group G of the four elements e, α, β, γ by the following multiplication table:

	e	α	β	γ
e	e	α	β	γ
α	α	e	γ	β
β	β	γ	e	α
γ	γ	β	α	e

Take four colors A, B, C, D to color the map R and suppose that these colors are obtained from one of them, say A, by the operations of the elements of G as

$$eA = A, \quad \alpha A = B, \quad \beta A = C, \quad \gamma A = D.$$

The countries in R are represented by the vertices of R'. Hence the result will be established if we are able to assign colors to the vertices of R' so that no two vertices of the same triangle get like colors. We now proceed to assign the colors A, B, C, D to the vertices of R'. Let V_1, V_2, $\cdots$ be the vertices. Let V_1 receive color A. If the side V_1V_j has color θA ($\theta = \alpha, \beta, \gamma$) then place the color θA at V_j. Do so till all the vertices adjacent to V_1 are exhausted. Then take any colored vertex V_j ($j \neq 1$) and repeat the process, the only change to be noticed is that at V_j there is the color θA and not A. Repeat the process till all the vertices of R' are colored. It now remains to show that this scheme of assigning colors to the vertices does not lead to a contradiction, *i.e.*, no vertex gets two different colors. Consider two adjacent vertices say V_1 and V_2 which lie on a triangle T_0 having vertices V_1, V_2, V_3. Consider a path π between V_1 and V_2. Let π have a sides of the color αA, b sides of color βA and c sides of color γA. Then the *index* of π is the element θ given by

$$\theta = \alpha^a \beta^b \gamma^c.$$

We shall show that θ is invariant for all paths between V_1 and V_2. Let b_0 be the boundary of the triangle T_0. Each cycle through V_1 and V_2 can be obtained as the sum of prime-cycles (which are all triangles) one of them being b_0. Consider one such cycle β_m which is obtained by the sum of m prime cycles. Now β_m is the boundary of the open map M_m which is formed by the m triangles mentioned above. In M_m take a triangle T_1 with boundary b_1 such that β_m and b_1 are adjacent. We can decompose the cycle β_m into the two paths:

(i) the side V_1V_2,

(ii) the path which is made up of all the remaining sides in β_m.

Let the index of the first be ϕ (*i.e.*, ϕA is the color of the side V_1V_2). Let the index of the second path be ψ. From the map M_m delete T_1 to obtain the map M_{m-1} with boundary β_{m-1}. Clearly $\beta_m = \beta_{m-1} + b_1$. Let the sides in b_1 be x, y and z.

Now two cases arise:

Case 1: $\beta_m \times b_1$ has two sides, say y and z. Now β_m and β_{m-1} are different because of the sides of T_1. But the index of the path along the cycle β_{m-1} is equal to the index of the path along the cycle β_m because the contribution to the index due to any two sides of T_1 is equal to that due to the remaining third side of T_1.

Case 2: $\beta_m \times b_1$ has only one side, say z. A similar consideration is valid in this case.

Thus we conclude that the index of the path along β_{m-1} is the same as the index of the path along β_m. We repeat the process and reach the path along $\beta_1 = b_0$. Now the contribution due to sides V_1V_3 and V_3V_2 together is equal to

that due to the side V_1, and V_1, V_2 have the same index. What is true of V_1 and V_2 is true of any other pair of vertices of the network which proves the lemma.

If a triangular network is colored with three colors so that each side of a triangle has different color then we shall say that the network is *properly* colored. If a network has the property that it can be properly colored then we shall say that it is *properly colorable.*

COROLLARY 4. *Every cycle of a properly colored triangular network has index e.*

Proof. A cycle is a closed path from say V_i to V_i. Thus the index of a cycle must be such that it keeps the color at any of its vertices unaltered.

COROLLARY 5. *In a properly colored triangular network the cycles having sides of one color are even.*

Proof. The index of the cycle is e. Suppose all the sides of the cycle are of color α. Then $\alpha^m = e$ where m is the number of sides in the cycle. Hence m is even.

LEMMA 4. *If four colors are needed to color a trivalent map R then the triangular network of R′ is properly colorable.*

Proof. The above lemma is the converse of Lemma 3. Let the vertices V_i of R' be assigned colors A, B, C, D as they are found in R. Color the sides of the network of R' in the following scheme with colors α, β, γ:

	A	B	C	D
A	e	α	β	γ
B	α	e	γ	β
C	β	γ	e	α
D	γ	β	α	e

If V_i and V_j have colors X and Y then color the side V_iV_j with the color θ found at the intersection of X-row and Y-column in the above table. For example, if V_i is B and V_j is D then the side V_iV_j gets the color β. Since every side in the network of R' connects only one pair of vertices, it follows that each side of the network is assigned the color only once. It is now necessary to show that each triangle has one side of each color.

Suppose this is not true. Let a triangle T of vertices V_1, V_2, V_3 get two or more sides of the same color. Two cases are possible:

Case 1: V_1V_2, V_2V_3, V_3V_1 are all of the same color say α. Since V_1, V_2, V_3 are on the same triangle the colors at them are all different. If V_1 is A then

V_2 is $\alpha A = B$. But this is not possible. Hence our supposition that T has sides of the same color is wrong.

Case 2: Let V_1V_2 and V_1V_3 have the color α. In a similar way to that of Case 1, we may show that this is not possible.

Lemmas 3 and 4 apply to closed networks, but they also apply to an open network. For, given an open network, we first convert it into a closed network by introducing more triangles so that the "gaps" are filled in. Not until the open network has been converted into a closed network do we start the coloring. Then we may apply Lemmas 3 and 4 so that the closed network gets properly colored. Now, removing the additional triangles will not disturb the color scheme of the rest, which is certainly properly colored. Hence we get

THEOREM 2. *A necessary and sufficient condition that four colors suffice to color a trivalent map M (whether on a closed or an open surface) is that the triangular network of M' be properly colorable.*

In view of Theorem 2, it appears that a further study of the network of the dual of a trivalent map may be of interest.

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A GENERALIZED FIBONACCI SEQUENCE

A. F. HORADAM, The University of New England, Armidale, Australia

1. Introduction. Recently in this MONTHLY [1], [2], [3] there have appeared several problems and results involving the Fibonacci sequence, and consequently one is prompted to offer some comments on a generalized theory.

In the following development $a = \frac{1}{2}(1+\sqrt{5})$, $b = \frac{1}{2}(1-\sqrt{5})$ are the roots of $x^2 - x - 1 = 0$ so that $a + b = 1$, $a - b = \sqrt{5}$, $ab = -1$. (The values of a and $-b$ are, of course, associated with the classical geometrical problem of the golden section.) For the Fibonacci sequence

$$(\alpha) \qquad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots,$$

defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geqq 3)$ with $F_1 = F_2 = 1$, it is well known (Daniel Bernoulli, 1732) that the nth term (Fibonacci number) is $F_n = (a^n - b^n)/\sqrt{5}$.

2. A generalized Fibonacci sequence. Suppose we preserve the recurrence relation but alter the first two terms to produce the generalized Fibonacci sequence defined by:

$$(\beta) \qquad H_n = H_{n-1} + H_{n-2} (n \geqq 3), \qquad H_1 = p,\ H_2 = p + q,$$

where p, q are arbitrary integers. That is, the generalized sequence is

$$(\gamma) \qquad p,\ p+q,\ 2p+q,\ 3p+2q,\ 5p+3q,\ 8p+5q,\ 13p+8q,\ \cdots.$$

Employing the usual method for difference equations, we deduce, after a little calculation, that

$$(\delta) \qquad H_n = \frac{1}{2\sqrt{5}}(la^n - mb^n),$$

where $l = 2(p-qb)$, $m = 2(p-qa)$, so that $l+m = 2(2p-q)$, $l-m = 2q\sqrt{5}$, $\frac{1}{4}lm = p^2 - pq - q^2 = e$ (say).

Certain results follow almost immediately from (β) and/or (δ), *viz.*,

$$(1) \qquad H_n/H_{n-1} \to a, \qquad H_n/H_{n-i} \to a^i, \qquad H_n/F_n \to p - qb \quad (\text{as } n \to \infty);$$

$$(2) \qquad 2^{n+1}H_n = \frac{l-m}{\sqrt{5}} \sum_{i=0}^{[\frac{1}{2}n]} 5^i C^n_{2i} + (l+m) \sum_{i=0}^{[\frac{1}{2}n-\frac{1}{2}]} 5^i C^n_{2i+1};$$

$$(3) \qquad H_{n+2} - 2H_n - H_{n-1} = 0, \qquad H_{n+1} - 2H_n + H_{n-2} = 0;$$

$$(4) \qquad \sum_{i=0}^{n-1} H_{2i+1} = H_{2n} - q, \qquad \sum_{i=1}^{n} H_{2i} = H_{2n+1} - p;$$

$$(5) \qquad \sum_{i=1}^{n} (H_{2i-1} - H_{2i}) = -H_{2n-1} + p - q.$$

Writing $\sigma_n = \sum_{i=1}^n H_i$, $\tau_n = \sum_{i=1}^n \sigma_i$, we have

$$(6) \qquad \sigma_n = H_{n+2} - H_2, \qquad \tau_n = H_{n+4} - (n+2)H_2 - H_1.$$

From (γ) we observe that

$$(7) \qquad H_{n+1} = qF_n + pF_{n+1}, \qquad H_{n+2} = pF_n + (p+q)F_{n+1}.$$

Putting $n = r$ in (7) and using (β), we find in turn

$$H_{r+3} = (p+q)F_r + (2p+q)F_{r+1} = H_2F_r + H_3F_{r+1},$$
$$H_{r+4} = (2p+q)F_r + (3p+2q)F_{r+1} = H_3F_r + H_4F_{r+1}, \cdots,$$

and, in general,

$$(8) \qquad H_{n+r} = H_{n-1}F_r + H_nF_{r+1} \qquad (n \geqq 3).$$

In the ensuing results it is perhaps preferable to use (δ) throughout, in which

case the following identities may be advantageously used:

$$(\epsilon)\qquad \begin{aligned} a - \frac{1}{a^3} &= - b\sqrt{5}, & b - \frac{1}{b^3} &= a\sqrt{5};\\ a + \frac{1}{a} &= \sqrt{5}, & b + \frac{1}{b} &= - \sqrt{5};\\ a^2 - a - 1 &= 0, & b^2 - b - 1 &= 0. \end{aligned}$$

(No attempt at any meaningful order in these results is implied.)

$$H_{n-1}^2 + H_n^2 = (2p - q)H_{2n-1} - eF_{2n-1}, \tag{9}$$

$$H_{n+1}^2 - H_{n-1}^2 = (2p - q)H_{2n} - eF_{2n}, \tag{10}$$

$$H_{n-1}H_{n+1} - H_n^2 = (-1)^n e, \tag{11}$$

$$H_nH_{n+r+1} - H_{n-s}H_{n+r+s+1} = (-1)^{n+s}eF_sF_{r+s+1}, \tag{12}$$

$$H_n^3 + H_{n+1}^3 = 2H_nH_{n+1}^2 + (-1)^n e, \tag{13}$$

$$H_{n+1-r}H_{n+1+r} - H_{n+1}^2 = (-1)^{n-r}eF_r^2. \tag{14}$$

(Observe that (11) is a special case of (14) when $r=1$ and n is replaced by $n-1$.) Putting $r=n$ in (14), we derive

$$H_{n+1}^2 + eF_n^2 = pH_{2n+1}. \tag{15}$$

Numerous other results may be deduced but, as we are not concerned here with an exhaustive list, we mention only the useful "Pythagorean" theorem:

$$\left\{2H_{n+1}H_{n+2}\right\}^2 + \left\{H_nH_{n+3}\right\}^2 = \left\{2H_{n+1}H_{n+2} + H_n^2\right\}^2, \tag{16}$$

and the remarkable fact that

$$\frac{H_{n+r} + (-1)^rH_{n-r}}{H_n} = F_{r+1} + (-1)^rF_{r-1} = \left(\frac{a}{2}\right)^r + (-1)^r\left(\frac{2}{a}\right)^r, \tag{17}$$

i.e., the expression on the left-hand side of (17) is independent of p, q, and n.

Taking $p=3$, $q=1$, for instance, so that (γ) becomes

$$(\zeta)\qquad 3, 4, 7, 11, 18, 29, 47, 76, 123, \cdots$$

and putting $n=3$, we find that (16) yields $396^2+203^2=445^2$. Again using these values for p and q, and setting $n=7$, $r=2$ in (17), we have $(123+18)/47=3=F_3+F_1$. Note that the simple results $3^2+4^2=5^2$, $5^2+12^2=13^2$, occur when $p=1$, $q=0$, $n=1$, and $p=1$, $q=0$, $n=2$, respectively. When $p=1$, $q=0$, $n=3$, we obtain $8^2+15^2=17^2$ after simplifying and dividing throughout by 4. An interesting question is: Does (16) exhaust all the Pythagorean number triples?

Searching through the available literature on generalizations of (α), one sees that, broadly speaking, the work may be generalized in two main directions. Either the recurrence relation can be generalized and extended (and this has been done in a variety of ways) or the recurrence relation is preserved but the first two Fibonacci numbers are altered—this has been our approach. (Naturally, these two techniques could be combined.) The research to which this article most closely approximates seems to be that of Tagiuri ([5], p. 404), who wrote in Periodico di Matematiche, vol. 16, 1901, on the subject. The results (7), (12), (14), (17) above are due to Tagiuri, who used a slightly different notation from that used here. Doubtless, a good deal of work along these lines has been done by mathematicians not recorded in [5].

3. Special cases. The Fibonacci sequence (α) is obtained from (γ) by putting $p=1$, $q=0$. Making this simplification, so that $l=m=2$, $e=1$, and (δ) reduces to $F_n=(a^n-b^n)/\sqrt{5}$, we obtain, for instance, from (9), (11), and (15), respectively, the well-known results

$$F_{n-1}^2 + F_n^2 = F_{2n-1}, \tag{9'}$$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{11'}$$

$$F_{n+1}^2 + F_n^2 = F_{2n+1}, \tag{15'}$$

attributed to Catalan, Simson, and Lucas, respectively. Obviously, (15′) is obtainable from (9′) by replacing n by $n+1$, but we cannot say the same about (15) and (9).

All the commonly known properties of the Fibonacci sequence are thus deducible from the generalized sequence as special cases when $p=1$, $q=0$. For example, if p is arbitrary and $q=1$ in (γ), we obtain the variant of the Fibonacci sequence discussed by Guest [4]. With his further modification $p=10$, we find that his $F_{11}=89=e$ in our notation and all the results given by him then occur as special cases of the general theory when $p=10$, $q=1$.

A question that may well be asked is: Under what conditions (relating to p and q) is the Fibonacci sequence (α) repeated? Obviously, from (γ) for $H_1=H_2=p$ (*i.e.*, $q=0$), each term of (α) is merely multiplied by p as Guest observed. Furthermore, we must avoid the case when q and p are consecutive Fibonacci numbers for if $p=F_n$, $q=F_{n-1}$, we find that the new sequence is the Fibonacci sequence with the first $n-1$ terms missing, *i.e.*,

$$F_n,\ F_n+F_{n-1},\ 2F_n+F_{n-1},\ 3F_n+2F_{n-1},\ 5F_n+3F_{n-1},\ \cdots,$$

which is the same as $F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, \cdots$. In particular, if $p=8$, $q=5$, the new sequence is 8, 13, 21, 34, 55, *i.e.*, (α) with the first five numbers missing.

On the other hand, if $p=F_{n-1}$, $q=F_n$, we do not obtain the Fibonacci sequence. With $p=5$, $q=8$, for example, we get the sequence 5, 13, 18, 31, 49, 80, 129, $\cdots$.

When $q=np$ (n an integer), we have the sequence

$$p\{1,\ 1+n,\ 2+n,\ 3+2n,\ 5+3n,\ 8+5n,\ 13+8n,\ \cdots\},$$

i.e., p times the generalized sequence with $p=1$, $q=n$.

Turning now to some recent problems and remarks on the ordinary Fibonacci sequence, we can generalize Ivanoff's result [2] thus:

$$\begin{aligned}\sum_{i=0}^{n} C_i^n H_{n-i} &= \frac{1}{2\sqrt{5}}\left\{ l\sum_{i=0}^{n} C_i^n a^{n-i} - m\sum_{i=0}^{n} C_i^n b^{n-i}\right\}\\ &= \frac{1}{2\sqrt{5}}\left\{l(1+a)^n - m(1+b)^n\right\}\\ &= \frac{1}{2\sqrt{5}}\left\{la^{2n} - mb^{2n}\right\} = H_{2n},\end{aligned}$$

where in the next to the last step we have used (ϵ).

Danese's result [3] is now seen to be a special case of (12) when n is replaced by $n+h$ and $s=h$, $r=k-h-1$ (with $p=1$, $q=0$, of course). The last two results in Ganis [1] are particular cases of Danese's result, as noted in [3]. Venkannayah's problem [3] carries over directly to the generalized case, but Everman's problem [3] is a special case (remembering that $F_5=5$) of a theorem due to Tagiuri, *viz.*, that F_r is a multiple of F_s provided that r is a multiple of s, which is not valid in the generalized sequence. For example, $H_6/H_3=29/7$.

Of the remaining results quoted in [1] (and due originally to Simson, Lucas, and Piccioli), the first and third are generalized in (11) and (6), respectively, and the second, $F_{n+1}=C_0^n+C_1^{n-1}+C_2^{n-2}+\cdots$, could be considered to be generalized in (7).

This paper developed out of an interest in the Fibonacci sequence and a desire to extend the results of Guest's stimulating article. Ever since Fibonacci (Leonardo of Pisa) wrote his Liber Abbaci in 1202, his intriguing sequence has fascinated men through the centuries, not only for its inherent mathematical riches, but also for its applications in art and nature. Indeed, it is almost true to say that the research generated by its nearly amounts to the quantity of offspring generated by the mythical pair of rabbits who started Fibonacci off on the problem!

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5. L. E. Dickson, History of the Theory of Numbers, vol. 1, New York, 1952, pp. 393–407. (For Tagiuri's work, see p. 404.)

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

MOMENTS OF A FUNCTION ON THE CANTOR SET

D. D. WALL, IBM Corporation

If a set of points x comprises a countable set x_i, then the average value of a function $f(x)$ on these points would be defined as

$$u_0(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i).$$

If the points x comprise a set S of positive measure m, then $u_0(f)$ would be defined as $u_0(f) = (1/m)\int_S f(x)dx$. For the Cantor set neither of these alternatives holds, so a new definition is needed. The appended reference considers a very similar problem, but we will use a different approach.

Observe that as the variable x ranges over points of the Cantor set (on the unit interval), the variables $\frac{1}{3}x$ and $\frac{2}{3}+\frac{1}{3}x$ range over those points of the Cantor set on the first third and last third of the unit interval, respectively. This suggests the relation

$$(1) \qquad u_0(f) = \tfrac{1}{2}u_0(f_1) + \tfrac{1}{2}u_0(f_2), \quad \text{where} \quad f_1 = f(\tfrac{1}{3}x) \quad \text{and} \quad f_2 = f(\tfrac{2}{3} + \tfrac{1}{3}x).$$

Now we can define $u_0(f)$ as a linear functional, with $u_0(1)=1$, which satisfies relation (1). As a linear functional, if $g(x)=a_0+a_1x \cdots +a_nx^n$, then $u_0(g) =a_0u_0(1)+a_1u_0(x)+ \cdots +a_nu_0(x^n)$, which is consonant with the notion of average or moment. By means of this definition, we obtain values of $u_0(x^n)$, $n=1, 2, \cdots$, as solutions of successive equations.

These results may be used in turn, to obtain average values of polynomial functions and also "higher" moments.

Examples.

$$u_0(x) = \bar{x} = \frac{1}{2}\cdot\frac{1}{3}\,\bar{x} + \frac{1}{2}\left(\frac{2}{3} + \frac{1}{3}\,\bar{x}\right), \qquad \bar{x} = \frac{1}{2}\,.$$

$$\begin{aligned} u_0(x^2) = \overline{x^2} &= \frac{1}{2}\,u_0\left\{\left(\frac{1}{3}\,x\right)^2\right\} + \frac{1}{2}\,u_0\left\{\left(\frac{2}{3} + \frac{1}{3}\,x\right)^2\right\} \\ &= \frac{1}{2}\left\{\frac{1}{9}\,u_0(x^2) + \frac{4}{9}\,u_0(1) + \frac{4}{9}\,u_0(x) + \frac{1}{9}\,u_0(x^2)\right\} \\ &= \frac{1}{2}\left\{\frac{2}{9}\,\overline{x^2} + \frac{4}{9} + \frac{2}{9}\right\}, \qquad \text{since } u_0(x) = \bar{x} = \frac{1}{2}\,; \end{aligned}$$

whence $\overline{x^2}=3/8$.

$$u_0(x^3) = \overline{x^3} = \frac{1}{2}\cdot\frac{1}{27}\overline{x^3} + \frac{1}{2}\left(\frac{8}{27} + \frac{4}{9}\cdot\frac{1}{2} + \frac{2}{9}\cdot\frac{3}{8} + \frac{1}{27}\overline{x^3}\right), \qquad \overline{x^3} = \frac{5}{16}\cdot$$

$$u_0(x^4) = \overline{x^4} = \frac{1}{2}\cdot\frac{1}{81}\overline{x^4} + \frac{1}{2}\left(\frac{16}{81} + \frac{32}{81}\cdot\frac{1}{2} + \frac{8}{27}\cdot\frac{3}{8} + \frac{8}{81}\cdot\frac{5}{16} + \frac{1}{81}\overline{x^4}\right);$$

whence $\overline{x^4} = 87/320, \cdots$.

$$u_0(e^{-x}) = \overline{e^{-x}} = 1 - \frac{1}{2} + \frac{3}{8}\cdot\frac{1}{2!} - \frac{5}{16}\cdot\frac{1}{3!} + \frac{87}{320}\cdot\frac{1}{4!} - + \cdots = .646, \cdots.$$

The average values can be used to obtain higher-order moments:

$$u_1(x) = \overline{x^2}/\bar{x} = 3/4 \quad u_2(x) = \overline{x^3}/\bar{x} = 5/8 \quad u_1(x^2) = \overline{x^3}/\overline{x^2} = 5/6, \cdots.$$

Note: $u_0(x^n) = m_n$ of the reference.

Reference

G. C. Evans, Calculation of moments for a Cantor-Vitali function, this MONTHLY, vol. 64, 1957, pp. 22–27.

A NOMOGRAPHIC SOLUTION OF THE QUARTIC

ELBERT JOHNSON and C. R. WYLIE, JR., University of Utah

Since the general quartic equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{1}$$

contains five homogeneous or four essential coefficients in addition to the unknown x, it is clear that no simple nomogram can be constructed for its solution. However, if λ is suitably chosen, the substitution $x = \lambda t$ will reduce (1) to a form in which only three arbitrary coefficients appear, and an interesting nomographic solution for the transformed equation becomes possible. Specifically, if we set

$$x = \begin{cases} \sqrt{(a_3/a_1)}t & a_1a_3 > 0, \\ \sqrt{(-a_3/a_1)}t & a_1a_3 < 0, \end{cases} \tag{2}$$

we obtain

$$Ut^4 + t^3 + Vt^2 \pm t + W = 0, \tag{3}$$

where

$$U = \frac{a_0}{a_1}\sqrt{\left(\pm\frac{a_3}{a_1}\right)}, \qquad V = \frac{a_2}{a_1}\sqrt{\left(\pm\frac{a_1}{a_3}\right)}, \qquad W = \pm\frac{a_4}{a_3}\sqrt{\left(\pm\frac{a_1}{a_3}\right)},$$

the plus sign applying if $a_1a_3>0$, the minus sign if $a_1a_3<0$.* Obviously, the coefficients in (3) can be obtained from those in (1) by easy slide rule calculations. We now write (3) in the form $Ut^4+(t^3\pm t)=-Vt^2-W=Z$ and consider the possibility of constructing nomograms for the component equations

$$Ut^4+(t^3\pm t)-Z=0, \tag{4.1}$$

$$Vt^2+Z+W=0, \tag{4.2}$$

which can be used simultaneously to solve (3), and hence (1).

Equation (4.1) contains just three variables and the construction of a nomogram for its solution presents no problem. Equation (4.2) contains four variables, but if we treat $(Z+W)$ as a single quantity this equation, too, can be solved nomographically. The necessary nomograms can be constructed in infinitely many ways, but since it turns out to be desirable to have the scales of Z and $(Z+W)$ linear and congruent, our choice is essentially limited to the nomograms derived from the following determinantal equivalents of (4.1) and (4.2):

$$\begin{vmatrix} \dfrac{t^3\pm t}{t^4+1} & \dfrac{t^4}{t^4+1} & 1 \\ -U & 1 & 1 \\ Z & 0 & 1 \end{vmatrix} = 0, \tag{5.1}$$

$$\begin{vmatrix} 0 & \dfrac{t^2}{t^2+1} & 1 \\ V & 1 & 1 \\ Z+W & 0 & 1 \end{vmatrix} = 0. \tag{5.2}$$

These nomograms are shown in Figure 1† and are to be used in the following way. First, the nomograms are placed so that the Z- and $(Z+W)$-scales fall along the same line and the origin of the Z-scale falls at the value W on the $(Z+W)$-scale. Then lines passing respectively through the appropriate points on the U- and V-scales and intersecting on the common $(Z,\ Z+W)$-scale are determined so that they intersect the respective t-scales (choosing the proper t-scale in the upper nomogram according as $a_1a_3>0$ or $a_1a_3<0$) in the same value of t. Since the Z- and $(Z+W)$-scales are uniformly and congruently graduated, it is obvious that values of t determined as we have just described will be solutions of (3) and will lead, through (2), to solutions of (1).

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* If $a_1a_3=0$, (1) initially contains only three essential coefficients and can therefore be handled directly by an obvious modification of the method we shall use on (3).

† Because of the near coincidence of the two t-curves around $t=0$ and $t=\infty$ in the upper nomogram, it would be more practical, though less elegant, to display these scales in separate nomograms.

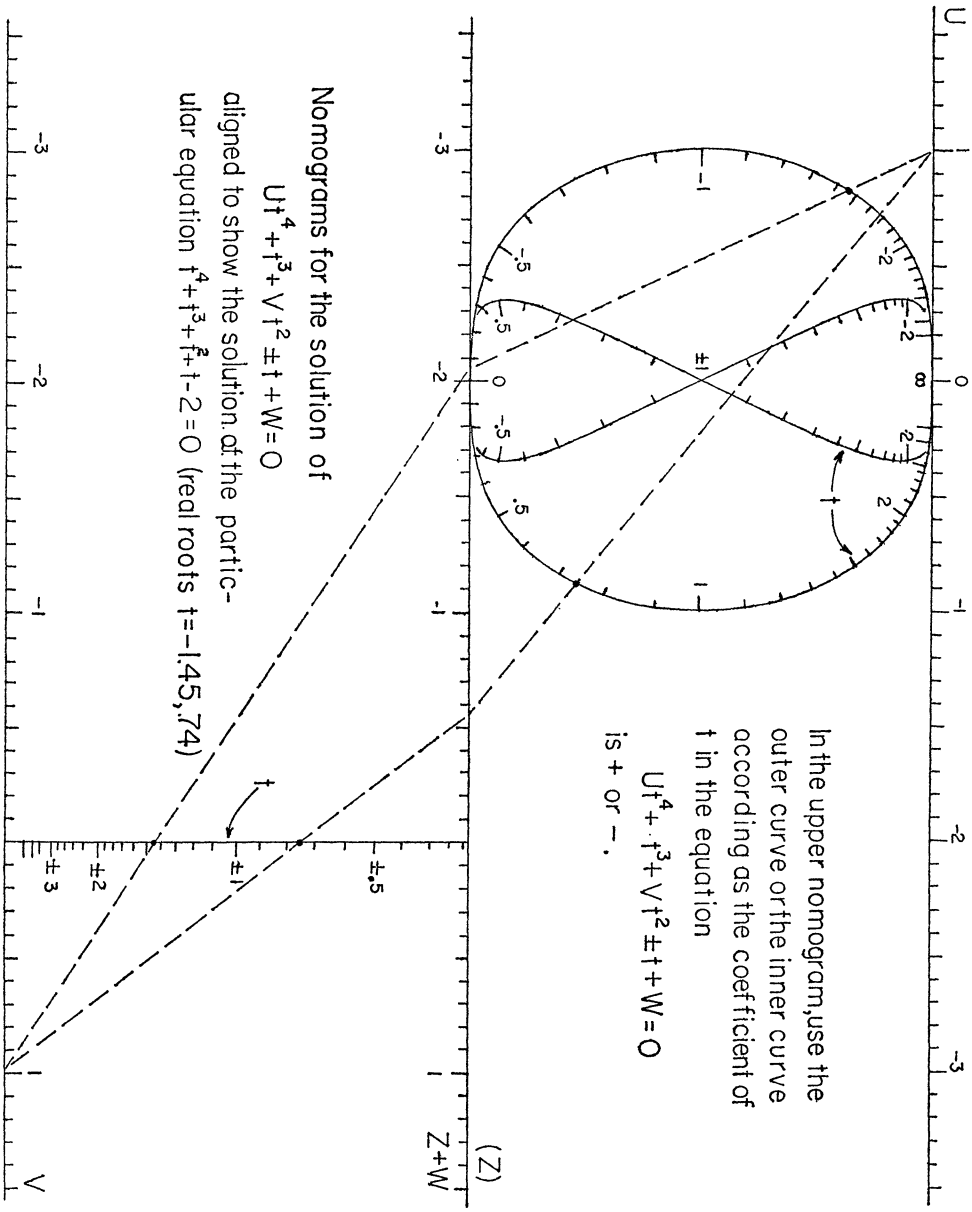

FIG. 1

This process of trial and error can be systematized by means of the apparatus illustrated in Figure 2. This consists of a shallow bed, $LMNO$, bearing sliding panels, $ABCD$ and $EFGH$, guided as shown, to which prints of the nomograms are attached so that the linear scales coincide with the edges of the panels.

Clips bearing small pins placed so that their centers fall directly on the outer scales can be moved along the edges of the bed until the pins coincide with the points P_U and P_V. Between the panels a triangular bar bearing a pin at one end slides back and forth along the common Z- and $(Z+W)$-scales. Finally, two transparent arms, slotted to fit over the pins at P_U and P_V are hinged at the pin P_Z. When the clips have been properly located and the upper panel has been shifted through the distance W relative to the lower, it is a simple matter to draw the center bar back and forth until hair lines inscribed on the two arms are observed to cut the two t-scales in points associated with the same value of t, and a solution is found. A working model of this device has been constructed and its utility confirmed.

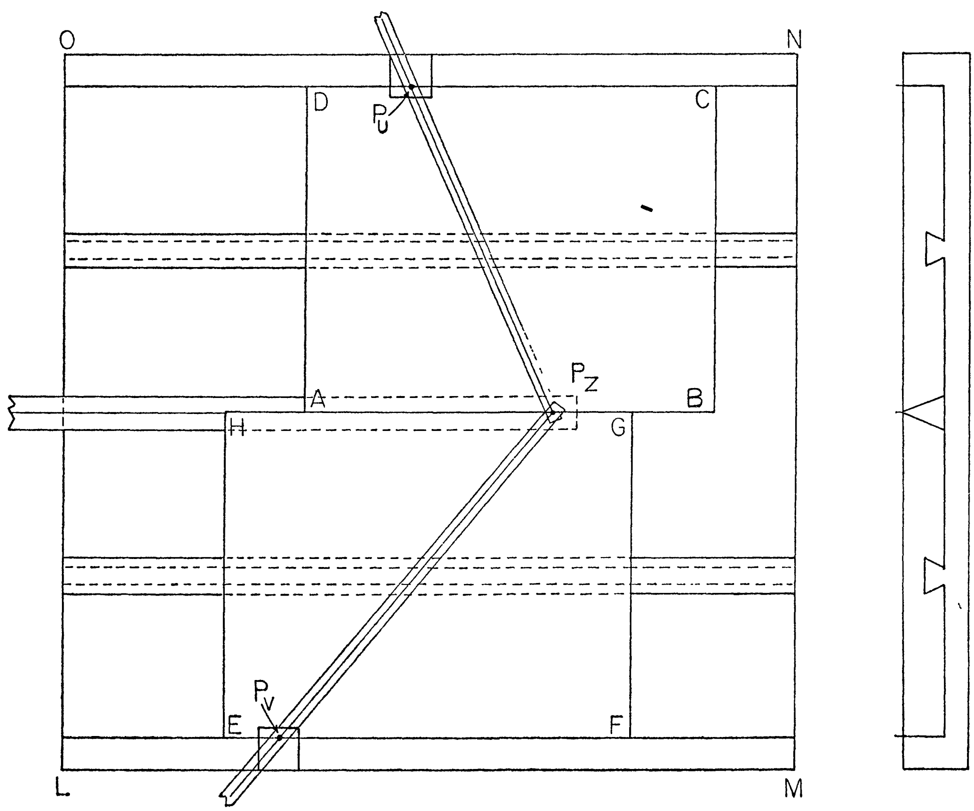

FIG. 2

In conclusion, it is worth pointing out that the procedure discussed in this note, in conjunction with the mechanical device we have just described, can also be used to solve the more general four-variable equation, $UT_1(t)+VT_2(t)+T_3(t)+W=0$, of which (3) is just a special case.

AN EXISTENCE THEOREM FOR INFINITE MATRICES

L. MIRSKY, University of Sheffield, England

1. Our principal object in the present note is to find necessary and sufficient conditions for the existence of infinite matrices subject to certain linear constraints. We shall, in fact, establish the following result.

THEOREM 1. *Let d_k, ρ_k, σ_k $(k \geqq 1)$ be real numbers. Then there exists an infinite matrix with nonnegative elements, with diagonal elements $d_1, d_2, \cdots$, with row-sums $\rho_1, \rho_2, \cdots$, and with column-sums $\sigma_1, \sigma_2, \cdots$ if and only if*

$$0 \leqq d_k \leqq \min(\rho_k, \sigma_k) \qquad (k \geqq 1), \tag{1}$$

$$\sup_{j \geqq 1} (\rho_j + \sigma_j - 2d_j) \leqq \sum_{k=1}^{\infty} (\rho_k - d_k) = \sum_{k=1}^{\infty} (\sigma_k - d_k) \leqq \infty.$$

A more general result which can be deduced from Theorem 1 is as follows.

THEOREM 2. *Let m be a nonnegative integer, and let d_k, ρ_k, σ_k $(k \geqq 1)$ be real numbers. Then the conditions*

$$0 \leqq d_k \leqq \min(\rho_k, \sigma_{m+k}) \qquad (k \geqq 1),$$

$$\sup_{j \geqq 1} (\rho_j + \sigma_{m+j} - 2d_j) \leqq \sum_{k=1}^{\infty} (\rho_k - d_k) = \sum_{k=1}^{m} \sigma_k + \sum_{k=1}^{\infty} (\sigma_{m+k} - d_k) \leqq \infty \tag{2}$$

are necessary and sufficient for the existence of an infinite matrix with nonnegative elements, with row-sums $\rho_1, \rho_2, \cdots$, with column-sums $\sigma_1, \sigma_2, \cdots$, and with mth superdiagonal $d_1, d_2, \cdots$.

In Sections 2 and 3 we shall give a proof of Theorem 1. The derivation of Theorem 2 from Theorem 1 is straightforward, and we shall therefore omit the details of the argument. In Section 4 the special case of infinite doubly stochastic matrices will be examined. Finally, a further specialization will lead to the discussion of *finite* doubly stochastic matrices in Section 5.

2. The proof of Theorem 1 will be seen to depend on the following simple lemma.

LEMMA. *Let ρ_k, σ_k $(k \geqq 1)$ be nonnegative numbers such that*

$$\sum_{k=1}^{\infty} \rho_k = \sum_{k=1}^{\infty} \sigma_k = s < \infty, \tag{3}$$

$$\rho_k + \sigma_k \leqq s \qquad (k \geqq 1). \tag{4}$$

Then there exist numbers y_k, z_k $(k \geqq 2)$ such that

$$\sum_{k=2}^{\infty} y_k = \rho_1, \tag{5}$$

$$\sum_{k=2}^{\infty} z_k = \sigma_1, \tag{6}$$

$$0 \leqq y_k \leqq \sigma_k \qquad (k \geqq 2), \tag{7}$$

$$0 \leqq z_k \leqq \rho_k \qquad (k \geqq 2), \tag{8}$$

$$y_k + z_k \geqq \rho_k + \sigma_k + \rho_1 + \sigma_1 - s \qquad (k \geqq 2). \tag{9}$$

We write $\tau = \sum_{k=2}^{\infty} \max(0, \rho_k + \sigma_k + \rho_1 - s)$, so that $\tau = \sum_{k=2}^{\infty} \delta_k(\rho_k + \sigma_k + \rho_1 - s)$, where each δ_k is 1 or 0. We shall show that

$$\tau \leqq \rho_1. \tag{10}$$

When all δ's are 0, this is obvious; when just one δ is positive, this follows by (4). When the number of positive δ's is at least two, we have

$$\begin{aligned} \tau &= \sum_{k=2}^{\infty} \delta_k(\rho_k + \sigma_k) + (\rho_1 - s) \sum_{k=2}^{\infty} \delta_k \\ &\leqq \sum_{k=2}^{\infty} (\rho_k + \sigma_k) + 2(\rho_1 - s) \\ &= s - \rho_1 + s - \sigma_1 + 2(\rho_1 - s) = \rho_1 - \sigma_1 \leqq \rho_1. \end{aligned}$$

Thus (10) is proved. Hence, in view of (4), there exist numbers y_k $(k \geqq 2)$ satisfying (5) and

$$\max(0, \rho_k + \sigma_k + \rho_1 - s) \leqq y_k \leqq \sigma_k \qquad (k \geqq 2). \tag{11}$$

Now put $\tau' = \sum_{k=2}^{\infty} \max(0, \rho_k + \sigma_k + \rho_1 + \sigma_1 - s - y_k)$, so that

$$\tau' = \sum_{k=2}^{\infty} \epsilon_k(\rho_k + \sigma_k + \rho_1 + \sigma_1 - s - y_k),$$

where each ϵ_k is 1 or 0. If the number of positive ϵ's is at least two, then

$$\begin{aligned} \tau' &= \sum_{k=2}^{\infty} \epsilon_k(\rho_k + \sigma_k - y_k) + (\rho_1 + \sigma_1 - s) \sum_{k=2}^{\infty} \epsilon_k \\ &\leqq \sum_{k=2}^{\infty} (\rho_k + \sigma_k - y_k) + 2(\rho_1 + \sigma_1 - s) \\ &= s - \rho_1 + s - \sigma_1 - \rho_1 + 2(\rho_1 + \sigma_1 - s) = \sigma_1. \end{aligned}$$

Thus we have

$$\sum_{k=2}^{\infty} \max(0, \rho_k + \sigma_k + \rho_1 + \sigma_1 - s - y_k) \leqq \sigma_1; \tag{12}$$

and, in view of (11), this inequality is still valid when the number of positive ϵ's is less than two.

Finally, by (12) and (4), there exist numbers z_k $(k \geqq 2)$ which satisfy (6), (8), and (9). This establishes the lemma.

3. We now come to the proof of Theorem 1. Suppose, in the first place, that there exists a matrix of the type described in the theorem. Then (1) is obviously satisfied. Moreover, for any $j \geqq 1$, $\rho_j + \sigma_j - 2d_j$ is the sum of the nondiagonal elements in the jth row and jth column, while

$$\sum_{k=1}^{\infty} (\rho_k - d_k) = \sum_{k=1}^{\infty} (\sigma_k - d_k) \quad (\leqq \infty)$$

is the sum of *all* nondiagonal elements. Hence (2) follows.

Next, suppose that (1) and (2) are given. We have to show that there exists a matrix with the requisite properties. It may clearly be assumed without loss of generality that $d_k = 0$ $(k \geqq 1)$. Thus ρ_k, σ_k $(k \geqq 1)$ are given nonnegative numbers such that

$$\rho_j + \sigma_j \leqq \sum_{k=1}^{\infty} \rho_k = \sum_{k=2}^{\infty} \sigma_k \leqq \infty \qquad (j = 1, 2, \cdots). \tag{13}$$

We denote by s the common value of the sums of the infinite series in (13). We have now to show that there exists an infinite matrix with nonnegative elements, with zero diagonal elements, with row-sums $\rho_1, \rho_2, \cdots$, and with column-sums $\sigma_1, \sigma_2, \cdots$.

Case 1. $s = \infty$.

It is clearly possible to choose numbers x_{1k}, x_{k1} $(k \geqq 2)$ such that

$$\sum_{k=2}^{\infty} x_{1k} = \rho_1, \qquad \sum_{k=2}^{\infty} x_{k1} = \sigma_1, \tag{14}$$

$$0 \leqq x_{1k} \leqq \sigma_k, \qquad 0 \leqq x_{k1} \leqq \rho_k \qquad (k \geqq 2). \tag{15}$$

This gives us the first row and the first column of the required matrix (x_{ij}). Since $\sum_{k=2}^{\infty} (\rho_k - x_{k1}) = \sum_{k=2}^{\infty} (\sigma_k - x_{1k}) = \infty$, we can now proceed inductively to the construction of all rows and columns.

Case 2. $s < \infty$.

In virtue of the lemma, we can choose numbers x_{1k}, x_{k1} $(k \geqq 2)$ which satisfy (14), (15), and $x_{1k} + x_{k1} \geqq \rho_k + \sigma_k + \rho_1 + \sigma_1 - s$ $(k \geqq 2)$. This gives us the first row and the first column of the required matrix (x_{ij}). Moreover

$$\sum_{k=2}^{\infty} (\rho_k - x_{k1}) = \sum_{k=2}^{\infty} (\sigma_k - x_{1k}) = s - \rho_1 - \sigma_1,$$

$$(\rho_k - x_{k1}) + (\sigma_k - x_{1k}) \leqq s - \rho_1 - \sigma_1 \qquad (k \geqq 2);$$

and, using the lemma repeatedly, we obtain an inductive procedure for constructing all rows and columns of the required matrix.

4. A (finite or infinite) square matrix is called *doubly stochastic* if its elements are nonnegative and if all row-sums and column-sums are equal to 1. Taking $\rho_k=\sigma_k=1$ $(k\geqq 1)$ in Theorem 1, we obtain the following result.

THEOREM 3. *The numbers $d_1, d_2, \cdots$ are the diagonal elements of some infinite doubly stochastic matrix if and only if*

$$0 \leqq d_k \leqq 1 \qquad (k \geqq 1), \tag{16}$$

$$2\left(1 - \inf_{j\geqq 1} d_j\right) \leqq \sum_{k=1}^{\infty} (1 - d_k) \leqq \infty. \tag{17}$$

Actually, a slightly stronger conclusion can be established by means of a more specialized argument. It can, in fact, be shown that if (16) and (17) are given, then $d_1, d_2, \cdots$ are the diagonal elements of a doubly stochastic matrix (d_{ik}) which is either the direct sum of finite doubly stochastic matrices or else satisfies $d_{ik}=0$ for $i\neq k$ and $\min(i, k)>m$, where m is a suitable number which depends on the d_k.*

Again, it is easy to see that any sequence $d_1, d_2, \cdots$ which satisfies (16) differs only slightly from the sequence of diagonal elements of some doubly stochastic matrix. To make this notion more precise, we define an *I-sequence* as any sequence $\{d_k\}$ which satisfies (16). Further, an I-sequence $\{d_k\}$ will be said to be a *D-sequence* if $d_1, d_2, \cdots$ are the diagonal elements of some doubly stochastic matrix. If $x=\{x_k\}$ is a bounded sequence, we define its norm by

$$\|x\| = \sup_{k\geqq 1} |x_k|. \tag{18}$$

THEOREM 4. *The set of D-sequences is everywhere dense, with respect to the norm* (18), *in the set of I-sequences.*

Let $x=\{x_k\}$ be any I-sequence and denote by ϵ any number satisfying $0<\epsilon<\frac{1}{2}$. We shall show that there exists a D-sequence $y=\{y_k\}$ such that $\|x-y\|<\epsilon$.

If $\sum_{k=1}^{\infty}(1-x_k)$ diverges, then, by Theorem 3, x is itself a D-sequence and we put $y_k=x_k$ $(k\geqq 1)$. If, on the other hand, the series converges, then $x_k\to 1$ $(k\to\infty)$ and so $x_k>\frac{1}{2}$ for $k>m$, say. If

$$y_k = \begin{cases} x_k & (k \leqq m), \\ x_k - \dfrac{\epsilon}{k} & (k > m), \end{cases}$$

then $\sum_{k=1}^{\infty}(1-y_k)$ diverges and therefore, by Theorem 3, $y=\{y_k\}$ is a D-sequence. This completes the proof.

5. Let $n\geqq 2$ be an integer. Taking $\rho_k=\sigma_k=1$ $(1\leqq k\leqq n)$ and $d_k=\rho_k=\sigma_k=0$ $(k>n)$ in Theorem 1, we obtain

* The two cases are not, of course, mutually exclusive.

THEOREM 5. *The numbers* $d_1, \cdots, d_n$ *are the diagonal elements of a doubly stochastic* $n\times n$ *matrix if and only if*

$$0 \leqq d_k \leqq 1 \qquad (1 \leqq k \leqq n), \tag{19}$$

$$2\left(1 - \min_{1\leqq j\leqq n} d_j\right) \leqq \sum_{k=1}^{n} (1 - d_k). \tag{20}$$

This result was derived some years ago by A. Horn ([2], Theorem 9). We shall conclude our discussion by sketching yet a further proof depending on quite different ideas.

A vector $d=(d_1, \cdots, d_n)$ will be said to belong to the set $\mathfrak{D}$ if (19) and (20) are satisfied. We denote by $\mathfrak{E}$ the subset of $\mathfrak{D}$ consisting of those vectors all of whose components are 0 or 1. Thus a vector (of order n) belongs to $\mathfrak{E}$ if and only if all its components are 0 or 1 and the number of zero components is different from one.

It is a matter of routine verification that $\mathfrak{E}$ is the set of extreme points of $\mathfrak{D}$.† Hence, by a standard result (cf. [1], p. 24), any $d\in\mathfrak{D}$ can be represented in the form $d=t_1e_1+ \cdots +t_re_r$, where the t's are nonnegative numbers with sum 1 while the e's are vectors in $\mathfrak{E}$. But it is clear that e_k is the vector of diagonal elements of some $n\times n$ permutation matrix P_k. Hence d is the vector of diagonal elements of the doubly stochastic matrix $t_1P_1+ \cdots +t_rP_r$. This establishes one half of Theorem 5. The converse inference is, of course, obvious.

References

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ON THE ANTICENTER OF A GROUP

H. H. JOHNSON, Princeton University

The concept of the anticenter $AC(G)$ of a group G was introduced by N. Levine [1]. In this paper we prove that the operation of forming the anticenter is idempotent, *i.e.*, $AC(G)=AC(AC(G))$. Using this the anticenter of all finitely generated abelian groups is determined. The anticenter is defined as follows:

DEFINITION 1. $R(G)=\{a\in G \mid ab=ba$ *implies there exists* $c\in G$ *such that* $a=c^j$, $b=c^k$ *for some integers* j *and* $k\}$.

DEFINITION 2. $AC(G)=\{a_1a_2 \cdots a_n \mid a_i\in R(G)\}$.

We need the facts that $AC(G)$ is a normal subgroup of G, that $AC(G)=R(G)$ when G is abelian, and that for finite abelian groups $AC(G)=G$ if and only if G is cyclic [1].

† A point x of a convex set $\mathfrak{X}$ is said to be *extreme* if it cannot be represented in the form $x=\frac{1}{2}(x_1+x_2)$, where $x_1, x_2\in\mathfrak{X}$, $x_1\neq x_2$.

THEOREM 1. $AC(AC(G))=AC(G)$.

Proof. We shall prove that $R(G)\subset R(AC(G))$. If $a\in R(G)$, suppose $b\in AC(G)$ and $ab=ba$. Then there exists $c\in G$ such that $a=c^j$, $b=c^k$. We must show that there is an element d in $AC(G)$ with $a=d^u$, $b=d^v$. If s is the least positive integer for which c^s is in $AC(G)$, then $s\leqq j$ and $s\leqq k$. Moreover, s divides j and k, for otherwise $j=n+ms$ where $1\leqq n\leqq s-1$. Then $c^j=c^nc^{ms}$, and since c^j and c^s belong to $AC(G)$, c^n would belong to $AC(G)$, $1\leqq n\leqq s-1$, contrary to the definition of s. Similarly for k. Hence if $d=c^s$, we have $a=d^u$ and $b=d^v$, $d\in AC(G)$.

COROLLARY 1. *If G is finite and abelian then $AC(G)=R(G)$ is cyclic.*

This is an immediate consequence of [1], Theorems 3 and 4.

THEOREM 2. *If H is a subgroup of G, then $R(G)\cap H\subset R(H)$. If G is abelian, $AC(G)\cap H\subset AC(H)$.*

Proof. If $a\in R(G)\cap H$ and $b\in H$ such that $ab=ba$, then there exists $c\in G$ such that $a=c^j$, $b=c^k$. If s is the least positive integer such that $c^s\in H$, then, as in Theorem 1, $s|j$ and $s|k$, so $a=(c^s)^u$ and $b=(c^s)^v$ for some $c^s\in H$. The second statement follows from [1], Theorem 4.

THEOREM 3. *Let G be any finite abelian group with canonical decomposition as the direct product $G_1\times\cdots\times G_z$ of cyclic groups G_i, where G_i has order $p_i^{\sigma_i}$, $i=1,\cdots,z$. If every prime p_i, $i=1,\cdots,z$ divides the orders of at least two of the direct factors of G then*

$$AC(G)=R(G)=1,$$

where 1 denotes the identity subgroup of G. If, on the other hand, $p_1,\cdots,p_y$ are those primes each of which divides the order of exactly one direct factor of G then

$$AC(G)=R(G)=G_1\times\cdots\times G_y.$$

Proof. Suppose p is a prime associated with more than one direct factor. Let $H_1,\cdots,H_n$ be those direct factors having orders $p_1^{e_1},\cdots,p_n^{e_n}$ respectively, $e_j>0$. Then

$$R(G)\cap H_1\times\cdots\times H_n\subset R(H_1\times\cdots\times H_n).$$

We shall show that $R(H_1\times\cdots\times H_n)=1$. $R(H_1\times\cdots\times H_n)$ is cyclic by Corollary 1. Let its generator be $r=a_1^{f_1}\cdots a_n^{f_n}$ where a_1 is the generator of H_1, a_2 that of H_2, etc. We shall prove that $r=1$.

Since G is abelian, $ra=ar$, so there exists an element $a_1^{h_1}\cdots a_n^{h_n}$ such that

$$r=a_1^{f_1}\cdots a_n^{f_n}=(a_1^{h_1}\cdots a_n^{h_n})^j,\qquad a=(a_1^{h_1}\cdots a_n^{h_n})^k.$$

Hence, $h_1k\equiv 1 \bmod p_1^{e_1}$, $h_2k\equiv 0 \bmod p_2^{e_2},\cdots,h_n\equiv 0 \bmod p_n^{e_n}$. Hence, $p_2^{e_2}|h_2,\cdots,p_n^{e_n}|h_n$. Then $p_2^{e_2}|f_2\cdots,p_n^{e_n}|f_n$, so $r=a_1^{f_1}$. But, similarly, we can show that $p_1^{e_1}|f_1$. Hence $r=1$, which proves the first assertion of the theorem.

Consider now $R(G)$ with its generator r_1. If, in the canonical representation of r_1 as powers of generators of $G_1, \cdots, G_z$, the generators of $H_1, H_2, \cdots$, or H_n occur with positive exponent, then by forming $(r_1)^g$ where g is the product of all $p_i^{\sigma_i}$ of primes different from p, we obtain a nontrivial element of $H_1 \times \cdots \times H_n$, belonging to $R(G)$. This is impossible. Hence, $R(G)$ is contained in the product of the G_i omitting $H_1, \cdots, H_n$.

By repeating this argument with any other prime which belongs to more than one direct factor G_i, we see that $R(G) \subset G_1 \times \cdots \times G_y$.

Let G_i be generated by a_i and have order $p_i^{\sigma_i}$, $i=1, \cdots, z$. We shall prove that $a_i \in R(G)$ for $i=1, \cdots, y$. This will prove that $R(G) = G_1 \times \cdots \times G_y$. By symmetry it suffices to show that $a_1 \in R(G)$.

For any element $a_1^{m_1} \cdots a_z^{m_z} \in G$ we must show the existence of an element $a_1^{\mu_1} \cdots a_z^{\mu_z} \in G$ and integers j, k such that

$$a_1 = (a_1^{\mu_1} \cdots a_z^{\mu_z})^j, \qquad a_1^{m_1} \cdots a_z^{m_z} = (a_1^{\mu_1} \cdots a_z^{\mu_z})^k.$$

This leads to equations

$$\begin{aligned}
&(1) & \mu_1 j &\equiv 1 \mod p_1^{\sigma_1}, \\
& & \mu_2 j &\equiv 0 \mod p_2^{\sigma_2}, \\
& & &\vdots \\
& & \mu_z j &\equiv 0 \mod p_z^{\sigma_z}, \\
&(2) & \mu_1 k &\equiv m_1 \mod p_1^{\sigma_1}, \\
&(3) & \mu_2 k &\equiv m_2 \mod p_2^{\sigma_2}, \\
& & &\vdots \\
&(z+1) & \mu_z k &\equiv m_z \mod p_z^{\sigma_z}.
\end{aligned}$$

Let $j = p_2^{\sigma_2} \cdots p_z^{\sigma_z}$. Since G_1 is the only direct factor of G whose order is a power of p_1, then $(j, p_1^{\sigma_1}) = 1$. Therefore there exist integers μ_1 and ν such that $1 = \mu_1 j + p_1^{\sigma_1}\nu$ and then the z equations of (1) are satisfied for arbitrary $\mu_2, \cdots, \mu_z$. Also

$$m_1 = \mu_1 m_1 j + p_1^{\sigma_1} \nu m_1,$$

so any integer $k \equiv m_1 j \mod p_1^{\sigma_1}$ will satisfy (2). Let $k = m_1 j + p_1^{\sigma_1}$. Then $(k, p_i^{\sigma_i}) = 1$ for $2 \leq i \leq z$, so the equations (3), $\cdots$, $(z+1)$ can be solved for $\mu_2, \mu_3, \cdots, \mu_z$.

THEOREM 4. *If G is a finitely generated infinite abelian group which contains at least one nontrivial element of finite order then $R(G) = 1$.*

Proof. If $R(G)$ possesses a nontrivial element a of finite order and b is any element of infinite order, then $a = c^j$, $b = c^k$ implies that c has both finite and infinite order, which is impossible. In the same way, if $R(G)$ contains an element of infinite order and a is an element of G of finite order, the same argument holds.

THEOREM 5. *Let G be a finitely generated infinite abelian group without torsion. Then $R(G)=G$ if G is cyclic and $R(G)=1$ otherwise.*

Proof. If G is cyclic, $R(G)=G$ by [1], Theorem 2. If $a_1, a_2, \cdots, a_n$ constitute a basis of G, with $n>1$, suppose $R(G)$ contains the element $r=a_1^{m_1}\cdots a_n^{m_n}$. Then $a_1r=ra_1$ implies that there exists $a_1^{\mu_1}\cdots a_n^{\mu_n}=c$ with $r=c^j$ and $a_1=c^k$. Then $\mu_1k=1, \mu_2k=\cdots=\mu_n^k=0$; hence $\mu_2=\cdots=\mu_n=0$, so $m_2=\cdots=m_n=0$. Similarly, $m_1=0$. Hence $r=1$ so $R(G)=1$.

Reference

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ON THE DEFINITENESS OF CERTAIN QUADRATIC FORMS ARISING IN A CONJECTURE OF L. J MORDELL

DINA GLADYS S. THOMAS, University of Colorado

H. S. Shapiro proposed the problem: Prove that

$$(1) \qquad R_n[X_1, \cdots X_n] = \sum_1^n \frac{X_i}{X_{i+1}+X_{i+2}} \geqq \frac{n}{2},$$

where the X_i are nonnegative numbers satisfying the cyclic relation $X_{n+k}=X_k$.

The relation has been proved for $n\leqq 6$. However, Zulauf* showed that (1) is false for $n\geqq 14$ and even. Mordell conjectured that (1) is false for $n\geqq 7$; however, the conjecture has not been proven. K. Goldberg of the National Bureau of Standards has tested (1) for $n=7$ for a large number of sets of values of the X_i, using a computing machine. These calculations failed to support Mordell's conjecture.

At the suggestion of S. Chowla, I have attempted to apply a generalization of Zulauf's argument to the cases $n=8$, 10, and 12, and a set of X_i falling in the neighborhoods of 0 and 1; that is, for $n=2s$ ($s=4$, 5 and 6) and $X_{2r-1}=1+a_r\epsilon$, $X_{2r}=b_r\epsilon$ for $\epsilon=$ a small, positive number. In this case, R_n becomes $n/2$ plus a quadratic form (neglecting terms of higher order). The case treated by Zulauf produced a negative number in place of the generalized quadratic form found here. If the quadratic form were indefinite, Mordell's conjecture would have been supported. Unfortunately, the quadratic form turns out definite, so for this set of X_i, Mordell's conjecture is false.

With the above X_i, we have

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* Note on a conjecture of L. J. Mordell, Abh. Math. Sem. Univ. Hamburg, vol. 22, 1958, p. 240.

ON THE COMMUTIVITY OF THE CLOSURE AND INTERIOR OPERATORS IN TOPOLOGICAL SPACES

NORMAN LEVINE, The Ohio State University

If E is the set of rationals in the space of reals R, then $c \operatorname{Int} E=\phi$ and $\operatorname{Int} cE=R$ where c denotes the closure operator and Int denotes the interior operator. It is thus evident that the closure and interior operators do not permute.

DEFINITION. *A set E in a topological space X has property Q if and only if $c \operatorname{Int} E=\operatorname{Int} cE$.*

It is the intent of this paper to give a characterization of property Q in a general topological space and, as a corollary, a characterization in a connected space.

LEMMA 1. *For E a subset of a topological space X, E has property Q if and only if $\mathcal{C}E$ has property Q. ($\mathcal{C}$ denotes the complement operator.)*

Proof. Necessity. Let E have property Q. Then $\operatorname{Int} c\mathcal{C}E=\mathcal{C}c\mathcal{C}c\mathcal{C}E=\mathcal{C}c \operatorname{Int} E$ $=\mathcal{C} \operatorname{Int} cE=\mathcal{C} \operatorname{Int} c\mathcal{C}\mathcal{C}E=\mathcal{C}\mathcal{C}c\mathcal{C}c\mathcal{C}\mathcal{C}E=c \operatorname{Int} \mathcal{C}E$. *Sufficiency.* Let $\mathcal{C}E$ have property Q. Then $E=\mathcal{C}\mathcal{C}E$ and thus, by the above, E has property Q.

LEMMA 2. *For E a subset of a topological space X, E has property Q implies that cE has property Q.*

Proof. $c \operatorname{Int} cE=c \operatorname{Int} c\mathcal{C}\mathcal{C}E=c\mathcal{C}c\mathcal{C}c\mathcal{C}\mathcal{C}E=c\mathcal{C}c \operatorname{Int} \mathcal{C}E=c\mathcal{C} \operatorname{Int} c\mathcal{C}E$ (by Lemma 1) $=c\mathcal{C}\mathcal{C}c\mathcal{C}c\mathcal{C}E=c \operatorname{Int} E=\operatorname{Int} cE$ (since E has property Q) $=\operatorname{Int} ccE$.

The converse of Lemma 2 is false. For if E is the set of rationals in the reals R, then E does not have property Q, but $cE=R$ and R does have property Q.

LEMMA 3. *For E a subset of a topological space X, E has property Q implies that $\operatorname{Int} E$ has property Q.*

Proof. E has property Q and thus $\operatorname{Int} E=\mathcal{C}c\mathcal{C}E$ has property Q by Lemmas 1 and 2.

The converse of Lemma 3 is false. For let E be the set of rationals in the reals R. Then E does not have property Q, but $\operatorname{Int} E=\phi$ and ϕ does have property Q.

LEMMA 4. *Let E be a subset of a topological space X. If E or $\mathcal{C}E$ is nowhere dense, then E has property Q.*

Proof. Let E be nowhere dense in X. Then $\operatorname{Int} cE=\phi$. But $\operatorname{Int} E\subset\operatorname{Int} cE=\phi$ and thus $\operatorname{Int} E=\phi$. Hence $c \operatorname{Int} E=\phi$ and E has property Q. If $\mathcal{C}E$ is nowhere dense, then $\mathcal{C}E$ has property Q and by Lemma 1, E has property Q.

The converse of Lemma 4 is false. Let X: a, b and let the open sets be ϕ, (a), (b), and X. Then (a) has property Q since (a) is both open and closed, but (a) is not nowhere dense nor is $\mathcal{C}(a)$ nowhere dense in X.

LEMMA 5. *Let A be closed in X and $B \subset A$. Then $cB = c_A B$ where c_A is the closure operator in the subspace A.*

Proof. Since $B \subset A$, we have $cB \subset cA = A$. But $c_A B = A \cap cB = cB$.

LEMMA 6. *Let A be open in X and $B \subset A$. Then* $\operatorname{Int} B = \operatorname{Int}_A B$ *where* Int_A *denotes the interior operator in the subspace A.*

Proof. Let $x \in \operatorname{Int} B$. Then $x \in G \subset B$ where G is open in X. It follows that $x \in G \cap A \subset B$ and thus $x \in \operatorname{Int}_A B$. Conversely, let $x \in \operatorname{Int}_A B$. Then $x \in O \cap A \subset B$ where O is open in X. Then $x \in \operatorname{Int} B$ since $O \cap A$ is open in X.

LEMMA 7. *Let A be open and closed in a topological space X. Let $B \subset A$ and $C \subset \mathcal{C}A$. Then* $\operatorname{Int}(B \cup C) = \operatorname{Int} B \cup \operatorname{Int} C$.

Proof. $\operatorname{Int} B \subset \operatorname{Int}(B \cup C)$ and $\operatorname{Int} C \subset \operatorname{Int}(B \cup C)$. Thus $\operatorname{Int} B \cup \operatorname{Int} C \subset \operatorname{Int}(B \cup C)$. Now let $x \in \operatorname{Int}(B \cup C)$. Then $x \in G \subset B \cup C$ where G is open in X. Case 1. $x \in A$. Then $x \in G \cap A \subset (B \cup C) \cap A = B$. Thus $x \in \operatorname{Int} B$. Case 2. $x \in \mathcal{C}A$. Then $x \in G \cap \mathcal{C}A \subset (B \cup C) \cap \mathcal{C}A = C$. Then $x \in \operatorname{Int} C$.

LEMMA 8. *Let A be open and closed in a topological space X and P a nowhere dense set in X. Then $P \cap A$ is nowhere dense in A.*

Proof. $\operatorname{Int}_A c_A P \cap A = \operatorname{Int} cP \cap A$ (by Lemmas 5 and 6) $\subset \operatorname{Int} cP = \phi$ since P is nowhere dense.

We now give a characterization of property Q in a general topological space.

THEOREM 1.* *Let E be a subset of a topological space X. Then E has property Q if and only if $E = (A - P) \cup (P - A)$ where A is both open and closed and P is nowhere dense.*

Proof. Sufficiency. Let $E = (A - P) \cup (P - A)$ where A is open and closed and P is nowhere dense. Now $A - P \subset A$ and $P - A \subset \mathcal{C}A$. Hence

$$\begin{aligned} c \operatorname{Int} E &= c \operatorname{Int} \{(A - P) \cup (P - A)\} \\ &= c\{\operatorname{Int}(A - P) \cup \operatorname{Int}(P - A)\} && \text{(by Lemma 7)} \\ &= c\{\operatorname{Int}_A (A - P) \cup \operatorname{Int}_{\mathcal{C}A}(P - A)\} && \text{(by Lemma 6)} \\ &= c \operatorname{Int}_A (A - P) \cup c \operatorname{Int}_{\mathcal{C}A} (P - A) \\ &= c_A \operatorname{Int}_A (A - P) \cup c_{\mathcal{C}A} \operatorname{Int}_{\mathcal{C}A} (P - A) && \text{(by Lemma 5).} \end{aligned}$$

Similarly $\operatorname{Int} cE = \operatorname{Int}_A c_A(A - P) \cup \operatorname{Int}_{\mathcal{C}A} c_{\mathcal{C}A}(P - A)$. Now $A - P = A - A \cap P$ and $P \cap A$ is nowhere dense in A by Lemma 8. Then $A - P$ is the complement relative to A of a set nowhere dense in A and thus $c_A \operatorname{Int}_A(A - P) = \operatorname{Int}_A c_A(A - P)$ by Lemma 4. Moreover $P - A = P \cap \mathcal{C}A$ is nowhere dense in $\mathcal{C}A$ by Lemma 8. Thus $c_{\mathcal{C}A} \operatorname{Int}_{\mathcal{C}A}(P - A) = \operatorname{Int}_{\mathcal{C}A} c_{\mathcal{C}A}(P - A)$ by Lemma 4 and it now follows that $c \operatorname{Int} E = \operatorname{Int} cE$.

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* The author is indebted to the referee for conjecturing this theorem.

Necessity. Suppose E has property Q. Then let $A = c \operatorname{Int} E = \operatorname{Int} cE$. A then is both open and closed in X. We now show that $E - A$ and $A - E$ are each nowhere dense. $\operatorname{Int} c(E-A) \subset \operatorname{Int} c(E) = A$, but also $\operatorname{Int} c(E-A) \subset \operatorname{Int} c(\mathcal{C}A) = \mathcal{C}A$. Hence $\operatorname{Int} c(E-A) = \phi$. In addition, $\operatorname{Int} c(A-E) \subset \operatorname{Int} cA = A$ and also $\operatorname{Int} c(A-E) \subset \operatorname{Int} c\mathcal{C}E = \mathcal{C}c\mathcal{C}c\mathcal{C}E = \mathcal{C}c \operatorname{Int} E = \mathcal{C}A$. Hence $\operatorname{Int} c(A-E) = \phi$. Let $P = (A-E) \cup (E-A)$. Then P is nowhere dense and the proof will be complete when we show that $E = (A-P) \cup (P-A)$.

To this end we observe that

$$(A-P)\cup(P-A) = (A\cap \mathcal{C}P)\cup(P\cap\mathcal{C}A)$$
$$= [A\cap\mathcal{C}\{(A\cap\mathcal{C}E)\cup(E\cap\mathcal{C}A)\}]\cup[\{(A\cap\mathcal{C}E)\cup(E\cap\mathcal{C}A)\}\cap\mathcal{C}A]$$
$$= [A\cap\mathcal{C}(A\cap\mathcal{C}E)\cap\mathcal{C}(E\cap\mathcal{C}A)]\cup[E\cap\mathcal{C}A] = [A\cap(\mathcal{C}A\cup E)\cap(\mathcal{C}E\cup A)]\cup[E\cap\mathcal{C}A]$$
$$= [(A\cap E)\cap(\mathcal{C}E\cup A)]\cup[E\cap\mathcal{C}A] = (A\cap E)\cup(E\cap\mathcal{C}A) = E.$$

COROLLARY. *Let X be a connected topological space and E a subset of X. Then E has property Q if and only if E is nowhere dense or $\mathcal{C}E$ is nowhere dense.*

Proof. By the above theorem, E has property Q if and only if E can be written as $(A-P)\cup(P-A)$ where A is both open and closed and P is nowhere dense. Since X is connected, the only sets which are both open and closed in X are X itself and ϕ. Thus E has property Q if and only if $E = (\phi - P)\cup(P-\phi)$ or $E = (X-P)\cup(P-X)$. In the former case E is nowhere dense and in the latter case $\mathcal{C}E$ is nowhere dense.

LEMMA 9. *Let E, F, and G be any sets. Let Δ be the symmetric difference operator. Then* (1) $E\Delta F = F\Delta E$, (2) $(E\Delta F)\Delta G = E\Delta(F\Delta G)$ *and* (3) $E\cap(F\Delta G) = (E\cap F)\Delta(E\cap G)$.

The results in the above lemma are well known and proofs will not be given.

THEOREM 2. *Let E and E^* have property Q in a topological space X. Then $E\cap E^*$ has property Q.*

Proof. $E = A\Delta P$, and $E^* = A^*\Delta P^*$ where A and A^* are both open and closed and P and P^* are both nowhere dense by Theorem 1. Then

$$\begin{aligned} E\cap E^* &= E\cap(A^*\Delta P^*) \\ &= (E\cap A^*)\Delta(E\cap P^*) \quad \text{(by Lemma 9)} \\ &= ((A\cap A^*)\Delta(P\cap A^*))\Delta((A\cap P^*)\Delta(P\cap P^*)) \quad \text{(by Lemma 9)} \\ &= (A\cap A^*)\Delta\{(P\cap A^*)\Delta(A\cap P^*)\Delta(P\cap P^*)\} \quad \text{(by Lemma 9)} \\ &= A^{**}\Delta P^{**}, \end{aligned}$$

where $A^{**} = A\cap A^*$ and $P^{**} = (P\cap A^*)\Delta(A\cap P^*)\Delta(P\cap P^*)$. But A^{**} is both open and closed and P^{**} is nowhere dense. Thus $E\cap E^*$ has property Q by Theorem 1.

COROLLARY. *Let E and E^* have property Q in a topological space X. Then* (1) $E \cup E^*$ *has property Q,* (2) $E - E^*$ *has property Q and* (3) fr E *has property Q where* fr *denotes the frontier operator.*

Proof. $E \cup E^* = \mathcal{C}\mathcal{C}(E \cup E^*) = \mathcal{C}\{\mathcal{C}E \cap \mathcal{C}E^*\}$. But $\mathcal{C}E$ and $\mathcal{C}E^*$ have property Q and thus $\mathcal{C}E \cap \mathcal{C}E^*$ has property Q by the above theorem. Then $E \cup E^*$ has property Q by Lemma 1. (2) follows from the identity $E - E^* = E \cap \mathcal{C}E^*$ and Lemma 1 and Theorem 2. To show (3) we note that fr $E = cE \cap c\mathcal{C}E$ and cE has property Q as well as $c\mathcal{C}E$ by Lemmas 1 and 2. Thus fr E has property Q by Theorem 2.

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NOTE ON A PAPER OF KLAMKIN CONCERNING STIRLING NUMBERS*

H. W. GOULD, West Virginia University

M. S. Klamkin [5] found the closed summation formula

$$K_n = \sum_{k=1}^{\infty} k^n x^k = (1 - x)^{-n-1} \sum_{k=0}^{n} x^{k+1}(1 - x)^{n-k}\Delta^k 1^n, \tag{1}$$

and remarked that the following relations allow K_n to then be expressed in terms of generalized Bernoulli numbers or Stirling numbers of the *first* kind:

$$\Delta^k 0^n + \Delta^{k+1} 0^n = \Delta^k 1^n, \tag{2}$$

$$\Delta^k 0^n = \frac{n!}{(n - k)!} B_{n-k}^{(-k)}, \tag{3}$$

$$S_n^m = \binom{n - 1}{m - 1} B_{n-m}^{(n)}, \tag{4}$$

where S_n^m is a Stirling number of the first kind as defined in [4].

There is considerable difficulty in passing from (3) to (4) however. This was also detected by Paasche [6]. From (4) we would have to write

$$B_{n-k}^{(-k)} = \frac{S_{-k}^{-n}}{\binom{-k - 1}{-n - 1}}, \tag{5}$$

which appears to be meaningless because S_b^a has no interpretation for both a and b negative, and there is a similar difficulty for the binomial coefficient.

If we look a bit deeper we may, however, show that the passage from (3) to (4) leads to no formula expressing $\Delta^k 0^n$ in terms of S_n^m but a mere tautology.

* Presented to the Allegheny Mountain Section of the Association, Pittsburgh, May 2, 1959. See this MONTHLY, vol. 66, 1959, p. 640.

To see this clearly, we note first some facts about Stirling numbers. In another standard combinatorial notation [2] the Stirling numbers of first kind, S_1, and of second kind, S_2, are related to those defined by Jordan in the following manner:

$$S_n^k = (-1)^{n-k} S_1(n-1, n-k), \tag{6}$$

and

$$\mathfrak{S}_n^k = S_2(k, n-k) = \frac{1}{k!} \Delta^k 0^n, \tag{7}$$

the latter part of (7) being of course the well-known fact that Stirling numbers of the second kind are nothing but higher differences of zero.

Now [2] it is possible to make sense of S_{-k}^{-n} by use of (6) and the further information that

$$S_1(-n-1, k) = S_2(n, k), \qquad S_2(-n-1, k) = S_1(n, k). \tag{8}$$

Indeed we find by substitutions that

$$\begin{aligned} S_{-k}^{-n} &= (-1)^{-n+k} S_1(-k-1, -k+n) = (-1)^{n-k} S_2(k, n-k) \\ &= (-1)^{n-k} \Delta^k 0^n. \end{aligned} \tag{9}$$

We may also find an interpretation for the binomial coefficient in (5) if we make use of the properties

$$\binom{x}{y} = \binom{x}{x-y} \quad \text{and} \quad \binom{-x}{y} = (-1)^y \binom{x+y-1}{y}. \tag{10}$$

Manipulative substitutions with these show that either

$$\binom{-k-1}{-n-1} = (-1)^{n-k} \binom{n}{k} \quad \text{or} \quad (-1)^{n-k-1} \binom{n}{k}.$$

Using this and (9) in (5) to simplify (3) we obtain

$$\Delta^k 0^n = \frac{n!}{(n-k)!} \cdot \frac{\pm \Delta^k 0^n}{\binom{n}{k} k!} = \pm \Delta^k 0^n,$$

and, choosing the plus sign, we see that the passage is indeed a tautology.

Now, it *is* possible to express the Stirling numbers of the second kind in terms of the Stirling numbers of the first kind as the author [2] has shown. The converse problem was first solved by Schlömilch, as pointed out in [2]. The formulas given in [2] then allow a solution to the present paradox and afford a method of expressing K_n here in terms of either S_1 or S_2. The existence of formulas relating S_1 to S_2 or conversely as shown in [2] depends on the fact that

the functions $\log(1+x)$ and e^x-1 are inverses of one another. The writer is indebted to the referee for pointing out that Jabotinski [3] has given general theorems which relate the coefficients in two inverse functions.

The writer would like to indicate here that after writing [2] he found that Goldberg [1] had also found a solution to the problem of expressing the Stirling numbers in terms of each other.

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CONCAVE FUNCTIONALS AND A PROBLEM OF BELLMAN

ARTHUR WOUK, Sylvania Applied Research Laboratory, Waltham, Massachusetts

In [1], p. 42, Bellman poses the problems of maximizing functions of real variables of the form

$$F(x_1, \cdots, x_n) = \sum_{i=1}^{n} \phi(x_i) \tag{1}$$

and

$$F(x_1, \cdots, x_n; y_1, \cdots, y_n) = \sum_{i=1}^{n} \phi(x_i, y_i),$$

subject to

$$x_i,\ y_i \geqq 0, \qquad \sum_{i=1}^{m} x_i = C_1, \qquad \sum_{i=1}^{n} y_i = C_2,$$

where the function ϕ is strictly concave and monotone increasing in each of its arguments. In this note the problem is stated for real concave functionals on a partially ordered vector space and we show that the maximum occurs when the vectors x_i are all equal.

Let X be a partially ordered real linear vector space ([2], pp. 10–16), X^+ the positive cone in X and ϕ a real-valued functional on X^+ which is *concave* there. For any C in X^+, let R_n be the set of n-tuples of vectors x_i in X^+ such that $x_1+x_2+\cdots+x_n=C$, let F be as in (1), and set

$$f_n(C) = \max_{R_n} F(x_1, \cdots, x_n).$$

Then we will see that $f_n(C) = n\phi(1/nC)$. For we notice that $f_1(c) = \phi(C)$ while

$$f_n(C) = \max_{0 \leqq y \leqq C} \{\phi(y) + f_{n-1}(C - y)\}, \qquad n \geqq 2, \tag{2}$$

and, from the concavity of ϕ, we see that

$$\frac{1}{n}\phi(y) + \frac{n-1}{n}\phi\left(\frac{1}{n-1}(C - y)\right) \leqq \phi\left(\frac{1}{n}C\right).$$

The induction argument is now quite clear and the claim is proven. From this last line we note that the maximum is unique if ϕ is *strictly concave.*

If ϕ is monotone nondecreasing, we may modify the definition of R_n by requiring only $x_1 + \cdots + x_n \leqq C$. Equation (2) can still be established easily, and the inductive proof is still valid.

This solves Bellman's problems, and leads to a number of interesting and sometimes familiar corollaries. For example, if X is the space of real continuous functions on a closed interval $[a, b]$, then it is easy to verify that the functional $\phi(x) = \min_{[a,b]} x(t)$ is concave and monotone nondecreasing with respect to the natural ordering $x(t) \geqq y(t)$ if $x(t) - y(t) \geqq 0$, $a \leqq t \leqq b$. Here R_n is the set of nonnegative functions $x_i(t)$ with $x_1(t) + x_2(t) + \cdots + x_n(t) \leqq C(t)$ for a fixed nonnegative function $C(t)$ and we conclude that

$$\max_{R_n} \sum_{i=1}^{n} \min_{[a,b]} (x_i(t)) = \min_{[a,b]} C(t).$$

If we take X to be a normed linear space and $\phi(x) = -\|x\|$, we obtain $\min \sum_{i=1}^{n} \|x_i\| = \|C\|$ whenever C, x_i are nonnegative vectors such that $x_1 + \cdots + x_n = C$; that is, the straight line is the shortest distance between two points.

Lastly, let x be a real variable, $\phi(x) = -x \log x$, and R_n the n-tuples $x_i \geqq 0$, $x_1 + \cdots + x_n = 1$. We conclude from the strict concavity of ϕ that

$$\max_{R_n} \left(-\sum_{i=1}^{n} x_i \log x_i\right) = \log n$$

and that the maximum is unique. This, in information theory, is the well-known assertion that the maximum information content of a source having n-elements is $\log n$, and is achieved only when all elements have equal probability* ([3], p. 15).

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* This application was pointed out to me by H. Young.

ON EXTREMALS OF FINITE SUMS AND DEFINITE INTEGRALS

A. SALENIUS, Sperry Gyroscope Co., Great Neck, N. Y.

1. Introduction. In many physical problems it is desirable to optimize a functional involving finite differences of the unknown function. Such problems often arise when the functional is a definite integral and the independent variable is time. The unknown function is thus involved in the functional at various delayed times. The problem parallels the problem of the calculus of variations in which the functional involves derivatives rather than differences. In the case when the functional is a finite sum involving finite differences of the unknown function, others [1] have shown that a necessary condition takes the form of a difference equation which is analogous to Euler's equation in the calculus of variations. The present note extends the method to functionals which are finite sums involving differences higher than the first and to functionals which are infinite sums, or definite integrals, involving finite differences of the unknown functions.

2. Finite sum involving a first difference. Let $\Delta y(i) \equiv y(i+k) - y(i)$, where k is a fixed integer and i is a variable integer such that $0 \leqq i \leqq n$. Assume that $F(i, y(i), \Delta y(i))$ possesses continuous second derivatives with respect to i, y, and Δy in the region defined by $0 \leqq i \leqq n$. Let it be required that $y(i)$ be a function such that the finite sum $I = \sum_{i=0}^{n} F(i, y, \Delta y)$ be an extremum.

A necessary condition is that $y(i)$ satisfy

$$\frac{\partial I}{\partial y(i)} = F_y(i) - F_{\Delta y}(i) + F_{\Delta y}(i-k) = 0,$$

or $F_y(i) - \Delta F_{\Delta y}(i-k) = 0$, which, in general, is a second-order difference equation, and in which $F(i) = 0$ for $i < 0$ and $i > n+k$.

3. Finite sum involving first and second differences. If the finite sum integrand is a function of second differences, $\Delta^2 y(i) \equiv \Delta y(i+k) - \Delta y(i)$, as well as first, or $I = \sum_{i=0}^{n} F(i, y, \Delta y, \Delta^2 y)$ then, for an extremum, $y(i)$ must satisfy

$$\frac{\partial I}{\partial y(i)} = F_y(i) - F_{\Delta y}(i) + F_{\Delta y}(i-k) + F_{\Delta^2 y}(i) - 2F_{\Delta^2 y}(i-k) + F_{\Delta^2 y}(i-2k) = 0,$$

or $F_y(i) - \Delta F_{\Delta y}(i-k) + \Delta^2 F_{\Delta^2 y}(i-2k) = 0$, which, in general, is a fourth-order difference equation, and in which $F(i) = 0$ for $i < 0$ and $i > n+2k$.

4. Finite sum involving higher differences. By induction on the above results it can be shown that if $I = \sum_{i=0}^{n} F(i, y, \Delta y, \cdots, \Delta^r y)$ is an extremum then $y(i)$ must satisfy the difference equation

$$\sum_{j=0}^{r} (-1)^j \Delta^j F_{\Delta^j y}(i - jk) = 0,$$

where $\Delta^0 \equiv 1$, which, in general, is of order $2r$, and in which $F(i) = 0$ for $i < 0$ and $i > n + rk$.

5. Definite integral involving a first difference. Assume

$$I = \int_a^b F(x, y, \Delta y)dx, \tag{1}$$

where $\Delta y \equiv y(x+\tau) - y(x)$, possesses continuous second derivatives with respect to x, y, and Δy in the region (a, b).

We note that

$$\begin{aligned} I = & \sum_{i=0}^{n-1} \int_{R_i} F(x(i), y(x(i)), \Delta y(x(i)))d(x(i)) \\ & + \int_{a+n}^{b} F(x(n), y(x(n)), \Delta y(x(n)))d(x(n)), \end{aligned} \tag{2}$$

where

$$x(i) = \begin{cases} a + i\tau + \xi, & 0 \leqq \xi < \tau, \\ 0, & \xi < 0, \xi \geqq \tau, \end{cases}$$

and

$$R_i\colon a + i\tau \leqq x(i) < a + (i+1)\tau.$$

If $y(x)$ makes the integral in (1) stationary it must also make the sum of the integrals in (2) stationary.

By considering each $x(i)$ as an independent variable, a necessary condition that the $y(x(i))$ make (2) stationary is that the $y(x(i))$ satisfy the equations

$$\frac{\partial I}{\partial y(x(i))} = F_y(x(i)) - F_{\Delta y}(x(i)) + F_{\Delta y}(x(i) - \tau) = 0,$$

or

$$F_y(x(i)) - F_{\Delta y}(x(i) - \tau) = 0. \tag{3}$$

Since $x = x(i)$ for $a + i\tau \leqq x < a + (i+1)\tau$, then (3) may be written simply as

$$F_y(x) - F_{\Delta y}(x - \tau) = 0. \tag{4}$$

Thus, if $y(x)$ makes the integral in (1) an extremum it must satisfy (4). It is observed that in (4), $F(x) = 0$ for $x < a$ and $x > b + \tau$.

6. Definite integral involving higher differences. By methods similar to those above it can be shown that for the integral

$$\int_a^b F(x, y, \Delta y, \cdots, \Delta^r y)dx$$

to be an extremum a necessary condition is that the function $y(x)$ satisfy the difference equation

$$\sum_{j=0}^{r} (-1)^j \Delta^j F_{\Delta^j y}(x - j\tau) = 0, \tag{5}$$

where $\Delta^0 \equiv 1$ and in which $F(x) = 0$ for $x < a$ and $x > b + r\tau$.

7. Discussion. The solutions of the above difference equations make the corresponding functionals stationary, but they are not necessarily extrema. However, in most physical problems the existence of an extremum can be intuitively recognized. If there exist several solutions to the difference equation, it is relatively easy to verify which makes the corresponding functional an extremum.

The boundary conditions are usually specified at the end points and usually consist of values of the unknown function and/or values of first and/or higher differences of the function. In many problems, however, boundary conditions may be specified within the interval in the same manner.

If we define $E^h f(x) \equiv f(x + h\tau)$, we may write (5) as

$$\sum_{j=0}^{r} (-1)^j (\Delta E^{-1})^j F_{\Delta^j y}(x) = 0.$$

Euler's equation for the extremals of the corresponding functional involving derivatives is

$$\sum_{j=0}^{r} (-1)^j D_x^j F_{y^{(j)}}(x) = 0.$$

Thus, we see that the equations are analogous with the operator ΔE^{-1} corresponding to the operator D_x.

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THE SILOV BOUNDARY FOR A LINEAR SPACE OF CONTINUOUS FUNCTIONS

H. S. BEAR,* University of Washington

The theorem of Silov (see [4], p. 80) states that if A is an algebra of real or complex valued continuous functions on a compact Hausdorff space X, then there is a smallest closed subset F of X on which each function in A assumes its maximum modulus. The set F is called the Silov boundary of A in X. Arens and Singer ([1], Theorem 2.4) have shown the existence of such a boundary for a subfamily A of continuous functions on a topological space X provided that A is a multiplicative semigroup, the functions in A vanish at infinity, and X has a

* The author wishes to acknowledge financial support from the National Science Foundation.

basis of sets of the form $\{x: |f_i(x)| < \epsilon, i=1, \cdots, n\}$ for functions $f_1, \cdots, f_n$ in A. Recently Bauer [2], and Bishop and de Leeuw [3] have shown the existence of more refined (nonclosed) boundaries for linear spaces of continuous functions on a compact space. The proofs of [2] and [3] use a fair amount of machinery from the theory of linear topological spaces. Our purpose here is to give an elementary proof of the existence of a closed boundary for linear subspaces of $C(X)$. The theorem can be deduced from that of Arens and Singer by taking exponentials, but the present proof is simpler in view of the topological requirement of the Arens-Singer theorem.

Let $C(X)$ denote the space of all continuous real- or complex-valued continuous functions on the compact Hausdorff space X. Let $\|f\|$ denote the maximum modulus of f on X. The weak topology on X induced by a family $\mathfrak{F}$ of functions on X has as a basis at x_0 the sets

$$\{x: |f_i(x) - f_i(x_0)| < \epsilon, i = 1, \cdots, n\}, \tag{1}$$

where $f_1, \cdots, f_n$ is any finite set of functions in $\mathfrak{F}$. Recall ([4] Theorem 5G, p. 12) that if $\mathfrak{F} \subset C(X)$, and $\mathfrak{F}$ contains the constant functions and separates points of X (*i.e.*, if $x \neq y$, there is some $f \in \mathfrak{F}$ such that $f(x) \neq f(y)$), then the weak topology induced by $\mathfrak{F}$ coincides with the given compact topology of X.

THEOREM. *If X is compact Hausdorff, and H is a point-separating linear subspace of $C(X)$, containing the constant functions, then there is a (obviously unique) closed set F such that*

(i) *for each $f \in H$, there is some $x \in F$ such that $|f(x)| = \|f\|$, and*

(ii) *if B is any closed set with property* (i) *then $F \subset B$.*

Proof. Let $\mathfrak{N}$ be a maximal nest of closed sets having property (i), and let $F = \bigcap \mathfrak{N}$. Clearly F is closed. By the definition of $\mathfrak{N}$, $M(f) = \{x: |f(x)| = \|f\|\}$ intersects each set in $\mathfrak{N}$, and since $\mathfrak{N}$ is a nest, $M(f)$ intersects any finite number of sets in $\mathfrak{N}$. By the finite intersection property of the compact space X, $M(f)$ intersects F, which says that F has property (i).

Now suppose that B is a closed set with property (i), and that $x_0 \in F - B$. Let V be a neighborhood of the form (1) such that $x_0 \in V \subset X - B$. Since H is a linear space containing the constants, the functions $g_i = f_i - f_i(x_0)$ are in H and

$$V = \{x: |g_i(x)| < \epsilon, i = 1, \cdots, n\}. \tag{2}$$

Since $F = \bigcap \mathfrak{N}$, where $\mathfrak{N}$ is a maximal nest, the set $F - V$ does not have property (i), and there is a function $f \in H$ such that $\|f\| > \sup\{|f(x)| : x \in F - V\} = \|f\|_{F-V}$. Let $k = mf$, where m is so large that $\|k\| - \|k\|_{F-V} > \|g_1\| + \cdots + \|g_n\|$. If $x \in V$ and α is any complex number with $|\alpha| = 1$, then $|k(x) + \alpha g_i(x)| < \|k\| + \epsilon$. If $x \in F - V$, then $|k(x) + \alpha g_i(x)| < \|k\|_{F-V} + \|g_i\| < \|k\|$. Therefore, for any i, any α with $|\alpha| = 1$, and any $x \in F$, $|k(x) + \alpha g_i(x)| < \|k\| + \epsilon$. Since F has property (i),

$$|k(x) + \alpha g_i(x)| < \|k\| + \epsilon \tag{3}$$

for all $x \in X$. Let t be any point of X such that $\|k\| = |k(t)|$. Pick α so that

$|k(t)+\alpha g_i(t)| = \|k\| + |g_i(t)|$. From (3) we conclude that $|g_i(t)| < \epsilon$. Since g_i can be any of the functions occurring in (2), it follows that $t \in V$. That is, k only assumes its maximum on V, and hence not on B, which contradicts the assumption that B has property (i). Therefore, if B has property (i), $F \subset B$ as claimed.

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AN ELEMENTARY PROOF OF THE FORMULA $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$

YOSHIO MATSUOKA, Kagoshima University, Japan

The formula in the title has been well known, but its various known proofs are less elementary, (see for example, [1], p. 219, 360, [3], p. 237, 267, 324, [4], Problem 99, p. 196, [5], p. 379). The following proof is quite elementary in character.

For any positive integer n, we consider

$$\int_0^{\pi/2} \cos^{2n} t dt.$$

Applying integration by parts twice, we obtain

$$\begin{aligned}\int_0^{\pi/2} \cos^{2n} t dt &= \left[t \cos^{2n} t\right]_0^{\pi/2} + 2n \int_0^{\pi/2} t \cos^{2n-1} t \sin t dt \\ &= n\left[t^2 \cos^{2n-1} t \sin t\right]_0^{\pi/2} - n \int_0^{\pi/2} t^2\left[-(2n-1)\cos^{2n-2} t \sin^2 t + \cos^{2n} t\right] dt \\ &= -2n^2 \int_0^{\pi/2} t^2 \cos^{2n} t dt + n(2n-1) \int_0^{\pi/2} t^2 \cos^{2n-2} t dt \\ &= -2n^2 I_{2n} + n(2n-1) I_{2n-2},\end{aligned}$$

where $I_{2n} = \int_0^{\pi/2} t^2 \cos^{2n} t dt$. Hence

$$-2n^2 I_{2n} + n(2n-1) I_{2n-2} = \int_0^{\pi/2} \cos^{2n} t dt.$$

As is known, (see, for example, [2], p. 226)

$$\int_0^{\pi/2} \cos^{2n} t dt = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2},$$

where, as usual,

$$(2n)!! = 2\cdot 4 \cdots (2n-2)(2n), \quad 0!! = 1;$$
$$(2n+1)!! = 1\cdot 3 \cdots (2n-1)(2n+1), \quad (-1)!! = 1.$$

Thus we have

$$-2n^2 I_{2n} + n(2n-1)I_{2n-2} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2},$$
$$\frac{(2n)!!}{(2n-1)!!}I_{2n} - \frac{(2n-2)!!}{(2n-3)!!}I_{2n-2} = -\frac{\pi}{4}\frac{1}{n^2}.$$

This implies that

$$\frac{(2n)!!}{(2n-1)!!}I_{2n} - \frac{0!!}{(-1)!!}I_0 = \sum_{k=1}^{n}\left[\frac{(2k)!!}{(2k-1)!!}I_{2k} - \frac{(2k-2)!!}{(2k-3)!!}I_{2k-2}\right]$$
$$= -\frac{\pi}{4}\sum_{k=1}^{n}\frac{1}{k^2},$$

and hence that

$$\frac{(2n)!!}{(2n-1)!!}I_{2n} = \frac{\pi^3}{24} - \frac{\pi}{4}\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi}{4}\left[\frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2}\right].$$

It is sufficient to prove that

(1) $$\lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!}I_{2n} = 0.$$

Now we have

$$I_{2n} = \int_0^{\pi/2} t^2 \cos^{2n} t\,dt \leqq \left(\frac{\pi}{2}\right)^2 \int_0^{\pi/2} \sin^2 t \cos^{2n} t\,dt$$
$$= \frac{\pi^2}{4}\left[\int_0^{\pi/2}\cos^{2n} t\,dt - \int_0^{\pi/2}\cos^{2n+2} t\,dt\right]$$
$$= \frac{\pi^3}{8}\left[\frac{(2n-1)!!}{(2n)!!} - \frac{(2n+1)!!}{(2n+2)!!}\right] = \frac{\pi^3}{8}\frac{(2n-1)!!}{(2n+2)!!}.$$

Therefore

$$0 < \frac{(2n)!!}{(2n-1)!!}I_{2n} \leqq \frac{\pi^3}{8}\frac{1}{2n+2}.$$

Thus we establish (1); hence the formula is proved.

Finally, the author wishes to express his thanks to a referee for valuable criticism.

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

AN ABSTRACT FORMULATION OF A PROBLEM RELATED TO GOLDBACH'S CONJECTURE

A. A. MULLIN, University of Illinois

0. Introduction. Goldbach's conjecture [1] states that every even integer greater than four can be represented as the sum of two odd prime numbers. This paper gives an abstract formulation of a problem that is at least as general as Goldbach's conjecture. Specifically the problem deals with a minimal condition associated with subsets of odd integers in the additive semigroup of strictly positive integers.

1. Definitions.

DEFINITION 1.1. *Let $(A, *)$ denote a nonempty set A together with a closed binary composition law "$*$" defined on A. Call $(A, *)$ an algebraic system. By a mutant of $(A, *)$ is meant a subset M of A that satisfies the condition that $M * M \subseteq \overline{M}$, where $M * M = \{a * b : a \in M, b \in M\}$ and $\overline{M}$ is the set of all of the elements of A not in M. Clearly M must be a proper subset of A. If and only if all of the elements of A are idempotent with respect to "$*$" let the empty set be the only mutant of $(A, *)$. With this convention every algebraic system has a mutant.*

DEFINITION 1.2. *A mutant M of $(A, *)$ is said to be a maximal mutant of $(A, *)$ if there is no mutant of $(A, *)$ which properly contains M. If and only if all of the elements of A are idempotent with respect to "$*$" let the empty set be the only maximal mutant of $(A, *)$. With this convention the Hausdorff extremal principle [2] guarantees that every algebraic system has a maximal mutant.*

DEFINITION 1.3. *Let N be a maximal mutant of $(A, *)$. If there exists a nonempty class of mutants $M_i \subseteq N$ of $(A, *)$ such that $M_i * M_i \supseteq \overline{N}$, for all $i \in I$, then $\bigcap_{i \in I} M_i$ is said to be a potential minimax mutant in N of $(A, *)$. When N is potential minimax mutant in N of $(A, *)$ it is said to be the trivial potential minimax mutant in N of $(A, *)$.*

DEFINITION 1.4. *Let N be a maximal mutant with a potential minimax mutant in N of $(A, *)$. A minimax mutant in N of $(A, *)$ is a potential minimax mutant in N of $(A, *)$ for which there is no potential minimax mutant in N of $(A, *)$ that is properly contained in it. When N is a minimax mutant in N of $(A, *)$, it is the only minimax mutant in N of $(A, *)$. Then N is said to be the trivial minimax mutant in N of $(A, *)$.*

2. Propositions.

LEMMA. *No nontrivial potential minimax mutant in N of $(A, *)$ can be a maximal mutant of $(A, *)$.*

Proof. No proper subset of a mutant can be a maximal mutant.

THEOREM 2.1. *The additive semigroup $(A, +)$ of strictly positive integers has a nontrivial minimax mutant in N, where N is the set of all positive odd integers.*

Proof. Clearly N is a maximal mutant of $(A, +)$. Since $N+N=\overline{N}$ then $N+N\supseteq\overline{N}$. Put $N^*=N\cap\{\bar{n}\}$, where $3<n\in N$. It is easy to show that this proper subset of N satisfies the condition $N^*+N^*\supseteq\overline{N}$. Now apply the Hausdorff extremal principle [2] to establish the existence of a nontrivial minimax mutant in N of $(A, +)$. When applying the Hausdorff extremal principal observe that any finite set is a suitable lower bound for the required simple ordering (set inclusion) of potential minimax mutants in N of $(A, +)$.

THEOREM 2.2. *Not every maximal mutant of the additive semigroup $(A, +)$ of strictly positive integers has a potential minimax mutant in it.*

Proof. Clearly the set $S=\{2+4n: n=0, 1, 2, \cdots\}$ is a maximal mutant of $(A, +)$. But $S+S\subseteq E$, where E is the set of all positive even integers. Hence no odd integer can be represented as the sum of two elements of S.

Conclusions. In terms of the present terminology which involves general concepts with many other properties than the ones presented here [3], if Goldbach's conjecture is a valid theorem, it represents, among other things, another existence proof for a nontrivial potential minimax mutant in the set of all positive odd integers with respect to the additive semigroup of positive integers.

References

1. R. G. Archibald, Goldbach's Theorem, Scripta Math., vol. 3, 1935, pp. 44–50 and pp. 153–161; further references to Goldbach's conjecture may be found there.

2. J. L. Kelley, General Topology, Princeton, 1955, pp. 32–33. We consider the "maximal principle" and the "minimal principle" as lumped together into the "extremal principle."

3. A. A. Mullin, Some remarks on a relative anti-closure property, Notices Amer. Math. Soc., vol. 7, 1960, p. 480; also, see Z. Math. Logik Grundlagen, vol. 7, Sept., 1961.

ON THE APPROXIMATION OF IRRATIONALS

H. E. CHRESTENSON, Reed College

This note contains an unexpected application of Lebesgue's theorem on the differentiability of a function of bounded variation to the theory of approximation of irrationals by rationals. For an exposition of Lebesgue's theorem the reader is referred to [3], Chapter 1. The Carus Monograph by Boas contains an excellent exposition of the difficult part of this theorem, *i.e.*, its proof for monotone functions ([1], p. 134).

THEOREM ([2], Theorem 198). *Let χ be a positive-valued function defined on the positive integers. Let E be the set of irrationals ξ such that $|\xi-p/q|<1/q\chi(q)$ for infinitely many q and appropriate p. If the series $\sum_{q=1}^{\infty} 1/\chi(q)$ converges then E has Lebesgue measure zero.*

We have adopted the convention that an expression like p/q shall always denote a quotient of integers with $q>0$ and $(p, q)=1$.

Proof. Without loss of generality we restrict our attention to the interval $[0, 1]$ where we define f by $f(x)=0$ if x is irrational, $=1/q\chi(q)$ if $x=p/q$. Since there are fewer than q fractions with denominator q (except for $q=1$, when there are two), any sum of the form $\sum_{k=1}^{n} |f(x_k)-f(x_{k-1})|$ with $0=x_0<x_1<\cdots<x_n=1$ is easily bounded by $2\sum_{q=1}^{n} 1/\chi(q)$, so f is of bounded variation. Thus, by Lebesgue's theorem, f is differentiable almost everywhere and hence differentiable for almost all irrationals.

Let ξ be an irrational for which $f'(\xi)$ exists. Then by considering a sequence of irrationals converging to ξ we find $f'(\xi)=0$. Now consider rationals p/q, close to ξ. We have

$$\frac{f(p/q)-f(\xi)}{p/q-\xi}=\frac{1}{q\chi(q)(p/q-\xi)},$$

and this quantity is small when p/q is close to ξ. Thus, except for finitely many q, $q\chi(q)|p/q-\xi|>1$, and ξ cannot be in the set E. Thus E contains only points where f' does not exist and hence E has measure zero.

This is certainly not the simplest proof of this theorem (see [2]) but with appropriate hints the construction of this proof becomes a good exercise for students acquainted with Lebesgue's theorem.

References

1. R. P. Boas, Jr., A Primer of Real Functions, Carus Monograph No. 13, New York, 1960.
2. G. H. Hardy and E. M. Wright, An Introduction to The Theory of Numbers, Oxford, 1938.
3. F. Riesz and B. Sz.-Nagy, Functional Analysis, New York, 1955.

A NOTE ON INDETERMINATE FORMS

GEORGE C. WATSON, North Carolina State College

In the teaching of indeterminate forms in sophomore calculus, it has been noted that in many textbooks, in the handling of the form 0^0, that examples of the type $\lim_{x\to 0} x^x$ and $\lim_{x\to 0} (\sin x)^{\tan x}$, which invariably tend to a limit of 1, are the only ones given. The good student is therefore inclined to ask, if such seems always the case, why is it necessary to say that $a^0=1$, provided that $a\neq 0$. This has led to the construction of the following simple example.

Consider

$$\lim_{x\to 0^+} x^{a/\ln x},\ a \text{ some constant.}$$

Then, by the usual procedure with such forms, let

$$z = x^{a/\ln x}.$$

Thence, taking logarithms,

$$\ln z = \frac{a}{\ln x}\cdot \ln x = a,$$

and hence

$$\lim_{x\to 0^+} \ln z = a.$$

Making the allowable interchange of operations,

$$\lim_{x\to 0^+} z = \lim_{x\to 0^+} x^{a/\ln x} = e^a,$$

and the limit may be varied at will by the choice of a. This is usually satisfactory for the good student.

In an effort to generalize this relatively simple result, let us examine the following theorem.

THEOREM. *If $f(x)$ is continuous and of the order of x at the origin (i.e., $\lim_{x\to 0} (f(x)/x)=b$, $0<b<\infty$), if $g(x)$ is continuous at the origin, and if $h(x)$ is of the order of* $\ln x$ *in a right-hand neighborhood of the origin* ($\lim_{x\to 0^+} (h(x)/\ln x)=c$, $0<c<\infty$), *and if these functions have the necessary derivative properties for the use of L'Hospital's rule, then*

$$\lim\ [f(x)]^{g(x)/h(x)} = e^{g(0)/c}. \tag{1}$$

Proof. Since $f(x)/x$ is given to be an indeterminate form of 0/0 type, one application of L'Hospital's rule will yield

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{f'(x)}{1} = b. \tag{2}$$

Similarly, it is known that $h(x)/\ln x$ is an indeterminate form of ∞/∞ type, and hence

$$\lim_{x\to 0^+} \frac{h(x)}{\ln x} = \lim_{x\to 0^+} \frac{h'(x)}{1/x} = c;$$

that is,

$$\lim_{x\to 0^+} xh'(x) = c. \tag{3}$$

Now, following the usual procedure, let

$$z = [f(x)]^{g(x)/h(x)}.$$

Then

$$\ln z = \frac{g(x)}{h(x)} \ln f(x),$$

and

$$\begin{aligned} \lim_{x\to 0^+} \ln z &= \lim_{x\to 0^+} \frac{g(x)}{h(x)} \ln f(x) = \lim_{x\to 0^+} g(x) \cdot \lim_{x\to 0^+} \frac{\ln f(x)}{h(x)} \\ &= g(0) \cdot \lim_{x\to 0^+} \frac{f'(x)/f(x)}{h'(x)} = g(0) \cdot \lim_{x\to 0^+} \frac{f'(x)(x/f(x))}{xh'(x)} . \end{aligned} \tag{4}$$

If we now make use of (2) and (3), then (4) becomes

$$\lim_{x\to 0^+} \ln z = \frac{g(0)\cdot(b)\cdot(1/b)}{c} = \frac{g(0)}{c} .$$

Thence, making use of the allowable interchange of limit and logarithm, we get

$$\lim_{x\to 0^+} z = \lim_{x\to 0^+} [f(x)]^{g(x)/h(x)} = e^{g(0)/c},$$

and (1) is established.

As an example of this more general result, consider

$$\lim_{x\to 0^+} [\sin bx]^{aJ_0(x)/Y_0(x)},$$

where a and b are constants, $J_0(x)$ is the well-known first solution of the Bessel equation of order zero $(x^2y''+xy'+x^2y=0)$, and $Y_0(x)$ is a second solution of this equation, which, for definiteness, may be taken in the Neumann form, *viz.*,

$$Y_0(x) = J_0(x) \ln x + \frac{x^2}{2^2} - \frac{x^4}{2^2\cdot 4^2}\left[1 + \frac{1}{2}\right] + \cdots .$$

The functions $\sin bx$, $aJ_0(x)$, and $Y_0(x)$ have the properties ascribed to $f(x)$,

$g(x)$, and $h(x)$. Applying the theorem directly, we can show that

$$\lim_{x\to 0^+} [\sin bx]^{aJ_0(x)/Y_0(x)} = e^a.$$

Since a is arbitrary, we may produce any limit we please. It is interesting to note that the parameter b apparently does not affect the result. In this last example c is unity, and $J_0(0)=1$.

(Note: This paper was presented before the Mathematics Section of the North Carolina Academy of Science, May 7, 1960.)

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE ADVANCED PLACEMENT PROGRAM IN MATHEMATICS

E. P. VANCE, Oberlin College AND R. S. PIETERS, Phillips Academy

It is well known that there is an increasing demand from our society for scientists and other personnel who are competent in mathematics. This demand is not being completely met, but, finally, something significant is being done in the direction of meeting it. One of the many significant responses to this need is the College Entrance Examination Board Advanced Placement Program in mathematics.

This country has been faced for some time with the important responsibility of developing more effectively the potentialities of students who have capabilities and interests in mathematics. More specifically, our country's immediate problem is one of spotting these students, showing them what mathematics really is, helping them to develop interest in mathematics, and in one way or another giving them a program that will be effective in producing the competence desired.

Unfortunately, over the first half of the twentieth century, the content and organization of high school mathematics for college preparatory students in this country remained essentially unchanged—we refer to the traditional presentation of algebra, geometry, and trigonometry. It is true that certain modifications of teaching methods and some minor changes in topical emphasis occurred as a result of the 1923 Committee on Mathematical Requirements and the 1944 Commission on Post-War Plans of the National Council of Teachers of Mathematics. But nothing really significant took place.

However, the pressure had built up, and the recent developments and positive actions that have taken place in the 1952–60 period have not come as a surprise, however significant, important and helpful they may be. What has

taken place in this period was inevitable. But the first of all these programs to, in a sense, take the leadership and set the direction for the change in secondary school curriculum was the program which is now known as the Advanced Placement Program of the College Entrance Examination Board.

In 1952 grants by the Ford Foundation made possible three different studies of the problem of enriching and/or accelerating the education of gifted children in school and college. Of these studies the one most directly of concern to us now is the School and College Study of Admission with Advanced Standing which was under the Directorship of Dr. William Cornog, then Principal of the Central High School of Philadelphia, and more recently Superintendent of Schools at Winnetka, Illinois. Professor H. W. Brinkmann of Swarthmore College was chosen as the head of the mathematics subject matter committee. The group invited to participate in this project consisted of twelve different colleges* and was supplemented by a group of twelve representatives of schools, including teachers, principals, and superintendents. The group spent that academic year discussing the particular problem presented, that of designing a course at the secondary school level that would be the equivalent of one year of college work.

The problem presented was not a simple one. New frontiers in mathematics were creating almost unlimited opportunities for growth in mathematical knowledge and its applications to the physical sciences and engineering, to the social and biological sciences, and to business and industry. Instead of emphasis on the applications and the utility aspect of mathematics at this level, the nature of the subject, its generality and open-ended quality, its widely divergent fields, its system of logic and the fact of its continuous and lively growth needed to be emphasized. In order to accomplish this, the committee felt that the entire secondary school curriculum should be changed, and drew up a tentative syllabus† for such a three year program, terminating in "the Advanced Placement Course," one full year of college calculus with analytic geometry. Although this is not always the first year of college mathematics for the well-prepared student in the typical good college or university at the present time, a year of such mathematics does come into the college curriculum at some level. This will continue to be the case for many years to come.

When the first year of mathematics taught in the twelve original colleges of the School and College Study was considered, only four of these colleges covered precisely this material as a freshman course. Some also taught algebra and trigonometry. Others introduced considerable mathematical logic, notion of proof, and elementary set theory, before continuing with the calculus. The designed course, however, seemed to be the appropriate one, and appeared to be the most logical program to follow. The fact that such material does, in actual-

* Bowdoin, Brown, Carleton, Haverford, Kenyon, M.I.T., Middlebury, Oberlin, Swarthmore, Wabash, Wesleyan, and Williams.

† This syllabus appeared in the 1955 edition of The Advanced Placement Program: Course Description, College Entrance Examination Board. In the present 1960 edition, only the final year course syllabus is given.

ity, constitute one full year of college mathematics has made the problem of credit and placement a relatively easy matter. Some colleges have been able to place the advanced placement students in their regular first semester course in logic, from which the student would jump into the second semester sophomore course, and at the same time take advantage of the college's special curriculum, as well as the advanced placement course. In some cases, the students receive only one semester of placement or credit, while in others, students have covered a year and a half of a specific college course. One fact is clear. If the individual college desires to work with these students, the details of placing them at the appropriate place or level is quite possible. No other one year of college mathematics would in any sense make this placement possible. Thus, the Advanced Placement Syllabus has received almost universal acceptance and its present course description appears to be agreeable to all. The present mathematics syllabus, which appears in the *Advanced Placement Program: Course Description,** follows:

The following is a check list of topics to be covered by the end of the year. It is not to be assumed that these topics are necessarily to be taken up in the order printed here but the list does indicate the scope of the Advanced Placement Examination in Mathematics.

1. Analytic geometry review and extension: rectangular and polar coordinates, distance and slope, parallelism and perpendicularity of lines, equations and graphs, line and circle, and other conics.
2. Differential calculus of algebraic functions: the function concept, absolute values and inequalities, definition and basic properties of limits, fundamental ideas of continuity, slope of a curve, average and instantaneous rates of change, definition of the derivative, formal differentiation, implicit functions and implicit differentiation, differentiation of composite functions and of parametrically defined functions, higher order derivatives, the differential and its use in approximation, and Rolle's theorem and the theorem of the mean.
3. Applications of differential calculus: tangents and normals, curve tracing, maximum and minimum values, both relative and absolute, rate problems and related rates, velocity and acceleration of a particle along a straight line and along a curve, and properties of conic sections involving tangents.
4. Integral calculus of algebraic functions: the inverse of differentiation, integration of simple expressions, basic formulas, integration by substitution, simple differential equations with initial conditions, intuitive development of the definite integral as the limit of a sum, intuitive treatment of the fundamental theorem of the integral calculus, evaluation of simple definite integrals, and approximation of definite integrals by the trapezoidal rule.
5. Geometric and physical applications of integration: the area under a curve, the average (mean) value of a function, areas between curves, volumes of revolution, volumes by slicing, motion in a straight line, and work.
6. The calculus of elementary transcendental functions: exponents and logarithms; the exponential and logarithmic functions; the inverse relationship of these functions; the derivatives of e^u, $\log_e u$ and $\log_a u$; the integrals of $e^u du$ and du/u; trigonometric functions of real numbers; limit of $(\sin x)/x$ as $x \to 0$; the derivatives and integrals of $\sin u$ and $\cos u$; the derivatives and integrals of other trigonometric functions; parametric representations involving trigonometric functions for curves such as the ellipse, hyperbola, and cycloid; the applications as listed above, when the functions involved are transcendental, including

* College Entrance Examination Board, 475 Riverside Dr., New York, 27.

growth and simple harmonic motion; and use of simple trigonometric substitutions in integration.

To prepare for such a program in the 12th grade, consolidation and revision of much material formerly taught in the secondary school curriculum is necessary, as well as, in some cases, acceleration.

A most interesting observation of many of the secondary school programs which have been designed might be made. All seem to follow the same general pattern with similar emphasis on the deductive method and logical approach to the material, and all agree that the Advanced Placement Program, as it is now outlined, has an important place for some students in the general organization of the entire curriculum.

The final report of the Commission of the College Entrance Examination Board has recommended it, and suggests that it either follow the completion of the regular Commission Program or be presented after the first half of the twelfth year in certain special cases. Also, the Illinois Project and the School Mathematics Study Group material fit directly into the Advanced Placement Program and have had students do satisfactory work on the examination.

At the time the original program, under the Fund for the Advancement of Education, was turned over to the College Board (1955), and became the present Advanced Placement Program, some of those in the mathematical world, aware of what had taken place and appreciative of the possibilities for the unbelievable growth of the Program in mathematics, wished to have their feelings known. Specifically, at the Evaluation Conference held at Williams College, June 22–25, 1955 the invited group unanimously adopted the following significant resolution:

"After conferring together and evaluating, where possible, the impact of the 1954 report of the Mathematics Subcommittee, we, the members of this conference, are unanimously agreed that, in view of the relatively small number of students who have thus far benefited from the program carried on under the report; in view of the long-range planning required by most schools before a change in curriculum and a full test of the program can be made; in view of the considerable number of schools here represented who wish to undertake to implement the subcommittee's report in the coming year; and, in view of the potential enrichment of the mathematical training at all levels which may result, it is apparent that many worthwhile gains would be lost if the 1954 report were now cast loose and left to make its own way at this critical time.

We therefore respectfully recommend that the present subcommittee be continued. We recommend this in order that this experienced group may:

(1) carry out a vigorous continuation of the program,
(2) conduct systematic evaluations of the program's results in the future,
(3) publicize the facts regarding the program among schools and colleges,
(4) work with the College Entrance Examination Board in ways which will help the program fulfill its objectives, and
(5) cooperate with other organizations and agencies who are interested in like problems."

Now, five years later, we recognize the importance of the general acceptance of this resolution, how it has been carried out, and discover the almost unbelievable results. The College Entrance Examination Board has created a committee of mathematicians and teachers of mathematics who have not only outlined

the program and made out the syllabus, but have taken the responsibility of writing an appropriate examination for this material each year. Although changes may take place in the years ahead, in recent years this examination has been divided into two parts. Part I consists of thirty multiple choice questions to be covered in one hour. Part II consists of ten longer problems for which the candidate was expected to write out detailed solutions of as many as possible in two hours. The first part is machine-marked, while the second is marked by a group of competent college and secondary school teachers selected by the Educational Testing Service specifically for this work. The test is given in May at the secondary schools and is marked in June. The student's Part II paper, his grade, an outline of the actual course taken in secondary school, and the teacher's specific recommendation are then sent to the college which the student plans to attend. On the basis of this information the college decides whether credit and/or placement is to be given the student.

Below are listed the results of the candidates participating since 1955, which indicate the growth of the program. From these results, especially those for 1960, it should be apparent that students should *not* be permitted to take the examination unless they are prepared in the course listed in the syllabus. Five is the highest grade, while one is the lowest.

Grade	Number of Candidates each year					
	1955	1956	1957	1958	1959	1960
5	5	44	80	115	225	176
4	14	109	169	235	283	404
3	30	87	176	334	586	606
2	32	100	117	217	329	711
1	25	46	182	276	447	1011
Totals	106	386	724	1177	1870	2908

In addition, each year a follow-up study on the placement of candidates is made. The following table shows what took place in the 1585 cases (out of 1870) which were reported in 1959.

Grade	Number of Candidates	Number Awarded Credit and/or Placement	Number Awarded Neither
5	196	188	8
4	254	236	18
3	510	371	139
2	276	70	206
1	349	22	327
Totals	1585	887	698

These figures alone speak strongly enough for the success of the Advanced Placement Program in Mathematics, and for the mandate to continue it, with as much publicity as possible.

The Advanced Placement Program Conferences have been an integral part of the success of the program in mathematics. These have been held annually in June at Williams College, Phillips-Exeter Academy, Oberlin College, Wesleyan University, Ripon College, and Case Institute of Technology. Since the increase in attendance and the enthusiasm of the participants have been very significant, similar conferences are planned for the future. This June the conference will be at the University of Kansas. Both college and secondary school personnel who are interested in learning more about the program are encouraged to attend.

Since the program has been outlined as a three-year program, there are even now many schools which have entered into the work of the syllabus, but have not completed the three-year cycle, so that they have not been counted as formally working with the program, since they have as yet had no students take the test. With the increase in general emphasis on a stronger mathematics curriculum and the introduction of algebra in the middle, or beginning of the eighth grade, the numbers of successful candidates will increase at an even more rapid rate in the near future. Many observers feel that, in fact, within ten or fifteen years, what is now the Advanced Placement Program in Mathematics will become the regular standard course for the well-prepared college-bound secondary school student. If, in fact, such proves to be the case, the contribution of this program in the field of mathematics education at the secondary school level would be considerably more than originally imagined.

In order for the reader to have as clear a picture of the program as possible, the College Entrance Examination Board and Educational Testing Service have permitted Part I of the 1958 test and Part II of the 1959 test to be published. These test questions will appear in this Department next month.

THE ALL PROJECT (ACCELERATED LEARNING OF LOGIC)

Layman E. Allen, Robin B. S. Brooks, James W. Dickoff, and Patricia A. James, Yale University

In American education today there is a need for better methods of elementary-school instruction and a need for early development of the skills of deductive reasoning so important in learning mathematics. The ALL Project (Accelerated Learning of Logic), established at Yale University under a three-year grant from the Carnegie Corporation, is an attempt to help meet both of these needs by (1) adapting and developing further certain new methods of instruction and (2) introducing mathematical logic into the elementary-school curriculum. In the studies undertaken in the ALL Project there will be investigated (1) methods based on those developed by Professor B. F. Skinner, Harvard University (See B. F. Skinner, *Cumulative Record*, 1959, pp. 143–182), and (2) a series of games, which is currently being developed by the ALL Project research staff.

The primary objective of the ALL Project is to develop and test materials and methods for teaching mathematical logic to elementary-school children. In pursuing this objective, information about the following will be sought:

(1) The characteristics of optimum Skinner programs for specified age and ability groups.
(2) The lowest age level for which the developed techniques for teaching mathematical logic can profitably be used.
(3) The optimum way of using the developed techniques, specifying (a) the equipment to be used, (b) the appropriate combination of Skinner-program work, game playing, and other modes of instruction, and (c) the appropriate amount of time to be devoted to the subject-matter.
(4) The costs of teacher-training, equipment, and materials for the techniques developed.
(5) The impact of this instruction on students in terms of their performance in other subject-matters, particularly mathematics.

It is anticipated that the methods developed can be adapted to subject-matters other than logic, but this adaptation is not an immediate objective of the ALL Project.

The initial studies of the ALL Project will be divided into four stages:

(1) The initial preparation of materials and determination of methods for administering them.
(2) The initial trial of prepared materials and modification of materials and methods in the light of acquired experience.
(3) The trial of modified materials with a larger and more varied sample group of students.
(4) The final modification of materials and report of findings, evaluations, and recommendations.

A "Skinner learning program" (in the terminology used here) presents subject-matter which requires the learner to make written responses to the material as it is presented. This method of presentation insures the learner's active participation. Answers are presented immediately after a response is made; there may be a representation of those parts of the material where incorrect responses occur. The Skinner learning programs are intended to be self-contained and self-explanatory, so that the learner while working through a program requires neither supplementary texts nor guidance from a teacher. This method of presentation allows each learner to proceed at his own pace. Since there is immediate reinforcement of correct responses, and immediate correction of inappropriate responses, a Skinner learning program affords one of the chief advantages of a private tutor. A record of the learner's errors shows the teacher or programmer where the presentation is weak or obscure and where each learner has particular difficulty. Skinner learning programs for mathematical logic will be especially useful, because they can be used in training teachers as well as in teaching students; such training of teachers will probably be necessary before this novel subject-matter can be introduced into the elementary school curriculum.

The series of games that is being developed has as its goal the teaching—in a competitive and entertaining atmosphere—some of the symbol-manipulating skills of logic. The games are graduated in their complexity. The player is taught first how to form sentences in a special "language," and then how to show which

of these sentences are "true," where the "language" is a system of formal logic and the "true sentences" are theorems. The following factors seem likely to make the games especially suitable for facilitating learning:

(1) Practice in logic comes as a by-product of an activity which is enjoyable in itself.
(2) Although young children enjoy these games, the more intricate ones pose a genuine challenge even for adults.
(3) In the play of the games there is no waiting time. Each player proceeds at his own pace throughout the entire time of play. The more adept player is not delayed by those who play more slowly.
(4) Everyone else in these games learns from the best player. His strategies are displayed openly so that all others may learn to adopt them. In effect, the best players act as teachers, although they are not formally assigned this role by the rules of the games.
(5) The games are so ordered that each new game is slightly more intricate than the previous one and each later game uses the skills learned in earlier games.
(6) The games emphasize individual rather than collective decision making. Each player plans and executes his own strategy independently in order to achieve specified goals.
(7) The games are flexible both in the number of persons who may play (two or more) and in the length of time for a game to be played (five minutes or more).

The number of such games and the range of the subject-matter that can be taught by them is still an open question. It does seem clear, however, that even for mathematical logic, the number of such games will be quite large. So far, only the first twelve games have been developed to the stage of constructing prototypes of them. The initial games in the series have already been developed and tried informally with groups of children and adults, including several faculty members and graduate students of Yale University; their responses have been enthusiastic.

When the materials have been initially developed, they will be tested by administering them to a small sample of sixth graders. Teaching machines and programmed textbooks will be used for displaying the prepared materials; this presentation will be supplemented with the proposed games and perhaps, some other traditional instruction. This initial trial will help to determine:

(1) The appropriate machine-to-student ratio.
(2) In what proportion machine instruction, games, and traditional instruction should be combined.
(3) The approximate time to be spent on the subject-matter (how long sessions should be, how frequently they should occur, and how many sessions should be given).
(4) How materials need to be modified (a) for effective use in sixth grade and (b) for use with learners of different ages and abilities.

After the materials are modified in light of the initial trial, they will be given to a larger sample of students, including children younger than those in sixth grade.

The last phase of this initial stage will involve a final modification of materials and methods. A report will be prepared containing information on

(1) Administration procedures, teacher-training, and procurement and preparation of equipment and material.

(2) Estimated outlay in money and in time of both students and teachers, and
(3) The effect of this training in logic on the students' progress in acquiring other intellectual skills.

The plan is to work in schools where achievement tests are administered annually. Comparisons of the year-to-year improvement in achievement test scores in various subject-matters—*e.g.*, mathematics, language skills (grammar), and reading comprehension—will be made between students who are given this training in logic and those who are not. These comparisons should furnish valuable information about the desirability of arranging a large-scale controlled experiment for testing the effects of introducing mathematical logic in the elementary-school curriculum by means of the materials and methods developed in this study.

The principal investigator for the ALL Project is Layman E. Allen, Assistant Professor of Law, Yale University. Frederic B. Fitch, Professor of Philosophy, Yale University, will serve as senior consultant. The research staff includes Robin B. S. Brooks, James W. Dickoff, and Patricia A. James.

N.S.F. SUMMER INSTITUTES IN MATHEMATICS: THE VISITING FOREIGN STAFF PROGRAM

WADE ELLIS, Oberlin College

In 1958 (and even earlier), a handful of eminent foreign mathematicians were included in the staffs of the relatively few summer institutes for high school mathematics and science teachers. Based on the experiences gained in this way, a broader and possibly more purposeful Visiting Foreign Staff Program for summer institutes was conceived and supported in major part by the National Science Foundation. This program had three parts, *viz.*, 1) Biological Sciences, 2) Mathematics, and 3) Physical Sciences. In 1959 these programs were operated independently and with little mutual cooperation by three different colleges and universities. In 1960 there was a project coordinator with a resulting increase in the cooperation of the three parts again operated by three separate institutions. In 1961 the program is being operated by the American Association for the Advancement of Science.

As originally conceived, the mathematics part of the VFSP had as its central purposes the following: a) To acquaint eminent foreign mathematicians and mathematics educators with problems in mathematics education in the United States in the hope that constructive suggestions might be forthcoming; b) To explain to the VFSP personnel the several distinct (but in most cases correlated) new programs intended for curricula improvement (such as those of SMSG, UICSM, and so on) being formulated and tried in this country; c) To give the VFSP personnel some understanding of the overall summer institutes program, by description, observation, and participation, with the expectation that they would make significant and perhaps useful evaluative and constructive criticisms; d) To provide institute personnel, including participants, instructors, and

directors, with information concerning the situations and conditions in mathematics education in foreign countries; and e) To acquaint some of our teachers (and instructors) with the types and extents of preparation and other qualifications demanded of secondary school teachers of mathematics in other parts of the world.

Each visit for the foreign visiting lecturers in mathematics lasted for one week (longer in a very few cases). Each institute director was free to make whatever use he wished of the visitor's time. The most usual practices were i) to invite institute participants to seek informal contacts with the visitor, ii) to ask for one or more formal lectures on mathematics education in his country, iii) to ask for one or more lectures on mathematics, and iv) to leave much (and in some cases, all) of his time free for informal activities.

At the end of the summer, 1960, the VFSP personnel were reassembled, with consultants and resource personnel to compare experiences and write a formal report. In 1959 a report was written by the mathematics section alone and later distributed to all 1960 institute directors. In 1960, a joint report was issued by all three sections. The terminal activity was attendance at the August Conference of the National Council of Teachers of Mathematics at the University of Utah.

Although it is perhaps too early to evaluate the program, the following comments seem appropriate: a) In 1960, the program was more effective in all respects than in 1959; b) The reports issued were pertinent and contained some useful material; c) activity similar to the institute program has been carried out successfully in at least two foreign countries, and is contemplated in at least one other.

Report of Progress in Development of the Statewide Study of Instruction in Mathematics in California

1. **Delimitation of roles and responsibility.** The understanding of improvement of the quality of instruction in mathematics in the elementary and high schools throughout California required at the outset a careful determination of the respective responsibilities of the State Department of Education, of the administrators and teachers of school districts, and of the departments of mathematics in colleges and universities of the state.

The responsibility of the Department of Education through its Bureaus of Elementary and Secondary Education is (1) to assemble the best available information regarding what mathematics needs to be taught in the public schools, (2) to bring this body of content to the attentive consideration of mathematics teachers, (3) to encourage school districts to introduce and to experiment with revised subject-matter, (4) to enlist the active assistance of college mathematicians in providing such inservice re-education of teachers as may prove needful, and (5) finally, to coordinate the course offerings among and within districts so that pupil progress from grade to grade may be orderly and without impediment.

It is the responsibility of superintendents of school districts, of directors of curriculum, and of school principals (1) to free time for teachers from their classrooms so that they may thoroughly acquaint themselves with the recommended new materials, (2) to determine with their teachers what topics may immediately be incorporated into ongoing courses, (3) to postpone the introduction of unfamiliar concepts until the teachers

may have opportunity to master these, and (4) to provide means of in-service education so teachers may achieve this mastery.

The assistance of departments of mathematics in colleges and universities must be enlisted to enable teachers in public school service to gain insight into and familiarity with whatever processes and emphasis which a reformation of mathematical offerings in the elementary and high school grades necessitates. A development of a more appropriate sequence of undergraduate courses for a teaching major in school mathematics also deserves renewed attention.

Although the teaching of arithmetic in the elementary school grades urgently needs careful review, and the strengthening of the preparation of elementary school teachers in this subject is of primary importance, it is at the junior and senior high school levels that major changes of topic, sequence, and emphasis are most readily evident. Accordingly, the Bureau of Secondary Education has undertaken four steps to effect progress toward the achievement of improved offerings in secondary school mathematics.

2. Achievement of a consensus among mathematicians as to recommended course content. The first task undertaken by the Bureau was made necessary by the fact that until 1960, when the Conference Board of the Mathematical Sciences was formed, mathematicians have had no single organized voice to speak for the profession similar to the American Institute of Biological Sciences, for example, whose numerous member and affiliated societies have combined for a Biological Sciences Curriculum Study. Many competent high school teachers have been seriously concerned to know whether they should follow the recommendations of the School Mathematics Study Group, or the University of Illinois Committee on School Mathematics, or the Report of the Commission on Mathematics of the College Entrance Examination Board, or perhaps another group. In December 1959, therefore, the Bureau asked representatives from several of the principal centers to meet in California and to establish, if possible, a consensus of agreements on common elements for a sequence of courses.

William H. Meyer, University of Chicago, served as moderator of the sessions, and a statement was prepared with the cooperation of E. G. Begle, Yale University, Director of SMSG; Howard F. Fehr, Teachers College, Columbia University, for the CEEB report; John L. Kelley, University of California, Berkeley; R. L. Wilder, University of Michigan; and professors of mathematics from Sacramento State College, Stanford University, and the University of California, Los Angeles, and Santa Barbara. Following the meeting, Dr. Meyer circulated the statement among his colleagues for their criticisms and corrections. A final draft of recommendations appeared in the September, 1960, issue of *California Schools*, published by the Department of Education. Already it has stimulated review and study on the part of many school districts.

3. Establishment of connection with the State Curriculum Commission study. As a next step the Bureau of Secondary Education called a second meeting in May 1960, at which certain mathematicians and representatives of selected school districts were asked to advise the State Curriculum Commission as to procedures for an expanded study under its auspices. In order that members of the Curriculum Commission and the delegates from school districts might acquaint themselves with the findings of major studies of school mathematics, their recommendations were summarized in informal presentations by consultants present.

The further agenda of the May meeting invited the support of school districts "to undertake (1) consideration of the summaries of recommendations and instructional materials, (2) determination of what topics present in current mathematics courses should be lessened in emphasis or eliminated, and (3) arrangements for introduction of new focus on concepts and principles as rapidly as may be effected with reference to teacher confidence and security." Likewise the representatives of departments of mathe-

matics in the Universities and State Colleges of California were queried to ascertain what contributions their several institutions could make toward inservice education to promote the prompt utilization by elementary and high school teachers of the recommendations and new materials.

4. **June 20–30, 1960, workshop for teachers of secondary school mathematics.** A two-weeks' workshop was organized for late June to which districts were invited to send teachers. Eighty-seven teachers were designated by their superintendents to attend. At this workshop the participants were sectioned by secondary school grade levels: one section for teachers in grades 7–8, a second for grades 9–10, a third for grades 11–12, and a fourth which concerned itself with problems of the general-terminal student. Each section had two co-directors selected from districts already in the forefront of making advances in the reorganization of their mathematics courses.

Dr. Meyer was recalled to California to direct the actual workshop and from time to time other mathematical authorities spent several days with each section. These included Paul H. Daus, University of California, Los Angeles; Richard A. Dean, California Institute of Technology; and Professor Moredock, Sacramento State College.

Many of the participants at the June 20–30 workshop have organized study groups within their districts. Several have begun to plan regional week-end meetings for high school teachers in their areas. Probable locations of forthcoming meetings, in whose planning the Bureau has been asked to assist, are Santa Clara County, Shasta County, Sonoma County, Kern and Tulare counties, Stanislaus and Merced counties, Riverside and San Bernardino counties, and San Diego County.

(Abstracted from a report to members of the California State Curriculum Commission by Frank B. Lindsay.)

Resolutions by the Council of Chief State School Officers

At the annual meeting of the Council of Chief State School Officers, held in Santa Fe, New Mexico, November 18, 1960, a number of resolutions on education were passed. Those which seem of most direct interest and concern to mathematics education are the following:

Quality education. The Council believes that meeting the needs for more teachers and more school facilities, critical though this be in many parts of our nation, is not enough. The challenge of quality is even more significant than that of quantity. The Council urgently calls upon state and local fiscal authorities, legislative bodies, state departments of education, boards of education, and the supporting public to direct more energy and money into educational research and into the continued development of supervisory and administrative services at state and local levels.

Cost of higher education. The Council recognizes that there is greatly increased need for well-trained, well-informed and well-educated citizens. There is a shortage of trained manpower in almost every field of endeavor, making it imperative that provisions be made so that all competent high school graduates desiring to do so may have the advantages of college or university education.

The Council believes that tuition, student fees, and other costs to students in public institutions should be kept as minimal as possible so that no capable high school graduate will be denied the opportunity of a college education for economic reasons.

The Council also believes that the financial support of higher education should be greatly increased.

Amendment and extension of National Defense Education Act. The Council favors the appropriate amendment and extension of the National Defense Education Act, inas-

much as its operation as an emergency measure has been of considerable assistance to education and can continue to be of even more assistance in the future. The Council favors the following specific changes in the Act, and then its extension for a term of not less than four years. [The changes recommended for Titles III, IV, and VII only are included in this report.]

Title III—Materials and Equipment for Science, Mathematics and Modern Foreign Languages.

Amend Section 303 (a) (1) to include English, history, geography, economics, and government, along with the original fields of science, mathematics and modern foreign languages, as subjects of instruction in elementary and secondary schools that may be aided under this title.

Amend Section 305 to permit any funds for loans to private non-profit schools and unused by them to be made available for reallocation to the states for allotment under Section 302 (a) and payment under Section 304 (a).

In view of the expanded scope of the title, the Council favors an authorization of not less than $85 million annually under Section 301 for making payments to state educational agencies to carry out the programs described in paragraph (5) of Section 303 (a).

Title IV—Fellowships.

Provide a new section to authorize new types of fellowships, traineeships and leadership institutes for the preparation and in-service training of educational administrators in colleges and universities, state departments of education, and local school districts.

It is proposed that the appropriations for Title IV be increased approximately $40 million annually above the present authorization.

Title VII—Research and Experimentation in Mass Media.

Amend to include authorization for experimentation and research projects by state departments of education covering all or any part of the curriculum of the elementary and secondary schools, with authorization for appropriations of $15 million annually.

Shell Merit Fellowships

Introduction. The Shell Merit Fellowships for outstanding high school teachers of mathematics and science were established by the Shell Companies Foundation, Incorporated, as a part of their aid to education program. From the onset of the program in 1955, it was assumed that a greater impact would be made on the improvement of mathematics teaching in secondary schools if good teachers, well-trained academically (with a minimum of five years teaching experience) were trained to assume professional leadership at a local level. With this in mind, a special effort has been made to distribute the Fellowships to meritorious teachers systematically in each of the states and each of the provinces of Canada. Two universities, Stanford and Cornell, were selected to administer the Fellowships and to devise appropriate seminars to implement the policy.

Stanford University. At Stanford University the program for the mathematics teachers has been developed cooperatively with the Department of Mathematics and two of the seminars are taught there. The program over the years has varied somewhat but typically the work has consisted of courses on *Mathematical Methods in Science* or *Introduction to Modern Algebra* and a combined course on *Statistics and Probability.* These courses are intensive, meet five days per week, and continue for two months. The intent is to extend the preparation of the teachers rather than to review previously taken courses.

A seminar on *Recent Developments in Secondary School Mathematics* is given in the School of Education. The "new" programs in mathematics are examined and ways of bringing about curriculum changes in high schools are explored.

Four days a week a special seminar two to three hours in length is conducted by a research scientist or mathematician on new developments in his field of interest. George Pólya, Linus Pauling, J. van Overbeek, Emilio Segrè, Gerald L. Pearson, Konrad Krauskopf, George Forsythe, Samuel Karlin, Wilson J. Frank, Robert F. Mozley, and Paul H. Kirkpatrick were among the seminar leaders in the 1960 program. The purpose of these seminars is to develop within teachers something of the mood and feeling for modern scientific intellectual activity. It was our assumption that this is best done by meeting and talking with men active in research.

The teachers live on the campus, eat and study together. The opportunity to discuss problems of mathematics teaching with other mathematics teachers has proven to be a significant aspect of the Shell Merit Seminars.

Cornell University. The special program at Cornell has followed a special study guide developed by Professor Emeritus Walter B. Carver. This guide has been revised for the last two summers of the Program. The work seeks to present new insights in algebra, geometry, trigonometry, limits and sequences, linear equations, problem solving, exponents and logarithms, calculus and modern mathematics. The course has met for double periods each day for six weeks for a total of 60 class periods. Opportunities to prepare instructional aids have also been available.

The mathematics teachers have been involved in one or more special seminars on teaching mathematics and have studied mathematics films and references. They have met with teachers of chemistry and physics for some of the broader instructional considerations. In the last five years renowned mathematicians and mathematics educators have had a part in the Program. Among these have been: Isadore Blumens, John A. Brown, Kenneth E. Brown, Harold Fawcett, Howard F. Fehr, P. R. Halmos, Phillip S. Jones, John Mayor, Paul C. Rosenbloom and Robert Walker. Several supervisors of mathematics have been among the teachers.

The mathematics teachers have at times been a part of instructional activities related to physics and all of them have become acquainted with the facilities, the course work and some of the research activities that make up the study of engineering. This has been done to help mathematics teachers better advise their students. All the mathematics teachers have had instruction relating to computers. They have developed the programs to solve certain problems and have had the opportunity to use a computer with the help of a technician. The understanding of computers has been further strengthened by a visit to the main plant of the International Business Machines Corporation. An understanding of the way mathematics is used in research has been heightened by a visit to the research centers of The Corning Glass Works and the Ansco Company.

The teachers live in a university dormitory where special library facilities are available. They eat their noon lunches and enjoy some special dinners and other social events with the entire group. These activities provide many opportunities to share ideas informally but there is also a special evening seminar each week when the mathematics teachers share ideas in an organized manner.

(The statements were prepared by W. M. Upchurch, Jr., Shell Companies Foundation; Paul DeH. Hurd, Stanford University; and Philip S. Johnson, Cornell University.)

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1466. *Proposed by H. Y. Shee, Huwei Middle School, Huwei, Taiwan, China*

Let $A_1 \cdots A_n$ be a cyclic polygon. Let B_i be the foot of the perpendicular from A_i on $A_{i-1}A_{i+1}$, C_i the point of intersection of A_iA_{i+2} and $A_{i-1}A_{i+1}$, and D_i the foot of the perpendicular from C_i on A_iA_{i+1}, where it is understood that all subscripts are to be reduced to the least positive residue modulo n. Show that

$$\prod_{i=1}^{n} A_{i-1}B_i/B_iA_{i+1} = \prod_{i=1}^{n} A_iD_i/D_iA_{i+1} = 1.$$

E 1467. *Proposed by D. G. de Figueiredo, New York University*

Write down the remainder of the division of $(x+1)^n$ by $(x-1)^3$.

E 1468. *Proposed by B. H. Bissinger, Lebanon Valley College*

Let $b_1, \cdots, b_n$ be any rearrangement of the positive numbers $a_1, \cdots, a_n$. Then $a_1/b_1 + \cdots + a_n/b_n \geqq n$.

E 1469. *Proposed by P. G. Kirmser, Kansas State University*

Find a function $f(x)$ such that $f(2x) = e^{x^2} f(x) \cos x$ with $f(0) = 1$.

E 1470. *Proposed by W. E. Patten, South Boston, Virginia*

It is desired to form a $2 \times n$ rectangle from 1×2 rectangles (dominoes), or we may say, to cover the rectangle with dominoes. In how many distinct ways can this be done, where two solutions are distinct when they cannot be brought into coincidence by rotations and reflections?

SOLUTIONS

A Triangle Formed by Three Cevians

E 1436 [1960, 922]. *Proposed by Mok-Kong Shen, Karlsruhe, W. Germany*

Through the vertices of a given $\triangle ABC$ draw straight lines l, m, n, respectively, such that n and l intersect in D, l and m in E, m and n in F inside $\triangle ABC$ and

(1) $\triangle ABE = \triangle BCF = \triangle CAD = \triangle DEF = (1/4)\triangle ABC$,

(2) $\triangle DEF$ is similar to $\triangle ABC$ and has an area equal to a given fraction of $\triangle ABC$.

An amalgam of solutions by Michael Goldberg, Washington, D. C., L. D. Goldstone, Watervliet, N. Y., and Beckham Martin, Toledo, Ohio. (1) Let l, m, n cut BC, CA, AB in U, V, W. We shall find a solution where $BU/UC=CV/VA=AW/WB$. Since a solution is affine, we first take triangle ABC equilateral. Draw lines parallel to the sides one-fourth the way from the sides to the opposite vertices. Through each pair of vertices draw internal arcs of radii equal to the circumradius. The intersections of the arcs with the lines give the points D, E, F. Note that two different triangles DEF can be so obtained. There result two possible positions of U on BC, and consequently two possible values of the ratio $k=BU/UC$. In the general case, where ABC is any triangle, we merely find the points U, V, W on BC, CA, AB such that $BU/UC=CV/VA=AW/WB=k$.

(2) Suppose, in similar triangles ABC and DEF, that $\measuredangle A=\measuredangle D$, $\measuredangle B=\measuredangle E$, $\measuredangle C=\measuredangle F$, and that $\triangle ABC/\triangle DEF=s^2$. We may easily draw circles ADC and AEB. Let AM and AN be diameters of these circles. Then

$$\begin{aligned} DE &= AE - AD = AN\cos(AN, AE) - AM\cos(AM, AE) \\ &= MN\cos(MN, AE) = AB/s. \end{aligned}$$

It follows tht $\cos(MN, AE)=AB/(sMN)$. Since MN is fixed, line ADE can be drawn, and D, E, F located. By differently pairing the equal angles in the two similar triangles ABC, DEF, six solutions can be obtained.

Also solved by Leon Bankoff, J. W. Clawson, J. M. Elkin, Jane Evans, S. H. Greene, D. C. B. Marsh, and S. W. Saunders. Late solutions by Dennis Couzin, T. R. Curry, and Guy Torchinelli.

Editorial Note. An easy computation shows that, in (1), $k=(3\pm\sqrt{5})/2$. The problem in (2) is that of placing a given triangle so that each side will contain one of three given points; this is Prob. 383 in Petersen's *Methods and Theories for the Solution of Problems of Geometrical Construction.*

A Convergent and a Divergent Series

E 1437 [1960, 922]. *Proposed by Nickolas Konopliv, University of Minnesota*

Test each of the following infinite series for convergence:

$$x+\sin x+\sin(\sin x)+\sin[\sin(\sin x)]+\cdots, \tag{1}$$

$$x-\sin x+\sin(\sin x)-\sin[\sin(\sin x)]+\cdots, \tag{2}$$

where $0<x<\pi$.

Solution by D. C. B. Marsh, Colorado School of Mines. (1) Denote the $(n+1)$st term by $\sin_n x$, $\sin_0 x$ being x. For $0<x<\pi$, $\sin_2 x=r$ lies (properly) between 0 and 1. Furthermore

$$\sin_3 x = \sin r > r - r^3/6 = r/2 + r(3-r^2)/6 > r/2.$$

We can show inductively that $\sin_{n+1} x>r/n$ for $n\geqq 3$; for, from the hypothesis that $\sin_n x>r/(n-1)$, we find

$$\sin_{n+1} x = \sin(\sin_n x) > \sin[r/(n-1)]$$

$$\begin{aligned} &> r/(n-1) - r^3/6(n-1)^3 \\ &= r/n + [r/n(n-1)][1 - nr^2/6(n-1)^2] \\ &> r/n + [r/n(n-1)](1-r^2) > r/n. \end{aligned}$$

Thus the series dominates r times the harmonic series, and therefore diverges for all x, $0<x<\pi$.

(2) For $0<x<\pi$, the sequence $\{\sin_n x\}$ is monotone decreasing and bounded below. There is thus a limit L, with $L=\sin L$, implying $L=0$. By the alternating series test, (2) is seen to be convergent for all x, $0<x<\pi$.

Also solved by R. D. Adams, S. H. Greene, E. E. Lattman, C. S. Ogilvy, W. H. Ruckle, Norman Schaumberger, O. E. Stanaitis, C. E. Stenard, W. C. Waterhouse, and the proposer. Late solutions by P. R. Chernoff, Dennis Couzin, J. B. Muskat, J. L. Pietenpol, C. F. Pinzka, and I. D. Ruggles.

A Car Travelling Behind Another Car

E 1438 [1960, 922]. *Proposed by S. H. Gould, Mathematical Reviews, Providence, R. I.*

A car travelling at a speed of v miles per hour is required by law to remain at a distance (in Rhode Island one car-length for every 10 miles per hour) of cv miles behind the car ahead, where c is a constant depending upon the state. Find, as a function of time, the maximum allowable acceleration for a car starting from rest immediately behind an unobstructed car which accelerates at a constant rate of a miles per hour per hour.

Solution by R. W. Means, University of Santa Clara. The distance the second car travels as a function of time is described by the equation

$$x = (at^2)/2 - c(dx/dt).$$

Differentiating and letting $d^2x/dt^2=b$, which is the acceleration of the second car, one finds $b=a-c(db/dt)$. Solving this differential equation, with the given initial conditions, one obtains $b=a(1-e^{-t/c})$.

Also solved by L. R. Bragg, R. B. Brian, Gus Di Antonio, Michael Goldberg, S. H. Greene, Cornelius Groenewoud, Emil Grosswald, F. W. Herlihy, A. R. Hyde, Theodore Katsanis, M. E. Lakser, E. E. Lattman, L. A. MacColl, D. C. B. Marsh, M. V. Mielke, Norman Mines, D. A. Moran, Thomas O'Brien, David Rothman, Allen Rubenstein, David Sachs, Norman Schaumberger, R. T. Shannon, E. A. Sturley, Julius Vogel, W. C. Waterhouse, and R. H. Wilson, Jr. Late solutions by E. W. Brown, C. W. Dodge, J. L. Pietenpol, C. F. Pinzka, and E. L. Spitznagel, Jr.

For a Collection of Examples and Counterexamples

E 1439 [1960, 922]. *Proposed by J. M. Elkin, Long Island University*

Construct a bounded strictly monotonic function $f(x)$ such that $f'(x)$ exists for all real x and $\lim_{x\to\pm\infty} f'(x)\neq 0$.

Solution by D. A. Moran, University of Illinois. Let $f(n)=1-2^{-n}$ for nonnegative integers n. Extend the definition of f to all nonnegative real numbers to agree with the functions f_n defined on their respective domains as follows:

$f_n(x)$ is any monotonic differentiable function defined on $[n, n+1]$ which satisfies:

$$f_n(n) = f(n), \qquad f_n(n+1) = f(n+1),$$
$$f_n(n+1/2) = \{f(n) + f(n+1)\}/2,$$
$$f_n'(n) = f_n'(n+1) = 0, \qquad f_n'(n+1/2) = 1.$$

For example, certain arcs of two properly selected ellipses satisfy the conditions for the graphs of these functions in their respective intervals of definition.

Next, set $f(-x) = -f(x)$. Then f is defined for all real numbers, is bounded, is clearly strictly monotone, and $\lim_{x\to\pm\infty} f'(x) \neq 0$, since this limit does not exist, the derivative attaining the values 0 and 1 in each closed unit interval.

Also solved by M. T. Bird, J. L. Brown, Jr., P. M. Cohn, C. H. Cunkle, Bernard Fusaro, V. E. Hoggatt, Jr., D. C. B. Marsh, Hugh Noland, P. L. Renz, L. A. Ringenberg, David Rothman, W. H. Ruckle, Julius Vogel, W. C. Waterhouse, and the proposer. Late solutions by P. R. Chernoff, B. H. Dadbeh, J. B. Muskat, J. L. Pietenpol, E. L. Spitznagel, Jr., and D. C. Stevens.

Examples of undulating functions similar to the kind described above are

$$f(x) = 2^{(2^{n+1}-n-2)}(x-n)^{2^{n+1}} - 1 - 2^{-n}, \qquad n \leqq x \leqq n+1/2, \qquad n = 0, 1, 2, \cdots,$$
$$f(x) = -2^{(2^{n+1}-n-2)}(n+1-x)^{2^{n+1}} + 1 - 2^{-n-1}, \qquad n+1/2 \leqq x \leqq n+1, \qquad n = 0, 1, 2, \cdots,$$
$$f(x) = -f(-x), \qquad x < 0,$$

given by D. C. B. Marsh, and

$$f(0) = 0,$$
$$f(x) = f(n\pi) + 2^{-n}[1 - \cos^{1/(2n+1)}(x - n\pi)], \qquad n\pi < x \leqq (n+1)\pi, \qquad n = 0, 1, 2, \cdots,$$
$$f(x) = -f(-x), \qquad x < 0,$$

given by Bernard Fusaro.

A Closed-form Expression

E 1440 [1960, 922]. *Proposed by W. B. Carver, Cornell University*

Find a closed-form expression for $\sum_{s=1}^{n} s^3 r^{s-1}$, $r \neq 1$.

Solution by D. C. B. Marsh, Colorado School of Mines. Consider the more general expression, $f(m, n, r) = \sum_{s=1}^{n} s^m r^{s-1}$, with $r \neq 1$ and m a nonnegative integer. It is readily verified that

$$f(m, n, r) = (\partial/\partial r)[rf(m-1, n, r)], \qquad f(0, n, r) = (r^n - 1)/(r-1),$$

so that $f(m, n, r) = (Dr)^m[(r^n - 1)/(r-1)]$, where $D = \partial/\partial r$. In the proposed problem $m = 3$, and the explicit expression is

$$\frac{n^3 r^{n+3} - (3n^3+3n^2-3n+1)r^{n+2} + (3n^3+6n^2-4)r^{n+1} - (n+1)^3 r^n + r^2 + 4r + 1}{(r-1)^4}.$$

Also solved by R. L. Bohuslou, L. R. Bragg, D. A. Breault, J. L. Brown, Jr., Gus Di Antonio, Ragnar Dybvik, J. M. Elkin, N. H. Fisher, Jr., W. W. Funkenbusch, Michael Goldberg, L. D. Goldstone, S. H. Greene, Cornelius Groenewoud, Emil Grosswald, Frank Herlihy, J. C. Hickman, J. E. Homer, Jr., A. R. Hyde, M. V. Mielke, D. A. Moran, C. S. Ogilvy, F. D. Parker, Thomas

Porsching, B. E. Rhoades, David Rothman, W. H. Ruckle, R. B. Sher and D. J. Uherka (jointly), F. C. Smith, M. R. Soliman, D. R. Sondergeld, O. E. Stanaitis, E. A. Sturley, Patrick Twomey, W. C. Waterhouse, Walter Zayachkowski, David Zeitlin, and the proposer. Late solutions by E. W. Brown, P. R. Chernoff, C. W. Dodge, J. B. Muskat, J. L. Pietenpol, and C. F. Pinzka.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4965. *Proposed by H. S. Shapiro, New York University*

Let $P(t_1, \cdots, t_n)$ be a homogeneous polynomial with real coefficients and D the partial differential operator

$$D = P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right).$$

Prove that no nonzero polynomial solution u of the equation $Du=0$ can be divisible by $P(x_1, \cdots, x_n)$.

4966. *Proposed by G. Di Antonio, Duquesne University, Pittsburgh, Pa.*

Find the equation of all surfaces all of whose normals intersect a given line, say $x=y=z$.

4967. *Proposed by D. S. Mitrinovitch, University of Belgrade, Yugoslavia*

Evaluate the following definite integral

$$J(a, n, k) = \int_{-\pi}^{+\pi} \frac{\cos (n-k)x}{(1-2a\cos x + a^2)^n}\, dx.$$

Here n and k are positive integers, and a is a real constant for which the integral has meaning.

4968. *Proposed by Basil Gordon, University of California, Los Angeles*

If M is any square matrix, let $\phi_M(\lambda)$ denote its characteristic polynomial, and M_i the matrix obtained by deleting the ith row and ith column of M. Show that if A and B are $n\times n$ matrices over a field of characteristic 0 or $p\geqq n$ with

$\phi_{A_i}(\lambda)=\phi_{B_i}(\lambda)$ $(i=1, \cdots, n)$, then $\phi_A(\lambda)$ and $\phi_B(\lambda)$ differ by a constant.

4969. *Proposed by David Gale, Brown University, and D. J. Newman, Yeshiva University*

Let C be a closed convex curve, p a measurable function defined on C such that $0\leqq p\leqq 1$. Prove there is a chord of C whose length ζ satisfies $\int_C p dz=\zeta$.

4970. *Proposed by S. W. Golomb, Jet Propulsion Laboratory, California Institute of Technology*

Let $Q=\{q_i\}$ be an infinite subset of the primes, and let $g(x)$ denote the number of members of Q which do not exceed x. Call Q rare if $\sum 1/q_i$ converges, and call Q sparse if $g(x)=o(x/\log x)$ as $x\to\infty$. What causal relation, if any, exists between rarity and sparsity?

SOLUTIONS

Analytic Solutions of a Functional Equation

4910 [1960, 499]. *Proposed by J. S. Frame, Michigan State University*

Find all analytic solutions of the functional equation $f(2z)=2f(z)f'(z)$.

I. *Solution by J. W. Ellis, Louisiana State University, New Orelans.* The given equation can be differentiated repeatedly to give, for all $n\geqq 0$,

$$2^{n-1}f^{(n)}(2z) = \sum_{q=0}^{n}\binom{n}{q}f^{(q)}(z)f^{(n-q+1)}(z).$$

Clearly the origin is not a pole. Straightforward induction arguments will then give the following:

(a) If $f(0)=k\neq 0$, then $f^{(n)}(0)=k(2k)^{-n}$ for all n. Thus $f(z)=k\exp(z/2k)$.

(b) If $f(0)=0$, then $f^{(2n)}(0)=0$ for all n.

  (i) If $f'(0)=0$, then $f^{(2n+1)}(0)=0$, so that $f(z)\equiv 0$.

  (ii) If $f'(0)\neq 0$, it must equal 1, and $f^{(2n+1)}(0)=Q^n$, where Q is an arbitrary complex number. If $Q=0$, this means $f(z)=z$; if $Q\neq 0$, then $f(z)=c^{-1}\sin cz$, where $c^2=-Q$.

Direct substitution shows that these four types of function satisfy the given equation.

II. *Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minnesota.* This problem is not new. See E. Beke, *Matematikai ès Fizikai Lapok*, 48 (1941), pp. 387–392. This reference (with solutions is given in I) is cited in *Differentialgleichungen Lösungsmethoden und Lösungen*, E. Kamke, Third Edition, Chelsea, v. I, p. 660, eq. 10.6b).

Also solved by Stephen Andrea, Alan Beal, Robert Breusch, W. G. Brown, L. Carlitz, P. E. Chernoff, George Glauberman and Burton Fein, Michael Goldberg, Reuben Hersh, W. S. Lawton, Y. Matsuoka, J. G. Mauldon, D. L. Muench, D. J. Newman, Arnold Singer, Vencil Skarda, J. H. van Lint, J. S. White, and the proposer.

An Extension of Picard's Theorem

4911 [1960, 596]. *Proposed by Peter Ungar, New York University*

Let $f(z)=\lambda z+a_2z^2+\cdots$ be an entire function, where $|\lambda|>1$. Let N be any neighborhood of the origin. The images of N under successive iterations of f cover the whole plane with the possible exception of one point. The same is true if $\lambda=1$ and $f(z)\not\equiv z$.

Solution by J. H. van Lint, Technical University, Eindhoven, Netherlands. The iterations f_n of f are all regular on N. If there were two values a and b not taken by any of the functions f_n then these functions would form a normal family (Montel's theorem). Let C be a closed circle $|z|\leqq r$ which lies entirely in N. Since $f_n(0)=0$ for all n, the normal family (f_n) must be uniformly bounded in C (cf. Golusin, *Geometrische Funktionentheorie*), *i.e.* $|f_n(z)|\leqq M$ for $|z|\leqq r$ and all n. Now Cauchy's inequality gives

$$|\lambda^n| \leqq \frac{M}{r} \text{ for all } n \text{ in case } f_n(z) = \lambda^n z + \cdots,$$

$$n|a_\nu| \leqq \frac{M}{r^\nu} \text{ for all } n \text{ in case } f_n(z) = z + na_\nu z^\nu + \cdots \ (a_\nu \neq 0).$$

These inequalities contradict $|\lambda|>1$ and $a_\nu\neq 0$, respectively. This proves that there is at most one value a, not taken by any of the f_n on N.

Also solved by W. A. Veech and the proposer.

Range of Three Integers where a_1 Divides a_2a_3

4912 [1960, 596]. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Prove: there exists an absolute constant $c_1>0$ such that, for every n, there always exist integers a_i $(i=1, 2, 3)$, $n<a_i<n+n^{1/2}+c_1n^{1/4}$, for which a_1 is a divisor of a_2a_3. If $c_2>0$ is sufficiently small this is false; in fact, there are infinitely many values of n such that, for every triplet, $n<a_i<n+n^{1/2}+c_2n^{1/4}$ we must have that a_1 does not divide a_2a_3.

Solution by Koichi Yamamoto, Kyushu University and the University of Southern California. We prove the

THEOREM. (A) *If n is a positive integer, we can find a triplet a_1, a_2, a_3 of integers such that*

$$n < a_1 < a_2 < a_3 < n + n^{1/2} + 3n^{1/4}, \qquad a_1 \mid a_2a_3.$$

(B) *If ϵ is a positive constant, then there exist infinitely many positive integers n such that, for every triplet a_1, a_2, a_3 with*

$$n < a_i < n + n^{1/2} + (3 - \epsilon)n^{1/4} \qquad (i = 1, 2, 3),$$

we have that $a_1\nmid a_2a_3$.

Proof of (A). For a given n denote by m the integer satisfying

$$2(n+1)^{1/2} < m \leqq 2(n+1)^{1/2} + 1, \tag{1}$$

and by k the integral solution of

$$(m^2 - 4n)^{1/2} - 2 \leqq k < (m^2 - 4n)^{1/2}, \qquad k \equiv m \pmod 2. \tag{2}$$

The condition (1) implies

$$m^2 - 4n \geqq 5, \tag{3}$$

which assures the existence of a positive solution k of (2). Put $b = \frac{1}{2}(m+k)$, $c = \frac{1}{2}(m-k)$, $a_1 = bc$, $a_2 = (b+1)c$, $a_3 = b(c+1)$. Then it is evident from $k > 0$ that $a_1 < a_2 < a_3$ and $a_1 | a_2 a_3$.

It follows from (2) that $4a_1 = m^2 - k^2 > 4n$, and so, $a_i > n$.

Before proving $a_3 < n + n^{1/2} + 3n^{1/4}$, note the following consequences of (1) and (2):

$$m \leqq 2n^{1/2} + n^{-1/2} + 1, \qquad m^2 - 4n \leqq 2m + 3, \tag{4}$$

$$k \leqq (m^2 - 4n)^{1/2} \leqq (2m+3)^{1/2} \leqq 2n^{1/4} + \tfrac{1}{2}n^{-3/4} + \tfrac{5}{4}n^{-1/4}. \tag{5}$$

Noticing that $|(m^2 - 4n)^{1/2} - 3| \leqq k - 1$, except for the case where $m^2 - 4n = 5$, and using (4) and (5), we have

$$\begin{aligned} 4a_3 &= (m+1)^2 - (k-1)^2 \\ &\leqq (m+1)^2 - ((m^2 - 4n)^{1/2} - 3)^2 = 4n + 2m - 8 + 6(m^2 - 4n)^{1/2} \\ &\leqq 4n + 4n^{1/2} + 12n^{1/4} - 6 + 2n^{-1/2} + \tfrac{15}{2} n^{-1/4} + 3n^{-3/4}. \end{aligned}$$

We see that $2n^{-1/2} + \frac{15}{2}n^{-1/4} + 3n^{-3/4} < 6$ for $n \geqq 9$. Thus $a_3 < n + n^{1/2} + 3n^{1/4}$ for $n \geqq 9$, except for the case where $m^2 - 4n = 5$. In this case we have $k = 1$ and

$$\begin{aligned} 4a_3 &= (m+1)^2 = 4n + (m^2 - 4n) + 2m + 1 = 4n + 2m + 6 \\ &\leqq 4n + 4n^{1/2} + 8 + 2n^{-1/2} < 4n + 4n^{1/2} + 12n^{1/4}. \end{aligned}$$

It is easy to check $a_3 < n + n^{1/2} + 3n^{1/4}$ directly, for $n \leqq 8$. This completes a proof of (A).

Proof of (B). Let $0 < \epsilon < \frac{1}{4}$ and let δ be a rational number such that

$$1 + \tfrac{1}{4}\epsilon < \delta < 1 + \tfrac{3}{4}\epsilon, \tag{6}$$

and consider a family of positive integers q such that δq is integral. We confine ourselves to numbers n of the form

$$n = m^2, \qquad m = q^2 - \delta q \tag{7}$$

and prove that there are infinitely many numbers q such that

$$m^2 < a_i < m^2 + m + (3 - \epsilon)m^{1/2} \qquad (i = 1, 2, 3), \tag{8}$$

with distinct a_i, entails $a_1 \nmid a_2 a_3$. We prove first the

LEMMA. *Suppose q is sufficiently large. If there exists a triplet of integers a_i such that $a_1 | a_2a_3$ and the condition (8) is satisfied, then there exist two positive integers t and v such that*

$$1 \leqq t \leqq 8, \tag{9}$$

$$v - \tfrac{1}{2}t + \tfrac{1}{2}t^{1/2} < t^{1/2}q < v - \tfrac{1}{2}t + \tfrac{1}{2}t^{1/2} + \frac{3 - t^{1/2}}{2t^{1/2}} + \frac{(t-1)\epsilon}{2t^{1/2}}\,. \tag{10}$$

Indeed, if a_i satisfy (8) and $a_1 | a_2a_3$, then $a_1 = b_1c_1$, $a_2 = b_1c_2$, $a_3 = b_2c_1$. We note that $a_2 < a_1 < a_3$ is impossible because, then, $a_3 - a_2 = (b_2 - b_1)c_1 + b_1(c_1 - c_2) \geqq c_1 + b_1 \geqq 2(b_1c_1)^{1/2} = 2a_1^{1/2} > 2m$ and $a_3 - a_2 < m + (3-\epsilon)m^{1/2}$ imply $3 > m^{1/2}$ which is false for q sufficiently large. Similarly $a_3 < a_1 < a_2$ is impossible. There remain two possibilities: (I) $b_1 < b_2$, $c_1 < c_2$, and (II) $b_1 > b_2$, $c_1 > c_2$, and in either case we can assume $b_1 \geqq c_1$ without loss of generality. Thus $b_1 \geqq (b_1c_1)^{1/2} = a_1^{1/2} > m$. Define v by $b_1 = m + v$. Define u in case (I) by $a_1 = m^2 + u$, and in case (II) by $a_2 = m^2 + u$. Now, in case (I) we have $a_1 + b_1 = b_1(c_1 + 1) \leqq b_1c_2 = a_2$, while in case (II) we have $a_2 + b_1 = b_1(c_2 + 1) \leqq b_1c_1 = a_1$. In either case, $m^2 + u + b_1 = m^2 + m + u + v \leqq$ some a_i; and hence, from (8) we have

$$u + v < (3 - \epsilon)m^{1/2}. \tag{11}$$

Since a_1 or $a_2 = m^2 + u = m^2 - v^2 + v^2 + u$ is a multiple of $b_1 = m + v$, so also is $v^2 + u$ a multiple of b_1:

$$v^2 + u = t(m + v), \qquad t > 0. \tag{12}$$

Since $tm < t(m+v) = v^2 + u \leqq (u+v)^2 < (3-\epsilon)^2 m$, we have $t < 9$, which is (9).

Next, it follows from (7) and (12) that

$$\begin{aligned}(v - \tfrac{1}{2}t)^2 &= v^2 - tv + \tfrac{1}{4}t^2 = tm - u + \tfrac{1}{4}t^2 = t(q^2 - \delta q) - u + \tfrac{1}{4}t^2 \\ &= t(q - \tfrac{1}{2}\delta)^2 + \tfrac{1}{4}t^2 - \tfrac{1}{4}t\delta^2 - u < t(q - \tfrac{1}{2}\delta)^2 + O(1),\end{aligned}$$

as $q \to \infty$. Hence if q is sufficiently large we have

$$\begin{aligned}v - \tfrac{1}{2}t &< t^{1/2}(q - \tfrac{1}{2}\delta) + O(q^{-1}) < t^{1/2}(q - \tfrac{1}{2}\delta) + \tfrac{1}{8}t^{1/2}\epsilon \\ &< t^{1/2}q - \tfrac{1}{2}t^{1/2}(1 + \tfrac{1}{4}\epsilon) + \tfrac{1}{8}t^{1/2}\epsilon = t^{1/2}q - \tfrac{1}{2}t^{1/2},\end{aligned}$$

by using (6). This proves the first inequality in (10). The second inequality is proved similarly by using (6), (7), (12) and (11). Note that $v - (t+1)/2 \geqq 0$, which is necessary for extraction of square roots from the inequality, is a consequence of (12) and (11):

$$\begin{aligned}v(v - (t+1)) &= v^2 - (t+1)v = tm - (u+v) \\ &\geqq tm - (3-\epsilon)m^{1/2} > m - 3m^{1/2} > 0,\end{aligned}$$

for sufficiently large q. The lemma is thus proved.

Now the lemma guarantees that if q is sufficiently large and if there exists a triplet a_i such that (8) is satisfied and $a_1 | a_2a_3$, then the fractional part $\{t^{1/2}q\}$

$=t^{1/2}q-[t^{1/2}q]$ of the number $t^{1/2}q$ lies in the open interval

$$I_t=\left(-\tfrac{1}{2}t+\tfrac{1}{2}t^{1/2},\ -\tfrac{1}{2}t+\tfrac{1}{2}t^{1/2}+\frac{3-t^{1/2}}{2t^{1/2}}+\frac{(t-1)\epsilon}{2t^{1/2}}\right)\ (\text{mod } 1)$$

for some t with $1\leqq t\leqq 8$. If $t\neq 1$, then the length of I_t is

$$l_t=\frac{3-t^{1/2}}{2t^{1/2}}+\frac{(t-1)\epsilon}{2t^{1/2}}<1.$$

Note in particular $I_1=(0,1)$, $I_4=(0,\tfrac{1}{4}+\tfrac{3}{4}\epsilon)$ (mod 1), and hence the condition

$$\{t^{1/2}q\}\notin I_t \tag{13}$$

is satisfied for $t=1$ and $t=4$. We note further, since $\epsilon<1/4$, that $l_2+l_8<1$.

Now, since the five irrational numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$ are linearly independent with respect to the rational number field, we find, by a theorem of H. Weyl (*Math. Ann.*, vol. 77, 1916, pp. 313–352, Satz 4, p. 319) on the multidimensional uniform distribution (mod 1) of irrational numbers, that the "probability" that q, a multiple of the denominator of δ, satisfies the condition (13) for all $t=1, 2, \cdots, 8$ is greater than

$$(1-l_2-l_8)(1-l_3)(1-l_5)(1-l_6)(1-l_7)>0.$$

Thus there exist infinitely many q's for which (13) is true for $t=1, 2, \cdots, 8$, *i.e.* for which

$$(q^2-\delta q)^2<a_i<(q^2-\delta q)^2+q^2-\delta q+(3-\epsilon)(q^2-\delta q)^{1/2}$$

for $i=1, 2, 3$ entails $a_1\nmid a_2a_3$. This proves the assertion (B).

Also solved by Robert Breusch and J. H. van Lint.

Editorial Note. The proposer remarks that it may be shown trivially that for any $c_3>0$ there exist infinitely many n such that three distinct integers a_i can be found satisfying $n<a_i<n+n^{1/2}+c_3n^{1/4}$ and such that $a_1|a_2a_3$.

Distinct Products of Pairs of Integers

4913 [1960, 596]. *Proposed by Paul Erdős, Technion, Haifa, Israel*

Let w be any positive integer, and let a_i, a_j be integers satisfying

$$w^2\leqq a_i<(w+1)^2,\qquad w^2\leqq a_j<(w+1)^2.$$

Prove that all products a_ia_j are distinct.

Solution by Jack Silver, Undergraduate, Montana State University. We first show that for $w^2\leqq a_i<(w+1)^2$, $i=1, 2, 3, 4$, $a_1a_2\leqq a_3a_4$ implies $a_1+a_2\leqq a_3+a_4$. If $a_i=w^2+b_i$ and $a_1+a_2>a_3+a_4$, then $b_1+b_2-b_3-b_4\geqq 1$. Therefore $(w^2+b_1)\cdot(w^2+b_2)\leqq(w^2+b_3)(w^2+b_4)$ yields

$$(b_1+b_2-b_3-b_4)w^2\leqq b_3b_4-b_1b_2<(b_1+b_2-b_4)b_4-b_1b_2,$$
$$w^2<(b_1-b_4)(b_4-b_2)\leqq\tfrac{1}{4}(b_1-b_2)^2\leqq w^2,$$

where we have used $b_1-b_2\leqq 2w$ and the fact that $xy\leqq\frac{1}{4}(x+y)^2$. But this is a contradiction.

Thus, if $a_1a_2=a_3a_4$, then $a_1+a_2=a_3+a_4$, whence $a_1^2+a_3a_4=a_1(a_1+a_2)$ $=a_1(a_3+a_4)$, or $(a_1-a_3)(a_1-a_4)=0$, so that a_1 equals a_3 or a_4. The desired result is now immediate.

The result is sharp. We cannot replace $<$ by $\leqq$ as the counterexample $w^2\cdot(w+1)^2=(w^2+w)\cdot(w^2+w)$ shows.

Also solved by A. C. Aitken, W. J. Blundon, Robert Breusch, L. J. Burton, Bro. Joseph Heisler, J. A. Holbrook, W. H. Jobe, Eileen Jones and Edward Barbeau, John B. Kelly, J. Kestelman, W. S. Lawton, Joe Lipman, D. C. B. Marsh, D. J. Newman, E. J. F. Primrose, J. L. Selfridge, Wu Ta-Sun, J. H. van Lint, Benjamin Weiss.

Generalization of a Familiar Combination Formula

4914 [1960, 596]. *Proposed by Leo Moser, University of Alberta*

For any positive integers m, n, prove that K is an integer, where

$$K=\frac{(mn)!1!2!\cdots(m-1)!1!2!\cdots(n-1)!}{1!2!\cdots(m+n-1)!}.$$

Solution by Leonard Carlitz, Duke University. Put

$$m=ap+r\quad(0\leqq r<p),\qquad n=bp+s\qquad(0\leqq s<p),$$
$$m+n=(a+b+\epsilon)p+t\qquad(0\leqq t<p,\ \epsilon=0\text{ or }1),$$

where p denotes a prime power. Then

$$\begin{aligned}\sum_{k=1}^{m-1}\left[\frac{k}{p}\right]&=\left[\frac{p}{p}\right]+\cdots+\left[\frac{2p-1}{p}\right]+\left[\frac{2p}{p}\right]+\cdots\\&=p+2p+\cdots+(a-1)p+ra\\&=\tfrac{1}{2}pa(a-1)+ra=\tfrac{1}{2}a(pa-p+2r)=\tfrac{1}{2}a(m-p+r).\end{aligned}$$

Hence it will suffice to show that

$$\left[\frac{mn}{p}\right]+\tfrac{1}{2}a(m-p+r)+\tfrac{1}{2}b(n-p+s)\geqq\tfrac{1}{2}(a+b+\epsilon)(m+n-p+t),$$

that is

$$2\left[\frac{mn}{p}\right]\geqq an+bm+(a+b)t-ar-bs+\epsilon(m+n-p+t),$$

or

$$(*)\qquad (a+b)(r+s)+2\left[\frac{rs}{p}\right]\geqq(a+b)t+\epsilon(m+n-p+t).$$

If $\epsilon=0$, so that $r+s=t$, (*) obviously holds.

If $\epsilon=1$, so that $r+s=p+t$, (*) becomes

$$(a+b)p+2\left[\frac{rs}{p}\right] \geqq (a+b)p+2t,$$

that is $[rs/p] \geqq t$. But since $(p-r)(p-s)>0$, $rs>p(r+s-p)=pt$, so that $[rs/p] \geqq t$, completing the proof.

Also solved by Robert Breusch, L. M. Kaplan, Richard Karch, and K. A. Post.

Note by the proposer. The given expression is a solution to the following problem: In how many ways can the integers $1, 2, \cdots, mn$ be placed in an $m \times n$ matrix so that the numbers in every row and column are monotone increasing? With this interpretation, K must obviously be integral.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Frontiers of Numerical Mathematics. Edited by Rudolph E. Langer. The University of Wisconsin Press, Madison, Wis., 1960. xi+132 pp. $3.50 (Photo-offset)

Eight invited speakers were asked to set forth the mathematical problems that stand astride the advance of their specialities. These papers were delivered at a symposium conducted by the Mathematics Research Center, U. S. Army and the National Bureau of Standards held at the University of Wisconsin in October, 1959. *Frontiers of Numerical Mathematics* contains the eight papers, each followed by the comments of some of the eighteen discussants who were there.

The titles and the authors of the papers are: *Stress Analysis in the Plastic Range*, William Prager; *Some Mathematical Problems of Nuclear Reactor Theory*, Garrett Birkhoff; *Numerical Problems of Contemporary Celestial Mechanics*, Sdenek Kopal; *Aeroelasticity*, Lee Arnold; *Operations Research*, Philip M. Morse; *Mathematical Bottlenecks in Theoretical Chemistry*, J. O. Hirschfelder; *Magnetohydrodynamics*, S. Chandrasekhar; *The Solution of Systems of Partial Differential Equations Arising in Meteorology*, J. Smagorinsky.

The papers are provocative and deserve scanning for new ideas. The place of the high-speed computer is quite uniformly considered as that of a useful tool following rather than replacing theory. However, Professor Morse mentions that simulation on a computer may be an aid to the development of improved mathematical models.

Advances in technology accentuate difficulties of old problems, as for example the orbit computations of a man-launched satellite, which since its launching "has already completed more revolutions around the earth than the number of

years which has elapsed since the dawn of human civilization."

The papers are not uniform in length nor in detail. However, the variety of views expressed in the well chosen chapters on applied mathematics is most stimulating. The photo-offset printing is well done.

HERBERT A. MEYER
University of Florida

Introduction to Probability and Random Variables. By George P. Wadsworth and Joseph G. Bryan. McGraw-Hill, New York, 1960. vii+202 pp. $8.75.

This book starts with a chapter on preliminary mathematics. It is a very good idea and well done. Often students come ill-prepared, and this way the material or any portion can be taken at will. Also the presentation of techniques does not clutter up the theory. I think such a chapter should be even more complete, including matrices. The lack of matrices throughout the book is a definite hindrance and permits only a limited presentation of multivariate theory at a time when it should be stressed more than before. Bayes's theorem is included, which is good in view of the tendency to omit it from elementary texts. Queueing theory and other modern applications are introduced, illustrating that this book is a new presentation rather than just a rewrite of standard texts.

In conjunction with the presentation of theory there are given many and practical examples, which are excellent. This should help overcome the essential debility in many books at this level, in which, although the theory is presented well, the student gets a "Yes, but what of it?" feeling. A sampling of the exercises shows that they are very good ones. They are appropriate in difficulty, cover the material well, and include some practical applications without becoming trivial.

With an additional hundred pages on the mathematics of statistical methodology, this book would be a competitor at the Hoel-Fraser level for a mathematical statistics course. With a hundred pages less, it would be a competitor for Cramér's elementary text. As it is, it is just appropriate for a mathematics department course for undergraduates as an introduction to probability and random variables, just as the title says. All in all, it is a well-written book and should serve extremely well if used for the appropriate purpose.

ROBERT H. RIFFENBURGH
University of Hawaii

Lectures on the Theory of Functions of a Complex Variable, Vol. I, *Holomorphic Functions.* By G. Sansone and J. C. H. Gerretsen. Noordhoff, Groningen, 1960. xii+481 pp.+index. Dfl. 45.– (paper) or Dfl. 48.75 (cloth).

This readable book is the first part of a two-volume English edition of Sansone's *Lezioni sulla Teoria delle Funzioni di una Variabile Complessa* (1947) with a new text containing revisions and rearrangement of the Italian work. It starts from first principles but does not emphasize logical foundations. The book is

primarily designed for beginners interested in a detailed account of special topics of the classical theory. No attempt is made to present a scholarly work and the reader will notice that a bibliography is not provided. Among the topics discussed are: power series, elementary functions, Cauchy integral theorem, residue theory, Weierstrass factor theorem, Mittag-Leffler theorem, elliptic functions, integral functions of finite order, Dirichlet series, Riemann zeta function, Laplace integral and asymptotic series.

HIROSHI YAMAUCHI
University of Hawaii

Rings of Continuous Functions. By Leonard Gillman and Meyer Jerison. The University Series in Higher Mathematics, Van Nostrand, Princeton N. J., 1960. ix+300 pp. $8.75.

A very interesting book on a subject which offers large vistas for further research and which will undoubtedly receive increasing recognition on the roster of graduate courses offered in our universities. Through a succession of preliminary editions developed in formal seminars and in the more critical atmosphere of office and home, the two authors have over the past 5 years performed the enormous task of the preparation, separation, ordering, and finally presentation of a plenitude of cohesive material. The result is a clear book which allows the reader to reach rapidly good depth and a wide angle view.

Mise en scène: A topological space X, usually completely regular, the ring $C(X)$ of all real continuous functions on X, the ring $C^*(X)$ of all bounded real continuous functions on X. Fundamental problems studied concern the relation of the algebraic properties of the rings (*e.g.* ideal structure) and topological properties of X (*e.g.* compactness). The work of Stone and Čech on compactification is central and is studied in great detail. The elaboration of this material is considered in the middle part of the book. The first third is devoted to the preparation of sharp tools concerning zero-sets, filters, ideals (fixed and free), ordered rings, and residue fields. Then come the chapters on the compactification βX of X, the theorem of Gelfand-Kolmogoroff, and material on the real compact spaces of Hewitt. The final third of the book is devoted to largely independent topics. We find a section on discrete spaces and the relation to nonmeasurable cardinals; a chapter on uniform spaces including Shirota's theorem relating realcompact spaces to spaces with complete uniform structure; and Katětov's work connecting the dimension of X to the analytic dimension of $C^*(X)$.

The sixteen chapters are followed by substantial exercises which are valuable both to the beginner and to the expert. You will find here, for example, the universal Gegenbeispiel, Tychonoff's plank, as well as the eye-filling wonders of the space $\beta N-N$. Modulo the reviewer's congenital enthusiasm it would seem, nevertheless, that this is an *opus magna laude dignum.*

EDGAR R. LORCH
Columbia University

BRIEF MENTION

Mathematical Tables, Royal Society:

Volume V, Representations of Primes by Quadratic Forms. Prepared by Hansraj Gupta, M. S. Cheema, A. Mehta, and O. P. Gupta. Cambridge University Press, 1960. 135+xxiii, \$8.50.

Volume VI, Tables of the Riemann Zeta Function. By C. B. Haselgrove, in collaboration with J. C. P. Miller. xxii+80 pp., \$9.50.

Volume VII, Bessel Functions, Part III, Zeros and Associated Values. By F. W. J. Oliver. xv+lix+79 pp., \$9.50.

Analogue and Digital Computers. By M. G. Say, A. C. D. Haley and W. E. Scott. Philosophical Library, 1960. viii+308 pp., \$15.00.

Although the publication date and the price might suggest this was an up-to-date book containing last-minute information, a typical sentence discloses the truth, "A typical size of store is about 10,000 digits (about 320 words of 32 digits, or 256 words of 40 digits)." Page 216.

Proceedings of the 1959 Computer Applications Symposium. Armour Research Foundation, Illinois Institute of Technology, 155 pp., \$3.00.

Fourteen invited papers presented on October 28–29, 1959. The first day's papers were devoted to business and management applications, while those on the following day stressed engineering and scientific applications. The interesting discussion of the pros and cons of automatic programming which occurs in the panel discussion makes the book worth investigating in and of itself.

Digital Computers and Nuclear Reactor Calculations. By Ward C. Sangren. Wiley, New York, 1960. xi+208 pp., \$8.50.

The book begins with an introduction to reactor problems, numerical analysis, digital computers, and their programming, followed by more detailed discussions of the important high-speed digital reactor calculations.

Theoretical Hydrodynamics. By Milne-Thomson. 4th Ed., Macmillan, New York, 1960. 650 pp., \$11.00.

Vector dynamics with a good bit of solid mathematics interspersed among the air foil and hydrodynamic applications. Not a modern book in a strict sense of the word, but a solid one.

Contributions to the Theory of Nonlinear Oscillations. Volume V. Princeton University Press, 1960. 284+append., \$5.00. Princeton Annals of Mathematics Studies, Vol. 45.

A collection of advanced papers which will undoubtedly be reviewed separately in *Mathematical Reviews* and other journals.

The Mathematics of Radiative Transfer. By I. W. Busbridge. Cambridge University Press, 1960. 133 pp., \$5.00. Number 50 in Cambridge Tracts in Mathematics and Mathematical Physics.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., *University of Buffalo*

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Harvey Cohn, University of Arizona, represented the Association at the inauguration of Dr. G. H. Durham as President of Arizona State University on March 11, 1961.

Professor Mary K. Landers, Hunter College, representated the Association at the inauguration of Dr. L. L. Jarvie as President of the New York City Community College on February 16, 1961.

Mr. F. A. Lee, Jr., Marion Institute, represented the Association at the inauguration of Dr. C. A. Anderson as President of Judson College on February 3, 1961.

Professor J. H. Wahab, Louisiana State University at New Orleans, represented the Association at the inauguration of Dr. H. E. Longnecker as President of Tulane University on April 15, 1961.

Catholic University of America: Dr. Inge Christensen and Mr. Gustav Hensel have been appointed Instructors.

Massachusetts Institute of Technology: Professor I. E. Segal, University of Chicago, has been appointed Professor; Dr. J. G. Glimm, Institute for Advanced Study, and Dr. James Munkres, Princeton University, have been appointed Assistant Professors; Drs. R. B. Darst, Louisiana State University, S. Y. Husseini, and Ronald Jacobowitz, Princeton University, J. I. Richards, Harvard University, and F. S. Van Vleck, University of Minnesota, have been appointed Instructors; Associate Professor G. B. Thomas, Jr., has been promoted to Professor; Assistant Professors H. P. McKean, Jr., and D. B. Ray have been promoted to Associate Professors; Dr. D. J. Benney has been promoted to Assistant Professor; Professors R. D. Douglass and D. J. Struik retired with the title of Professor Emeritus; Institute Professor Norbert Wiener retired with the title of Institute Professor Emeritus; Associate Professor S. D. Zeldin retired with the title of Associate Professor Emeritus.

Michigan State University: Professor J. S. Frame has resigned as Head of the Department of Mathematics retaining the position of Professor of Mathematics; Professor C. P. Wells has been appointed Head of the Department of Mathematics; Assistant Professors J. E. Adney, Purdue University, Ti Yen, Illinois Institute of Technology, and R. P. Gilbert, University of Pittsburgh, have been appointed Assistant Professors; Assistant Professors J. G. Hocking, R. H. Oehmke, and M. L. Tomber have been promoted to Associate Professors.

University of British Columbia: Drs. Mario Benedicty, University of Pittsburgh, C. W. Clark and Maurice Sion, University of California, Berkeley, and Z. A. Melzak, Bell Telephone Laboratories, Murray Hill, New Jersey, have been appointed Assistant Professors; Drs. D. F. Rearick and R. C. Thompson, California Institute of Technology, and Lorraine Schwartz, University of California, Berkeley, have been appointed Instructors; Mr. J. F. Scott-Thomas, Massachusetts Institute of Technology, has been appointed Lecturer; Assistant Professor H. A. Thurston has been promoted to Associate Professor; Mr. Elöd Macskásy and Dr. Rimhak Ree have been promoted to Assistant Professors.

University of Kentucky: Dr. A. L. Duquette, University of Colorado, has been appointed Assistant Professor; Professor Stanislaw Balcerzyk, University of Torun, Poland, has been appointed Research Instructor.

University of Nevada: Assistant Professor Ronald Macauley, University of Washington, has been appointed Assistant Professor; Assistant Professor R. N. Thompson has been promoted to Associate Professor.

Dr. C. H. Boll, Lockheed Aircraft Corporation, Sunnyvale, California, has accepted a position with Loral Electronic Corporation, Tustin, California.

Dr. Evelyn B. Collins, Staff Assistant at the Space Computing Center of International Business Machines, Washington, D. C., has joined the technical staff of Space Technology Laboratories, Los Angeles, California.

Dr. E. H. Crisler, Bendix Aviation Corporation, South Bend, Indiana, has accepted a position as a member of the Senior Staff of Hughes Aircraft Company, Culver City, California.

Dr. W. E. deMalignon, University of South Dakota, has been appointed Assistant Professor at San Diego State College.

Professor J. W. Givens, Wayne State University, has been appointed Professor of Engineering Sciences at Northwestern University.

Mr. L. K. Grodman, International Business Machines, Poughkeepsie, New York, has accepted a position with Mitre Corporation, Bedford, Massachusetts.

Dr. Ali Kyrala, Goodyear Aircraft Corporation, Litchfield Park, Arizona, has been appointed Associate Professor at Arizona State University.

Mrs. Carolyn S. Sharif, Austin Public Schools, Austin, Texas, has been appointed Teacher at Sunset High School, Dallas, Texas.

Professor K. W. Wegner, Carleton College, has been appointed Chairman of the Department of Mathematics to replace Professor K. O. May who has returned to full-time teaching.

Professor Emeritus H. C. Feemster, York College, died in February, 1961. He was a Charter Member of the Association.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the Summer of 1961:

Michigan State University, June 19 to September 1: Professor Larcher, projective geometry, theory of matrices and groups; Professor Hocking, introduction to sets and abstract spaces; Professor Gilbert, theory of functions of a complex variable; Professor Wasserman, partial differential equations. June 19 to July 26: Professor Hertzog, advanced calculus, theory of numbers; Professor Stewart, advanced topics in matrices. July 27 to September 1: Professor Doyle, advanced calculus, topology; staff, advanced differential equations, theory of polynomials.

University of Colorado, June 16 to July 21; July 24 to August 25: Professor Dunton, theory of numbers; Professor Zirakzadeh, foundations of geometry; Professor Schmidt, topology, advanced calculus; Professor Hodges, infinite processes, history of mathematics; Dr. Oliver, teaching of secondary school mathematics; Professor Fischer, first session, Professor Jones, second session, modern algebra; Professor McKelvey, foundations of analysis; Dr. Hirsch, theory of groups.

University of Illinois, June 19 to August 12: Professor Ketchum, functions of real variables, introduction to numerical analysis; Professor Mendel, introduction to higher analysis (real variables); Professor Peters, elementary geometry from a modern viewpoint; Professor Ribenboim, group theory; Professor Blyth, advanced statistics; Professor Rotman, topological spaces. In addition the following courses will be given: fundamental concepts of algebra, linear transformations and matrices, introduction to higher algebra, advanced calculus, differential equations and orthogonal functions, complex variables and application.

University of North Carolina at Chapel Hill, June 6 to July 18: Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Professor MacNerney, linear algebra; Professor Linker, differential equations; Professor Mackie, theory of numbers; Professor Artzy, foundations of geometry; Professor Pettis, some recent results in algebra. July 19 to August 26: Professor Lasley, analytic geometry from higher point of view; Dr. Patty, theory of equations; Dr. Buckholtz, advanced calculus; Professor Wells, topics in analysis; Professor Whyburn, advanced differential equations.

University of Virginia, June 19 to August 12: Dr. Henderson, foundations of geometry, Professor Williams, foundations of algebra, introductory analysis; Professor Malbon, differential equations and applied mathematics; mathematics for teachers; Professor Paige, advanced analysis.

University of Wisconsin, Professor Sanderson, applied differential equations, elementary plane topology; Professor Losey, applied mathematical analysis, projective geometry; Professor MacDuffee, matrices and their applications, advanced topics in algebra; Professor Immel, higher analysis, introduction to the theory of probability; Professor Rothman, advanced calculus, modern views of mathematics; Professor Struble, introduction to measure and integration; Professor Jones, advanced topics in point-set topology; Professor Davis, theory and operation of computing machines, advanced topics in real variable theory.

West Virginia University, June 12 to July 21: Professor Cochran, astronomy for teachers, fourier series and partial differential equations; Professor Cunningham, advanced calculus; Professor Peters, introduction to algebraic theories, group theory. July 24 to August 30: Professor Bragg, theory of numbers, linear algebra; Mrs. Easton, special topics; Professor Stewart, advanced calculus, higher plane curves; Professor Vest, operational methods in partial differential equations.

TRAVEL GRANTS FOR ATTENDANCE AT THE INTERNATIONAL CONGRESS OF MATHEMATICS

Travel grants will be made to a number of mathematicians who wish to attend the International Congress of Mathematicians in Stockholm, on August 15–22, 1962. It is hoped that funds available through various sources may provide travel assistance for a considerable number of mathematicians.

There will be a greater effort than in the past to give aid to younger people. As grants will be made only to those who have filed applications, it is urgent that any who wish to receive a grant should fill out and file an application. Younger people are urged to file applications so that their cases can be considered. Applications can be obtained from the Division of Mathematics, National Academy of Sciences, National Research Council, Washington 25, D. C. by requesting an application for a travel grant to the 1962 International Congress.

The deadline for filing of applications is November 1, 1961, and an attempt will be made to announce the grants by January 1, 1962. Awarding of grants will be made only to those persons whose applications have been received, in good order, by November 1. The selection will be made by a committee consisting of the regular Committee on Travel Grants of the Division of Mathematics of the National Academy of Sciences—National Research Council enlarged to include representatives of societies affiliated with the Division and representatives of various governmental agencies.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE EDITOR OF THE MATHEMATICS MAGAZINE

The Board of Governors of the Association has elected Professor Robert E. Horton of Los Angeles City College as Editor of the *Mathematics Magazine* for a three-year term beginning January 1, 1961.

Professor Horton has been Editor of the Problems and Questions Department of the *Mathematics Magazine* since 1953 and its Editor-in-Chief since 1959. He has been an engineer with Douglas Aircraft Co. 1938–39, a mathematics teacher at Black-Foxe Military Institute 1939–41, a mathematics teacher in the Los Angeles City Schools in 1941, a mathematics instructor at the University of California, Los Angeles 1946–47, a mathematics instructor at Los Angeles City College since 1947 and since 1960 Assistant Dean of Instruction at Los Angeles City College. He has published numerous articles in the *Mathematics Magazine* and other publications.

HENRY L. ALDER, *Secretary*

THE NOVEMBER MEETING OF THE MINNESOTA SECTION

The annual fall meeting of the Minnesota Section of the Mathematical Association of America was held on November 5, 1960, at the University of North Dakota, Grand Forks, North Dakota. Professor Charles Hatfield, University of North Dakota, presided at the morning session, and the Section Chairman, Professor Fulton Koehler, University of Minnesota, presided at the afternoon session. There were 63 persons registered for the meeting, of whom 44 were members of the Association.

During the brief business meeting, the Section Chairman discussed the arrangements for conducting the 1961 High School Mathematics Contest.

One feature of the day's program was a panel discussion on the use of television in the teaching of mathematics. The panelists were Professors Bernard Derwort, St. Thomas College, John Dyer-Bennet, Carleton College, and Jack Indritz, University of Minnesota. Professor Fulton Koehler was the moderator. The panelists, each of whom had had actual experience with this new medium, agreed that an increased utilization of audio-visual devices is inevitable, but split on whether this is desirable, with Professor Indritz tending to welcome the new techniques, Professor Dyer-Bennet being quite critical, and Professor Derwort remaining somewhat neutral.

The following papers were presented:

1. *Mathematics in Taiwan*, by Professor K. W. Wegner, Carleton College.

Describing his year as a Fulbright Lecturer at National Taiwan University in 1959–60, the speaker mentioned the lack of any graduate offerings in mathematics in Taiwan, the small number of teachers with advanced degrees, the high standards in the courses given, the intense competition for admission to college, the great difficulties encountered by students in getting to foreign countries for graduate work, the low salaries of teachers, and the activities of the research organization Academica Sinica. He pointed out that the basic sciences and mathematics are losing the best students to engineering because of the lack of opportunities at graduation.

2. *On approximation of alternating series*, by Professor O. E. Stanaitis, St. Olaf College.

The remainder of an alternating series is evaluated by special inequalities obtained from Euler's summation formula. This leads to fast, accurate approximations of alternating series. It has been shown, for example, that the series $S=\sum_{n=1}^{\infty}(-1)^{n+1}/n^2$ can be approximated correctly to six decimal places by adding only ten terms of the series. In order to secure the same accuracy without evaluation of the remainder it would be necessary to calculate a thousand terms of the series.

3. *A Monte Carlo experiment*, by Mr. R. M. Collins, Jr., Minnesota Mutual Life Insurance Co., St. Paul, Minnesota.

The Monte Carlo technique was used to simulate claim experience in a group of lives. To simulate the "exposure" of each life, a random number was compared with the probability of death. If the number was less than or equal to that probability, a death occurred. This was done for the entire group n times to simulate n years of exposure. The result was a claim distribution for the group. The random number supply consisted of an initial table of 1000 numbers from which succeeding tables were generated by an accumulation process. The experiment was performed using a Datatron 205.

4. *Inner and outer quotients*, by Professor Joong Fang, St. John's University, Collegeville, Minnesota.

Some authors in elementary analysis venture to present complex quotients as vectors, but all authors in vector analysis fail to mention "vector quotients" which, however, do exist, and are found to be as follows: 1. The inner quotient: $B=cA/|A|^2+C\times A$ where A, B, and C are vectors, and $A\cdot B=c$ (c a scalar); in particular, if $c=1$, then $B=A^{-1}$; *i.e.*, $A\cdot A^{-1}=1$. 2. The outer quotient: $B=C\times A/|A|^2+cA$ if and only if $A\cdot C=0$, where $A\times B=C$, $(A\neq 0)$ and c is a scalar.

5. *Some singular cases of the implicit function theorem*, by Professor W. S. Loud, University of Minnesota (by invitation).

Singular cases of the implicit function theorem, in which the fundamental Jacobian is zero, arise in the study of periodic solutions of perturbed differential equations. The problem of solving the system $F(x, y, z)=G(x, y, z)=0$ for x and y as functions of z is treated, where at $(0, 0, 0)$, $F=G=F_xG_y-G_xF_y=0$. The cases are analyzed in which a knowledge of the partial derivatives of F and G at $(0, 0, 0)$ through order three is sufficient to resolve the problem of existence and local behavior of solutions. The technique can be extended to more general systems.

6. *A definition of functions on the strength of an operative logic*, by Professor Joong Fang, St. John's University, Collegeville, Minnesota.

This is a sequel to *Inverse functions vs. "converse" functions* (this MONTHLY, vol. 66, p. 947), which is further founded here on an operative logic (in the spirit of E. Lorenzen's *Einführung in die operative Logik und Mathematik*). The *order* of operators with respect to operands is articulated by the proto-logical "sense" of the relation with respect to its "referent" and "relatum," (in the spirit of B. Russell's *Introduction to Mathematical Philosophy*). A function as such, (*i.e.*, an *operative* set of any logical type) is then generally denoted by a set of dyadic n-tuples, $(x_1, \cdots, x_n)$, where the elements of the set, each of which represents a set of the first type, are always grouped into two in compliance with the principle of *ordered* one-to-one correspondence.

MURRAY BRADEN, *Secretary*

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twenty-third annual meeting of the Northern California Section of the Mathematical Association of America was held at San Jose State College, January 14, 1961. Professor S. P. Hughart, Chairman of the Section, presided at the afternoon session and at Session I (research papers) in the morning. Professor D. W. Blakeslee, Vice-Chairman of the Section, presided at Session II (papers on the teaching of mathematics). There were 150 persons in attendance, including 110 members of the Association.

At the business meeting the resignation of Professor Roy Dubisch as Secretary-Treasurer was accepted and a resolution was passed expressing appreciation for his services during the past years. Professor B. J. Lockhart, U. S. Naval Postgraduate School, was elected Secretary-Treasurer for a three-year term and Professor G. E. Latta, Stanford University, was elected Vice-Chairman. The Chairman for next year is Professor D. W. Blakeslee, San Francisco State College, and Professor S. P. Hughart, Sacramento State College, is the Program Chairman.

By invitation of the Section, two addresses were given. Professor Daniel Zelinsky, Northwestern University and University of California, Berkeley, spoke on *Functors*, and Professor Günter Lumer, Stanford University, spoke on *Differential Equations: Heuristics and Modern Methods.*

The following papers were presented:

1. *On the construction of T-forms*, by Professor Dmitri Thoro, San Jose State College.

Theorems are obtained for the construction of T-forms with given invariants and leading coefficient. These results are then used to determine integers represented by certain indefinite ternary quadratic forms.

2. *On a semantic construction of intuitionistic logic*, by Professor V. H. Dyson, San Jose State College.

On the basis of a Gentzen-type formalization of Heyting's first order predicate calculus, PC_H^1, a notion of validity is introduced with respect to binary trees. This is shown to be equivalent to the topological interpretation over the totality of open subsets of Cantor's discontinuum. Then an intuitionistically correct proof is given for the equivalence between the weak completeness of PC_H^1 and a certain second order formula. This formula seems to be of basic significance in various contexts with intuitionistic logic and analysis. The extent of its validity is as yet unknown. Thus the major open problem resulting from this paper is an investigation of intuitionistically meaningful notions of a function in the light of this formula.

3. *The Diophantine equation* $x^2+y^2+z^2=m^2$, by Mr. R. S. Spira, University of California, Berkeley.

This equation was first solved by L. E. Dickson, giving the parametric solution: $x=2(ut-vw)$, $y=2(uw-vt)$, $z=u^2+v^2-w^2-t^2$, $m=u^2+v^2+w^2+t^2$. Skolem (1941) gave an incorrect algorithm to find u, v, w, t given x, y, z, m. A correct form of this algorithm is given in this paper. Steiger (1956) empirically found a set of seven conditions on the parameters to assure each primitive solution being obtained once and only once. It is shown in this paper that these conditions can be satisfied using the correct form of Skolem's algorithm, but that the uniqueness proof involves a difficult unsolved problem connected with Brahmagupta's product formula: $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$.

4. *An experimental test of an heuristic suggestion due to Pólya*, by Professor C. M. Larsen, San Jose State College.

In *How to Solve It*, G. Pólya presents a list of heuristic suggestions for solving problems, including the following: Do you know a related problem? Can you use its result? Can you use its method? To investigate the utility of this hint, two sets of problems and solutions were prepared. One set made explicit reference to the hint; the second set, otherwise the same, did not. These materials were then given to matched pairs of students. Their scores on the problems were approximately equal, but students who had the hint explicitly available solve the problems in significantly less time.

5. *Some remarks on definition by induction*, by Professor C. A. Hayes, University of California, Davis.

The principle of definition by induction is one usually glossed over rather casually, yet it is of great importance and should be thoroughly understood by high school mathematics teachers. The speaker's aim is to cast some light into this corner of mathematics and perhaps provide teachers with an approach suitable for presentation to bright high school students.

6. *Inequalities in calculus*, by Professor L. H. Lange, San Jose State College.

Several examples of the use of some traditional inequalities in solving extremum problems without the formal machinery of calculus are discussed.

7. *Change in student attitudes towards mathematics in the last five years*, by Professor D. A. Norton, University of California, Davis.

A questionnaire given five years ago to about five hundred entering freshman at the University of California at Davis testing several hypotheses concerning the formation of attitudes towards mathematics was repeated this fall on a group of about eight hundred entering freshman for the purpose of testing change in attitudes. The study revealed, as expected, that five years of propaganda has produced a decided positive shift in the student attitudes toward mathematics. This shift was considerably greater among the male students than among the female.

8. *The formula for inverting a matrix, or, when is an adjoint not an adjoint*, by Dr. A. B. Novikoff, Stanford Research Institute.

The formula for the entries of A^{-1}, where A is a nonsingular $n \times n$ real matrix is one of a class of similar formulas all involving (i) pattern of plus and minus ones; (ii) the computation of minors; (iii) the taking of a transpose; and (iv) one lone operation of division, namely by the determinant of A to some power. In the special case of the formula for A^{-1}, an auxiliary matrix of "cofactors" is introduced and called (by some authors) the "adjoint" of A. These formulas are all derived by exterior algebra, and it is shown how the transpose operation results.

ROY DUBISCH, *Secretary*

PROPOSED AMENDMENTS TO THE BY-LAWS OF THE M.A.A.

At the business meeting of the Association to be held at Oklahoma State University in Stillwater, Oklahoma, on Wednesday, August 30, 1961, motions will be made to amend the By-Laws as follows:

A. That ARTICLE II be amended to read:

1. There shall be two classes of members, ordinary and institutional.

2. Any person interested in the field of collegiate mathematics shall be eligible for election to ordinary membership in the Association.

3. Any institution, academic or corporate, interested in the support of collegiate mathematics shall be eligible for election to institutional membership in the Association.

4. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission. In the case of individuals the application shall be endorsed by two ordinary members of the Association.

5. (The same as ARTICLE II, Section 3, in the present By-Laws).

B. That the last two words of ARTICLE VI, Section 5, be revised to read: "ordinary members."

C. That ARTICLE VII be amended to read:

1. Ordinary members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to ordinary membership of individuals and classes of applicants without payment of the initiation fee.

2. The annual dues of each ordinary members shall be five dollars ($5), including a subscription to the official journal.

3. The fees, dues, and privileges of institutional members of the Association shall be established from time to time by the Board of Governors.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, that member shall be dropped from the list after due notice.

5. (The same as ARTICLE VII, Section 4, in the present By-Laws.)

6. Any ordinary member who . . . (otherwise the same as ARTICLE VII, Section 5, in the present By-Laws).

HENRY L. ALDER, *Secretary*

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to February 1, 1961)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary, a Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee, advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor, the Secretary and the Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Assocation, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of four members, including the Secretary and the Treasurer.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section. A Governor who has moved permanently from the Section by which he was elected shall be considered to have ended his term of office on the Board. If the Governor has moved from the Section because he is no longer employed there, it shall be interpreted that he has moved permanently from the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary, a Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary and the Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. In the absence of the President, the First Vice-President (or in his absence the Second Vice-President) shall have and exercise the powers of the President. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary shall have the usual duties pertaining to his office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President.

12. The Treasurer shall have the usual duties pertaining to his office including the collection of dues and the supervision and sefekeeping of the funds of the Association.

13. (a) There shall be an Executive Director who shall be a paid employee of the Association. He shall have charge of the central office of the Association and shall carry out such other duties as may be assigned to him by the Board. He shall be responsible to the Board and shall attend meetings of the Board, the Executive Committee, and the Finance Committee, but he shall not be *ex officio* a member of these bodies.

(b) The Executive Director shall be elected by the Board under terms and conditions of employment fixed by the Finance Committee.

ARTICLE IV—MEETINGS

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

ARTICLE V—SECTIONS

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted. The by-laws of each Section when organized and any subsequent changes in these by-laws must be approved by the Board. The Board shall maintain general supervision over the activities of all Sections.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

ARTICLE VI—OFFICIAL PUBLICATIONS

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to nonmembers for less than the annual dues of individual members.

ARTICLE VII—DUES

1. Members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the admission fee.

2. The annual dues of each member shall be five dollars ($5), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of two dollars ($2).

ARTICLE VIII—AMENDMENTS TO THE ARTICLES OF ASSOCIATION AND BY-LAWS

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting. The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Summer Meeting at Oklahoma State University, Stillwater, Oklahoma, on August 29, 30 and 31, 1961. The Register will be conducted in the Terrace Room from 9:00 to 5:00 on each of these three days.

There is no charge for registering either to job applicants or to employers, except when the Late Registration Fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $1 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position description forms, which must be completed and returned to Providence not later than August 4, 1961, in order to be included free of charge in the listings at the meeting in Stillwater, Oklahoma. Forms which arrive after this closing date, but before August 14, will be included in the Register Display at the meeting for a Late Registration Fee of $3.00, and will also be included in the printed listings, but not until ten days after the meeting. The printed listings will be available for distribution both during and after the meeting.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, University of Vermont, Burlington, June 20, 1961.
NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.
PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Long Beach State College, March 9, 1962.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

FILM MANUALS

Two Film Manuals have been prepared as supplements to the films of the same name which were produced by the M.A.A. Committee on Production of Films. Each Manual contains an approximation to the words spoken in the film, supplementary material to amplify the treatment of the subject and a number of problems.

MAA Film Manual #1, **MATHEMATICAL INDUCTION,** by Leon Henkin

MAA Film Manual #2, **THEORY OF LIMITS,** by E. J. McShane

Copies of the manuals may be purchased at $1.00 each from:

Harry M. Gehman, *Executive Director*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

RALPH D. JAMES, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. JAMES, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

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CYCLIC POLYGONS

W. B. CARVER, Cornell University

In the April 1960 issue of the MONTHLY, problem E 1411 was concerned with a cyclic pentagon. The theorem to be proved was a proportion, each term of the proportion being the product of the lengths of five line segments determined by the pentagon. The proposer's solution* gave a proof which he generalized to a similar proportion for a cyclic n-gon, the terms of the proportion being products of n segments. In this paper the problem is attacked by the use of complex coordinates, and a large number of similar relations are obtained for the pentagon and for the n-gon, the number increasing enormously with n. Many of the theorems proved are simpler than the proportion of the problem, the theorems stating equality between two products rather than a proportion between four products.

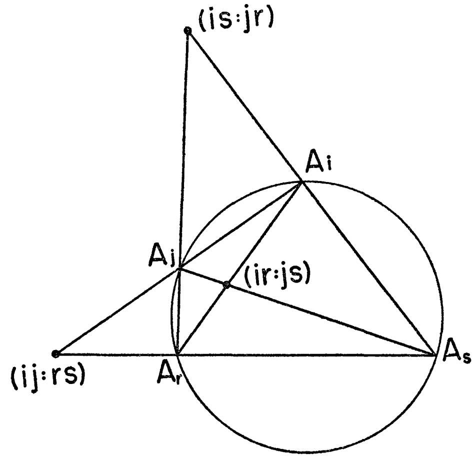

FIG. 1

Consider n distinct points on a circle, $n \geqq 5$, and name them *in any order*, A_i, $i=1, \cdots, n$. We then have a simple n-gon (not in general convex) with vertices at A_i and with n sides, $A_1A_2, A_2A_3, \cdots, A_nA_1$. For a fixed set of n points there are $\frac{1}{2}(n-1)!$ such simple n-gons. The n points, taken in pairs, determine $\frac{1}{2}n(n-1)$ lines A_iA_j, including the n sides. In what follows all subscripts are to be reduced, modulo n, to the set $1, \cdots, n$; and equations *in these subscripts* are to be regarded as congruences, modulo n.

The following notations will be used (see Fig. 1):

* This MONTHLY, vol. 67, 1960, p. 1029.

ij will denote the distance (positive) between the points A_i and A_j $(ij=ji)$.

$(ij:rs)$, where i, j, r, s are any four distinct subscripts, will denote the point of intersection of the lines A_iA_j and A_rA_s. (The case in which these lines are parallel will be treated separately. Where $(ij:rs)$ is used it is assumed that the lines are not parallel.) There are $\frac{1}{8}n(n-1)(n-2)(n-3)$ of these intersections, but not necessarily all distinct when $n>5$.

$i(ij:rs)$ will denote the distance (positive) between the points A_i and $(ij:rs)$, and similarly for $j(ij:rs)$, $r(ij:rs)$, and $s(ij:rs)$.

$\prod^n [\ ]$ will denote a product of n factors, the first factor being given in the brackets [ ], and each subsequent factor obtained by increasing each subscript of the preceding factor by one. (The superscript n will be omitted from this symbol when the meaning is clear without it.)

Thus if $n=5$ and the points A, B, C, D, E of problem E 1411 are respectively A_1, A_2, A_3, A_4, A_5, then the points Q, R, S, T, P will be respectively (13:24), (24:35), (35:41), (41:52), (52:13); and the proportion to be proved may be written

$$\frac{\prod [32]}{\prod [14]} = \frac{\prod [2(13:24)]}{\prod [4(13:24)]}. \tag{1}$$

We may take the circle as the unit circle in the complex plane, and then each point A_i will correspond to a complex turn t_i, $|t_i|=1$. If a and c are any two complex numbers, the distance between the corresponding points is $\sqrt{\{(a-c)(\bar{a}-\bar{c})\}}$, and the conjugate of any turn is its reciprocal, $\bar{t}=1/t$. Using these two facts one deduces readily that the distance between A_j and A_r is

$$jr = \pm (t_j - t_r)\sqrt{\left(\frac{-1}{t_jt_r}\right)}.$$

The expression on the right is real, and the sign is chosen to make it positive. (This holds also for the $\pm$ signs in the next six equations.) Then

$$\prod^n [jr] = \pm \sqrt{\{(-1)^n\}} \prod^n \left[\frac{t_j - t_r}{t_j}\right].$$

Similarly

$$\prod^n [is] = \pm \sqrt{\{(-1)^n\}} \prod^n \left[\frac{t_i - t_s}{t_i}\right],$$

$$\frac{\prod^n [jr]}{\prod^n [is]} = \pm \frac{\prod^n [t_j - t_r]}{\prod^n [t_i - t_s]}. \tag{2}$$

The equations of the lines A_iA_j and A_rA_s in conjugate coordinates are*

$$x + t_it_jy = t_i + t_j \quad \text{and} \quad x + t_rt_sy = t_r + t_s;$$

and solving for x, one finds that the point of intersection of these lines corresponds to the complex number

$$\frac{t_it_j(t_r + t_s) - t_rt_s(t_i + t_j)}{t_it_j - t_rt_s}.$$

(If the lines are parallel, $t_it_j - t_rt_s = 0$, a case to be treated separately.) Then the distance between this point and the point A_r is found to be

$$r(ij:rs) = \pm \frac{(t_r - t_i)(t_r - t_j)}{t_it_j - t_rt_s}\sqrt{\left(\frac{-t_s}{t_r}\right)}.$$

Hence

$$\prod^n [r(ij:rs)] = \pm \sqrt{\{(-1)^n\}} \prod^n \left[\frac{(t_r - t_i)(t_r - t_j)}{t_it_j - t_rt_s}\right].$$

Similarly

$$\prod^n [i(ij:rs)] = \pm \sqrt{\{(-1)^n\}} \prod^n \left[\frac{(t_i - t_r)(t_i - t_s)}{t_it_j - t_rt_s}\right],$$

and

$$\text{(3)} \qquad \frac{\prod^n [r(ij:rs)]}{\prod^n [i(ij:rs)]} = \pm \frac{\prod^n [(t_r - t_i)(t_r - t_j)]}{\prod^n [(t_i - t_r)(t_i - t_s)]} = \pm \frac{\prod^n [t_j - t_r]}{\prod^n [t_i - t_s]}.$$

From (2) and (3) we have the geometrical relation

$$\frac{\prod^n [jr]}{\prod^n [is]} = \frac{\prod^n [r(ij:rs)]}{\prod^n [i(ij:rs)]}$$

for all values of $n \geqq 5$, and all choices of the four subscripts i, j, r, s from the set $1, \cdots, n$.

In exactly the same way we prove three more similar theorems, and the four theorems may be conveniently given in the form

$$\text{(4)} \qquad \frac{\prod^n [jr]}{\prod^n [is]} = \frac{\prod^n [r(ij:rs)]}{\prod^n [i(ij:rs)]} = \frac{\prod^n [j(ij:rs)]}{\prod^n [s(ij:rs)]},$$

* W. B. Carver, The conjugate coordinate system for plane Euclidean geometry, this MONTHLY, vol. 63, No. 9, Part II, 1956.

$$(5)\qquad \frac{\prod^n [js]}{\prod^n [ir]} = \frac{\prod^n [s(ij:rs)]}{\prod^n [i(ij:rs)]} = \frac{\prod^n [j(ij:rs)]}{\prod^n [r(ij:rs)]}.$$

One sees next that, for certain choices of the subscripts i, j, r, s, some of the ratios in (4) and (5) become equal to one, giving equality between the numerators and denominators of some of the fractions. In particular

(I) If $i+s=j+r$, $\prod^n [js] = \prod^n [ir]$, and therefore

$$\prod^n [s(ij:rs)] = \prod^n [i(ij:rs)], \qquad \prod^n [j(ij:rs)] = \prod^n [r(ij:rs)].$$

(I′) If $i+r=j+s$, $\prod^n [jr] = \prod^n [is]$, and therefore

$$\prod^n [r(ij:rs)] = \prod^n [i(ij:rs)], \qquad \prod^n [j(ij:rs)] = \prod^n [s(ij:rs)].$$

(II) If $i+j=r+s$, $\prod^n [js] = \prod^n [ir]$, $\prod^n [jr] = \prod^n [is]$, and therefore

$$\prod^n [i(ij:rs)] = \prod^n [j(ij:rs)] = \prod^n [r(ij:rs)] = \prod^n [s(ij:rs)].$$

(III) *If n is even* and if $i+j=2r=2s$, $\prod^n [ir] = \prod^n [jr]$, $\prod^n [is] = \prod^n [js]$, and therefore

$$\prod^n [i(ij:rs)] = \prod^n [j(ij:rs)].$$

Whenever i, j, r, s are chosen to satisfy any of the conditions (I), (I′), (II), or (III), we therefore have simpler theorems, theorems stating equality between two products instead of a proportion between four products.

Consider now the case of the pentagon. For $n=5$, any choice whatever of i, j, r, s will satisfy one of the conditions (I), (I′), or (II). If we take i, j, r, s as 1, 3, 2, 4, we have from (4)

$$(6)\qquad \frac{\prod [32]}{\prod [14]} = \frac{\prod [3(13:24)]}{\prod [4(13:24)]}.$$

Also, since condition (I) is satisfied, we have

$$(7)\qquad \prod [4(13:24)] = \prod [1(13:24)],$$

$$(8)\qquad \prod [3(13:24)] = \prod [2(13:24)].$$

From (6) and (8) it follows that

$$\frac{\prod [32]}{\prod [14]} = \frac{\prod [2(13:24)]}{\prod [4(13:24)]},$$

which proves the theorem of problem E 1411 in the form given by (1). In the

notation of the problem we have proved, not merely the proportion, but the simpler relations,

$$CQ\cdot DR\cdot ES\cdot AT\cdot BP = BQ\cdot CR\cdot DS\cdot ET\cdot AP,$$

$$DQ\cdot ER\cdot AS\cdot BT\cdot CP = AQ\cdot BR\cdot CS\cdot DT\cdot EP.$$

Figures 2 and 3 show these relations. In each of Figures 2–8, the product of the lengths of the dotted line segments is equal to the product of the solid segments.

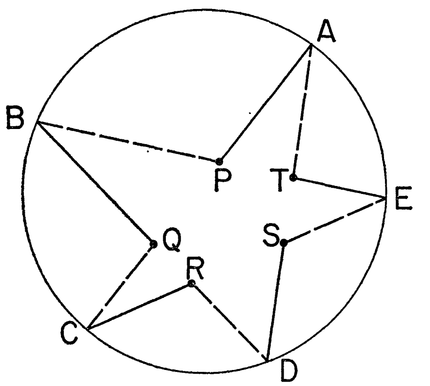

FIG. 2

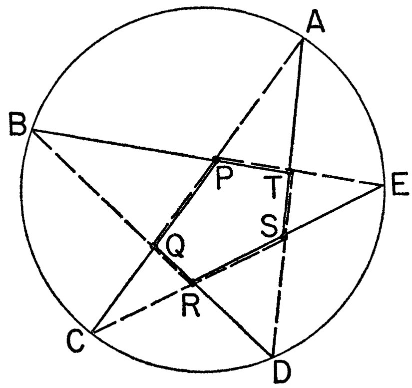

FIG. 3

For $n=5$ there are fifteen points $(ij:rs)$. If we take i, j, r, s as 1, 2, 4, 3, we have a different set of intersections (12:43), (23:54), (34:15), (45:21), and (51:32), condition (I′) is satisfied, and we get a new set of relations similar to (6), (7), and (8), but involving different sets of segments. But a different situation arises when we take i, j, r, s as 1, 4, 3, 2. We have the intersections (14:32), (25:43), (31:54), (42:15), and (53:21), condition (II) is satisfied, and we have equality of four products,

$$\prod [1(14:32)] = \prod [4(14:32)] = \prod [3(14:32)] = \prod [2(14:32)],$$

which is equivalent to three independent relations like (7) and (8). Figures 4 and 5 show the four sets of segments, Figure 4 those of $\prod[1(14:32)]$ and $\prod[3(14:32)]$, and Figure 5, those of $\prod[4(14:32)]$ and $\prod[2(14:32)]$.

We see then that when a simple pentagon has been fixed by assigning the names A_1, A_2, A_3, A_4, A_5 *in some order* to five points on a circle, we have seven different theorems similar to (7) and (8); two such theorems when i, j, r, s are so chosen that condition (I) holds, two different theorems when condition (I′) holds, and three more theorems for condition (II). (Here, and in all that follows, we are not counting the proportional relations like (6) but only the simpler relations like (7) and (8).)

In Figures 2, 3, 4, 5 the names A_i are assigned to the five points to give a *convex* pentagon. But the convex pentagon is only one of twelve different simple pentagons with vertices at these same points, and for each simple pentagon we get seven different theorems. However, we do not get a total of 84 different theorems, but only 42, due to the fact that each theorem is obtained twice. For example, in Figure 6 we use the same points on the circle as in Figure 4, but we assign the names A_i to them in a different order. Then the dotted segments $\prod[3(14:32)]$ in Figure 6 and $\prod[1(14:32)]$ in Figure 4 are the same, and similarly for the solid segments $\prod[4(14:32)]$ in Figure 6 and $\prod[3(14:32)]$ in Figure 4; and hence we do not have two different theorems, but the same theorem occurring twice.

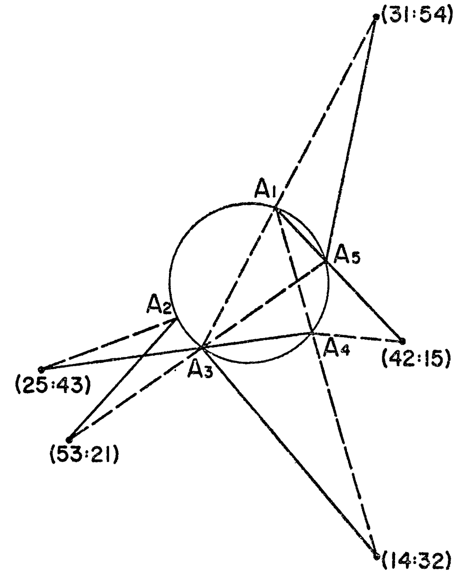

FIG. 4

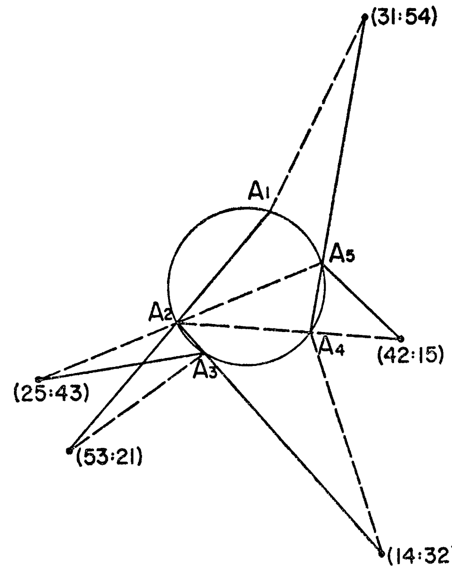

FIG. 5

Since there are $\frac{1}{2}(n-1)!$ different simple n-gons with their vertices at n fixed points on a circle, and since there are $\frac{1}{8}n(n-1)(n-2)(n-3)$ intersections $(ij:rs)$, the number of geometrical relations similar to (7) and (8) increases very rapidly with n. The whole situation is more complicated for composite values of n than for prime values.

For $n=7$ there are 360 simple heptagons with vertices at 7 fixed points, there are 105 intersections $(ij:rs)$, each geometric relation occurs *three* times, and the total number of different geometric relations appears to be 2520. The case $n=6$ presents more difficulties than the case $n=7$, but the total number of different geometric relations is 660. A pretty example is that of the convex hexagon with $(ij:rs)=(13:25)$ so that condition (III) holds. Figure 7 shows this case.

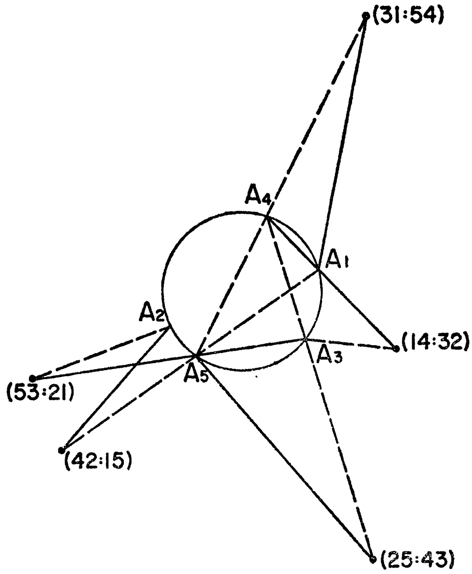

FIG. 6

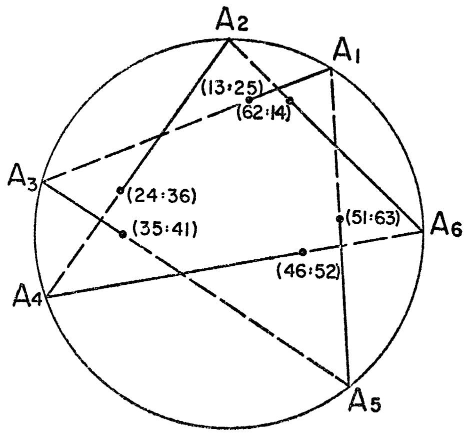

FIG. 7

It remains to consider the case where the point $(ij:rs)$ does not exist because the lines A_iA_j and A_rA_s are parallel. When such a symbol as $j(ij:rs)$ occurs in the numerator of a fraction in (4) or (5) a similar symbol, say $r(ij:rs)$, will occur in the denominator. If one starts with A_iA_j *not parallel* to A_rA_s and moves one of the four points on the circle so that the lines become more and more nearly parallel, the lengths of the segments $j(ij:rs)$ and $r(ij:rs)$ increase indefinitely, but *their ratio obviously approaches one.* Therefore the correct theorem for the parallel

case is obtained by dropping out the symbols $j(ij:rs)$ and $r(ij:rs)$ from the numerator and denominator of the fraction. If several pairs of lines are parallel, we similarly drop out the corresponding factors for each pair of parallels. As an example, take the five points A_i as the points $(-25, 0)$, $(15, -20)$, $(25, 0)$, $(-15, -20)$, and $(-7, 24)$ on the circle $x^2+y^2=625$, with lines A_1A_3 and A_2A_4 parallel and also lines A_4A_1 and A_5A_2. If we take $(ij:rs)$ as $(13:24)$ so that condition (I) holds, (8) may be written

$$\frac{3(13:24)\cdot 4(24:35)\cdot 5(35:41)\cdot 1(41:52)\cdot 2(52:13)}{2(13:24)\cdot 3(24:35)\cdot 4(35:41)\cdot 5(41:52)\cdot 1(52:13)} = 1.$$

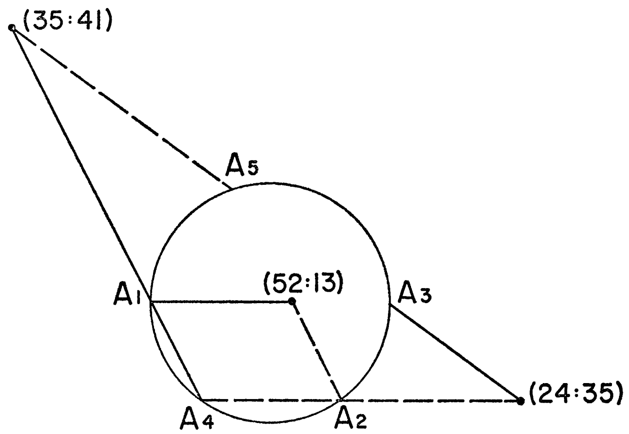

FIG. 8

If we simply drop the pairs of meaningless symbols $3(13:24)$, $2(13:24)$ and $1(41:52)$, $5(41:52)$ from the numerator and denominator, we have the simple relation

$$4(24:35)\cdot 5(35:41)\cdot 2(52:13) = 3(24:35)\cdot 4(35:41)\cdot 1(52:13).$$

Figure 8 shows the segments.

This last example might be given to a high school geometry class in the form:

Let $ABCD$ be any parallelogram with a circle through A, B, and C. Let the lines CD and AD cut the circle at E and F respectively, and let the line EF cut the line BA at G and the line BC at H. Prove that $AD\cdot FH\cdot GB=DC\cdot BH\cdot GE$.

CIRCULAR PROBABILITY PROBLEMS

WILLIAM C. GUENTHER, The Martin Company and University of Wyoming

1. Introduction. If a circle C_1 of radius R is dropped upon a fixed circle C_2 of radius D, several interesting and useful probability problems arise. Two of these involve (a) the probability that C_1 covers a randomly selected point within C_2, and (b) the probability that C_1 covers a randomly selected point on the circumference of C_2. The latter of these two, which will be referred to as the arc-length problem, will be considered in detail. The first, which might be called the area problem, is discussed in [3].

It is assumed that the center of C_1 is aimed at the center of C_2 with aiming errors being circularly normally distributed with unit standard deviation. If the center of C_2 is chosen as the origin, then the center of C_1, say (x, y), has as its probability density function

$$f(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

The density of $r = \sqrt{(x^2+y^2)}$ is, of course, $g(r) = re^{-r^2/2}$, $r > 0$.

2. The arc-length problem. For a given value of r, let $L(r)$ be the length of the arc of C_2 covered by C_1. Then for this value of r the probability sought is $L(r)/(2\pi D)$. Hence the desired probability is

$$P(R, D) = \int_0^\infty g(r) \frac{L(r)}{2\pi D} dr.$$

By using a little elementary geometry, one easily finds that

$$\text{(a)} \quad D > R: L(r) = \begin{cases} 2D \arccos\left(\dfrac{D^2 + r^2 - R^2}{2rD}\right), & D - R < r < D + R; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{(b)} \quad D = R: L(r) = \begin{cases} 2D \arccos\left(\dfrac{r}{2D}\right), & 0 < r < 2D; \\ 2D\pi, & r = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{(c)} \quad D < R: L(r) = \begin{cases} 2D \arccos\left(\dfrac{D^2 + r^2 - R^2}{2rD}\right), & R - D < r < R + D; \\ 2D\pi, & 0 \leqq r \leqq R - D; \\ 0, & \text{otherwise.} \end{cases}$$

Hence it follows that

$$P(R, D) = \frac{1}{\pi}\int_{D-R}^{D+R} re^{-r^2/2} \arccos\left(\frac{D^2 + r^2 - R^2}{2rD}\right) dr, \qquad D > R;$$

$$P(R, D) = \frac{1}{\pi}\int_{0}^{2D} re^{-r^2/2} \arccos\left(\frac{r}{2D}\right) dr, \qquad D = R;$$

$$P(R, D) = 1 - e^{-(R-D)^2/2} + \frac{1}{\pi}\int_{R-D}^{R+D} re^{-r^2/2} \arccos\left(\frac{D^2 + r^2 - R^2}{2rD}\right) dr, \qquad D < R.$$

3. Evaluation of integrals. First consider the case $D=R$ since it is the simplest of the three. Integration by parts yields

$$P(R, D) = \frac{1}{\pi}\left[\frac{\pi}{2} - \int_0^{2D} \frac{e^{-r^2/2}}{\sqrt{(4D^2 - r^2)}} dr\right].$$

Now let $r=2D \sin \frac{1}{2}\theta$. The result is

$$\begin{aligned} P(R, D) &= \frac{1}{2}\left[1 - \frac{1}{\pi}\int_0^{\pi} e^{-2D^2 \sin^2 \frac{1}{2}\theta} d\theta\right] \\ &= \frac{1}{2}\left[1 - \frac{1}{\pi} e^{-D^2}\int_0^{\pi} e^{D^2 \cos \theta} d\theta\right] = \frac{1}{2}\,[1 - e^{-D^2} I_0(D^2)], \quad D > 0, \end{aligned} \tag{1}$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero. Tables of $e^{-x}I_0(x)$ are available. Watson [4] has a table covering x from 0 to 16 in increments of .02. For larger x,

$$e^{-x}I_0(x) \cong \frac{1}{\sqrt{(2\pi x)}}\left[1 + \frac{1^2}{1!8x} + \frac{1^2 \cdot 3^2}{2!(8x)^2} + \cdots\right],$$

which is a rapidly converging series [1].

When $D \neq R$ the evaluation is more difficult. Next consider the case $D>R$. Integration by parts yields

$$P(R, D) = \frac{1}{\pi}\int_{D-R}^{D+R} \frac{e^{-r^2/2}(D^2 - R^2 - r^2)}{r\sqrt{[(2rD)^2 - (D^2 + r^2 - R^2)^2]}} dr.$$

Letting $r^2=u^2(D^2-R^2)$, the integral becomes

$$\frac{1}{\pi}\int_{k^{-1}}^{k} \frac{e^{-cu^2}(1 - u^2)du}{u\sqrt{[(k^2 - u^2)(u^2 - k^{-2})]}} = \frac{1}{2\pi}\int_{k^{-2}}^{k^2} \frac{e^{-ct}(1 - t)dt}{t\sqrt{[(k^2 - t)(t - k^{-2})]}},$$

where $k^2=(D+R)/(D-R)$, $c=\frac{1}{2}(D^2-R^2)$. This in turn may be written as two integrals

$$\frac{1}{2\pi}\int_a^b \frac{e^{-ct}dt}{t\sqrt{[(b - t)(t - a)]}} - \frac{1}{2\pi}\int_a^b \frac{e^{-ct}dt}{\sqrt{[(b - t)(t - a)]}} = F - G,$$

where b and a have replaced k^2 and k^{-2} to facilitate writing. In G the substitution $t=\frac{1}{2}(b+a)-\frac{1}{2}(b-a)\cos\theta$ yields

$$G = e^{-\frac{1}{2}c(a+b)}\cdot\frac{1}{2\pi}\int_0^{\pi} e^{\frac{1}{2}c(b-a)\cos\theta}d\theta = \tfrac{1}{2}e^{-\frac{1}{2}c(a+b)}I_0\left[\tfrac{1}{2}c(b-a)\right].$$

Next note that $dF/dc=-G$ and $F=\int_c^\infty G\,dc$. Thus

$$F = \int_c^\infty \left\{\tfrac{1}{2}e^{-\frac{1}{2}(a+b)u}I_0\left[\tfrac{1}{2}(b-a)u\right]\right\}du.$$

It is well known that

$$I_0(x) = \sum_{i=0}^{\infty}\frac{(\frac{1}{2}x)^{2i}}{(i!)^2}\,.$$

Making this replacement in F yields

$$\begin{aligned} F &= \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}(a+b)u}I_0\left[\tfrac{1}{2}(b-a)u\right]du - \frac{1}{2}\int_0^c e^{-\frac{1}{2}(a+b)u}I_0\left[\tfrac{1}{2}(b-a)u\right]du \\ &= \frac{1}{2\sqrt{(ab)}} - \frac{1}{2}\int_0^c\left\{e^{-\frac{1}{2}(a+b)u}\sum_{i=0}^{\infty}\frac{\left[\frac{1}{4}(b-a)\right]^{2i}u^{2i}}{(i!)^2}\right\}du. \end{aligned}$$

The value of the first integral may be found in any treatise on Bessel functions. In the second integral let $v=\frac{1}{2}(a+b)u$. Then

$$\begin{aligned} F &= \frac{1}{2\sqrt{(ab)}} - \frac{1}{a+b}\sum_{i=0}^{\infty}\left[\frac{b-a}{2(a+b)}\right]^{2i}\frac{(2i)!}{(i!)^2}\int_0^{\frac{1}{2}c(a+b)}\frac{v^{2i}e^{-v}}{(2i)!}dv \\ &= \frac{1}{2} - \frac{D^2-R^2}{2(R^2+D^2)}\sum_{i=0}^{\infty}\left[\frac{RD}{R^2+D^2}\right]^{2i}\binom{2i}{i}I\left(\frac{R^2+D^2}{2\sqrt{(2i+1)}},\,2i\right), \end{aligned}$$

where $I(u,\,p)$ is the incomplete gamma function which is tabulated in [**2**]. Finally

$$\begin{aligned} P(R,\,D) &= \frac{1}{2}\Bigg\{1 - e^{-(R^2+D^2)/2}I_0(RD) \\ &\quad - \frac{D^2-R^2}{R^2+D^2}\sum_{i=0}^{\infty}\left[\frac{2RD}{R^2+D^2}\right]^{2i}\left[\frac{1}{2^{2i}}\binom{2i}{i}\right]I\left(\frac{R^2+D^2}{2\sqrt{(2i+1)}},\,2i\right)\Bigg\}. \end{aligned} \tag{2}$$

Only a few changes are necessary for the case $D<R$. Integration by parts yields

$$P(R,\,D) = 1 - \frac{1}{\pi}\int_{R-D}^{R+D}\frac{e^{-r^2/2}(r^2+R^2-D^2)}{r\sqrt{[(2rD)^2-(D^2+r^2-R^2)^2]}}dr.$$

Letting $r^2=u^2(R^2-D^2)$ the result is

$$P(R, D) = 1 - \frac{1}{\pi}\int_{k^{-1}}^{k} \frac{e^{-cu^2}(1+u^2)}{u\sqrt{[(k^2-u^2)(u^2-k^{-2})]}}\,du$$
$$= 1 - \frac{1}{2\pi}\int_{k^{-2}}^{k^2} \frac{e^{-ct}(1+t)}{t\sqrt{[(k^2-t)(t-k^{-2})]}}\,dt,$$

where $c=\frac{1}{2}(R^2-D^2)$, $k^2=(R+D)/(R-D)$. Thus $P(R, D)=1-[F+G]$, where the roles of R and D have been interchanged. This reduces to (2). The term containing the infinite series is now added since $R>D$.

Although each case was handled separately, (2) holds for all three situations.

From a practical point of view $R=0$ does not make sense. On the other hand, $D=0$ is an important special case but for this situation $L(r)$ is meaningless. We note that $P(R, 0)=\Pr[r<R]=1-e^{-R^2/2}$.

4. **A sample table.** By using (2) the entries for Table 1 were prepared. Linear interpolation was used to determine the third decimal place for the incomplete gamma values. Convergence of the series is quite rapid. For values of R and D used for the table no more than nine terms were required to eliminate the contribution to the third decimal place and often considerably less was sufficient.

TABLE 1

$P(R, D)$

D \ R	.5	1.0	1.5	2.0	2.5	3.0	3.5
0	.12	.39	.68	.86	.96	.99	1.00
.5	.10	.36	.63	.83	.94	.98	1.00
1.0	.07	.27	.51	.73	.88	.96	.99
1.5	.04	.16	.36	.58	.77	.90	.96
2.0	.02	.08	.21	.40	.61	.79	.90
2.5	.01	.03	.10	.23	.42	.62	.80
3.0	.00	.01	.04	.11	.25	.43	.63
3.5	.00	.00	.01	.05	.12	.26	.44

References

1. N. W. McLachlan, Bessel Functions for Engineers, New York, 1955.
2. Karl Pearson, Tables of the Incomplete Γ-Function, New York, 1955.
3. Herbert Solomon, Distribution of the measure of a random two-dimensional set, Ann. Math. Statist., vol. 24, 1953, pp. 650–656.
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A GENERAL CHAIN RULE WITHOUT COMPONENTS FOR DERIVATIVES IN VECTOR SPACES*

J. V. LEWIS, University of New Mexico

1. Introduction. Ordinarily the chain rule for derivatives of composite functions in vector spaces must be stated in terms of partial derivatives and components. For example let **f**, **g**, and **h** be functions with the domain of **f** in m-dimensional vector space E_m, the range of **f** and domain of **g** in n-dimensional vector space E_n, and the range of **g** in p-dimensional vector space E_p. Let $\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ for every $\mathbf{x} \in E_m$. The regular chain rule is used only for real-valued functions g and h (case $p=1$) or for each component of **h** separately. It is usually written as

$$\frac{\partial h}{\partial x_r} = \sum_{k=1}^{n} \frac{\partial g}{\partial y_k} \frac{\partial f_k}{\partial x_r}, \qquad r = 1, \cdots, m;$$

or as the scalar vector product

$$\frac{\partial h}{\partial x_r} = \nabla g \cdot \frac{\partial \mathbf{f}}{\partial x_r}, \qquad r = 1, \cdots, m,$$

where **f** and h are evaluated at **x** and g at **y** where $\mathbf{y} = \mathbf{f}(\mathbf{x})$. The chain rule developed here gives the derivative of the vector-valued composite function **h** in any direction as a product of a derivative of **g** and a derivative of **f** and is capable of a simple intuitive interpretation.

2. General chain rule. Use the following notation:

$D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$ = derivative of **f** in direction **u** at **x**,
$\mathbf{v}$ = unit vector in direction $D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$,
$D_{\mathbf{v}}\mathbf{g}(\mathbf{y})$ = derivative of **g** in direction **v** at **y**,
$\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ for every **x** in domain **f**,
$D_{\mathbf{u}}\mathbf{h}(\mathbf{x})$ = derivative of **h** in direction **u** at **x**.

Under the conditions given in the theorem in Section 5,

$$D_{\mathbf{u}}\mathbf{h}(\mathbf{x}) = D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \left| D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) \right|,$$

where $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Thus the derivative of the composite **h** in any direction is the magnitude of the rate of change of **f** in that direction times the rate of change of **g** in the direction in which **f** is changing.

3. Illustrations. *Case* $m = n = p = 2$. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ and

$$\mathbf{f}(\mathbf{x}) = (-x_1, 2(x_2 + 1)x_1) \qquad \text{for every } \mathbf{x} \in E_2,$$
$$\mathbf{g}(\mathbf{y}) = (-y_2 y_1, y_1) \qquad \text{for every } \mathbf{y} \in E_2,$$

* Presented to the meeting of the Southwestern Section of the Association, April, 1960.

$$\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x})) \qquad \text{for every } \mathbf{x} \in E_2,$$
$$\mathbf{h}(\mathbf{x}) = (2(x_2+1)x_1^2, -x_1) \qquad \text{for every } \mathbf{x} \in E_2.$$

Choose $\mathbf{x}=(2, 0)$, $\mathbf{u}=(2/\sqrt{5}, 1/\sqrt{5})$. The derivative of $\mathbf{h}$ in direction $\mathbf{u}$ at $\mathbf{x}$ will be found. This is the ordinary derivative with respect to s at $s=0$ (along the line consisting of points of the form $\mathbf{x}+s\mathbf{u}=(2+(2s)/\sqrt{5}, s/\sqrt{5})$ for real s. $\mathbf{f}$ transforms this line into a parabola consisting of points of the form (Fig. 1)

$$\mathbf{f}(\mathbf{x}+s\mathbf{u}) = (-2[1+s/\sqrt{5}], 4[1+s/\sqrt{5}]^2)$$

for real s. Finally, $\mathbf{g}$ transforms the parabola into a cubic consisting of points of the form (Fig. 2)

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+s\mathbf{u})) = (8[1+s/\sqrt{5}]^3, -2[1+s/\sqrt{5}])$$

for any real s.

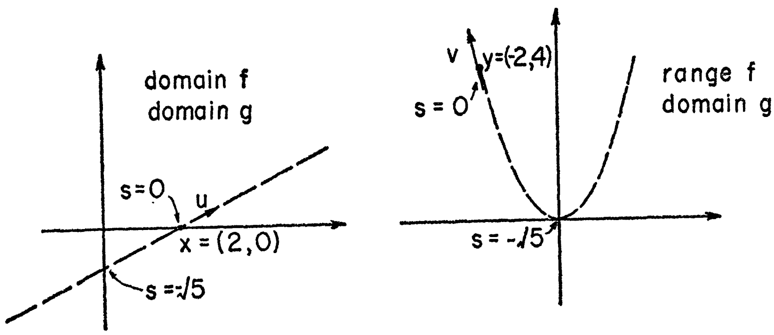

FIG. 1

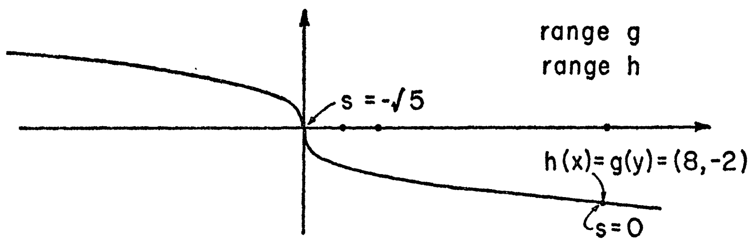

FIG. 2

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = (-2/\sqrt{5}, 8/\sqrt{5}), \qquad |D_{\mathbf{u}}\mathbf{f}(\mathbf{x})| = 2\sqrt{(17/5)};$$
$$\mathbf{v} = (-1/\sqrt{17}, 4/\sqrt{17}), \qquad D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) = (12/\sqrt{17}, -1/\sqrt{17}), \text{ where } \mathbf{y} = \mathbf{f}(\mathbf{x});$$
$$D_{\mathbf{u}}\mathbf{h}(\mathbf{x}) = D_{\mathbf{v}}\mathbf{g}(\mathbf{y})\,|D_{\mathbf{u}}\mathbf{f}(\mathbf{x})| = (12/\sqrt{17}, -1/\sqrt{17})\cdot 2\sqrt{(17/5)} = (24/\sqrt{5}, -2/\sqrt{5}),$$

which is the direction tangent to the cubic at $\mathbf{h}(\mathbf{x})=(8, -2)$.

Case $m=n=p=1$. The general chain rule takes an unusual form in one dimension because there are only two directions $+1$ and -1 and we customarily take derivatives only in the $+1$ direction. Let

$$\begin{array}{lll} f(x) &= e^{-x} & \text{for every } x \in E_1, \\ g(y) &= 2y - 1 & \text{for every } y \in E_1, \\ h(x) &= g(f(x)) & \text{for every } x \in E_1, \\ h(x) &= 2e^{-x} - 1 & \text{for every } x \in E_1. \end{array}$$

$$\begin{array}{ll} D_{+1}f(0) = -1, & f'(0) = -1; \\ D_{-1}g(1) = -2, & g'(1) = 2; \\ D_{+1}h(0) = D_{-1}g(1)\,|\,D_{+1}f(0)\,|, & h'(0) = g'(1)f'(0); \\ D_{+1}h(0) = (-2)(+1), & h'(0) = (+2)(-1); \\ D_{+1}h(0) = -2, & h'(0) = -2. \end{array}$$

The ordinary chain rule uses derivatives in only the positive direction whereas the general chain rule obtains the correct result by taking the derivative of g in the negative direction since f is decreasing.

4. Properties of directional derivatives and the differential. To facilitate proving the general chain rule we state some of the usual definitions and properties of the directional derivative and the differential.

DEFINITION. *The derivative of* $\mathbf{f}$ *in direction* $\mathbf{u}$ (*a unit vector*) *at* $\mathbf{x}$ *is*

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = \lim_{s\to 0} \frac{\mathbf{f}(\mathbf{x} + s\mathbf{u}) - \mathbf{f}(\mathbf{x})}{s}$$

whenever this limit exists.

DEFINITION. *A function* $\mathbf{f}$ *has a differential* $d\mathbf{f}_{\mathbf{x}}$ *at* $\mathbf{x}$ *provided* $d\mathbf{f}_{\mathbf{x}}$ *is a linear function and for every* $\epsilon > 0$ *there is radius* $\delta > 0$ *so that for every* $\mathbf{y}$ *for which* $0 < |\mathbf{y}-\mathbf{x}| < \delta$, *the following inequality holds*

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(\mathbf{y} - \mathbf{x})| < \epsilon |\mathbf{y} - \mathbf{x}|.$$

Properties. Let $\mathbf{f}$ *have a differential* $d\mathbf{f}_{\mathbf{x}}$ *at* $\mathbf{x}$. *Then*

(1) *If* $\mathbf{u}$ *is a unit vector in domain of* $\mathbf{f}$ *and* $\mathbf{t} = |\mathbf{t}|\mathbf{u}$, $d\mathbf{f}_{\mathbf{x}}(\mathbf{u}) = D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$, $d\mathbf{f}_{\mathbf{x}}(\mathbf{t}) = D_{\mathbf{u}}\mathbf{f}(\mathbf{x})\,|\mathbf{t}|$.

(2) *For all vectors* $\mathbf{s}$ *and* $\mathbf{t}$ *and all real numbers* a *and* b $d\mathbf{f}_{\mathbf{x}}(a\mathbf{s}+b\mathbf{t}) = a\,d\mathbf{f}_{\mathbf{x}}(\mathbf{s}) + b\,d\mathbf{f}_{\mathbf{x}}(\mathbf{t})$.

(3) *For each* $\mathbf{x}$ *there exists a constant* M *so that* $|D_{\mathbf{u}}\mathbf{f}(\mathbf{x})| < M$ *for every unit vector* $\mathbf{u}$ *in the domain of* $\mathbf{f}$.

(4) *There exists a constant* N *and a radius* $r > 0$ *so that for every* $\mathbf{y}$ *in the domain of* $\mathbf{f}$ *if* $0 < |\mathbf{y} - \mathbf{x}| < \mathbf{r}$, *then* $|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| < N|\mathbf{y} - \mathbf{x}|$.

5. THEOREM. *Let* f, g, h *be functions with domain* $\mathbf{f} \subset E_m$, *range* $\mathbf{f} \subset$ *domain* $\mathbf{g} \subset E_n$, *range* $\mathbf{g} \subset E_p$. *Let* $\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ *for every* $\mathbf{x} \in$ *domain* f. *Let* f *have a differential at a particular point* x *and* g *have a differential at* $\mathbf{y} = \mathbf{f}(\mathbf{x})$. *Let* u *be a unit vector in* E_m *and* v *be a unit vector in* E_n *in the direction of* $D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$. *Then*

$$D_{\mathbf{u}}\mathbf{h}(\mathbf{x}) = D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \, | D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) | .$$

Proof. As a preliminary, use the properties to choose a constant $M > 0$ and for each given $\epsilon_1 > 0$ choose $r > 0$ so that for every s, every z and every w:

(a) If $|s| \leqq r$, then $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})| \leqq \epsilon_1 |s|$.
(b) If $|s| \leqq r$, then $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})| \leqq M|s|$.
(c) If $|\mathbf{z}-\mathbf{y}| \leqq r$, then $|\mathbf{g}(\mathbf{z}) - \mathbf{g}(\mathbf{y}) - d\mathbf{g}_{\mathbf{y}}(\mathbf{z}-\mathbf{y})| \leqq \epsilon_1 |\mathbf{z}-\mathbf{y}|$.
(d) If w is a unit vector in E_n, $|D_{\mathbf{w}}\mathbf{g}(\mathbf{y})| \leqq M$.

It must be shown that

$$\frac{\mathbf{g}(\mathbf{f}(\mathbf{x} + s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x}))}{s} - D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \, | D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) |$$

tends to **0** as s tends to 0. Consider the product of this expression with s and later divide by s. Also note that

$$s D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \, | D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) | = d\mathbf{g}_{\mathbf{y}}(s D_{\mathbf{u}}\mathbf{f}(\mathbf{x})) = d\mathbf{g}_{\mathbf{y}}(d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})).$$

Now write

$$\begin{aligned} &\mathbf{g}(\mathbf{f}(\mathbf{x} + s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - s D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \, | D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) | \\ &\quad = \mathbf{g}(\mathbf{f}(\mathbf{x} + s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})) \\ &\quad = \mathbf{g}(\mathbf{f}(\mathbf{x} + s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x} + s\mathbf{u}) - \mathbf{f}(\mathbf{x})) \\ &\qquad + d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x} + s\mathbf{u}) - \mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})) = \mathbf{T}_1 + \mathbf{T}_2, \end{aligned}$$

where

$$\begin{aligned} \mathbf{T}_1 &= \mathbf{g}(\mathbf{f}(\mathbf{x} + s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x} + s\mathbf{u}) - \mathbf{f}(\mathbf{x})), \\ \mathbf{T}_2 &= d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x} + s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})). \end{aligned}$$

Then $|\mathbf{T}_1| \leqq \epsilon_1 |\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})|$ provided $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})| \leqq r$ by (c). By (b), $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})| \leqq M|s|$ provided $|s| \leqq r$. Choose δ as the smaller of r and r/M. Then for $|s| < \delta$, $|\mathbf{T}_1| \leqq \epsilon_1 M |s|$. Also $\mathbf{T}_2 = D_{\mathbf{w}}\mathbf{g}(\mathbf{y}) |\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})|$, where w is a unit vector in the direction of $\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})$, and $|\mathbf{T}_2| \leqq M |\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})|$ by (d). By (a) for $|s| < r$, $|\mathbf{T}_2| \leqq M\epsilon_1 |s|$. Thus,

$$\text{if } |s| \leqq \delta, \quad \text{then} \quad |\mathbf{T}_1 + \mathbf{T}_2| \leqq 2M\epsilon_1 |s| .$$

Given $\epsilon > 0$ choose $\epsilon_1 < \frac{1}{2}(\epsilon/M)$. If $0 < |s| < \delta$, then

$$| \mathbf{g}(\mathbf{f}(\mathbf{x} + s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - s D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \, | D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) | \, | = | \mathbf{T}_1 + \mathbf{T}_2 | \leqq 2M\epsilon_1 |s| < \epsilon |s|.$$

If $0 < |s| < \delta$, then, dividing by $|s|$,

$$\left| \frac{g(f(x + su)) - g(f(x))}{s} - D_y g(y) \mid D_u f(x) \mid \right| < \epsilon.$$

Thus the limit of the expression on the left hand side of the above inequality as s tends to 0 is 0. Consequently, $D_u h(x) = D_v g(y) | D_u f(x) |$.

NECESSARY AND SUFFICIENT CONDITIONS FOR PRIME PAIRS

ROBERT D. LARSSON,* Clarkson College of Technology

The author has found a new formulation for the sieve of Eratosthenes, in an attempt to gain fresh insights into some of the problems of prime numbers. In particular, it has been of use in finding new information regarding the existence of an infinite number of prime pairs. The new viewpoint for the sieve will be given first, followed by the theorems concerning prime pairs.

I. PROBLEM. *Given the set of primes $p_1, p_2, \cdots, p_n, \cdots$, where $p_1 = 2$, $p_2 = 3, \cdots$, find the primes which lie between p_n and p_n^2.*

Procedure. Compute r_i for

$$p_n \equiv r_i \pmod{p_i}, \qquad i = 2, \cdots, n - 1,$$

where r_i is the least positive residue. We see that $p_n - r_i + kp_i$ is divisible by p_i for $i = 2, \cdots, n-1$ and $k = 1, 2, \cdots$. Now form the set of even integers in the set $kp_i - r_i$. Call this set E. Let E' be the set of positive even integers not contained in E. Set E' has a least element, say b_1. Then $p_n + b_1$ is the next prime p_{n+1}.

Proof. $p_n + b_1$ is not divisible by p_i for $i = 1, \cdots, n-1$ by construction. $p_n + b_1$ is not divisible by p_n unless E' is empty, or $b_1 = p_n^2 - p_n$. But either of these possibilities would require that there be no primes between p_n and p_n^2. This would contradict Bertrand's conjecture [1] that for any positive integer n there is a prime p such that $n < p \leqq 2n$. And certainly $2n < n^2$ for $n > 2$. Now $p_n + b_i$ is a prime for all

$$b_i < p_n^2 - p_n, \qquad b_i \text{ in } E'.$$

The set $p_n + b_i$, $i = 1, 2, \cdots$, represents all primes between p_n and p_n^2, where b_i takes on all values in the set E' which are less than $p_n^2 - p_n$. This set can be ordered of course.

* National Science Foundation Faculty Fellow.

Note the suitability of this method for a program for a high-speed computer, especially for large primes.

II. From **I** we have

$$p_n \equiv r_i \pmod{p_i} \quad \text{or} \quad p_n - r_i = k_i p_i,$$

where $k_i = [p_n/p_i] \geqq 1$. Therefore $p_n - k_i p_i = r_i$ and $p_i - r_i = (k_i+1)p_i - p_n$. But $p_n + (p_i - r_i)$ is divisible by p_i for $i = 1, \cdots, n-1$, where $r_1 \equiv 1$.

Now $2 + p_n = p_{n+1}$ if and only if $p_i - r_i \neq 2$ for any i. Note that $p_1 - r_1 \equiv 1$. But if $p_i - r_i = 2$ then $k_i + 1$ is odd, *i.e.*, $k_i = 2m_i$. Therefore $k_i + 1 \geqq 3$. But $p_{n+1} < 2p_n$ by Bertrand's conjecture. But the sieve (see the first three lines of the proof of I) only requires that we test whether a number is prime or not by dividing by those primes which are less than the square root of the number. Therefore we need test as to whether $k_i = 2m_i$ for those $p_i < \sqrt{(2p_n)}$.

Therefore the necessary and sufficient conditions that for a given p_n that $p_n + 2$ is a prime are as follows:

For each p_i *such that*

$$[p_n/p_i] = 2m_i \quad \textit{and} \quad 3 \leqq p_i < \sqrt{(2p_n)},$$

then $p_i - r_i > 2$, *i.e.*, $(2m_i + 1)p_i - p_n > 2$.

COROLLARY. *If* $[p_n/p_i] = 2m_i + 1$ *for* $3 \leqq p_i < \sqrt{(2p_n)}$, *then* $p_n + 2$ *is a prime.*

The hypothesis holds for $p_n = 3, 5, 11, 17$. It fails for p_n equal to the first of a prime pair up to and including the prime 3557. Since as n becomes larger the hypothesis fails for an i relatively small compared to n, it is my conjecture that there are no other prime pairs beyond 17, 19 that satisfy the hypotheses of this corollary.

III. Using congruences, the necessary conditions can be restricted to the following basic ones, which I have not found listed elsewhere. They can of course be expanded.

For $p_n > 3$:

$$p_n \equiv 1 \pmod 3 \quad \text{or} \quad p_n \equiv 2 \pmod 3.$$

But if $p_n + 2 = p_{n+1}$, then $p_n \not\equiv 1 \pmod 3$. Therefore

$$p_n \equiv 2 \pmod 3 \quad \text{or} \quad p_n = 3m + 2.$$

But since p_n is an odd number, then $3m$ is also odd. Therefore

$$p_n = 3(2r+1) + 2 = 6r + 5. \qquad r = 0, 1, 2, \cdots$$

where $r \not\equiv 5k, 5k+3, 7k, 7k+5$. $k = 0, 1, 2, \cdots$.

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THE ROLE OF INDUSTRIAL MEMBERS IN THE MATHEMATICAL ASSOCIATION OF AMERICA*

H. O. POLLAK, Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

Mathematical activity—like all of Gaul—may be divided into three major areas: Education, Research, and Applications. Each is of great importance, and in fact three major organizations, the Association, the Society, and SIAM, concentrate in these respective areas. And yet much of the strength of the mathematical fabric comes from the interaction among these three. The interaction between education and research has traditionally been provided by common personnel at the universities; on the high school level it has been weak for many years, but has recently been conspicuous in efforts such as the School Mathematics Study Group. The interaction between research and applications has become much stronger through the growth of computing and the increased use of mathematicians in industry, as well as through a number of new graduate programs in applied mathematics at the universities. My topic today, the role of the industrial member of the Association, is one aspect of the third of these interactions, that between applications and education.

When we consider the role of the industrial member of the Association, we can look at his activity from two points of view, that of the man and of the industry which employs him on the one hand, and that of the section and of the Association as a whole on the other. First of all, what will be the attitude of the industry? The company will undoubtedly be happy to have the man participate in any and all activities of the Association. Why? The crudest answer, of course, is publicity for the industry, pure and simple. But this is far from the whole truth. The best of industry has always felt its obligations to the universities, and it has become increasingly popular for companies to give scholarships, sponsor research contracts, and the like. Recently, however, it has become clear that money is not enough; industry has begun to contribute *people* back to education, both as visiting professors and lecturers, and as participants in the study and preparation of new curricula. Perhaps this is an indication of a somewhat troubled conscience, for industry drains a great many highly trained people from education. The best of industry also encourages its mathematicians to be mathematicians in the fullest sense of the word; if the mathematician feels that participation in the Association is part of his fulfillment, more power to him.

This brings us to the man himself. The same conscience operates, if anything more strongly, in him. He too feels that he should give something of himself back directly to the educational process which has prepared him. In addition, many mathematicians in industry, whether they admit it or not, miss teaching just a little bit, and greatly enjoy the opportunity to participate in many of the Association's activities.

* Presented to the meeting of Section Officers at the Forty-first Summer Meeting of the Association, East Lansing, Michigan, August 30, 1960.

How does the Association benefit by this participation? Basically, it is as simple as this: Competent, dedicated, manpower is always welcome. There are many places where this is useful; not the least contribution to the activities of the section which the industrial mathematician can make is to give lectures at meetings. Novel applications of mathematics to, for example, economics, or communications, or military strategy, are fascinating, and make fine program material. One key point is that the depth of the mathematics required in interesting applications is frequently not so great but that the speaker can reach important results in a half hour without losing his audience along the way. Another point is that, because of his experience in explaining mathematics to his colleagues with less specialized training, the industrial mathematician should, despite his lack of teaching experience, command a quality of exposition which compares favorably to that of the average professor.

To the goal of the Association as a whole, the excellence of mathematics teaching, the industrial mathematician can also contribute in several ways. A large fraction of the people to whom mathematics is taught in college are, for instance, engineers who end up in industry. How good was their mathematical education? One way to find out is to ask the user, and the mathematician "on the spot" can often tell you a great deal about the mathematics which it is useful and important to know in his own organization. Finally, while no one knows exactly just why and how students learn mathematics, motivation from the physical world does play a part with some of them. By making known some of the exciting applications of mathematics in real situations, the industrial mathematician contributes to the enjoyment of mathematics by everyone.

A NOTE ON A GENERALIZATION OF BOOLEAN MATRIX THEORY

MICHAEL YOELI,* Syracuse University

1. Introduction. Let $Q, +, \cdot$ be an associative semiring [1], containing elements 0 and 1 with the following properties (for any $a \in Q$):

$$a + 0 = a, \qquad a \cdot 0 = 0 \cdot a = 0,$$

$$a \cdot 1 = 1 \cdot a = a, \qquad a + 1 = 1 + a = 1.$$

We shall call such a system $Q, +, \cdot$ a *Q-semiring*. Any distributive lattice L, $+, \cdot$ with 0 and 1, and especially any Boolean algebra obviously form examples of Q-semirings.

Another example, discussed by Shimbel [2] in connection with transportation and similar networks, is the following: Q is the set of nonnegative integers, together with $+\infty$; $a+b$ is defined as min (a, b), and $a \cdot b$ as the arithmetic sum

* On leave from Technion, Israel.

of a and b; $+\infty$ and the integer 0 take the role of the 0- and 1-elements, respectively.

This note discusses matrices over Q-semirings, generalizing results of Boolean and distributive lattice matrix theory [3], [4].

2. Some properties of Q-semirings. From the above definition of a Q-semiring the laws of absorption $a+a\cdot b=a$ and $a+b\cdot a=a$ follow immediately:

$$a + a\cdot b = a\cdot(1 + b) = a\cdot 1 = a,$$

$$a + b\cdot a = (1 + b)\cdot a = 1\cdot a = a.$$

Substituting 1 for b, we derive the law of additive idempotency: $a+a=a$. A Q-semiring thus forms a semilattice [5] with respect to addition, with 0 and 1 as zero and universal elements, respectively. It, therefore, forms a partially ordered system, $a \geqq b$ being defined by $a+b=a$. Evidently, $a\geqq b$ implies $xa\geqq xb$ and $ay\geqq by$ for all x, y. Repeated application of this rule leads to:

$$a_1 \cdots a_k \geqq b_1\cdot a_1 \cdots b_k\cdot a_k\cdot b_{k+1}. \tag{1}$$

3. Q-semiring matrices. Let Q, $+$, $\cdot$ be a Q-semiring, and Q_n the set of all $n\times n$ matrices $A=(a_{ij})$, $B=(b_{ij})$, $\cdots$, with elements in Q. In Q_n we define $A+B$, AB and $A\geqq B$ by:

$$A + B = (a_{ij} + b_{ij}), \qquad AB = \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right),$$

$$A \geqq B \text{ if and only if } a_{ij} \geqq b_{ij} \text{ for every } i, j.$$

$A\geqq B$ evidently implies $A+B=A$ and conversely. One immediately verifies that Q_n, $+$ forms a semilattice with the matrix 0 (all entries 0) as zero element, the matrix E (all entries 1) as universal element, and with $A\geqq B$ as the corresponding partially ordering relation. With respect to addition and multiplication, Q_n forms an associative semiring, with the matrix I (all diagonal entries 1, all others 0) as multiplicative identity. Again, $A\geqq B$ implies $XA\geqq XB$ and $AY\geqq BY$ for all X, $Y\in Q_n$.

4. Q-nets. Hohn, Seshu, and Aufenkamp [6] have introduced the general concept of a net (*i.e.*, a weighted, directed graph) and have developed matrix methods for the analysis of such nets. In this paper, which is concerned with matrix theory, we introduce the concept of a "Q-net," essentially in order to facilitate the study of Q-semiring matrices. However, the results obtained form interesting generalizations of theorems on switching nets [3], [4] and transportation networks [2].

A *Q-net* is defined as a directed graph consisting of n vertices $v_1, \cdots, v_n$, with exactly one branch b_{ij}, $i\neq j$, from each v_i to each other v_j, each branch b_{ij} being weighted by an element w_{ij} of a given Q-semiring. The matrix $C=(c_{ij})$ defined by

$$c_{ij} = \begin{cases} 1 & \text{if } i = j, \\ w_{ij} & \text{if } i \neq j, \end{cases}$$

is called the *connection matrix* of the Q-net.

A *directed path* of a Q-net of length r from v_i to v_j is defined as a sequence of r branches of the form $b_{ik_1}, b_{k_1k_2}, \cdots, b_{k_{r-1}j}$. If $i=j$ the path is called *closed*, otherwise *open*. If $i, k_1, \cdots, k_{r-1}, j$ are all distinct, the path is called *proper*. An open path, which is not proper, is called *redundant*. The *weight* w of a directed path is defined as the product of its branch weights:

$$w = w_{ik_1} \cdot w_{k_1k_2} \cdot \cdots \cdot w_{k_{r-1}j}.$$

Given any two distinct vertices v_i and v_j, $i \neq j$, of a Q-net, the *tranmission* t_{ij} from v_i to v_j is defined as the sum of the weights of all directed paths from v_i to v_j. The *transmission matrix* T of a Q-net is defined by $T = (t_{ij})$, where $t_{ii} = 1$, $i = 1, \cdots, n$, and t_{ij}, $i \neq j$, is the above defined transmission.

THEOREM 1. *Let C and T be the connection and transmission matrices, respectively, of a Q-net with n vertices. Then*

$$C^m = T \text{ for any } m \geqq n - 1. \tag{2}$$

The proof of this theorem is based on the following lemmas of which Lemma 1 is evident.

LEMMA 1. *The maximal length of any proper path is $n-1$.*

LEMMA 2. *Let w be the weight of an open redundant path p from v_i to v_j. Then there exists a proper path from v_i to v_j with weight w', such that $w' \geqq w$.*

Proof. A proper path p' from v_i to v_j may be obtained from the original redundant path p by eliminating some of its branches. Applying (1) to weights w' and w of paths p' and p we obtain $w' \geqq w$.

LEMMA 3. *The (i, j)-entry, $i \neq j$, of C^m, $m \geqq 1$, is the sum of the weights of all directed paths from v_i to v_j with length $r \leqq m$.*

Proof. The (i, j)-entry of C^m is

$$c_{ij}^{(m)} = \sum_{(k)} c_{ik_1} c_{k_1k_2} \cdots c_{k_{m-1}j}.$$

The weight of any directed path of length m will be a term of this sum. Furthermore, $C \geqq I$ implies $C^m \geqq C^r$, *i.e.*, $c_{ij}^{(m)} \geqq c_{ij}^{(r)}$, for any $r \leqq m$. It follows that $c_{ij}^{(m)}$ includes the weight of any path of length $r \leqq m$ from v_i to v_j. Conversely, any term of $c_{ij}^{(m)}$ represents the weight of a directed path of length $r \leqq m$ from v_i to v_j. Lemma 3 is thus proved.

Proof of Theorem 1. By Lemma 3 and the definition of T, $C^m \leqq T$, for any $m \geqq 1$. Now let p be a directed path from v_i to v_j, $i \neq j$, of length $r > n-1$ and weight

w. Then, by Lemma 1, p is redundant and by Lemma 2 there exists a proper path p' from v_i to v_j with weight $w' \geqq w$. By Lemma 1, p' is of length $r' \leqq n-1$. Hence, by Lemma 3, $T = C^{n-1}$.

Furthermore, $C \geqq I$, implies $C^m \geqq C^{n-1}$, for any $m \geqq n-1$. We thus have, for any $m \geqq n-1$, both $C^m \leqq T$ and $C^m \geqq C^{n-1} = T$, whence (2).

Theorem 1 generalizes similar results obtained in [2], [3], and [4]. A restricted form of this Theorem was first stated by A. G. Lunts [7].

5. Matrix theorems. We shall now use Theorem 1 to prove the following theorem on Q-semiring matrices:

THEOREM 2. *Let A be an $n \times n$ matrix over a Q-semiring with $A \geqq I$. Then $I \leqq A \leqq \cdots \leqq A^{n-1} = A^n = A^{n+1} = \cdots$.*

Proof. $A \geqq I$ implies $A^k \geqq A^{k-1}$. Because of $A \geqq I$, we may consider A as the connection matrix of a Q-net. The rest of Theorem 2 therefore follows immediately from Theorem 1.

The corresponding theorem on Boolean matrices is due to Lunts [7] (see also [3], Th. 3.2.1). Introducing the notation

$$A^* = \sum_{k=1}^{\infty} A^k,$$

we obviously have: $A \geqq I$ implies $A^* = A^{n-1}$.

THEOREM 3. *Let A and B be $n \times n$ matrices over a Q-semiring with $A \geqq I$, $B \geqq I$. Then*

$$(A + B)^* = (A^*B^*)^* = (B^*A^*)^*. \tag{3}$$

Proof. $A^* \geqq A$ and $B^* \geqq I$ implies $A^*B^* \geqq A$. Similarly $A^*B^* \geqq B$. Therefore $A^*B^* \geqq A+B$; whence

$$(A^*B^*)^* \geqq (A + B)^*. \tag{4}$$

Now $A^* \leqq (A+B)^*$ and $B^* \leqq (A+B)^*$. Therefore $A^*B^* \leqq ((A+B)^*)^2 = (A+B)^*$; whence

$$(A^*B^*)^* \leqq ((A + B)^*)^* = (A + B)^*. \tag{5}$$

Combining (4) and (5) we obtain (3). Theorem 3 is a modification of an analogous theorem of the algebra of relations [8].

6. Determinants and adjoints. We shall now restrict ourselves to commutative Q-semirings. Let A be an $n \times n$ matrix over a commutative Q-semiring. We define its determinant $|A|$ by

$$|A| = \sum a_{1h_1} \cdots a_{nh_n},$$

where the summation is taken over all permutations $(h_1, \cdots, h_n)$ of $(1, \cdots, n)$.

This definition, coinciding with the definition of permanent in ordinary matrix theory [9], generalizes the concept of Boolean determinant [10]. One easily verifies that the usual expansion methods of determinants also apply to the above definition, just as they apply to permanents in ordinary matrix theory.

The adjoint $\hat{A}$ of A is defined by $\hat{A}=(A_{ij})$, where A_{ij} is the cofactor of a_{ji} in $|A|$.

THEOREM 4. *Let A be an $n\times n$ matrix over a commutative Q-semiring with $A\geqq I$. Then $\hat{A}=A^*$.*

Proof. We consider the Q-net having A as its connection matrix. We then have to prove that the transmission matrix $T=(t_{ij})$ of the net is given by $T=\hat{A}$, *i.e.*, $A_{ii}=1$, and the transmission t_{ij} from v_i to v_j, $i\neq j$, equals A_{ij}. That $A_{ii}=1$ follows immediately from $a_{ii}=1$. Now, consider any pair (i,j), $i,j=1,\cdots,n$, with $i\neq j$.

Let H be the set of permutations

$$h=\begin{pmatrix}1 & \cdots & n\\ h_1 & \cdots & h_n\end{pmatrix}$$

with $h_j=i$. Let $h=u\cdot v\cdot\cdots$ be the decomposition of h into cyclic permutations, u having the form $u=(j,i,\cdots,j)$. Let U denote the set of all such permutations u. Writing A_h instead of $a_{1h_1}\cdots a_{nh_n}$, we have $A_h=A_u\cdot A_v\cdots$; therefore $A_u\geqq A_h$. Because $H\supseteq U$, we have

$$\sum_{h\in H}A_h=\sum_{u\in U}A_u.$$

Evidently, $\sum_{u\in U}A_u=a_{ji}t_{ij}$, whereas $\sum_{h\in H}A_h$ consists of all terms of $|A|$ containing a_{ji}. t_{ij} is, therefore, the cofactor of a_{ji} in $|A|$, *i.e.*, $t_{ij}=A_{ij}$; whence $T=\hat{A}$. Theorem 4 is thus proved. Furthermore, we have also covered the proof of the following:

COROLLARY. *Let C and T be the connection and transmission matrices of a Q-net. Then $T=\hat{C}$.*

This corollary is a generalization of a result due to Aranovich [10]. Its applicability to transportation networks, for example, is noteworthy.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

NOTE ON COMPLETE SEQUENCES OF INTEGERS

J. L. BROWN, JR., Ordnance Research Laboratory, Pennsylvania State University

Recently, Hoggatt and King [1] have defined an arbitrary sequence, $\{f_i\}_{i=1}^{\infty}$, of positive integers to be complete if, and only if, every positive integer n can be represented in the form $n=\sum_1^{\infty}\alpha_i f_i$, where each α_i is either zero or unity. Here the α_i are determined by n, although, in general, this determination is not unique; further, for each n, there exists an $N(n)$ such that $\alpha_i=0$ for $i>N(n)$.

The purpose of this note is to give a simple necessary and sufficient condition for the completeness of such number sequences and to show that the Fibonacci numbers are characterized by certain properties involving completeness. In the following, α_i and β_i will always denote quantities which are either zero or unity for each value of the subscript, and n will represent an arbitrary positive integer.

LEMMA. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of positive integers (not necessarily distinct) with $f_1=1$ and satisfying $f_{p+1}\leqq 1+\sum_1^p f_i$ for $p=1, 2, \cdots$. Then for $0<n<1+\sum_1^k f_i$, there exist $\{\alpha_i\}_{i=1}^{k}$ such that $n=\sum_1^k \alpha_i f_i$.*

Proof. The lemma obviously holds for $k=1$; assume that it holds for $k=N$. Then we must show that $0<n<1+\sum_1^{N+1} f_i$ implies the existence of $\{\beta_i\}_{i=1}^{N+1}$ such that $n=\sum_1^{N+1}\beta_i f_i$. We need only consider values of n satisfying $1+\sum_1^N f_i \leqq n<1+\sum_1^{N+1} f_i$ since the case $n<1+\sum_1^N f_i$ is covered by the induction hypothesis. Thus, $n-f_{N+1}\geqq 1+\sum_1^N f_i-f_{N+1}\geqq 0$ by assumption. If $n-f_{N+1}=0$, the conclusion follows; otherwise, $0<n-f_{N+1}<1+\sum_1^N f_i$ implies the existence of $\{\alpha_i\}_{i=1}^{N}$ such that $n-f_{N+1}=\sum_1^N \alpha_i f_i$. The lemma is immediate on transposing f_{N+1} and identifying $\beta_i=\alpha_i$ for $i=1, \cdots, N$ and $\beta_{N+1}=1$.

THEOREM 1. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a nondecreasing sequence of positive integers with* $f_1=1$. *Then* $\{f_i\}$ *is complete if and only if* $f_{p+1}\leqq 1+\sum_1^p f_i$ *for* $p=1, 2, \cdots$.

Proof. The sufficiency follows directly from the lemma. To show necessity, assume there exists $n_0\geqq 1$ such that $f_{n_0+1}>1+\sum_1^{n_0} f_i$. Then $f_{n_0+1}>f_{n_0+1}-1 > \sum_1^{n_0} f_i$ which implies that the positive integer $f_{n_0+1}-1$ cannot be represented in the form $\sum_1^{\infty} \beta_i f_i$.

Example. For the Fibonacci sequence, $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n=1, 2, \cdots)$, completeness follows immediately from the well-known result [2], $F_{p+1}<F_{p+2}=1+\sum_1^p F_i$ $(p=1, 2, \cdots)$.

COROLLARY. *If* $\{f_i\}$ *is an arbitrary nondecreasing complete sequence of positive integers, then* $f_i\leqq\phi_i$ $(i=1, 2, \cdots)$ *where* $\phi_i=2^{i-1}$.

Proof. The corollary obviously holds for $i=1$. Assuming it for $i=n$, we have $f_{n+1}\leqq 1+\sum_1^n f_i\leqq 1+\sum_1^n \phi_i=\phi_{n+1}$, which proves the assertion.

From the relation $\phi_{n+1}=1+\sum_1^n \phi_i$ it is seen that $\{\phi_i\}$ is a complete sequence; the preceding corollary then shows that $\{\phi_i\}$ is the fastest growing complete sequence in the sense that any other nondecreasing complete sequence is dominated by the $\{\phi_i\}$ termwise. The representation $n=\sum_1^{k(n)} \alpha_i\phi_i$ afforded by $\{\phi_i\}$ for an arbitrary integer n is such that $\alpha_k\alpha_{k-1}\cdots\alpha_1$ is the binary representation of n. As a consequence, representation in terms of the $\{\phi_i\}$ is unique. Furthermore, this property characterizes the sequence $\{\phi_i\}$. For let $\{f_i\}$ be a nondecreasing complete sequence of positive integers such that $\{f_i\}$ is not identical to the sequence $\{\phi_i\}$ and let k be the smallest integer such that $f_k\neq\phi_k$. By Theorem 1, $f_k\leqq 1+\sum_1^{k-1} f_i=1+\sum_1^{k-1}\phi_i=\phi_k$.

Since $f_k\neq\phi_k$, we must have $f_k<1+\sum_1^{k-1} f_i$ and hence $f_k=\sum_1^{k-1}\alpha_i f_i$ by the lemma. But the integer f_k can also be represented by the single term f_k itself and thus has two distinct representations in terms of the $\{f_i\}$.

It is of interest [1] to determine when the completeness of a sequence is left unaltered by the removal of an arbitrary term of the sequence.

THEOREM 2. *Let* $\{f_i\}$ *be a nondecreasing complete sequence of positive integers. The condition,* $f_{p+1}\leqq 1+\sum_1^{p-1} f_i$ $(p=1, 2, \cdots)$, *is necessary and sufficient for the sequence to remain complete after the deletion of an arbitrary term.*

Proof. Straightforward by application of Theorem 1 to the deleted sequence.

Example. Since $F_{p+1}=1+\sum_1^{p-1} F_i$ for the Fibonacci sequence, this sequence retains the completeness property after the deletion of an arbitrary term.

THEOREM 3. *Let* $\{f_i\}$ *be a nondecreasing complete sequence of positive integers with* $f_{p+1}\geqq 1+\sum_1^{p-1} f_i$ $(p=1, 2, \cdots)$. *Then the deletion of two arbitrary terms, say* f_μ *and* $f_\nu(\mu\neq\nu)$, *is sufficient to destroy completeness.*

Proof. Assume without loss of generality that $\mu<\nu$. The case $\mu=1$, $\nu=2$ obviously destroys completeness since every complete sequence of positive integers

must contain a term equal to unity. For $\nu>2$, $f_{\nu+1}\geqq 1+\sum_1^{\nu-1} f_i=1+\sum_1^{\mu-1} f_i+\sum_{\mu+1}^{\nu-1} f_i+f_\mu>1+\sum_1^{\mu-1} f_i+\sum_{\mu+1}^{\nu-1} f_i$. But this violates the condition of Theorem 1 for completeness which requires that each term of the deleted sequence be less than or equal to one plus the sum of all preceding terms.

Note that the Fibonacci sequence $\{F_i\}$ satisfies the hypotheses of Theorem 3 and, therefore, possesses the stated property.

The following partial converse to Theorem 3 provides the basis for a characterization of the Fibonacci numbers.

THEOREM 4. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a complete sequence of positive integers such that* (i) *deletion of an arbitrary term does not affect completeness*; *and* (ii) $f_1=f_2=1$ *and* $f_{i+1}>f_i$ *for* $i\geqq 2$.

Then, the deletion of two arbitrary terms destroys completeness if, and only if,

$$f_{p+1} \geqq 1+\sum_{1}^{p-1} f_i \qquad (p=1,2,\cdots). \tag{*}$$

Proof. Condition (*) is sufficient by Theorem 3. To show necessity, assume that the deletion of any two terms of the sequence renders the deleted sequence incomplete and that (*) is not satisfied. Then there exists $n_0\geqq 2$ such that $f_{n_0+1}<1+\sum_1^{n_0-1} f_i$. Now, consider the sequence which results from the original sequence after deleting the specific terms f_{n_0} and $f_1=1$. For $p<n_0$,

$$f_{p+1} \leqq 1+\sum_{2}^{p} f_i \qquad \text{(by (i) and Theorem 1).} \tag{1}$$

$$f_{n_0+1} < 1+\sum_{1}^{n_0-1} f_i \leqq 1+\sum_{2}^{n_0-1} f_i, \quad \text{since} \quad f_1=1. \tag{2}$$

For $p>n_0$,

$$f_{p+1} \leqq 1+\sum_{1}^{n_0-1} f_i+f_{n_0}+\sum_{n_0+1}^{p-1} f_i \qquad \text{(by (i) and Theorem 2).}$$

Thus,

$$f_{p+1} < 1+\sum_{1}^{n_0-1} f_i+\sum_{n_0+1}^{p} f_i \qquad \text{(by (ii))}$$

and

$$f_{p+1} \leqq 1+\sum_{2}^{n_0-1} f_i+\sum_{n_0-1}^{p} f_i\cdot(p>n_0). \tag{3}$$

But inequalities (1), (2), and (3) together with Theorem 1 imply that the deleted sequence is complete, in contradiction to our assumption.

Lastly, we give a characterization of the Fibonacci sequence:

THEOREM 5. *Let* $\{f_i\}_{i=1}^{\infty}$ *be an arbitrary complete sequence of positive integers having at most two elements equal and possessing the following properties*:
(*A*) *deletion of an arbitrary member of the sequence does not destroy completeness.*
(*B*) *deletion of any two members of the sequence renders the deleted sequence incomplete.*

Then if $\{g_i\}_{i=1}^{\infty}$ *denotes the sequence obtained by rearranging the terms of* $\{f_i\}$ *in nondecreasing order*, $g_i = F_i$ *for* $i = 1, 2, \cdots$, *where* $\{F_i\}$ *is the Fibonacci sequence.*

Proof. Since the stated properties are invariant with respect to reordering, the sequence $\{g_i\}_{i=0}^{\infty}$ satisfies both (A) and (B). By Theorem 2, $g_{p+1} \leqq 1 + \sum_1^{p-1} g_i$ $(p = 1, 2, \cdots)$ while by Theorem 4, $g_{p+1} \geqq 1 + \sum_1^{p-1} g_i$ $(p = 1, 2, \cdots)$. Thus

$$g_{p+1} = 1 + \sum_{1}^{p-1} g_i \qquad \text{for } p = 1, 2, \cdots \tag{4}$$

and $g_1 = 1$, since every complete sequence must contain unity. (In addition, $g_2 = 1$, and because of our assumption that the sequence has at most two identical terms, the g_i are distinct and hence increasing for $i \geqq 2$.) Noting that (4) is the recurrence relation satisfied by the Fibonacci numbers, we have $g_i = F_i$ for every $i \geqq 1$.

Certain properties related to completeness have been analyzed by Daykin [3] in connection with the introduction and study of generalized Fibonacci numbers. Unique representation of an arbitrary integer in terms of these generalized numbers is possible if the admissible form of the representation is suitably constrained; moreover, the generalized Fibonacci sequences are the only sequences possessing this property.

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A DIVISIBILITY PROPERTY OF THE BINOMIAL COEFFICIENTS*

L. CARLITZ, Duke University

Let p be prime. It is familiar that the binomial coefficients $\binom{p}{r}$ $(0 < r < p)$ are all divisible by p. It is perhaps less familiar that, for fixed n, the binomial coefficients $\binom{n}{r}$ $(0 < r < n)$ are all divisible by p if and only if n is a power of p.

Let

$$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqq n_j < p) \tag{1}$$

and put $S(n) = n_0 + n_1 + \cdots + n_k$. Schäffer ([3], Lemma 3) has proved that the

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binomial coefficients $\binom{n}{(p-1)r}(0<(p-1)r<n)$ are all divisible by p if and only if $S(n)\leqq p-1$.

We shall prove the following theorem which includes both of these results.

THEOREM. *Let p be a fixed prime and k a fixed integer such that $k|p-1$. Then the binomial coefficients $\binom{n}{kr}(0<kr<n)$ are all divisible by p if and only if*

$$S(n) \leqq k. \tag{2}$$

Proof. Lucas has proved ([1], p. 52) that

$$\binom{n}{r} \equiv \binom{n_0}{r_0}\binom{n_1}{r_1} \cdots \binom{n_k}{r_k} \pmod p, \tag{3}$$

where n_j is defined by (1) and

$$r = r_0 + r_1p + \cdots + r_kp^k \qquad (0 \leqq r_j < p). \tag{4}$$

If $k|r$ then by (4), $k|r_0+r_1+\cdots+r_k$, so that (since $r>0$), $r_0+r_1+\cdots+r_k\geqq k$. Assume that n satisfies (2), so that $n_0+n_1+\cdots+n_k\leqq k$. Thus

$$n_0 + n_1 + \cdots + n_k \leqq r_0 + r_1 + \cdots + r_k. \tag{5}$$

It follows from (5) that $r_j\geqq n_j$ for at least one j and, indeed, $r_j>n_j$ for at least one j unless $n=r$. But by (3), $r_j>n_j$ implies $\binom{n}{r}\equiv 0 \pmod p$. This evidently proves the sufficiency of the condition (2).

Next assume that $S(n)>k$. Then we can find nonnegative integers r_0, $r_1, \cdots, r_k$ such that $r_j\leqq n_j$ $(j=0, 1, \cdots k)$ and $r_0+r_1+\cdots+r_k=k$. If we put $r=r_0+r_1p+r_2p^2+\cdots+r_kp^k$ it follows that $k|r$, $0<r<n$ and $\binom{n}{r}\not\equiv 0 \pmod p$. This completes the proof of the theorem.

The condition $k|p-1$ is evidently necessary. For example, if $p=5$, $k=3$, and $n=7$, we have $S(7)=3$ but $\binom{7}{6}=7\not\equiv 0 \pmod 5$.

We remark that McCarthy [2] has applied Schäffer's lemma to determine the Bernoulli polynomials $B_n(x)$ such that $pB_n(x)$ are Eisenstein polynomials with determining prime p. References to other results on divisibility properties of binomial and multinomial coefficients will be found in L. E. Dickson's *History of the Theory of Numbers*, vol. 1, Ch. IX. Washington, 1919.

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ON THE PERIODICITY OF HOMEOMORPHISMS OF THE REAL LINE

NEILL MCSHANE,* Yale University

It is interesting to note that a complete description of the periodic homeomorphisms of the real line can be derived using only the simplest properties of monotone continuous functions.

DEFINITION. *A homeomorphism $f(\cdot)$ is periodic of period n if $f(\cdot)$ composed with itself n times $[f(f(\cdots(f(\cdot))\cdots))$ with n terms$]$ is the identity function and n is the least such integer.*

LEMMA. *Every continuous $1-1$ map of the real line onto itself is either order-preserving or order-reversing.*

Proof. Let $r<s<t$ and $f(r)>f(s)<f(t)$ where $f(\cdot)$ is a continuous $1-1$ map of the real line onto itself. By the intermediate value theorem (see *Advanced Calculus* by R. C. Buck, New York, 1956, page 41, Theorem 12) for any u such that $f(s)<u<\min(f(r),f(t))$ there exist v and w, $r<v<s$ and $s<w<t$, such that $f(v)=f(w)=u$. Since $f(\cdot)$ is $1-1$ this is impossible. Similarly $r<s<t$ and $f(r)<f(s)>f(t)$ is impossible. Thus $f(\cdot)$ is either order-preserving or order-reversing.

COROLLARY. *Every continuous $1-1$ map of the real line onto itself is a homeomorphism.*

THEOREM. *Let $f(\cdot)$ be any homeomorphism of the real line such that $f(\cdot)$ composed with itself n times, n a positive integer, is the identity function. If $f(\cdot)$ is order-preserving it is the identity function. If $f(\cdot)$ is order-reversing then its period is two and $f(f(x))=x$ for all real x.*

Proof. If $f(\cdot)$ is order-preserving and if there exists x such that $f(x)>x$, then $f(f(x))>f(x)>x$. By induction the nth iterate $f^{(n)}(x)>x$ for every positive integer n. Thus $f(\cdot)$ can not be periodic under composition. Similarly $f(y)<y$ for some y implies $f(\cdot)$ not periodic. Every periodic homeomorphism other than the identity function is thus order-reversing.

Let $f(\cdot)$ be order-reversing and periodic of period n; $f(f(\cdot))$ is also periodic with period either n or $n/2$. The composition of two order-reversing functions is order-preserving. Thus, by the first paragraph, $f(f(\cdot))$ is the identity function.

There exist homeomorphisms of period two and a method for generating them. A simple use of the intermediate value theorem shows that every homeomorphism of period two has exactly one fixed point. Therefore every such homeomorphism $f(\cdot)$ can be generated by choosing for some point b a function $g(\cdot)$ defined for all $x\geqq b$, $1-1$, monotone decreasing, continuous, and satisfying $g(b)=b$ and $\lim_{x\to\infty} g(x)=-\infty$. Then the function $f(\cdot)$ defined by $f(x)=g(x)$ for $x\geqq b$ and $f(x)=g^{-1}(x)$ for $x<b$ is a homeomorphism of period two. If $f(\cdot)$ rather than $g(\cdot)$ is originally specified, $g(\cdot)$ can be defined readily. Let b be the fixed

* It is with deep regret that the Editor has learned of the death of Neill McShane in an automobile accident, March 11, 1961.

point of $f(\cdot)$ and let $g(x)=f(x)$ for $x \geqq b$.

Examples of homeomorphisms of period two are the functions $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$, where $f(x)=b-x$ for all real x and some real constant b; $g(x)=-1+\ln(-x)$ for $x<-1$ and $-e^{x+1}$ for $x \geqq -1$; and $h(x)=-x^n$ for $x \geqq 0$ and $(-x)^{1/n}$ for $x<0$. (Here n is a *positive* constant.)

CORRECTION

A. K. Rajagopal, *On some of the classical orthogonal polynomials*, this MONTHLY, vol. 67, 1960, pp. 166–169. Equation (3) should be $(d^n/dx^n)(\exp U) = (\exp U)(d/dx)+U')^n \cdot 1$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

WHEN DOES ZERO CORRELATION IMPLY INDEPENDENCE?

ROY LEIPNIK, Michelson Laboratory, China Lake, California

1. Statistics textbooks point out that zero correlation implies independence for normal distributions, but not for distributions in general. There is a simple class G of bivariate distributions for which the subclass M with this property can be explicitly determined, where M properly contains the bivariate normals.

2. Let G be the bivariate distributions with densities of the form

$$g(x, y) = r(x\cdot y) \exp\left(-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}\right), \qquad -\infty < x, y < \infty, \tag{1}$$

where the "linking function" r has the property that $r(u)-r(-u)$ does not change sign for $u \in [0, \infty)$.

Let I, Z, M, N_ρ be subsets of G defined respectively by

I: $r(u) = C|u|^\alpha, \quad \alpha > -1$;

Z: $r(u) = r(-u)$ almost everywhere;

M: $M = (G - Z) \cup I = G - (Z - I)$;

N_ρ: $r(u) = (1-\rho^2)^{1/2}/(2\pi d) \exp \rho u/d, \quad d = \sigma_1\sigma_2, \quad |\rho| < 1$.

Clearly the distributions in I are "independent" (that is, products of independent univariate distributions). The converse is readily proved, for if r belongs to an "independent" distribution, then as is easily shown $r(xy) = \text{const. } r(x)r(y)$, whose only positive measurable solutions are of the form $r(u) = C|u|^\alpha$. The con-

dition $\alpha > -1$ comes from the integrability of the density. These yield χ distributions with $\alpha+1$ degrees of freedom.

Since the marginal densities of distributions in G are even functions, the marginal first moments are zero if they exist. Thus the covariance of a distribution in G is, if it exists,

$$\begin{aligned}\sigma_{12} &= \int\int xyr(xy) \exp\left(-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}\right) dxdy \\ &= 2\int_0^\infty \int_0^\infty xy(r(xy) - r(-xy)) \exp\left(-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}\right) dxdy.\end{aligned}$$

Since $r(u)-r(-u)$ has one sign for $u\in[0, \infty)$, $\sigma_{12}=0$ if and only if $r(u) = r(-u)$ almost everywhere. Z, then, is the set of uncorrelated distributions, and properly contains I.

The distributions in $M=(G-Z)\cup I$ have the property that zero correlation implies independence, since the only even linking functions for these distributions have the independent form.

3. Clearly N_ρ is the set of normal distributions with correlation ρ, and since for these distributions

$$r(u) - r(-u) = (1 - \rho^2)^{1/2}/(\pi d) \sinh \rho u/d,$$

it follows that $N_\rho \subset G$.

If $N = \bigcup_\rho N_\rho$, over $|\rho| < 1$, then $N\cap Z = N\cap I = N_0$, and therefore the set N of bivariate normal distributions is a proper subset of M.

4. It is easy to define a class P of densities with the desired property, yet disjoint from N. Let s be a noneven linking function for a density in $G-Z$, take $\alpha\neq 0$, and choose a and b to make $r(u) = a|u|^\alpha + bs(u)a$ linking function. Let P be the corresponding densities. Note that $P\subset G$, $P\cap Z = P\cap I$, $P\subset M$, $P\cap N = 0$.

5. A necessary and sufficient condition for independence, derived by Lancaster [1] and Sarmatov [2], and cited by the reviewer of this note, is the vanishing of generalized correlation coefficients defined in terms of multi-orthonormal functions on the marginal distributions. This completely resolves the relationship between independence and zero-correlation, but probably requires graduate-level analysis for understanding.

6. The construction of G was prompted by a pedagogical discussion with Professor R. F. Tate, University of Washington, Seattle.

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$\int \sec\theta\, d\theta$

NORMAN SCHAUMBERGER, Bronx Community College

The integral of sec θ, which is usually handled by multiplying numerator and denominator by $\sec\theta+\tan\theta$, can be treated in the following interesting manner (suggested by Jesse Douglas of The City College of New York).

The student is aware that the sec θ and tan θ are closely related. This might suggest that we take differentials of each,

$$d(\sec\theta) = \sec\theta\tan\theta d\theta, \qquad d(\tan\theta) = \sec^2\theta d\theta.$$

If we now add the two expressions and factor the right side we obtain $d(\sec\theta+\tan\theta)=(\sec\theta+\tan\theta)\sec\theta d\theta$. Dividing by $\sec\theta+\tan\theta$ and integrating, we have the desired result.

ON THE EULER—CAUCHY EQUATION

BERTHOLD SCHWEIZER, University of Arizona

One of the standard examples appearing in almost every text on differential equations is the so-called Euler-Cauchy equation,

$$(1) \qquad a_nx^n\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1x\frac{dy}{dx} + a_0y = h(x).$$

This equation is solved by showing that "the change of variable, $x=e^t$" reduces it to a linear differential equation with constant coefficients. The procedure involves some juggling of derivatives and differentials which, although easily followed, is hardly understood (even by the better students) and certainly, as far as clarity is concerned, leaves much to be desired. This need not be the case. In this note we will show how, when looked at in the proper light, the procedure whereby (1) is solved—which is in fact nothing more than a trivial, but particularly nice, consequence of the rule for the differentiation of composite functions—can be presented in a straightforward and transparent manner. We do this *à la* Menger.

LEMMA.* *If* $g=f\,exp$, *then* $exp^n\cdot\mathbf{D}^nf\,exp=\mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathrm{n}+1)g$, *for* $n=1, 2, \cdots$.

Proof. We proceed by induction. First of all,

$$\mathbf{D}g = exp\cdot\mathbf{D}f\,exp.$$

* Substitution of functions is denoted by juxtaposition, and multiplication by a dot. Thus $g=f\,exp$ means $g(\mathrm{x})=f\,(exp\,\mathrm{x})=f(e^x)$, for any number x; $\mathbf{D}^nf\,exp$ is the function whose value at x is $\mathbf{D}^nf\,(exp\,\mathrm{x})$, or $f^{(n)}(e^x)$; j is the identity function and j^n is the nth power function, *i.e.*, $j(\mathrm{x})=\mathrm{x}$ and $j^n(\mathrm{x})=\mathrm{x}^n$; and $\mathbf{D}^0f=f$. Note also that $hj=jh=h$, for any function h.

We also adhere to the typographical convention of denoting operators in boldface, functions in italic, and numbers in roman type.

Now suppose that for some integer k,

$$exp^k \cdot \mathbf{D}^k f\, exp = \mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)g.$$

Then, differentiating, we have,

$$k\cdot exp^k \cdot \mathbf{D}^k f\, exp + exp^{k+1}\cdot \mathbf{D}^{k+1} f\, exp = \mathbf{D}^2(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)g;$$

whence

$$\begin{aligned} &exp^{k+1}\cdot \mathbf{D}^{k+1} f\, exp \\ &\quad = [\mathbf{D}^2(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1) - \mathbf{k}\mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)]g \\ &\quad = \mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k})g. \end{aligned}$$

THEOREM. *If f satisfies the linear differential equation,*

$$\sum_{k=0}^{n} a_k \cdot j^k \cdot \mathbf{D}^k f = h, \tag{2}$$

where $a_0, a_1, \cdots, a_n$ are constant functions, then $g=f\, exp$ satisfies the differential equation $\mathbf{L}g=h\, exp$, where $\mathbf{L}$ is the linear differential operator with constant coefficients,

$$\mathrm{a}_0\mathbf{D}^0 + \sum_{k=1}^{n} \mathrm{a}_k[\mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)].$$

Proof. Substituting *exp* into both sides of (2) yields,

$$\sum_{k=0}^{n} a_k \cdot exp^k \cdot \mathbf{D}^k f\, exp = h\, exp$$

which, on applying the lemma to the left-hand side, reduces to the desired result.

COROLLARY. *If g is a solution of $\mathbf{L}g=h\, exp$, then $f=g\, log$ is a solution of* (2).

By replacing the exponential function with other functions, this technique of solving the Euler-Cauchy equation may be applied in numerous other cases. For example, let $g=f\, sin$, so that

$$\begin{aligned} \mathbf{D}g &= \mathbf{D}f\, sin\cdot cos, \\ \mathbf{D}^2 g &= \mathbf{D}^2 f\, sin\cdot cos^2 - \mathbf{D}f\, sin\cdot sin \\ &= \mathbf{D}^2 f\, sin\cdot(1-sin^2) - \mathbf{D}f\, sin\cdot sin. \end{aligned}$$

In this case we find that if f satisfies the differential equation,

$$(1-j^2)\cdot \mathbf{D}^2 f - j\cdot \mathbf{D}f + {}^f = h$$

then $g=f\, sin$ satisfies

$$\mathbf{D}^2 g + g = h\, sin.$$

Since *sin* and *cos* are two independent solutions of $D^2g+g=0$, it follows that *sin arcsin* and *cos arcsin*, *i.e.*, j and $\sqrt{(1-j^2)}$, are two independent solutions of $(1-j^2)\cdot D^2f-j\cdot Df+f=0$.

Further examples can be constructed at will.

A FURTHER NOTE ON δ AND ϵ

ALLAN DAVIS, University of Utah

In Classroom Notes (*A note on δ and ϵ*, this MONTHLY, vol. 67, p. 780), Professor A. H. Sprague is concerned with the instructive exercise of finding, for given $\epsilon>0$, a corresponding $\delta>0$ in order to show from the $\delta-\epsilon$-definition that for given polynomial function f, $\lim_{x\to a}f(x)=f(a)$. Professor Sprague observes an advantage in expanding $f(x)-f(a)$ in powers of $x-a$, since it is $|x-a|$ which δ restricts. The method of obtaining a suitable δ which he finally presents, however, seems less simple and direct than the one illustrated, through examples, below.

Suppose one wishes to show that $\lim_{x\to 1}(x^3+5x^2)=6$. Thus, one wishes to demonstrate that for any $\epsilon>0$, a $\delta>0$ exists such that $|x^3+5x^2-6|<\epsilon$ whenever $0<|x-1|<\delta$.

Now,

$$\begin{aligned}|x^3+5x^2-6| &= |\{(x-1)^3+3x^2-3x+1\}+5x^2-6|\\ &= |(x-1)^3+8x^2-3x-5|\\ &= |(x-1)^3+\{8(x-1)^2+16x-8\}-3x-5|\\ &= |(x-1)^3+8(x-1)^2+13x-13|\\ &\leqq |x-1|^3+|8|\,|x-1|^2+|13|\,|x-1|.\end{aligned}$$

Choose δ in two stages. First, choose $\delta<1$. Then, if n is any positive integer, when $0<|x-1|<\delta$, so is $|x-1|^n<\delta$. Thus, however, when $0<|x-1|<\delta$,

$$|x-1|^3+|8|\,|x-1|^2+|13|\,|x-1| < \delta+8\delta+13\delta=22\delta.$$

Second, choose δ also so that $22\delta<\epsilon$. Apparently, if $\delta<1$ and $\delta<\epsilon/22$, then when $0<|x-1|<\delta$, $|x^3+5x^2-6|<\epsilon$.

As a second example, to show that $\lim_{x\to -2}(-3x^4-8x+2)=-30$, one has,

$$\begin{aligned}&|(-3x^4-8x+2)-(-30)|\\ &\quad= |-3(x+2)^4+24(x+2)^3-72(x+2)^2+88(x+2)|\\ &\quad\leqq |-3|\,|x+2|^4+|24|\,|x+2|^3+|-72|\,|x+2|^2+|88|\,|x+2|.\end{aligned}$$

First, choose $\delta<1$, then if $0<|x+2|<\delta$,

$$\begin{aligned}&|-3|\,|x+2|^4+|24|\,|x+2|^3+|-72|\,|x+2|^2+|88|\,|x+2|\\ &\qquad\qquad < 3\delta+24\delta+72\delta+88\delta=187\delta.\end{aligned}$$

Second, choose δ also so that $\delta<\epsilon/187$. For δ satisfying both choices,

$|(-3x^4-8x+2)-(-30)|<\epsilon$ whenever $0<|x+2|<\delta$.

For any polynomial function f, of degree at least 1, it is apparent that the method for choosing δ which is being used in the above examples, amounts only to taking $\delta<1$ and also $\delta<\epsilon/K$ where K is the sum of the absolute values of the coefficients in the expansion of $f(x)-f(a)$ in terms of $x-a$. As it turns out, students sometimes make this observation and then interest themselves in how the coefficients may be obtained by a more elegant procedure than the pedagogically rather effective—because simple—device of "insistence and adjustment" which was used in the first example. There have been classroom instances where students have invented or recalled the method for obtaining the coefficients through successive divisions of $f(x)-f(a)$ by $x-a$. Students repeating the course, to be sure, will advocate finding the coefficients via Taylor's formula.

CORRECTION

The proof of the arithmetic mean, geometric mean inequality indicated by P. H. Diananda, University of Malaya in Singapore, on page 1007, this MONTHLY, vol. 67, is to be found in D. S. Mitrinovic, *Zbornik matematickih problema*, t.I (2nd ed.), Beograd, 1958, pp. 232–233.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

ADVANCED PLACEMENT EXAMINATION OF THE COLLEGE ENTRANCE EXAMINATION BOARD

Last month Vance and Pieters reviewed the history of the development of the advanced placement examinations and pointed out their influence on new developments in mathematics and their current importance in mathematical education. Permission has been granted by the College Entrance Examination Board to publish the questions in Section I of the mathematics examination given in 1958, and in Section II of the 1959 examination.

Mathematics, Section I, 1958

Time—1 hour

Directions: Solve the following problems, using the blank pages for scratchwork. Indicate your answers on the answer sheet. No credit will be given for anything written in the examination book. Do not spend too much time on any one problem.

1. The equation of the circle with center $(2, -1)$ and radius 3 is

(A) $x^2-4x+y^2-2y=4$ (B) $x^2+4x+y^2-2y=4$ (C) $x^2-4x+y^2+2y=4$
(D) $x^2-4x+y^2+2y=14$ (E) $x^2+4x+y^2+2y=14$

2. The equation of the line whose intercepts are twice those of the line $2x-3y-6=0$ is

(A) $2x-3y-3=0$ (B) $4x-6y-12=0$ (C) $4x-6y-6=0$
(D) $2x-3y+12=0$ (E) $2x-3y-12=0$

3. A relative maximum value of the function $y=x^3-\frac{3}{2}x^2+2$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$ (E) 2

4. If $y=2/(4+x^2)$, then $dy/dx=$

(A) $\dfrac{2}{(4+x^2)^2}$ (B) $\dfrac{2x}{4+x^2}$ (C) $\dfrac{4x}{(4+x^2)^2}$ (D) $\dfrac{-4x}{(4+x^2)^2}$ (E) $\dfrac{-2x}{(4+x^2)^2}$

5. The distance from the point (2, 2) to the midpoint of the segment joining (2, 3) with $(-4, -1)$ is

(A) $2\sqrt{2}$ (B) 3 (C) $\sqrt{10}$ (D) $3\sqrt{2}$ (E) $\sqrt{13}$

6. The value of k that makes the pair of lines

$$6x-9y=5$$
$$kx-4y=8$$

perpendicular is

(A) -6 (B) $-\dfrac{8}{3}$ (C) $\dfrac{3}{8}$ (D) $\dfrac{8}{3}$ (E) 6

7. The number of points of inflection of the curve $y=x^5+5x^3+10x+1$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E)4

8. The area between $x^2y=1$ and the x-axis from $x=\frac{1}{2}$ to $x=1$ is

(A) $\dfrac{7}{24}$ (B) $\dfrac{1}{2}$ (C) 1 (D) $\dfrac{5}{3}$ (E) 2

9. The radius, x, of a sphere increases with the time, t. What is the radius of the sphere when the surface area $(S=4\pi x^2)$ and radius are increasing at the same rate?

(A) $\dfrac{1}{8\pi}$ (B) $\dfrac{1}{\pi}$ (C) π (D) 2π (E) 8π

10. $\int_1^e dx/x=$

(A) 0 (B) 1 (C) e (D) $e-1$ (E) $\log_e x$

11. $\int_2^5 dx/\sqrt{x-1}=$

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 3

12. The slope of the curve $x^2+2xy+3y^2=3$ at the point $(2, -1)$ is

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

13. An equation of the line tangent to the curve $y=\sin 2x+3\cos 2x$ at the point on the curve where $x=0$ is

(A) $2x-y=-3$ (B) $2x-y=-1$ (C) $6x+y=1$ (D) $2x+y=3$ (E) $x-2y=-3$

14. The equation of the parabola whose focus is $(-1, 3)$ and directrix is the line $y=-1$ is

(A) $x^2+2x=8y$ (B) $x^2+4x-y=-6$ (C) $24x^2=8y$ (D) $2x^2+5y=17$
(E) $x^2+2x-8y+9=0$

15. A point moves along the x-axis according to the law: $x=3t^2-t^3$. When the acceleration is zero, the velocity is

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

16. The derivative of $\log_e(x^5)$ is

(A) $5(\log_e x^4)$ (B) $5x^4(\log_e x)$ (C) $5x^3$ (D) $5x$ (E) $\frac{5}{x}$

17. If $x=2t-3$ and $y=4-5t^2$, then dy/dx is

(A) $-20t$ (B) $-10t$ (C) $-5t$ (D) $-5x$ (E) $-5(x+3)$

18. The second derivative of $f(x)=e^x+e^{-x}$ is

(A) e^x-e^{-x} (B) $2f(-x)$ (C) $f(x)$ (D) $-f(x)$ (E) $x^2f(x)$

19. The area bounded by the parabola $x=2y^2$ and the line $x=8$ is rotated around the x-axis. The volume of the solid of revolution swept out is

(A) 8π (B) 10π (C) 12π (D) 14π (E) 16π

20. $\int_0^1 xe^{x^2}dx$ is equal to

(A) e (B) e^2 (C) $\frac{1}{2}(e-1)$ (D) $2e$ (E) 1

21. $\lim_{x\to 0}(\tan 2x)/x$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) ∞

22. At what point on the curve $y=\sqrt{4-x}$ is the tangent parallel to the line $x+2y=7$?

(A) $\left(\frac{63}{16}, \frac{1}{4}\right)$ (B) $(3, -1)$ (C) $(0, 2)$ (D) $(3, 1)$ (E) $(4, 0)$

23. One of the following statements is false with regard to the graph of $y=\{x-1\}/\{x^2(x-2)\}$. Which one is false?

(A) The x-intercept is 1.
(B) There are two vertical asymptotes.
(C) There is one horizontal asymptote.
(D) There is no y-intercept.
(E) The graph is symmetric to the y-axis.

24. The minimum value of $y=2\sqrt{x-1}$ occurs at

(A) $x=-1$ (B) $x=0$ (C) $x=1$ (D) $x=2$ (E) none of these

25. The area under the curve $y=\tan x$ between $x=0$ and $x=\pi/3$ is

(A) $\log_e 2$ (B) $\cot\left(\frac{\pi}{3}\right)$ (C) $-\cot^{-1}\left(\frac{\pi}{3}\right)$ (D) $-\log\sin\left(\frac{\pi}{3}\right)$ (E) $\frac{1}{2}$

26. If $f(x)=(1-x)^3$ and $g(x)=1/x$, then the derivative of $f(g(x))$ is

(A) $3(1-x)^{-4}$ (B) $3\left(-\frac{1}{x^2}\right)^2$ (C) $3(1-x)^2$ (D) $3\left(1-\frac{1}{x}\right)^2$ (E) none of these

27. The area bounded by the curve $x=(y-1)^2$ and the straight line $x=y+1$ is

(A) $\frac{3}{2}$ (B) $\frac{9}{2}$ (C) 9 (D) $\frac{27}{2}$ (E) 27

28. The curve $y=f(x)$ between the points where $x=a$ and $x=b$ is revolved about the line $y=1$. If $f(x)>1$ for $a\leqq x\leqq b$, the volume of the solid generated is

(A) $\pi\int_a^b [f(x)]^2dx$ (B) $\pi\int_a^b ([f(x)]^2-1)dx$ (C) $\pi\int_a^b [(f(x))^2-2f(x)+1]dx$

(D) $\pi\int_a^b (f(x)+1)^2dx$ (E) none of these

29. If $y=x^2-x$, the derivative of y^2 with respect to x^2 is

(A) $2x-1$ (B) $2x^2-3x+1$ (C) $4x^3-6x^2+2x$ (D) 2 (E) $12x^2-12x+2$

30. The area in the first quadrant next to the y-axis and bounded by that axis and the curves $y=\cos x$ and $y=\sin x$ is rotated about the x-axis. The volume of the solid generated is

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{\pi}{4}-\frac{1}{2}$ (D) $\frac{\pi^2}{2}$ (E) $\frac{\pi^2}{4}-\frac{\pi}{2}$

Mathematics, Section II, 1959

(In this section students are asked to show all of their work and to indicate clearly the methods used, because grades are based on the correctness of methods as well as on the accuracy of final answers. Some credit is given for partial solutions.)

Time—2 hours

1. Find the area of the region bounded by the curves $y=x-2$ and $y=2x-x^2$.
2. Consider a parallelogram and any point in its plane. Prove by analytic geometry that if the sum of the squares of the distances of the point from two opposite vertices of the parallelogram is equal to the sum of the squares of the distances of the point from the other two vertices of the parallelogram, then the figure is a rectangle.
3. The area in the first quadrant bounded above by the curve $y=\sin x$, below by the x-axis, and on the right by the line $x=\pi/2$ is divided into two equal parts by the line $x=c$. Find c.
4. A light is on top of a pole h feet high. A ball is dropped from a point at the same height as the light but k feet horizontally away from it. How fast is the shadow of the ball moving along the ground half a second later? Assume that the ball falls a distance $s=16t^2$ feet in t seconds.
5. Discuss the graph of $y=(x^2+1)e^x$. Consider (a) the intercepts, (b) asymptototes, (c) maxima and minima, (d) points of inflection, (e) behavior for large $|x|$, (f) symmetries. Choose convenient scales and sketch the curve.
6. The graph of
$$y=\frac{ax+b}{(x-1)(x-4)}$$
has a horizontal tangent at the point $(2, -1)$. Find a and b, and show that the function has a relative maximum at this point.
7. Given a function $f(x)$ such that $f(1)=f(2)=4$, and such that $f''(x)$ exists and is positive throughout the interval $1\leqq x\leqq 3$:
(a) What can you say about the sign of $f'(3)$?
(b) Prove your statement, stating whatever theorems you use in your proof.
8. A straight fence 100 yards long stands on a ranch. The fence is to be left standing, and part or all of it is to be used in forming a rectangular corral, using an additional 260 yards of fencing for the other three sides. Find the maximum area which can be so enclosed.
9. The area bounded by the curve $y=e^{-x}$ and the lines $y=0$, $x=0$, and $x=10$ is rotated about the x-axis. Compute to three significant figures the volume of the solid of revolution so generated. Justify any approximations which you use.
10. $P(p, q)$ is a variable point on the parabola $y=x^2$. $V(0, 0)$ is the vertex of the parabola. $R(0, r)$ is the point where the perpendicular bisector of VP intersects the y-axis. Find $\lim_{p\to 0} r$.

Teacher Training Requirements in New York

On August 26, 1960, the New York State Board of Regents substantially raised the requirements for new high school teachers. The acceptable minimum of college credits in science, mathematics, and foreign language will be doubled. Dr. James E. Allen, State Commissioner of Education, is quoted in the *New York Times* of August 27 as saying that he considered this move the most dramatic and the most significant step so far in the direction of improving education. He said that, though the new requirements were not aimed at the often controversial "methods" courses, the inevitable effect will be to give them relatively less importance and to subordinate them to the "subject" courses. For science teachers, the science requirement under the new regulations is 42 semester hours. The new certification rules apply to teachers entering the high schools in 1963, thus giving the colleges time to make the necessary curriculum changes.

(American Institute of Physics *Educational Newsletter*, Vol. III, No. 10, August 31, 1960.)

Feasibility Study in Elementary and Junior High School Science

The work of the course-content improvement programs in biology, chemistry, geology, mathematics, and physics makes it desirable that similar attention be given to science education in the elementary and junior high schools. Study is needed on problems related to the proper placement of topics and the more specialized courses and whether fields other than the traditional ones, such as the whole area of the atmospheric and earth sciences, should be included at the elementary and junior high school levels.

Because of the widespread recognition of and interest in these problems, the American Association for the Advancement of Science has undertaken this year a preliminary study of science instruction in the elementary and junior high school years, with the support of NSF. The study was directed by a small planning group of leading scientists, teachers, and school administrators. Plans included three 2- or 3-day regional conferences, each involving approximately 40 participants. In a final report recommendations were made on the kind of studies that could contribute to the improvement of science instruction in elementary and junior high schools.

(*Science Education News*, September 1960.)

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Howard Eves, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1453. (*Corrected*) [1961, 177].

Let A be the sum of the digits of a natural number N, let $B=A+N$, let A' be the sum of the digits of the number B, and let $C=B+A'$. Find N if the digits of C are those of N in reverse order.

E 1471. *Proposed by J. L. Brenner, Stanford Research Institute, Menlo Park, California*

If A, B are invertible matrices of the same dimension, it is not always possible to solve $XY=A$, $YX=B$ for X, Y; A and B must be similar, since $X^{-1}AX=B$. Under what conditions on invertible A, B, C can one solve $XY=A$, $YZ=B$, $ZX=C$ for X, Y, Z?

E 1472. *Proposed by C. H. Cunkle, Utah State University, and W. H. Leser, Franklin and Marshall College*

In a recent text, a ring R is defined as an additive Abelian group, closed and associative under multiplication and for which the distributive laws hold, and a subset M of R is defined to be an ideal provided M is closed under addition from within ($x\in M$ and $y\in M$ implies $x+y\in M$) and closed under multiplication from within and without ($m\in M$ and $r\in R$ implies $mr\in M$ and $rm\in M$). Show that with this definition an ideal M need not be a subring of R.

E 1473. *Proposed by J. L. Pietenpol, Columbia University*

Show that there are infinitely many square triangular numbers.

E 1474. *Proposed by Alvin Hausner, City College of New York*

Show that the equation $m^{n^m}=n^{m^n}$ has no solutions in positive integers with $m\neq n$.

E 1475. *Proposed by R. H. C. Newton, Berkhamsted, England*

Let OP be a radius of a right section of a perfectly reflecting circular cylinder. Let Q be any point on OP, and let a ray leave Q and reflect from the inner surface of the cylinder. Does it ever return to Q, and if so, under what conditions?

SOLUTIONS

Las Vegas Deathbed Scene

E 1441 [1960, 1028]. *Proposed by Bart Park, Michigan College of Mining and Technology*

A resident of Las Vegas, on his deathbed, held the following conversation with a friend who liked to solve problems.

"Over the past few years I have saved every silver dollar that came into my hands. When I reached one hundred I tied them in a bag. The first three hundred accumulated pretty fast and I hoped to reach a thousand, but I didn't make it. The several bags, each containing one hundred dollars, are in the closet of the next room. What I want you to do is visit my minor son on his next and each succeeding birthday and give him the number of dollars which equals his age. I figure that on your last trip you will have just used up all of the money."

"That's interesting," said his friend. "Before I see the bags let me try to

figure out how many trips will be required." After an interval the friend said, "I'll have to know your son's age." He was told. Then he said, "I now know how many trips will be required." How many will he have to make?

Solution by G. A. Kandall, Massachusetts Institute of Technology. Let t represent the number of trips required ($t \geqq 1$) and let a represent the son's present age ($0 \leqq a < 21$). Then $ta + t(t+1)/2 = 100n$, where n is an integer, $4 \leqq n \leqq 9$. It is not difficult to determine that the only possible solutions are $(t, a, n) = (25, 3, 4)$, $(25, 7, 5)$, $(25, 11, 6)$, $(25, 15, 7)$, $(35, 2, 7)$, $(25, 19, 8)$, $(40, 2, 9)$. But according to the problem, a uniquely determines t. Hence $a \neq 2$. Therefore $t = 25$.

Also solved by R. G. Albert, R. H. Anglin, J. W. Baldwin, Leon Bankoff, Merrill Barnebey, W. F. Barnett, Jeanette Bickley, D. A. Breault, R. I. Canfield, F. H. Cleveland, D. I. A. Cohen, E. C. Coolidge, Gus Di Antonio, J. W. Ellis, J. F. Foley, Robert Frandmelcrot, Michael Goldberg, L. D. Goldstone, S. H. Greene, J. E. Homer, Jr., A. R. Hyde, P. G. Kirmser, D. C. B. Marsh, D. A. Moran, S. J. Myzel, C. S. Ogilvy, Sidney Penner, Jon Petersen, J. L. Pietenpol, John Rainwater, J. R. Retherford, L. A. Ringenberg, S. W. Saunders, E. L. Spitznagel, Jr., D. C. Stevens, R. S. Strichartz, W. B. Stovall, Jr., W. C. Waterhouse, Walter Zayachkowski, and the proposer. Late solutions by Mike Brown, W. P. Cooke, Jr. and H. B. Lambert (jointly), R. B. Grafton, Glen Luchau, J. B. Muskat, C. F. Pinzka, and Guy Torchinelli.

Almost half of these solutions were incomplete or disagreed in some way with the solution published above.

The Diophantine Equation $x^2+36=y^5$

E 1442 [1960, 1028]. *Proposed by Andrzej Makowski, Warsaw, Poland*

Prove that the equation $x^2+36=y^5$ has no solution in integers x and y.

Solution by Philip Franklin, Massachusetts Institute of Technology. Suppose $x^2+36=y^5$ with x an even integer. Then y must be an even integer. Since $x=4m$ or $4m+2$, $x^2\equiv 0$ or $4 \pmod{16}$, and $x^2+36\equiv 4$ or $8 \pmod{16}$. But $y=2k$ implies $y^5\equiv 0 \pmod{16}$. Thus there is no integral solution with x even.

Suppose that $x^2+36=y^5$ with x an odd integer. Then $x=2m+1$ implies $x^2\equiv 1 \pmod 4$ and $x^2+36\equiv 1 \pmod 4$. Since $y^5\equiv 1 \pmod 4$, $y\equiv 1 \pmod 4$ and $y=4k+1$. Observe next that $x^2+4=y^5-2^5$ is divisible by $y-2=4k-1$. Hence, by a known theorem, the integer $n=y^5-2^5$ has no proper representation as a sum of two squares. That is, $n\neq a^2+b^2$ with $(a, b)=1$. But, since x is odd, $(x,2)=1$, and $n=x^2+2^2$ would give a proper representation. This contradiction shows that there is no integral solution with x odd.

The problem suggests some associated results and a conjecture. The results, which are elementary, are:

(a) $x^2+9=y^3$ has no integral solutions.

(b) $x^2+81=y^5$ has no integral solutions.

The argument, similar to that given, uses $x^2+1=y^3-2^3$ and $x^2+49=y^5-2^5$.

(c) $x^2+4=y^5$ has no integral solutions.

Here the residues modulo 11 show that $x^2+u\not\equiv y^5 \pmod{11}$ if $u\equiv 4 \pmod{11}$. For example $u=(11m\pm 2)^2$. Hence (b) is a special case of this.

The following conjecture is of a more general nature (and, if true, is probably much more difficult to prove):

The equation $x^2+b^2=y^5$ has no positive integral solutions with $(x, y)=1$ if $b<38$.

Here the bound is suggested by $41^2+38^2=5^5$. For this bound, the condition $(x, y)=1$ is necessary, in view of $4^2+4^2=2^5$ and $55^2+10^2=5^5$.

Also solved by R. H. Anglin, W. J. Blundon, Leonard Carlitz, Gus Di Antonio, Michael Goldberg, L. D. Goldstone, S. H. Greene, Alvin Hausner, D. C. B. Marsh, Jon Petersen, and the proposer. Late solutions by R. B. Grafton, J. B. Muskat, W. C. Waterhouse, and C. C. Yalavigi.

The proposer pointed out that by an argument analogous to that used above, one can show that the equation

$$x^2 + 4(2^{2n-1} + 1) = y^{2n+1}, \qquad n > 2,$$

has no solution in integers x and y.

A Property of the Tetrahedron

E 1443 [1960, 1028]. *Proposed by N. A. Court, University of Oklahoma*

If two of the four lines joining the vertices of a tetrahedron to the orthocenters of the opposite faces are coplanar, the remaining two lines are also coplanar.

Solution by John Rainwater, University of Washington. Let A and B be two vertices, H_a and H_b the opposite orthocenters, P and Q the other two vertices, A' and B' the projections of A and B on PQ. Then A, H_b, A', and also B, H_a, B' are collinear. We *assume* (which the proposer failed to do) that $A \neq H_b$, $B \neq H_a$. Then: (AH_a, BH_b coplanar)$\leftrightarrow$(A, B, H_a, H_b coplanar)$\leftrightarrow$(AH_b, BH_a coplanar) $\leftrightarrow$(AA', BB' coplanar)$\leftrightarrow$($A'=B'$)$\leftrightarrow$(AB in a plane, namely ABA', perpendicular to PQ)$\leftrightarrow$(AB perpendicular to PQ). This last condition is symmetric in the pairs (A, B) and (P, Q).

Examples are easily constructed showing that the statement may be false when $A=H_b$ or $B=H_a$.

Also solved by V. F. Ivanoff, D. C. B. Marsh and the proposer.

Editorial Note. One must not confuse the lines joining the vertices of a tetrahedron to the orthocenters of the opposite faces with the altitudes of the tetrahedron. It is known that if two altitudes of a tetrahedron are coplanar, then the other two altitudes are also coplanar (see Art. 205 of Court's *Modern Pure Solid Geometry*, (1935)).

A Matrix Identity

E 1444 [1960, 1028]. *Proposed by Lincoln Teng, Willow Run Laboratories, University of Michigan*

Let A and B, two $n\times n$ matrices, be such that A and $AB-BA$ commute. Show that $n(A^kB-BA^k)=k(A^nB-BA^n)A^{k-n}$ for any integer $k>n$.

Solution by Leonard Carlitz, Duke University. We have first, for any positive integer k,

$$A^kB - BA^k = k(AB - BA)A^{k-1}. \tag{1}$$

Indeed, (1) is obvious for $k=1$, and assuming it holds for a particular k, then

$$
\begin{aligned}
A^{k+1}B - BA^{k+1} &= A(A^kB - BA^k) + (AB - BA)A^k \\
&= kA(AB - BA)A^{k-1} + (AB - BA)A^k \\
&= (k + 1)(AB - BA)A^k.
\end{aligned}
$$

Next, if $k>n>0$, then, by (1),

$$k(A^nB - BA^n)A^{k-n} = kn(AB - BA)A^{k-1} = n(A^kB - BA^k),$$

regardless of whether n is the order of the matrices.

Also solved by R. G. Albert, D. A. Breault, G. D. Chakerian, R. W. Cottle, L. D. Goldstone, S. H. Greene, Jiang Luh, D. C. B. Marsh, F. D. Parker, W. V. Parker, Jon Petersen, J. L. Pietenpol, Thomas Porsching, D. A. Robinson, D. W. Robinson, and W. C. Waterhouse. Late solutions by T. H. Slook, D. H. Skypek, and the proposer.

Relation (1) is known and appears, e.g., on p. 61 of P. R. Halmos, *Finite-Dimensional Vector Spaces* (2nd ed.). As was pointed out by D. A. Robinson, the same proof as given above establishes the general

THEOREM *If A and B are elements in a ring $\{R, +, \cdot\}$ such that A and $AB-BA$ commute and if k and n are positive integers such that $k\geqq n$, then $n(A^kB-BA^k)=k(A^nB-BA^n)A^{k-n}$.*

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4971. *Proposed by D. J. Newman, Yeshiva University*

Does the infinite matrix take l^2 vectors into l^2 vectors?

$$
\begin{vmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6}\cdots \\
0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{3}\cdots \\
0 & 0 & 1 & 0 & 0 & \frac{1}{2}\cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdots
\end{vmatrix}
$$

4972. *Proposed by P. T. Bateman, University of Illinois*

If $P(x)$ is a polynomial with real coefficients but no zeros in (0, 1), it is known that the power series for $(1-x)^{-r-1}P(x)$ has positive coefficients for all

sufficiently large positive integers r. How large must r be taken if $P(x)=1-20x+101x^2$?

4973. *Proposed by Albert Wilansky, Lehigh University*

The sequence $\{\cos nx\}$ is topologically free on $[0, \pi]$, *i.e.*, no one of its members is uniformly approximable by a linear combination of the others. Is the same true on $[0, a]$ if $0<a<\pi$?

4974. *Proposed by P. M. Cohn, The University, Manchester, England*

Does there exist a field (commutative or not) whose multiplicative group is a free group?

4975. *Proposed by Robert Spira, Berkeley, California*

Prove or disprove: If $\operatorname{Re}(z)\geqq 1$, then for any positive integer n,

$$|z^{n+1}-1| > |z^n|\cdot|z-1|.$$

The result has been checked for $n<10$.

4976. *Proposed by H. A. Pogorzelski, American Mathematical Society, Providence, R. I.*

We employ the notation $2\upharpoonright n$ in lieu of 2^n and represent the expressions $2\upharpoonright 2$, $2\upharpoonright(2\upharpoonright 2)$, $2\upharpoonright(2\upharpoonright(2\upharpoonright 2))$, $\cdots$ by $2 \upharpoonright\!\downharpoonleft 2$, $2\upharpoonright\!\downharpoonleft 3$, $2\upharpoonright\!\downharpoonleft 4$, $\cdots$, respectively; that is, $2\upharpoonright\!\downharpoonleft n$ is defined recursively by

$$2\upharpoonright\!\downharpoonleft 0=1, \qquad 2\upharpoonright\!\downharpoonleft(n+1)=2\upharpoonright(2\upharpoonright\!\downharpoonleft n).$$

(This operation is called tetration by R. L. Goodstein.) The associative law does not hold in general for $2\upharpoonright 2\upharpoonright 2\upharpoonright\cdots\upharpoonright 2$. Prove the following conjecture: If n is an integer $\geqq 3$, then the various ways of associating within the exponent-chain $2\upharpoonright 2\upharpoonright\cdots\upharpoonright 2$ of n 2's give a set of $2\upharpoonright(n-3)$ different integral values of which the minimum is $2\upharpoonright(2\upharpoonright(n-1))$ and the maximum is $2\upharpoonright\!\downharpoonleft n$.

SOLUTIONS

Sections of a Cylinder of Revolution

3610 [1944, 243]. *Proposed by W. H. Rasche.*

Determine the position of a cylinder of revolution which cuts the vertical and horizontal planes in equal ellipses similar to a given ellipse.

Solution by C. S. Ogilvy, Hamilton College, Clinton, N. Y. If the problem is taken to mean all three coordinate planes, we note that there is in general no solution; for only one line (essentially) makes equal angles with all three, and the shape of the ellipse is not variable. Therefore we choose the two planes XY and ZY without loss of generality.

If the given ellipse has eccentricity e, then the solution is that the axis of the cylinder must have direction components $(1, k, 1)$ (to within sign,) where $k^2=(2e^2-1)/(1-e^2)$. One way to achieve this is to choose the point

$P(\sqrt{(1-e^2)}, \sqrt{(2e^2-1)}, \sqrt{(1-e^2)})$. Then $|OP|=1$, and the lengths of the projections of OP on both the XZ and the YZ planes are each e. Thus the acute angle which OP makes with each of these two planes is $\cos^{-1} e$. But, as proved in problem E-1353 [1959, 726], this is exactly the required angle.

It is interesting to observe that real k requires $e \geqq 2^{-1/2}$, where of course equality means that the axis of the cylinder meets both planes at 45°. Any less eccentric ellipse in one plane cannot be matched in the other.

A Diophantine Equation

4666 [1955, 734; 1957, 54; 1960. 1034]. *Proposed by R. Venkatachalam Iyer, Trivandrum, India*

If $T_p = p(p+1)/2$, solve in integers the equation

$$\frac{1}{T_x} + \frac{1}{T_y} = \frac{1}{T_z}\cdot$$

Comment by J. J. Schäffer, Institute of Mathematics and Statistics, Montevideo, Uruguay. The alleged proof [1960, 1034] contains an unfortunate oversight which renders it invalid. After reducing the problem to the Diophantine equation

$$(ab - c^2)^2 = c^4 + \{(a - b)^2 - 2\}c^2 + 1,$$

the solver states that the right member is a quadratic expression in c^2, and so will be a perfect square, if *and only if* its discriminant $\{(a-b)^2-2\}^2-4$ vanishes, whence $a-b=\pm 2$ or 0.

The necessity of the condition, however, does not follow, since what is required is a perfect square number rather than the square of a polynomial in c^2. Furthermore, the conclusion is false, as taking $c=3$, $a-b=5$ will illustrate (also $c=4$, $a-b=5$ or 28, etc.)

These illustrations are not, unfortunately, counterexamples to the purported lemma that $a-b$ must be 0 or ± 2. The original problem, therefore, remains only partially solved without proof of the lemma or effective counterexample.

Subset of Measure Zero of the Unit Interval

4901 [1960, 382]. *Proposed by Albert Wilansky, Lehigh University*

Let the class of subsets of measure zero of the interval [0, 1] be partially ordered by inclusion and let M be a maximal chain. Is the cardinal number of M greater than that of the continuum? Is the union of the sets in M measurable?

Solution by S. Khabbaz and G. Stengle, Lehigh University. Let A be an arbitrary subset of [0, 1]. Then A can be initially ordered *i.e.* well ordered in such a way that every segment is countable. Thus A can be expressed as the union of a chain of countable sets, hence sets of measure 0. Choosing in particular A to be nonmeasurable, the first question is answered in the negative.

Let A be a set of measure zero having the cardinality c of the continuum.

Proposition c_{64} of Sierpinski, *Hypothèses du Continu* shows that A contains the union of a chain containing more than c subsets. Granting the continuum hypothesis, such a chain contains 2^c sets.

The authors are indebted to J. A. Schatz for suggesting the reference to Sierpinski.

Trinomials in $FG(2)$

4915 [1960, 597]. *Proposed by S. W. Golomb, California Institute of Technology*

Show that if the trinomial $f(x) = x^n + x^a + 1$ over $GF(2)$, with $0 < a < n$, has any repeated factors, then $f(x)$ is a perfect square.

I. *Solution by Joe Lipman, Harvard University.* Denote the highest (lowest) even (odd) power term in a polynomial by *he*, *ho*, *le*, *lo*, respectively. Put $f(x) = P_1(x) \cdot P_2(x)$ with $P_1(x)$ a square, *i.e.*, a sum of even powers. Any factor of $f(x)$ has at least two terms, one of which is 1. Consequently $P_1(x) \cdot P_2(x)$ has at least four terms, viz., $he_1 \cdot he_2$, $he_1 \cdot ho_2$, $le_1 \cdot le_2$, $le_1 \cdot lo_2$, unless $P_2(x)$ has no odd power terms, *i.e.*, unless $P_2(x)$ is also a perfect square. The conclusion follows.

II. *Solution by C. H. Franke, Rutgers—The State University, Newark, N. J.* It is sufficient to show that $(f, f') \neq 1$ implies n, a both even. Then if f has a repeated factor, $(f, f') \neq 1$ and $f(x) = (x^{n/2} + x^{a/2} + 1)^2$.

If n, a both odd, $f'(x) = x^{n-1} + x^{a-1}$ and $f(x) + x \cdot f'(x) = 1$ so that $(f, f') = 1$. If n is odd and a even, $f'(x) = x^{n-1}$ so that $(f, f') = 1$ since $f(0) \neq 0$: and similarly for n even, a odd.

Generalization is easy.

Also solved by W. J. Blundon, L. Carlitz, R. A. Cuninghame-Green, J. B. Kelly, A. G. Konheim, D. C. B. Marsh, H. F. Mattson, K. A. Post, R. F. Rinehart, and the proposer.

Relations Implying that two Matrices Be Normal

4916 [1960, 597]. *Proposed by Olga Taussky, California Institute of Technology*

Fuglede's theorem (*Proc. Nat. Acad. Sci.*, 36 (1950), 35–40) specialized to finite matrices over the complex numbers states: for A normal the relation $AB = BA$ implies $A^*B = BA^*$. Putnam's generalization (*Amer. Journ. Math.*, 73 (1951), 357–362) states: if A_1, A_2 are normal and if $A_1B = BA_2$, then $A_1^*B = BA_2^*$.

Find, conversely, a necessary and sufficient condition on a nonsingular n by n matrix B such that the relations $A_1B = BA_2$ and $A_1^*B = BA_2^*$ imply that A_1 and A_2 are normal.

Solution by the proposer. The condition is that B^*B does not have any multiple characteristic root. For, let this condition hold. Then multiply $A_1B = BA_2$ on the left by B^*. This implies

$$B^*A_1B = B^*BA_2 = A_2B^*B.$$

Let U be a unitary matrix which transforms B^*B into diagonal form. Since the elements in this diagonal matrix are different, A_2 must transform to diagonal form simultaneously. This implies that A_2 is normal. Similarly, it follows that A_1 is normal.

To show that the condition is necessary take a nonsingular matrix B which has at least one multiple characteristic root. A not-normal matrix A_2 can be determined such that $B^*BA_2 = A_2B^*B$. Then determine A_1 such that $A_2B^* = B^*A_1$. This implies $A_2B^*B = B^*A_1B = B^*BA_2$. Hence $A_1^*B = BA_2^*$ and $A_1B = BA_2$.

Two Definite Integrals

4917 [1960, 699]. *Proposed by L. Lewin, Enfield, Middlesex, England*

Prove

$$\int_0^{\pi/6} \log^2(2\sin x)dx = 7\pi^3/216. \tag{1}$$

$$\int_0^{\pi/6} x\log^2(2\sin x)dx = 17\pi^4/25920. \tag{2}$$

Solution by W. H. J. Fuchs, Cornell University. Let C be the closed curve obtained by stringing together

$$\begin{aligned} &C_1\colon\ z = x && 0 \leqq x \leqq 1;\\ &C_2\colon\ z = e^{i\theta} && 0 \leqq \theta \leqq \pi/6;\\ &C_3\colon\ z = e^{it/2}(2\cos t)^{1/2} && \pi/3 \leqq t \leqq \pi/2, \end{aligned}$$

with a small indentation at $z=1$ to exclude $z=1$ from the interior of C. By Cauchy's theorem,

$$\begin{aligned} 0 &= \int_C \log^2(1-z^2)dz/z\\ &= \int_0^1 \log^2(1-x^2)dx/x + i\int_0^{\pi/6}\{\log|2\sin\theta| + i(\theta-\pi/2)\}^2 d\theta\\ &\quad + \int_{\pi/3}^{\pi/2}\{\log(-e^{2it})\}^2\left(\frac{i}{2} - \frac{1}{2}\tan t\right)dt. \end{aligned}$$

Taking imaginary parts:

$$0 = \int_0^{\pi/6}\log^2(2\sin\theta)d\theta - \int_0^{\pi/6}(\theta - \tfrac{1}{2}\pi)^2 d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/2}(2t-\pi)^2 dt;$$

$$\int_0^{\pi/6}\log^2(2\sin x)dx = 7\pi^3/216.$$

In the same way, equating to 0 the real part of $\int_C \log^3 (1-z^2)dz/z$:

$$0 = \int_0^1 \log^3 (1 - x^2)dx/x + \int_0^{\pi/6} \left\{-3 \log^2 (2 \sin \theta) \cdot \left(\theta - \frac{\pi}{2}\right) + \left(\theta - \frac{\pi}{2}\right)^3\right\} d\theta$$
$$+ \frac{1}{2}\int_{\pi/3}^{\pi/2} (2t - \pi)^3 dt.$$

$$\int_0^1 \log^3 (1 - x^2)dx/x = \frac{1}{2}\int_0^1 (1 - y)^{-1} \log^3 y \cdot dy = \frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 y^n \log^3 y \cdot dy$$
$$= -\frac{1}{2}\sum_{k=1}^{\infty} 6k^{-4} = -\pi^4/30.$$

Since now all other terms are known, we find

$$\int_0^{\pi/6} \theta \log^2 (2 \sin \theta)d\theta = 17\pi^4/25920.$$

Note by P. J. de Doelder, Technical University, Eindhoven, Netherlands. The proposer's solution can be found in his *Dilogarithms and Associated Functions*, McDonald, London (1958). The first of the proposed integrals equals $-\frac{1}{2}Ls_3(\pi/3)$ by (6.57), and by a formula on p. 152: $Ls_3(\pi/3) = -7\pi^3/108$, which gives (1). Formula (7.72), p. 185, gives

$$Ls_4^{(1)}(\pi/3) = -\int_0^{\pi/3} y \log^2 (2 \sin \tfrac{1}{2}y)dy = -17\pi^4/6480,$$

from which (2) follows upon substitution of $y = 2x$.

Ab-integration in a Ring

4920 [1960, 700]. *Proposed by Smbat Abian, University of Pennsylvania*

An Ab-integration in a ring A is a mapping $a \to a^*$ of A into itself satisfying the following conditions:

$$(a + b)^* = a^* + b^*, \qquad (a^*b + ab^*)^* = a^*b^*, \qquad \text{for any pair } a, b \in A.$$

Is there a nontrivial Ab-integration in the field of real numbers?

Solution by D. C. Stevens, Cornell University. There is. Set $1^* = s$, and $(1/1^*)^* = t$, where s and t are algebraically disjoint over the rationals.

From the given first condition, it is easy to verify that $(rx)^* = rx^*$ for r a rational. Further, $(1 \cdot 1^* + 1^* \cdot 1)^*$ gives $2s^* = s^2$ and by induction, $(s^n)^* = s^{n+1}/(n+1)$ for integral $n \geqq 0$; also $(t/s)^* = t^2/2$.

Hence $s \leftrightarrow x$ (the usual variable in analysis), $t \leftrightarrow \log x$, and * corresponds to integration, is an isomorphism between certain functions and certain reals. For reals y which are not polynomials in s and t we may set $y^* = 0$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of The Association.

An Introduction to the Theory of Numbers. By Ivan Niven and H. S. Zuckerman. Wiley, New York, 1960. viii+250 pp. $6.25.

This is a well-written, carefully organized text for a first course in number theory at the senior or graduate level. There are chapters on divisibility, congruences, quadratic reciprocity, number-theoretic functions, diophantine equations, Farey fractions, continued fractions, distribution of primes, algebraic numbers, partitions and density of sequences. The content of the final three chateers is not ordinarily to be found in a book at this level. There are in fact many unusual and praiseworthy sections throughout the book, such as those conneciing the notions of elementary congruence theory and of modern algebra, that giving a proof of Legendre's necessary and sufficient conditions that $ax^2+by^2+cz^2=0$ have a nontrivial solution, and those on numerical computations. The collection of problems is an outstanding feature; there are over 500, carefully selected to illustrate the text or to lead the student in new directions. In summary, this should remain one of the standard texts in the subject for many years.

W. J. LEVEQUE
University of Michigan

Modern Probability Theory and its Applications. By Emanuel Parzen. Wiley, New York, 1960. xv+464 pp. College $9.00, Trade $10.75.

The book is designed primarily as a text for a basic probability course of a semester's duration, on the junior-senior level. A somewhat shorter course may be taught out of the book for students with only a year of calculus as background. On the other extreme, there are two chapters at the end of the book which are of a more sophisticated nature than normally taught in undergraduate courses.

The book starts with fundamental definitions, and works its way—at a rather leisurely pace—through discrete spaces, to random variables in one and n dimensions. It treats all the usual topics in a probability course, ending with a careful treatment of sums of independent random variables and limit theorems for sequences of random variables. The author places great emphasis on clarity of presentation, perhaps too much emphasis.

The book's greatest asset is the excellent collection of illustrative examples and varied exercises. They are selected from a wide variety of fields of interest to the pure mathematician or to scientists. While many of these are well known, there are also a pleasing number of new applications of probability theory.

The author has also been wise in his choice of when to prove theorems, and which theorems to state without proof. In many cases theorems are stated and proved in a form sufficiently strong for most applications and yet sufficiently far from the most general known theorem to allow elementary treatment.

The book's weakest part is the first chapter. It has become customary in probability books to leave a few basic terms undefined. While the author is to be commended for attempting to fill this gap, his efforts at defining such terms as "random phenomenon" and "probability" are not very successful. For example, according to Parzen's definition a sequence that alternates is a random phenomenon. However, if we observe it only every second time, it is not a random phenomenon. This leads to great difficulties.

Equally troublesome, from the point of view of applications, is the fact that one does not know whether an event is a random event until one has observed it infinitely often (since it is random only if the frequency of occurrence tends to a limit). Thus we can never know in practice whether probability theory is applicable to a sequence of experiments. If one reads these definitions literally, and if the universe should happen to have a finite future, then probability theory is in principle inapplicable. However, the value of the book is not destroyed if these early definitions are ignored.

The book may prove itself valuable as a college text, especially for prospective scientists and engineers. It will certainly be a very valuable reference work for applied mathematicians.

JOHN G. KEMENY
Dartmouth College

The Foundations of Mathematics. By Evert W. Beth. North-Holland, Amsterdam, 1959. xxvi+741 pp. $13.25.

Work in the foundations of mathematics has passed out of the realm of pure philosophy and logic into the domain of the mathematician. This is not to say that philosophical issues are unimportant, but rather to underline the increasing importance of the work being done in foundations for the mathematician as such. Indeed, as Professor Beth points out, there is a tendency for certain areas of work in foundations to be considered apart from their relevance to the broad problem of a precise and well-formulated foundation for mathematics. I need but mention two areas of importance to illustrate what I mean—axiomatic set theory and recursive functions.

The increasing amount of research in this interesting and important area calls for a book that may serve as a comprehensive survey of the field. The present book provides precisely what is needed, for it gives an over-all view of the development of foundation theory as well as brief accounts of recent results. The book begins with a short historical section, after which the number system is constructed on the basis of the form of a deductive system. Metamathematical questions lead to the analysis of the concept of existence in mathematics. Set theory, its applications to completeness theorems and models, is considered after

a survey of the paradoxes. The decision problem and the related theory of recursive functions are extensively developed. The final section is devoted to philosophical considerations expressing Professor Beth's views. The bibliography is quite good and should serve as an excellent guide to students seeking additional sources. An Index of Authors, an Index of Subjects, and a detailed Table of Contents make the material in the book readily accessible.

LOUIS O. KATTSOFF
Boston College

Classical Dynamics. By R. H. Atkin. Wiley, New York, 1959. ix+273 pp. $5.25.

This book has as its avowed aims those of teaching the student dynamics and to answer examination questions—English examination questions, that is. The reviewer, like the author, does not believe these aims to be mutually exclusive, but unlike the author (apparently), does not believe this textbook suitable to their attainment. The book is traditional nineteenth century dynamics at its most pedestrian level. Relieving features are the use of vector notation and occasional recourse to matrices. It largely neglects the physical aspects of dynamics, but does not have the saving grace of setting forth with any notable clarity or precision the mathematical structure of the subject. It cannot be said, however, that problem solving is neglected. On the contrary, one gets the impression that the treatise consists of solved examples with a thin, perfunctory connective tissue of discussion of general principles. For example: there are four central chapters labelled *Kinetics of particle motion* which consist of 43 pages of worked-out examples and 15 pages of theory. (In fairness, it should be said that the examples frequently do embody portions of what may be called theory.) Every chapter is concluded with a large number of exercises taken from Tripos papers and Love's *Theoretical Mechanics.*

The reviewer feels that this approach to dynamics is conducive neither to real understanding nor to the excitation of an interest in this discipline. Indeed, the bright mathematics student may be so repelled as to develop a lasting aversion for all applications of mathematics and so to exacerbate the all too existent applied-pure mathematics dichotomy present in Western Europe and America.

WILLIAM H. PELL
National Bureau of Standards

Pure Mathematics. By F. Gerrish. Vol. I, *Calculus.* xxv+1 to 361+23 pp. Vol. II, *Algebra, Trigonometry, Coordinate Geometry.* xxviii+363 to 758+25 to 48 pp. Cambridge University Press, New York, 1960. Vol. I, $5.00, Vol. II, $6.50.

These are British books intended to prepare the student for a certain degree at the University of London without being "cram" books. It would be out of place for an American reviewer writing for American readers to attempt to

judge how well they fulfill their purpose. Rather it is appropriate to describe the books and to comment on their usefulness in American situations.

They are not "cram" books. They are sophisticated and rigorous. Their conciseness and extensive coverage remind one of textbooks like Fine's *College Algebra*, long out of style in America. Their content is of the traditional type. Set theory and Modern Mathematics with capital M's are definitely missing.

One can hardly imagine these books being used as texts in America, even in courses with the same titles as the books. In this country some of the content of Volume II would be placed before Volume I, some would be placed after and some would not be used at all. Volume I covers approximately the content of a traditional American first course in the calculus including partial derivatives and a long chapter on differential equations, but excluding series and multiple integrals. Volume II contains much equation theory including complex numbers, determinants, and Cramer's rule, and a long, thorough chapter on series. Also in this volume a long course in coordinate geometry with much work on conics precedes the final chapter on spherical trigonometry and follows one on De-Moivre's theorem applied to roots of equations and to series with complex terms.

A bright student could use these books as sources of enrichment or a teacher could use them for reference. For instance, Volume I contains such materials as the proof that the arithmetic mean is always equal to or greater than the geometric mean, page 8, Euler's constant, page 123, the use of the binomial theorem to show that, as n increases without bound, $(1+1/n)^n$ has a limit between $2\frac{1}{4}$ and 3, page 57, the distinction between derivation and differentiation, page 89. Volume II contains such goodies as a neat proof of Ceva's and Menelaus's theorems by coordinate geometry, page 574, length of common perpendicular between two skew lines, page 728, many relations between circular and hyperbolic functions, pages 557–60. The problem of the circle orthogonal to three circles in all its cases is efficiently disposed of in one half of page 592.

These books could also be used for review by students already somewhat familiar with the material. The author claims that they are suitable for students working alone. The learning helps he gives are (a) an introductory review chapter in each volume, (b) helpful warnings of errors easy to make, (c) expositions of fine distinctions not mentioned in many texts and (d) conciseness and clarity in writing. Missing are the long heuristic discussions which help build concepts in modern American texts.

Mechanical features of the books worth mentioning are (a) use of different kinds of type and starred material for greater flexibility, (b) use of lower case delta instead of upper case delta in designating increments, (c) consecutive numbering of pages through the two volumes as if they were one, (d) designation of answers to odd-numbered problems at the backs of the books by page number as well as by section number, (e) long, useful indexes.

ANICE SEYBOLD
North Central College

Handbook of Laplace Transformation. By Floyd E. Nixon. Prentice-Hall, Englewood Cliffs, N. J., 1960. ix+115 pp. $4.50.

Nearly three-sevenths of the book is devoted to " . . . the most modern and extensive table of transform pairs available today, all number-coded for quick and easy reference." These give the inverse transforms of proper rational fractions. Besides derivations of relationships and illustrations of applications to ordinary, linear, differential equations the mathematics usually encountered is clearly illustrated. Although not advanced, the book should prove to be helpful.

Formula (3.10–13) should have the summation sign in the denominator. Assuming that $f(t-a)$ on pages 30 and 65 is automatically zero for $t<a$ is unconventional, as is the use of $\int_{0+}^{t} f(t)dt$ for $f^{(-1)}(t)$ on pages 52 and 65.

EARL LAFON
University of Oklahoma

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor J. D. Mancill, University of Alabama, represented the Association at the inauguration of Dean Wallace Colvard as President of Mississippi State University on April 15, 1961.

Professor G. M. Merriman, University of Cincinnati, represented the Association at the inauguration of Dr. J. M. Read as President of Wilmington College on April 30, 1961.

Professor C. T. Salkind, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Dr. J. R. Everett as Chancellor of the Municipal College System of the City of New York on April 24, 1961.

Emory University: Assistant Professor E. E. Grace will be on leave for the summer of 1961 and the academic year 1961–62 at the University of Wisconsin as a National Science Foundation Science Faculty Fellow; Professor Tomlinson Fort, University of Miami, has been appointed Professor; Dr. Mary Neff has been appointed Assistant Professor; Mr. Marshall Saade has been appointed Instructor.

Mr. A. A. Benvenuto, System Development Corporation, Santa Monica, California, has accepted the position of Operations Research Analyst at the System Development Corporation, Falls Church, Virginia.

Mr. F. L. Coling, Raytheon Manufacturing Company, Bedford, Massachusetts, has accepted a position as Engineer with the Douglas Aircraft Company, Missile Division, Santa Monica, California.

Mr. W. P. Durbin has accepted a position as Senior Aerosystems Engineer with Convair, Fort Worth, Texas.

Dr. T. C. Fry, Remington Rand, New York, New York, has accepted a position with the National Center for Atmospheric Research, Boulder, Colorado.

Mr. Peter Lees, Onelunga High School, Auckland, SES, New Zealand, has accepted a position as Quality Control Officer with the Waitomo Portland Cement Company, Te Kuiti, New Zealand.

Mr. Joaquin Loustaunau, Oklahoma State University, has accepted a position at the Instituto Tecnologico y de Estudios Superiores de Monterrey, Monterrey, Mexico.

Mr. J. P. Menard, National Bureau of Standards, Washington, D. C., has accepted a position as Assistant Director of the Computing Center at Syracuse University.

Mr. A. M. Peiser, M. W. Kellogg Company, New York, New York, has been appointed Staff Consultant in Mathematics.

Associate Professor Alice T. Schafer, Connecticut College, has been promoted to Professor.

Mr. C. J. Smith, Remington Rand UNIVAC, St. Louis, Missouri, has accepted a position as Associate Engineer in the Scientific Data Processing Department of the McDonell Automation Center, St. Louis, Missouri.

Miss Eugenia I. Trapp, International Business Machines, Los Angeles, California, has been appointed Staff Instructor at International Business Machines, New York, New New York.

Mr. Byron Cosby, Sr., Educational Service Bureau, Columbia, Missouri, died February 3, 1961. He was a Charter Member of the Association.

Mr. P. L. Poston, Great Lakes Mutual Life Insurance Company, died August 22, 1960. He was a member of the Association for eleven years.

Professor Emeritus W. D. Reeve, Columbia University, died February 16, 1961. He was a Charter Member of the Association.

Professor Emeritus S. E. Urner, Los Angeles State College, died January 30, 1961. He was a Charter Member of the Association.

PAU SCIENCE DIVISION SPONSORS OPERATION CLEAN-OUT-THE-ATTIC

Under a new program organized by the Division of Science Development of the Pan American Union, General Secretariat of the Organization of American States, thousands of issues of scientific and technical journals are being transferred from the overcrowded libraries of United States scientists to the understocked shelves of the libraries and information centers of Latin America. The rapid development of the sciences in that area make availability of journal files essential. Scientists of the U.S.A. are being asked to donate to the U.S.B.E. files of scientific and technical journals which they no longer wish to keep in their libraries. The value of the gift plus the cost of making it (mailing charges in this case) are income-tax deductible. Further information about the program may be obtained from the U. S. Book Exchange, 3335 V. St., N.E., Washington 18, D. C., or from the Division of Science Development, Pan American Union, Washington 6, D. C.

GRADUATE TRAINEESHIPS IN BIOMETRY

Training programs designed to prepare students in the application of statistical and mathematical methods to biological problems, particularly those related to health and medical sciences, now exist in more than 20 universities throughout the country. Supported by training grant funds from the Public Health Service, NIH, these programs provide unusual opportunities for careers in teaching, research, and consultation. Employment opportunities for biometricians are excellent, with the demand by governmental and voluntary health agencies, medical research and educational institutions, and industry running far in excess of the available supply of trained personnel.

Programs of study are individually designed to lead to doctoral degrees, and in special instances, to other academic degrees. Traineeship stipends are provided at various levels depending on previous education and experience of the trainee and include allowances for

dependents. Substantially full economic support or partial support may be provided, depending upon the proportion of time spent in training.

Interested applicants are encouraged to correspond with one or more of the Program Directors listed below because course offerings, as well as specific research problems for application of learned skills, vary from school to school.

Dr. Virgil Anderson
Purdue University
Lafayette, Indiana

Dr. George F. Badger
Western Reserve University
Cleveland 6, Ohio

Dr. Jacob E. Bearman
University of Minnesota
Minneapolis 14, Minnesota

Dr. Antonio Ciocco
University of Pittsburgh
Pittsburgh 13, Pennsylvania

Dr. Wilfrid J. Dixon
Medical Center, U.C.L.A.
Los Angeles 24, California

Dr. W. T. Federer
Cornell University
Ithaca, New York

Dr. John W. Fertig
Columbia University
New York 32, New York

Dr. Franklin A. Graybill
Colorado State University
Fort Collins, Colorado

Dr. Bernard G. Greenberg
University of North Carolina
Chapel Hill, North Carolina

Drs. John Gurland & T. A. Bancroft
Iowa State University
Ames, Iowa

Dr. Boyd Harshbarger
Virginia Polytechnic Institute
Blackburg, Virginia

Dr. Allyn Kimball
Johns Hopkins University
Baltimore 5, Maryland

Dr. Schuyler G. Kohl
State Univ. of N.Y., Col. of Med.
Brooklyn 3, New York

Dr. Robert F. Lewis
Tulane University
New Orleans 12, Louisiana

Dr. Eugene Lukacs
The Catholic University of America
Washington, D. C.

Dr. Paul Meier
University of Chicago
Chicago 37, Illinois

Prof. Felix Moore
University of Michigan
Ann Arbor, Michigan

Dr. Lincoln E. Moses
Stanford Medical School
Stanford, California

Dr. Hugo Muench
Harvard School of Public Health
Boston 15, Massachusetts

Dr. Robert Quinn
Vanderbilt University
Nashville 5, Tennessee

Prof. J. A. Rigney
North Carolina State College
Raleigh, North Carolina

Drs. W. W. Schottstaedt and James Hagans
University of Oklahoma
Oklahoma City 4, Oklahoma

Dr. Malcolm E. Turner, Jr.
Medical College of Virginia
Richmond, Virginia

Dr. Colin White
Yale School of Medicine
New Haven, Connecticut

Dr. J. Yerushalmy
University of California
Berkeley 4, California

For those unable to train during the academic year, an unusual opportunity is provided by a cooperative Graduate Session of Statistics in the Health Sciences sponsored by these Program Directors and made possible by a training grant from the PHS, NIH. For information concerning available stipends and course offerings at elementary, intermediate, or advanced levels for the summers of 1961 and 1962, write Dr. Jacob E. Bearman, University of Minnesota, Minneapolis, Minnesota.

REPORT OF THE SECOND CONFERENCE ON MATHEMATICAL EDUCATION IN SOUTH ASIA

The report of this conference held at Bombay in January 1960 contains invited addresses by a number of distinguished mathematicians (Artin, Stone, Krull, Lichnerowicz, Moise, Newman, Alexandrov, and Gnedenko) together with brief accounts of the discussion at the meetings of the working groups.

Copies of the report (xxiii+205 pages, paper bound) may be purchased for $2.00 by sending remittances to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York. The price quoted is a special rate for orders placed through the office of the Association and does not apply to orders placed through other agencies.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE PROPOSED DOCTOR OF ARTS DEGREE

At their meetings in Washington, D. C. on January 25, 1961 the Board of Governors of the Association and the Council of the American Mathematical Society both voted to approve, in principle, a report which recommended the establishment of a new graduate degree in mathematics. The proposal was described to both of these bodies at their meetings in East Lansing in August 1960, and a joint committee was asked to study the matter and report in January. The committee consisted of E. E. Moise, Chairman, M. M. Day, P. R. Halmos, and A. D. Wallace.

While the name of the degree was not considered part of the substance of the Committee's proposal, the Board of Governors approved a motion that the preferred name for the new degree be the Doctor of Arts.

A detailed report on the Committee's recommendations appears in the April, 1961, *Notices of the American Mathematical Society*. Reprints of this article and further information can be obtained by writing to Professor R. J. Wisner, Executive Director, Committee on the Undergraduate Program in Mathematics, Michigan State University Oakland, Rochester, Michigan.

HENRY L. ALDER, *Secretary*

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-eight annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi, on February 17–18, 1961, with Mississippi State University as host institution. The Friday afternoon meeting was held in two concurrent sessions. Professor J. H. Wahab, Louisiana Vice-Chairman, and Professor Elinor Walters, Mississippi Vice-Chairman, presided. Professor Z. L. Loflin, Chairman of the Section, presided at the Friday evening

and Saturday morning sessions. There were 206 persons registered, including 97 members of the Association.

The following officers were elected for the coming year: Chairman, Professor B. O. Van Hook, Mississippi Southern College; Vice-Chairman for Louisiana, Professor W. B. Temple, Louisiana Polytechnic Institute; Vice-Chairman for Mississippi, Professor S. R. Knox, Millsaps College; Secretary-Treasurer, Professor Z. L. Loflin, University of Southwestern Louisiana.

At the business meeting a report on the progress of the High School Contests was given by Professor N. A. Childress, Chairman of the Committee on Contests and Professor H. T. Karnes, Chairman for Louisiana. Professor Arthur Ollivier reported on the meeting of the Board of Governors.

The invited speaker for the meeting was Professor C. W. Curtis of the University of Wisconsin. His lecture on Friday evening was entitled, "Historical Remarks on Some American Algebraists," and his Saturday morning address was on "Lie Algebras and Linear Groups."

The following papers were presented:

1. *A clan on the swastika*, by Professor Haskell Cohen, Louisiana State University.

2. *The sieve process for pairs of primes*, by Professor P. C. Garcia, University of Southwestern Louisiana.

For n an even positive integer, the natural number x is said to be the bottom half of a prime n pair if both x and $x+n$ are prime. This paper is concerned with finding all the bottom halves of prime n pairs less than or equal to a given natural number N by a process similar to the sieve method due to Erastosthenes. Included are counting processes that allow one to compute the number of bottom halves of prime n pairs $\leqq N$

3. *Associativity property of finite groups*, by Professor W. E. Koss, Louisiana Polytechnic Institute.

If a set of n elements satisfy the closure, identity, and inverse properties of a group with respect to a well defined operation, then at least $2n(n-1)(n-2)$ products of the form $a_ia_ja_k$ must be checked if the exhaustion of the cases method is used to check for associativity. The method devised in this note reduces the problem to one where only $(n-1)^2$ permutation products of the form P_iP_j are needed.

4. *Curves of positive area*, by Professor Gail Young, Tulane University.

5. *A special application of interior ballistics*, by Professor D. E. Johnson, Louisiana Polytechnic Institute.

6. *Bessel polynomials*, by Professor C. W. Barnes, University of Mississippi.

7. *Differentiability of solutions of $Y'=f(X, Y)$* by Professor D. R. Scholz, Louisiana State University.

8. *On the convergence of the ratio of Fibonacci numbers*, by Professor W. M. Sanders, Mississippi Southern College.

Let $\{a_n\}$ be the Fibonacci sequence defined by the recursive formula $a_0=0$, $a_1=1, \cdots,$ $a_n=a_{n-1}+a_{n-2}$ for $n\geq 2$. The infinite series $\sum_{n-1} 1/a_n$, where a_n is the nth Fibonacci number, converges. This is established by the ratio test after having resolved the sequence of ratios $\{a_n/a_{n+1}\}$ into two bounded subsequences one of which is strictly monotone increasing and the other strictly monotone decreasing having the same limit. No explicit usage of the well-known fact that lim $(a_n/a_{n+1})=2/(1+\sqrt{5})$ is made.

9. *On the automorphism group of a class of finite p-groups*, by Professor O. J. Huval, University of Southwestern Louisiana.

In this article it is shown that if a finite p-group G possesses an abelian subgroup M whose

factor group is cyclic, then G has outer automorphisms. If in addition the exponent of the center of G is less than or equal to the index of M in G, then the order of the automorphism group of G is divisible by the order of G.

10. *On subgroups which commute with every subgroup*, by Professor R. B. Howe, Mississippi State University.

It is shown that the condition that a subgroup H of a group G commute with every subgroup of G is a necessary but not a sufficient condition that H be normal in G.

11. *Rings which are unions of fields*, by Professor J. W. Ellis, Louisiana State University, New Orleans.

Z. L. LOFLIN, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The forty-first regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of California, Santa Barbara, California, on March 11, 1961. Professor R. C. James, Chairman of the Section, presided. The registered attendance was 176, including 130 members of the Association.

At the business meeting Professor T. M. Apostol, Chairman of the Nominating Committee, reported that the following officers were elected by mail ballot for the next academic year: Chairman, Professor Clifford Bell, University of California, Los Angeles; Vice-Chairman, Professor P. A. White, University of Southern California; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. Professor Apostol reported also that the members of the Program Committee elected for the coming year were: Professor Howard Tucker, University of California, Riverside, Chairman; Dr. Richard Bellman, Rand Corporation; Professor Richard Dean, Caltech; Professor Anthony Mardellis, Long Beach State College; Mr. T. E. Sydnor, Pasadena City College.

Mr. B. K. Gold, Chairman of the Contest Committee, presented a report for the committee and a motion, which was approved, requesting the National Mathematics Contest Office to invite all high schools in the Southern California area to participate in the 1962 National Contest.

Professor W. E. Smith of Occidental College presented a resolution honoring two former governors of the Section, who are retiring this year: Professor P. H. Daus, University of California, Los Angeles, and Professor C. G. Jaeger, Pomona College. Professor Smith's resolution was approved with applause.

The following program was presented:

1. *The Fibonacci operator*, by Professor C. J. A. Halberg, Jr., University of California, Riverside.

A specific bounded linear operator on the sequence l_1 is considered. Certain techniques are used to determine the spectrum of this operator, and it is shown that the uniform norm of the nth power of the operator is equal to the $(n+2)$th term of the Fibonacci sequence.

2. *Wife selection problems*, by Professor T. S. Ferguson, University of California, Los Angeles.

N distinct concealed numbers are being shown to you one at a time in random order. You may stop the proceedings at any time by selecting the number being shown to you. The problem of finding a selection procedure to maximize some criterion may be interpreted in terms of selecting a wife, buying a house, choosing a new staff member, and so on. The well-known problem of maximizing the probability of selecting the largest of the N numbers when you know nothing whatsoever about their values is the starting point for the investigation of several related problems.

3. *The School Mathematics Study Group*, by Mr. William Wooton, Los Angeles Pierce College.

The School Mathematics Study Group has finished one major part of its work, that of providing a complete set of class-room tested model textbooks for grades seven through twelve. Projects still in progress are the production of monographs for students, study guides for teachers, and

model textual material for noncollege-bound students. Other projects include textbooks for elementary grades four through six, and an alternate treatment of high school geometry.

4. *From my problem-solving seminar*, by Professor George Pólya, Stanford University.

One of the teacher's tasks—perhaps his principal task—is to recognize and develop the right attitude to independent work (to problem solving) in his students. Therefore, the teacher's training should provide suitable opportunity for independent work, that is, "research" on the appropriate level. Such training of teachers is usually not done, but the speaker's seminar attempts to provide this opportunity. Three problems from the seminar are presented; one will appear in *The Mathematics Teacher*.

5. *Inequalities*, by Professor Robert Schatten, University of Southern California.

6. *Rational approximation problems*, by Dr. E. W. Cheney, Space Technology Laboratories.

An efficient computational algorithm, developed jointly by H. L. Loeb and the speaker, is presented for the following problem: Given elements, $F, G_1, \cdots, G_n, H_1, \cdots, H_m$, of the Banach space $C[0, 1]$, obtain the coefficients a_i and b_i which render the "rational" function $\sum a_iG_i/\sum b_iH_i$ an optimum approximation to F.

R. B. HERRERA, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.
MINNESOTA, Moorhead State College, Fall, 1961.
MISSOURI
NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.
NEW JERSEY, St. Peter's College, Jersey City, November 4, 1961
NORTHEASTERN
NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.
ROCKY MOUNTAIN
SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.
SOUTHERN CALIFORNIA, Long Beach State College, March 9, 1962.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

$$o = \frac{d^2x}{dt^2} + \frac{x}{r^3} + \frac{\partial R}{\partial x}$$

$$o = \frac{d^2y}{dt^2} + \frac{y}{r^3} + \frac{\partial R}{\partial y}$$

$$o = \frac{d^2z}{dt^2} + \frac{z}{r^3} + \frac{\partial R}{\partial z}$$

PLOTTING PATHWAYS IN SPACE

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THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to nonspecialists and scientific workers in other fields.

The complete list of Carus Mathematical Monographs is:

No. 1. *Calculus of Variations,* by G. A. Bliss.

No. 2. *Analytic Functions of a Complex Variable,* by D. R. Curtiss.

No. 3. *Mathematical Statistics,* by H. L. Rietz.

No. 4. *Projective Geometry,* by J. W. Young.

No. 6. *Fourier Series and Orthogonal Polynomials,* by Dunham Jackson.

No. 7. *Vectors and Matrices,* by C. C. MacDuffee.

No. 8. *Rings and Ideals,* by N. H. McCoy.

No. 9. *The Theory of Algebraic Numbers,* by Harry Pollard.

No. 10. *The Arithmetic Theory of Quadratic Forms,* by B. W. Jones.

No. 11. *Irrational Numbers,* by Ivan Niven.

No. 12. *Statistical Independence in Probability, Analysis and Number Theory,* by Mark Kac.

No. 13. *A Primer of Real Functions,* by Ralph P. Boas, Jr.

One copy of each monograph may be purchased by members of the Association for $2.00 each. Orders should be sent to Harry M. Gehman, Executive Director, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies and copies for non-members of Monographs 1-8 are priced at $4.00 each, and must be purchased from the Open Court Publishing Co., La Salle, Illinois.

In the case of Monographs 9-13, additional copies and copies for non-members may be purchased at $4.00 from John Wiley and Sons, 440 Park Avenue South, New York 16, New York.

GEORGE BANTA COMPANY, MENASHA, WISCONSIN

AN ELEMENTARY INTRODUCTION TO ELLIPTIC FUNCTIONS BASED ON THE THEORY OF NUTATION

SERGE J. ZAROODNY, Ballistics Research Laboratories, Aberdeen Proving Ground

Introduction. Elliptic functions are a classical branch of mathematics, and recur in many problems of physics and engineering—either directly, or as a skeleton of more complicated problems. Yet it cannot be denied that they are known nowhere as widely as they deserve. In many problems when the elliptic functions could advantageously be used, or at least inspected as a possibly useful first approximation, it is usually possible to find an even simpler (and not always adequate) skeleton of the theory in trigonometry.

Perhaps this is so because the elliptic functions are traditionally presented not as a refinement on trigonometry, but as a subject in the theory of functions of a complex variable; trigonometry, however, is never presented—and in fact, seldom reviewed—as such a subject. Indeed, there is no need to encumber a presentation of the circular functions to a beginner by a simultaneous presentation of the hyperbolic functions and of all relations between these two sets of functions. Nor would such a presentation be particularly instructive: for the poles of sin and cos are at an imaginary infinity (and hence cannot be circled), while those of tan, cot, sec and cosec lie simply on the real axis.

Moreover, the elliptic functions are traditionally presented not as solutions of differential equations, but as inversions of certain integrals. This is as though one introduced trigonometry by defining the arcsin as an "incomplete circular integral of the first kind," or worse, as though π were introduced as "twice the complete circular integral." The tradition evolved in the pre-computer era, when the numerical solution of differential equations was considered just another theoretical method, and not a very practical method at that, of specifying a new function. The solution was considered completed only when reduced to quadrature or series and interpolation; and it was therefore considered natural that much algebraic work had to be done before one resorted to computations. In contrast, now, when torrents of new functions are being generated on machines and the bottlenecks lie in the analytical work, the definition of a function by its differential equations is not only the more convenient, but appears to be a basic method.

A student naturally expects that the elliptic functions are a generalization of the circular functions, and are therefore more complicated; but he hopes that the new labor would be, in proportion, no more than, say, going from a circle to an ellipse, or from plane trigonometry to the spherical trigonometry. Instead, he finds an entirely new approach, and a new theory, which only retrospectively reduces to trigonometry—and a rather advanced trigonometry at that—as the poles of the elliptic analogues of sin and cos recede to imaginary infinity. The forward step (from circular to elliptic functions) requires turning this infinity into a large finite number. It would have been easier if this step involved an appearance of something small which was zero before; as a matter of fact, it is the latter which is provided—in various disguises—by his physical problems.

The historical necessity for economy in computations has left us other pedagogical hurdles. Thus, the differential equations which the new functions satisfy are usually not displayed prominently at the outset (so that it is only after much study that the student realizes that the generating equations had been served up as the differentiation formulas). Even so, often a desire to avoid (economically) the systems of equations, and to define each function by its single equation, results in an extensive use of transformations which are hardly essential. Thus, it is not easy for the student to recognize that equations such as

$$s'' = -(1+k^2)s + 2k^2s^3 \quad \text{or} \quad (s')^2 = (1-s^2)(1-k^2s^2) \tag{1}$$

are supposed to be the answer to the problem of the pendulum,

$$\ddot{x} + \sin x = 0, \tag{2}$$

through the relations

$$ks = \sin \tfrac{1}{2}x; \; k = \tfrac{1}{2}\dot{x}_0 \text{ or } 2/\dot{x}_0; \; x_0 = 0. \tag{3}$$

In this classical problem he further finds that even after the tables of s have been located, much drudgery remains in converting s to x; if a machine is to be resorted to, after all, it is much easier to put on it (2), rather than (3); certainly, various refinements of the problem are easier with (2) than with (1). In one important aspect his disappointment is even more legitimate. To him (2) is a complete statement of the physical system in question, expressed in some natural units which are independent of the initial conditions; and this statement should be augmented by a separate bit of information, the statement of the initial conditions. But with (1), he finds, the initial conditions in the equation are fixed, and a cognizance of the initial conditions of the physical problem requires working out the value of the modulus k; this is as though a change of the initial conditions changed the "system." In a sense, perhaps, this is precisely the essence of a nonlinear physical system; and the first equation of (1), in fact, shows that the nonlinearity of (2) can be taken care of by an introduction of a single nonlinear term, which is somewhat as if the expansion $\sin x = x - \frac{1}{6}x^3$ were exact. But, to the student, (1) seems to mix unnecessarily the statement of the system with the statement of the initial conditions. Thus the theory does not seem to show much for his trouble.

Of course, there are better reasons for the elliptic functions than helping our hypothetical student with his simple pendulum; there are many who would see in this pedagogical situation no more than a spur to further study, who would emphasize the logic of the historical development of the theory of elliptic functions, and who would point out that an approach on a higher level will in the long run be the more fruitful. But all these things will come to the student in good time. The path to arousing his interest in mathematics need not lie in keeping the solution of (2) difficult; for, all too often, the result is simply discouragement (if so simple a problem requires this much theory, he may well ask, what will the further developments require?). An insistence on the more powerful

methods of approach may equally well be likened by him to a prohibition of vaulting the wall of a labyrinth.

In fact, a certain amount of redundancy of effort (such as the exercises) is inavoidable in any study. While of all the disciplines the body of mathematics is most nearly as a "tree" of the circuit theory, our personal body of knowledge is certainly more as a network. Therefore, there seems to be room for a simplified and inductive, introductory, rearrangement of the theory of elliptic functions: one that would start from particularly simple physical problems involving these functions, which could gradually be expanded as needed to meet most of the existing theory, and which might have to be followed by a regular deductive course. Only a bare introduction of this sort is attempted here; and the only possibly novel idea is that a theoretical access to a computing machine might not be without some effect on the student's mathematical growth.

Of physical problems involving the elliptic functions a particularly neat one is that of the nutations of a free rigid body.

Physical background. Let us start with Euler's dynamical equations,

$$(4) \qquad I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2,$$

wherein we know that, given the initial conditions $(\omega_{10}, \omega_{20}, \omega_{30})$, the functions ω_1, ω_2, ω_3 of time t can readily be produced. The attractiveness of this problem lies in the fact that the principal three elliptic functions are no more than these three solutions, suitably scaled. Our object will be merely to systematize the possible extreme multiplicity of such functions.

We shall label the right-handed triad of the principal axes of inertia of our rigid body in the order of increasing moments of inertia:

$$(5) \qquad I_1 \leqq I_2 \leqq I_3;$$

and must note that the physical significance of the I_1, I_2, I_3 imposes upon them a further restriction: they must be such as can form a triangle,*

$$(5') \qquad I_1 \geqq I_3 - I_2.$$

In order to mark at once the intimate relation of the elliptic functions to the trigonometry, the student should work out the important two degenerate cases, in which the elliptic functions degenerate into the circular functions. These are the cases of axial symmetry: the *spindle* ($I_1 < I_2 = I_3$, $\omega_1 =$ constant) and the *disc* ($I_1 = I_2 < I_3$, $\omega_3 =$ constant). The relation (5′) limits the extreme case of the disc to the case of *thin disc* ($I_1 = I_2 = \frac{1}{2}I_3$). The extreme case of the spindle is the case of *thin rod* ($I_1 \to 0$), and the border-line case between a "fat spindle" and a "thick disc" is the *sphere* ($I_1 = I_2 = I_3$); these two extreme particularizations are so simple they are not interesting. Another particularization of the general case is

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* This is so because $I_1 = K_2 + K_3$, etc., where $K_2 = \int_m x_2^2 dm$, etc. The relation (5′) states simply that $K_2 + K_3 \geqq K_2 - K_3$.

a different generalization of the thin disc: it is the case of *flat body* ($I_1 \neq I_2$, $I_1 + I_2 = I_3$), and does not lead to the circular functions.

Alternatively, noting only that because of (5) only the second coefficient in parenthesis in (4) is positive—and that the other two are negative—the student may at once expect the axes 1 and 3 to possess some similar properties not shared by the axis of the intermediate moment of inertia, the axis 2.

A great deal of information about the functions ω_1, ω_2, ω_3 can be got simply by inspecting the trajectories of the vector $\boldsymbol{\omega}$ in our body-fixed coordinate system 1, 2, 3; they are called polhodes ("pole paths," the "pole" being really the instantaneous vector of the angular velocity of the body*). While (4) gives the polhodes as parametric curves with t as the parameter, it is instructive to eliminate this fourth variable, t, from these equations. Multiplying (4) by ω_1, ω_2, ω_3, adding and integrating, we have

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \text{const.} = 2T, \text{ say,} \tag{6}$$

viz., the conservation of the kinetic energy; while multiplying by $I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$, adding and integrating, we have

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{const.} = L^2, \text{ say,} \tag{7}$$

viz., the conservation of the magnitude of the angular momentum.

Strictly, there are two more constants of motion; for the vector $\mathbf{L}$ of angular momentum of our body is conserved in direction (in space), as well as in magnitude. But this does not show in our body-fixed system 1, 2, 3. Rather, equations (4) state that the 1, 2, 3 system is at all times so oriented that $\mathbf{L}$, given in that system by the variable components $L_1 = I_1\omega_1$, $L_2 = I_2\omega_2$, $L_3 = I_3\omega_3$, remains constant in the space-fixed, newtonian, coordinate system.

We might also note that (7) is somewhat more general than (6); for (7) asserts only an absence of exterior forces on our body, while (6) asserts the absence of both exterior and interior work-producing forces.

Both (6) and (7) are equations of ellipsoids for the vector $\boldsymbol{\omega}$. Rewriting them as

$$\left(\frac{\omega_1}{\sqrt{(2T/I_1)}}\right)^2 + \left(\frac{\omega_2}{\sqrt{(2T/I_2)}}\right)^2 + \left(\frac{\omega_3}{\sqrt{(2T/I_3)}}\right)^2 = 1,$$

$$\left(\frac{\omega_1}{L/I_1}\right)^2 + \left(\frac{\omega_2}{L/I_2}\right)^2 + \left(\frac{\omega_3}{L/I_3}\right)^2 = 1,$$

we see that the "momentum ellipsoid" has semiaxes L/I_1, L/I_2, L/I_3; while the "kinetic energy ellipsoid" has semiaxes $\sqrt{(2T/I_1)}$, $\sqrt{(2T/I_2)}$, $\sqrt{(2T/I_3)}$, and

* Polhodes should be viewed not so much as curves, as conical surfaces with the apex at the center of mass of the body. The nutation of the body can be viewed as a slipless rolling of these polhodes on stationary curves (or conical surfaces) called herpolhodes ("creeping pole paths"). In our force-free case the latter curves happen to lie in a plane.

thus is more nearly a sphere.*

The trajectory of ω now can readily be visualized as the intersection of these two ellipsoids. For our purposes the most interesting observation is simply that these intersections are closed curves; it is thus strongly suggested that the motion is periodic. Figure 1 illustrates a momentum ellipsoid (7), with several trajectories resulting from the intersection of this ellipsoid with several kinetic-energy ellipsoids (6); that is, all of these trajectories have the same angular momentum but different energies. The larger energies (that is, the larger kinetic-energy ellipsoids) are associated with trajectories near axis 1; the lowest, with those about axis 3. The trajectories are divided into two groups, those looping axis 1, and those looping axis 3 (in a space-fixed coordinate system, of course, it is either the axis 1 or the axis 3, that nutates about the stationary $\mathbf{L}$). A steady spin (constant ω) is possible about each one of the axis 1, 2, 3; but the spin about 1 and 3 is "stable" in a certain† sense, while that about 2 is "unstable" (if disturbed, the trajectory goes into a large loop either about 1 or about 3). In all these cases the two ellipsoids are tangent to each other at the axis of spin; but in the two stable cases one ellipsoid is wholly inside the other, except for the two points of tangency; while at the axis 2 the ellipsoids intersect, and the points of tangency are of a different type. The speed of ω along each trajectory, as given by (4), is obviously very low at the sharpened corners of the loops near axis 2; thus the body spends a long time spinning with one end of axis 2 near the stationary $\mathbf{L}$, then rapidly "nutates" till it reverses its position, etc. The dividing trajectory which appears as passing through the axis 2, approaches this axis asymptotically, or leaves it in the analogous manner. The diametrally opposite

* In this connection we might make some remarks which are rather getting ahead of our story; but the reader might return to this footnote later, as an exercise. In certain problems of which our present one is a skeleton, it is in fact convenient to transform the variables in such a way that one of these two ellipsoids is a sphere. In particular, in Russian texts on mechanics it is the "more fundamental" ellipsoid, (7)—rather than the fatter one, (6)—that is so transformed: one then inspects the motion, in the 1, 2, 3 system, of the (really stationary) vector $\mathbf{L}$. Alternatively, one could inspect the motion of a vector whose magnitude is $\sqrt{(2T)}$, and whose physical significance is more obscure. In either case Poinsot's delightful geometrical interpretation of the nutations of a free body (whereof the essence is that one inspects the motion of the vector ω) is so altered that it is practically lost. When (7) becomes a sphere, in particular, the other ellipsoid resembles the "moment of inertia ellipsoid," rather than the body itself: the relation between the two ellipsoids becomes, so to say, inverted. The sense of existence of a plane corresponding to the plane in which Poinsot's herpolhodes lie becomes easier: it becomes simply a plane tangent to the sphere. But the herpolhodes collapse into a point, and the rolling of this sphere on this plane is no longer slipless; to regain something like Poinsot's interpretation, it becomes necessary to slide the herpolhodes in that plane. However, in either case the polhodes can readily be produced when needed.

† This stability is taken in the Lyapunov's sense. It should be noted that this is one case where the Lagrangian criterion of stability (an equilibrium occurring at an energy extremum, and being stable or not as this extremum is a minimum or maximum) is not applicable. Here at an energy maximum (axis 1) there is stability, while the instability occurs (at axis 2) when there is no energy extremum. Yet, the two stable cases are not exactly alike; if the system is extended to include some dissipation of energy, axis 1 becomes unstable, while axis 3 becomes asymptotically stable.

loops correspond, of course, simply to a reversal of **L**.

Since all trajectories intersect the 1–3 plane, all of them will be included if we take the initial conditions in that plane, between axis 1 and 3:

$$\omega_{20} = 0. \tag{8}$$

We should then expect that ω_1 and ω_3 are even functions of time, while ω_2 is an odd function. Noting that all trajectories have symmetries about the planes 1–2, 2–3, and 3–1, we should also expect certain symmetries about the instants of time represented by an integer number of quarter-periods.

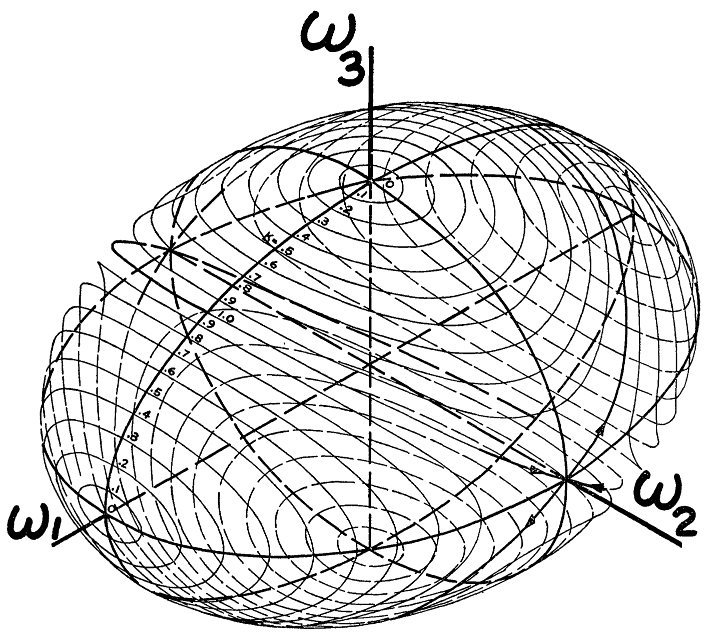

FIG. 1. Polhodes (the trajectories of the vector of angular velocity of the body in the body-fixed coordinates) of a free rigid body. In this case the ratios of the moments of inertia of the body are $I_1/I_2/I_3=3/4/5(K_1/K_2/K_3=1/2/3)$. All polhodes shown have the same angular momentum.

We may note in passing that for these intersections to exist the L^2 and the $2T$ must be not entirely independent of each other: the maximum and the minimum semiaxes of the ellipsoids must be restricted by $\sqrt{(2T/I_1)}<L/I_1$ and $\sqrt{(2T/I_3)}>L/I_3$, otherwise written

$$L^2/I_1 > 2T > L^2/I_3 \quad \text{or} \quad 2TI_1 < L^2 < 2TI_3. \tag{9}$$

The two groups of trajectories which loop, respectively, axis 1 and 3 are correspondingly identified by $\sqrt{(2T/I_2)}>L/I_2$ and $\sqrt{(2T/I_2)}<L/I_2$ (*i.e.*, at the axis 2 the kinetic energy ellipsoid is either outside, or inside, the momentum ellipsoid). In the case

(10) $$\sqrt{(2T/I_2)} = L/I_2$$

(*i.e.*, the two ellipsoids are tangent to each other at the axis 2), one has both

$$2TI_2 = I_1I_2\omega_1^2 + I_2^2\omega_2^2 + I_3I_2\omega_3^2 = L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

and therefore

(11) $$I_1(I_2 - I_1)\omega_1^2 = I_3(I_3 - I_2)\omega_3^2$$

or

(12) $$|\,\omega_3/\omega_1\,| = \sqrt{\left(\frac{I_1}{I_3}\cdot\frac{I_2 - I_1}{I_3 - I_2}\right)} = \text{const.} = J, \text{ say,}$$

which shows, incidentally, that the trajectory dividing these two groups of loops does lie in a plane of slope J. The extremes of J occur at the two cases of axial symmetry and the situation at these extremes can be summarized by the following table:

Illustrated by	FIG. 2	FIG. 3
If I_2 approaches	I_1	I_3
the body becomes a	disc	spindle
whose axis is	3	1
Both ellipsoids become spheroids which are	oblate	prolate
The slope J of the dividing trajectory approaches	0	∞
so that this trajectory approaches the plane	1–2	2–3
and there vanish the loops about the axis	1	3
leaving only the circular loops about the axis	3	1
with the energy at these axes being a	minimum	maximum
Also, in the extreme cases (not illustrated) of a	thin disc	thin rod
when I_1 approaches	$\frac{1}{2}I_3$	0
the ratio of the spheroid diameter to its polar axis is	2	0

The solutions of (4) for the borderline case of (10) or (12) are further illustrated in Figure 4 (p. 602) and are described by (22) below. The functions ω_1 and ω_3 then are essentially sech t and ω_2 is tanh t.

Curiously, the requirements (9) have nothing to do with (5′). The mathe-

matical relations discussed—and those about to be discussed— hold even when (5′) is not satisfied and equations (4) cease to have their physical analog.

The intersection of the ellipsoids (6) and (7), forming one of the polhodes of Figure 1, is further illustrated in Figure 5 (p. 603).

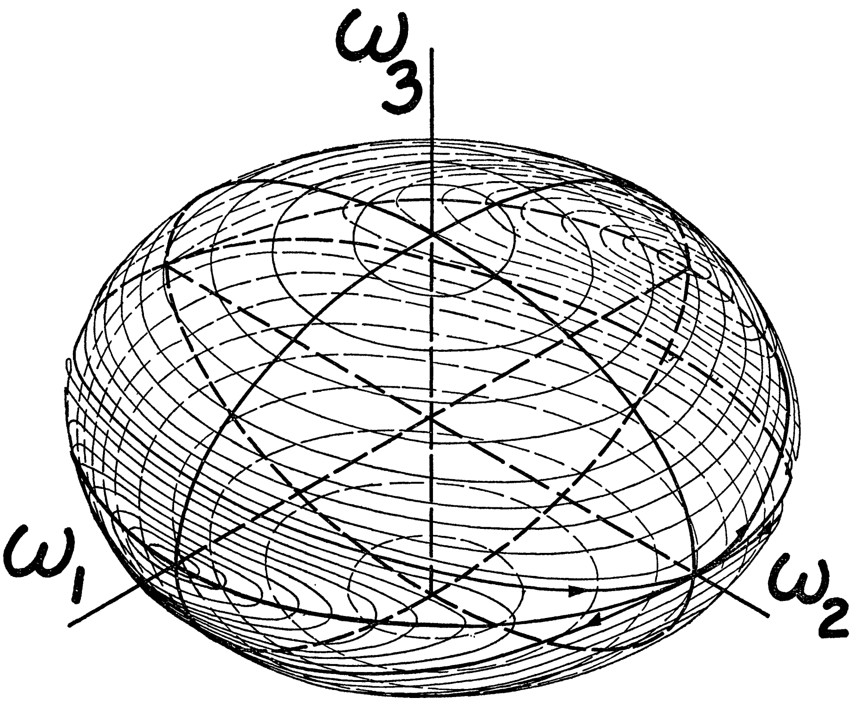

FIG. 2. Polhodes of a body which is nearly a disc ($I_1/I_2/I_3=3/3.2/5$). Note that the thickness (L/I_3) of the eventual oblate spheroid is at least one half of its diameter (L/I_2 or L/I_1) by (5′).

Generalization. The generalization of the solutions of (4) which results in the definition of elliptic functions amounts to no more than changing the scales of the variables ω_1, ω_2, ω_3, t; that is, to taking for these quantities some units other than radian/second and second. Let these quantities in the new units be W_1, W_2, W_3, u. The standard definition then is as follows.

It is natural to take for the even functions ω_1 and ω_3 the units of ω_{10} and ω_{30}, so that the maximum magnitude* of W_1 and W_3 is 1. For the unit of the odd function ω_2, let us take a quantity ω_{2M}, which we shall leave undefined for a while (presently it will turn out to be, usually, the maximum magnitude of ω_2). Finally, let us take the unit U of t in such a way that $\dot{\omega}_{20}$ in the new units is 1; and designate derivatives with respect to u by primes. Thus (Fig. 6, p. 605)

* That the minima and the maxima of (at least one of) W_1 and W_3 are equal in magnitude, follows from the symmetry of Figure 4. We avoid here the word "amplitude," for in the theory of elliptic functions it has a specialized meaning.

$$W_1 = \omega_1/\omega_{10}, \qquad W_2 = \omega_2/\omega_{2M}, \qquad W_3 = \omega_3/\omega_{30}, \qquad u = t/U,$$
$$\omega_1 = \omega_{10}W_1, \qquad \omega_2 = \omega_{2M}W_2, \qquad \omega_3 = \omega_{30}W_3, \qquad t = Uu, \tag{13}$$

where $(') = d/du = Ud/dt = U(\dot{\ })$ $(\dot{\ }) = d/dt = (1/U)(')$. Equations (4) become

$$W_1' = -\left\{U\frac{\omega_{2M}\omega_{30}}{\omega_{10}}\cdot\frac{I_3 - I_2}{I_1}\right\}W_2W_3,$$
$$W_2' = +\left\{U\frac{\omega_{10}\omega_{30}}{\omega_{2M}}\cdot\frac{I_3 - I_1}{I_2}\right\}W_3W_1, \tag{14}$$
$$W_3' = -\left\{U\frac{\omega_{10}\omega_{2M}}{\omega_{30}}\cdot\frac{I_2 - I_1}{I_3}\right\}W_1W_2.$$

Now we can proceed in two ways, depending upon whether we are more interested in loops about axis 1 or 3.

1. Let us first favor the axis 3. The first two coefficients in the parenthesis

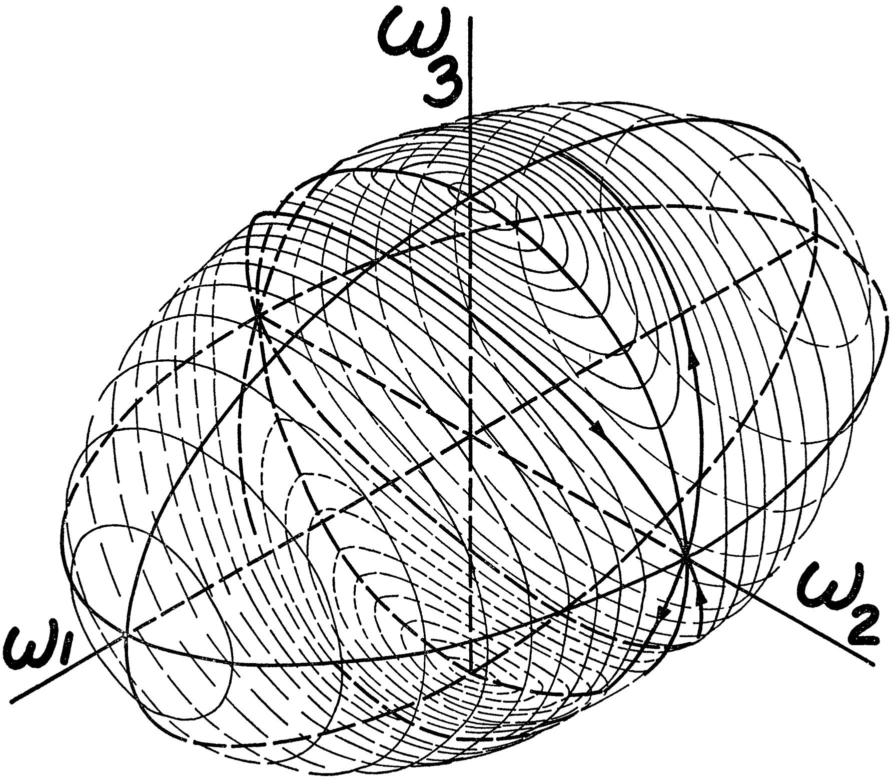

FIG. 3. Polhodes of a body which is nearly a spindle ($I_1/I_2/I_3=3/4.8/5$). Note that for a slender spindle ($I_1\to 0$) the diameter (L/I_2 or L/I_3) of the eventual prolate spheroid can be negligible in comparison with its axis (L/I_1).

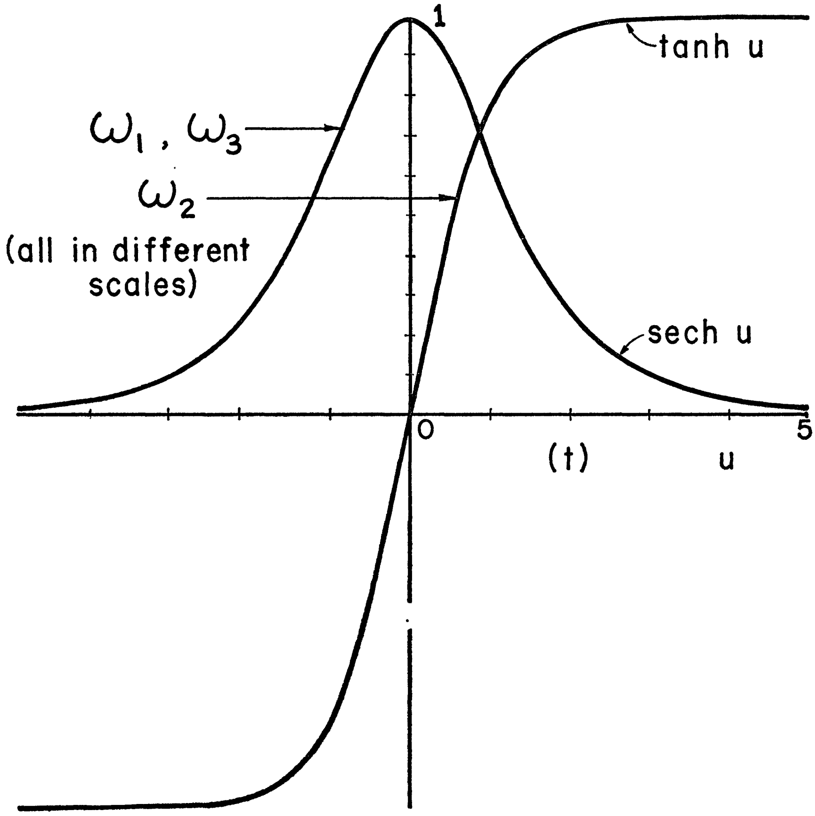

FIG. 4. The limiting case of unstable nutation shown starting from a point in the 1–3 plane. The trajectory approaches and/or comes out from the axis 2 asymptotically.

may be made equal if we select ω_{2M} in such a way that the ratio of these two coefficients is 1:

$$\omega_{2M} = \omega_{10}\sqrt{\left(\frac{I_3 - I_1}{I_3 - I_2}\cdot\frac{I_1}{I_2}\right)}. \tag{15}$$

Furthermore, these two coefficients will be 1, and hence W'_{20} will be 1, if

$$U = (1/\omega_{30})\sqrt{\left(\frac{I_1 I_2}{(I_3 - I_2)(I_3 - I_1)}\right)}. \tag{16}$$

The third coefficient then becomes

$$\left(\frac{\omega_{10}}{\omega_{30}}\right)^2 \cdot \frac{I_2 - I_1}{I_3 - I_2}\cdot\frac{I_1}{I_3} = k^2, \text{ say,} \tag{17}$$

and (4) acquire a relatively simple form

$$(18)\qquad \begin{aligned} &W_1' = -W_2W_3, \quad W_2' = +W_3W_1, \quad W_3' = -k^2W_1W_2;\\ &W_{10} = 1, \quad W_{20} = 0, \quad (W_{20}' = 1), \quad W_{30} = 1. \end{aligned}$$

The number k which characterizes (18) thus reflects both a "system" (a combination of I_1, I_2, I_3) and the initial conditions (ω_{10}/ω_{30}, with $\omega_{20}=0$), and

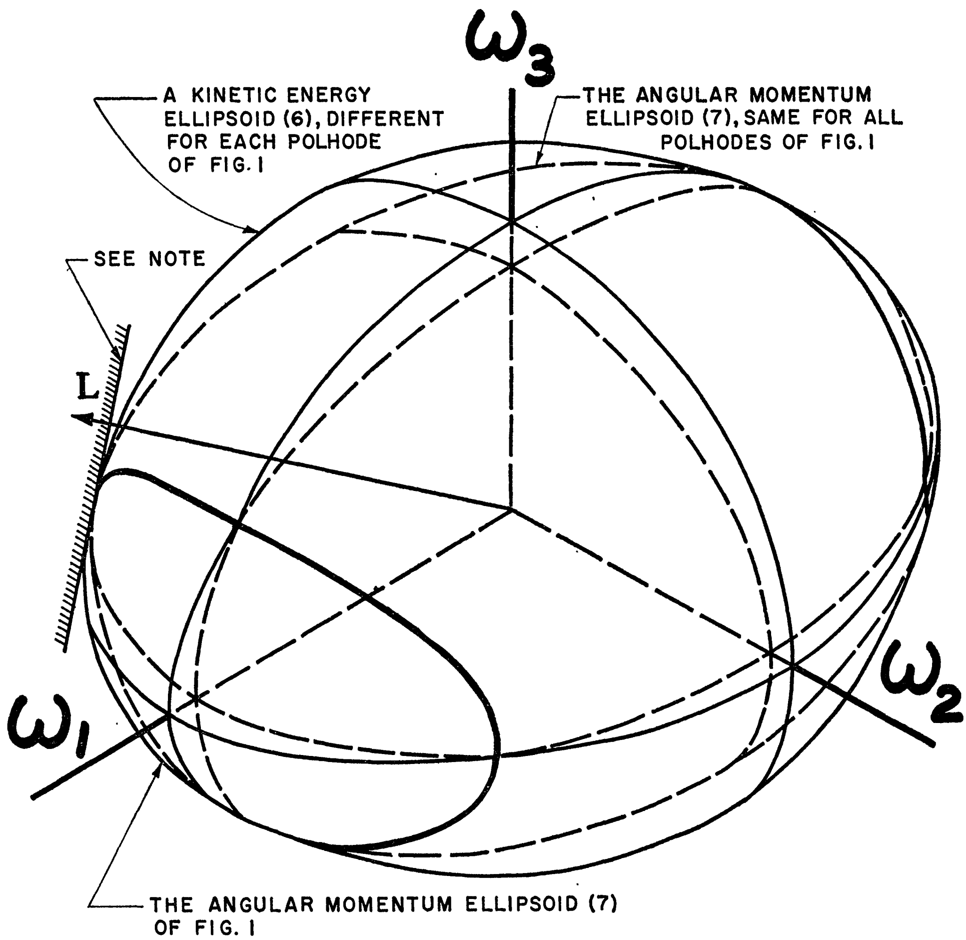

FIG. 5. Illustration of a pair of polhodes of Fig. 1 as an intersection of the pair of ellipsoids. (Note the herpolhode plane (a plane which is fixed, viz., not spinning with the coordinate system 1, 2, 3 of the body; normal to vector **L** of the angular momentum of the body; and tangent to the kinetic energy ellipsoid) for a particular polhode. This plane is drawn for a particular instant when it is seen edge on. The interesting property of the polhodes of a free rigid body is that, at all points on each polhode, the planes tangent to the kinetic energy ellipsoid are equidistant from the center of this ellipsoid. The motion of the body can therefore be visualized as a rolling of this ellipsoid, with its center stationary, on this plane. The herpolhodes encircle the point at which the vector **L** pierces this plane. Unlike the polhodes, the herpolhodes are in general not closed curves.

covers a large multiplicity of solutions of (4).

Now, the standard nomenclature is as follows:

$$\begin{array}{lll} & W_1 \text{ is called cn } u & \text{(read: cosine-amplitude of } u\text{),*} \\ (19) & W_2 \text{ is called sn } u & \text{(read: sine-amplitude of } u\text{),*} \\ & W_3 \text{ is called dn } u & \text{(read: delta-amplitude of } u\text{).} \end{array}$$

k is called the modulus (the word "parameter" has a different specialized meaning in the theory of elliptic functions) and is usually reckoned as $0 \leqq k \leqq 1$.

Thus the elliptic functions are defined by the following differential equations

$$\begin{array}{llll} & (\mathrm{cn}\ u)' = -\ \mathrm{sn}\ u\ \mathrm{dn}\ u, & \mathrm{cn}\ 0 = 1; \\ (20) & (\mathrm{sn}\ u)' = +\ \mathrm{cn}\ u\ \mathrm{dn}\ u, & \mathrm{sn}\ 0 = 0; \\ & (\mathrm{dn}\ u)' = -\ k^2\ \mathrm{cn}\ u\ \mathrm{sn}\ u, & \mathrm{dn}\ 0 = 1. \end{array}$$

They are really functions of two variables, u *and* k; *e.g.*, cn u, which perhaps could be written as $\mathrm{cn}_k u$, is really $\mathrm{cn}(k, u)$. However, from the viewpoint of our hypothetical student who has some access to a computing machine, the dependence of these functions upon the modulus k is clinched only when he can write the generating differential equations for these functions with k as the independent variable. Such equations, however, are practically never written; nor are the elliptic functions functions of the complex variable $u+ik$. Yet, certain aspects of these functions which are closely related to their variability with k arise when one considers the functions of complex, or just imaginary, u. It is for this reason, perhaps, that the elliptic functions are traditionally considered to "belong" to the theory of functions of complex variable.

The coefficient k^2 is 0 at axis 3 (where $\omega_1=\omega_2=0$), and ∞ at axis 1 (where $\omega_3=\omega_2=0$). It is readily noted, by reference to (10) and (12), that for the polhode which divides the two groups of loops $k^2=1$. Thus the loops about axis 3 are given (in this interpretation) by $0 \leqq k^2 < 1$, and the loops about axis 1, by $1 < k^2 \leqq \infty$.

Clearly, the positive number k^2 is more convenient than the modulus k itself, which theoretically can be negative. If we consider the "great ellipse" in the plane 1–3 of Figure 1, the possible negative values of k may be associated

* Unfortunately, the more natural terms, "elliptic cosine" and "elliptic sine," together with the more natural notation, are apparently pre-empted by the characteristic functions ce_n and se_n of the Mathieu functions theory (which, however, has no analog of the third member of the triplet, the function dn u). In fact, the function ce_1 sometimes does resemble the platykurtic shape of sn, while se_1 sometimes resembles the leptokurtic shape of cn. This standard nomenclature is rather puzzling, since the Mathieu theory is both more complicated, and more recent. Basically, Mathieu's is a linear theory (even though the characteristic functions of that theory, superposable only with the secular, Ince's, modes, may be viewed as bordering on the nonlinear theory). The theory of elliptic functions, on the other hand, is distinctly nonlinear; it is only with the inversion of the problem that some resemblance to the superposability of solutions occurs, in that a certain integral is "expandable" in terms of several rational functions and the three elliptic integrals.

simply with the other half of either loop. As we proceed around this ellipse (or, as we proceed, at any point of our ellipsoid, at right angles to polhodes), the values of k obviously repeat periodically. But as it happens, the quantity which increases monotonically as we so proceed is the imaginary part of u.

2. Had we favored axis 1, we would have made the second two coefficients in (14) unity, obtaining, say

$$\bar{\omega}_{2M} = \omega_{30}\sqrt{\left(\frac{I_3 - I_1}{I_2 - I_1}\cdot\frac{I_3}{I_2}\right)}, \tag{15'}$$

$$\overline{U} = (1/\omega_{10})\sqrt{\left(\frac{I_2 I_3}{(I_2 - I_1)(I_3 - I_1)}\right)}, \tag{16'}$$

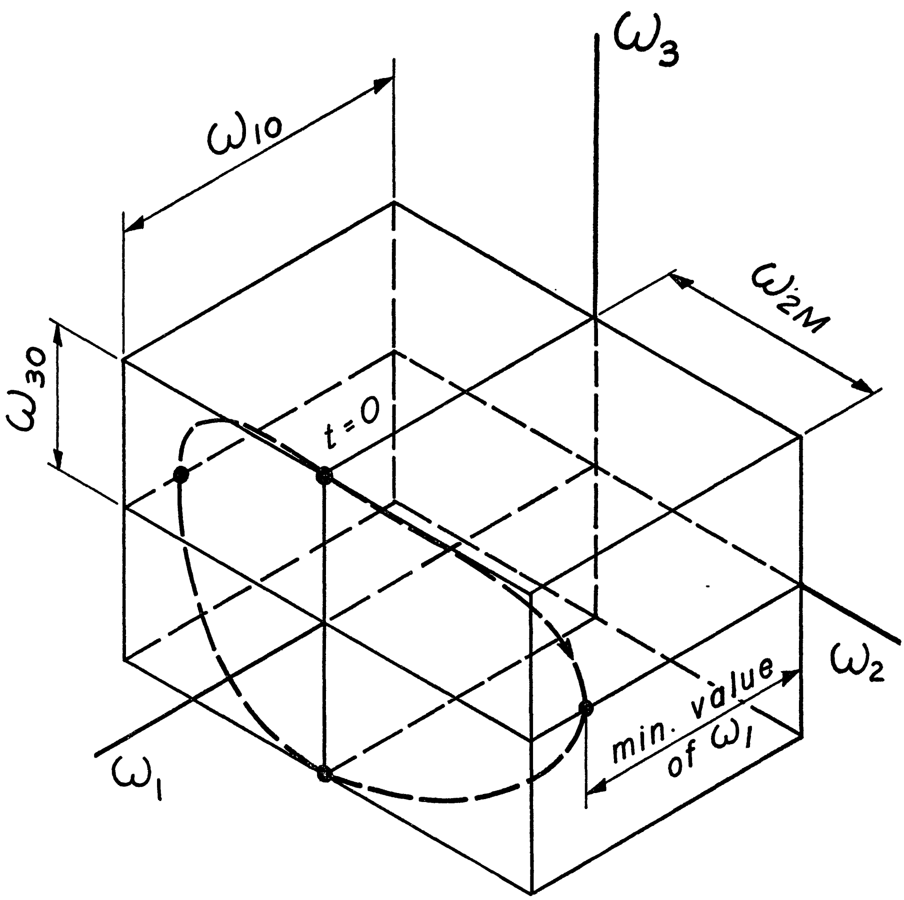

FIG. 6. Illustration of the units of ω_1, ω_2, ω_3, for a particular trajectory of Fig. 1 looping the axis 1. The unit of time is so chosen that the speed of the trajectory when crossing the axis 1 is unity. Note that for trajectories looping the axis 3, a different set of units is used.

and would have obtained another modulus,

$$\bar{k} = \frac{\omega_{30}}{\omega_{10}}\sqrt{\left(\frac{I_3 - I_2}{I_2 - I_1}\cdot\frac{I_3}{I_1}\right)}, \tag{17'}$$

which is readily noticed to be the reciprocal of k. Then the loops about axis 1 would be given by $0 \leqq \bar{k}^2 < 1$, and the loops about axis 3, by $1 < \bar{k}^2 \leqq \infty$. The equations would have been

$$d\overline{W}_1/d\bar{u} = -\bar{k}^2\overline{W}_2\overline{W}_3, \quad d\overline{W}_2/d\bar{u} = +\overline{W}_3\overline{W}_1, \quad d\overline{W}_3/d\bar{u} = -\overline{W}_1\overline{W}_2, \tag{18'}$$

so that with a modification of the identification (19), *i.e.*, with

(19′) $\overline{W}_1$ becoming dn $\bar{u}$, $\overline{W}_2$ remaining sn $\bar{u}$, $\overline{W}_3$ becoming cn $\bar{u}$,

we would have arrived at the same equations as (20), except for the reciprocal meaning of the modulus. It should be noted, though, that $\overline{W}_2 \neq W_2$, $\bar{u} \neq u$; for the re-working of (18′) into (18) requires putting $\bar{u}\bar{k} = u$, $\overline{W}_2/\bar{k} = W_2$, in addition to $\overline{W}_3 = W_1$, $\overline{W}_1 = W_3$, so that

$$\mathrm{cn}(1/k, u) = \mathrm{dn}(k, u/k), \quad \mathrm{sn}(1/k, u) = k\,\mathrm{sn}(k, u/k), \quad \mathrm{dn}(1/k, u) = \mathrm{cn}(k, u/k). \tag{19''}$$

The elliptic functions of modulus greater than 1, however, are usually not considered. Rather, one merely selects that one of (18) and (18′) which gives $k^2 < 1$ (thereby making the first generalization for loops about axis 3, and the second for those about axis 1; that is, referring each loop to its own center).* It will be noted that with our numerical approach there is no objection (except for compliance with this convention) to $k > 1$. The transformation (13) for (19′) is further illustrated in Figure 6.

Remarks on notation. While the generalization is a matter of choice of units, the choice of the "unitary" initial conditions is by no means imperative. Eagle† has proposed an alternative canonical form of these functions; in brief, it is

$$\mathrm{Sn}\, z = hk\,\mathrm{sn}\, u, \quad \mathrm{Cn}\, z = hk\,\mathrm{cn}\, u, \quad \mathrm{Dn}\, z = h\,\mathrm{dn}\, u, \quad u = hz, \quad h = K/\tau,$$

where K, as usual, is the quarter-period of sn u and cn u (or half-period of dn u), and τ is the long-needed abbreviation for $\frac{1}{2}\pi$ (to be pronounced "hi" for "half-pi"). Some of the advantages of his notation are that (20) becomes simply

$$(\mathrm{Cn}\, z)' = -\mathrm{Sn}\, z\,\mathrm{Dn}\, z, \quad (\mathrm{Sn}\, z)' = \mathrm{Cn}\, z\,\mathrm{Dn}\, z, \quad (\mathrm{Dn}\, z)' = -\mathrm{Cn}\, z\,\mathrm{Sn}\, z$$

and that many analogous simplifications occur later in the theory; that the real period of z is always 2π; and (for our purposes) that the variation of Sn z, etc., with k (and particularly for small k) does reflect, to a great extent, the variation of ω_1, ω_2, ω_3 with k. On the other hand, a student working with, say,

---

* Clearly, the same maximum value of ω_2, which is $\omega_{2M}(\mathrm{sn}_k\, u)_{\max} = \bar{\omega}_{2M}(\mathrm{sn}_{\bar{k}}\, \bar{u})_{\max}$, would be got from (18′) as from (18); but this is not the same as to say that $\bar{\omega}_{2M} = \omega_{2M}$, nor that $(\mathrm{sn}\, u)_{\max}$ is always 1.

† Albert Eagle, The Elliptic Functions As They Should Be, Cambridge, 1958.

an analog computing machine might find the classical notation the more practical. Thus, the choice of "unitary" initial conditions may correspond happily to the limitation of machine voltage to + or -100 volts; while with such machine the re-scaling of the variables is always necessary, the variation of K with k might be of essence, and re-scaling from u to z, merely an extra labor; the blend with trigonometry (for $k \to 0$) in the standard notation may be the more nearly suggestive of the frequently used "linearization" of more complicated problems; and the fact that for the dividing polhode these functions approach infinity (with h) may seem an unnecessary further departure from practical applications. A great deal of other elegance of Eagle's notation would be appreciated by a more profound student of elliptic functions, rather than by their occasional user concerned only in recognizing these functions as a "skeleton" of his problem; in brief, by treating these functions as a self-contained—and internally highly consistent—subject, Eagle deepens, rather than resolves, the estrangement of the theory of elliptic functions from "practical" problems.

Basic properties of elliptic functions. When k of (17) approaches 0 (whether because the body approaches an axial symmetry, or because the vector $\boldsymbol{\omega}$ approaches either one of axes 1 and 3), equations (20) with the same initial conditions reduce to those of circular functions:

$$
\begin{aligned}
W_1' &= -W_3W_2, & W_1 &= \operatorname{cn} u \to \cos u; \\
W_2' &= +W_3W_1, & W_2 &= \operatorname{sn} u \to \sin u; \\
W_3' &\to 0, & W_3 &= \operatorname{dn} u \to 1.
\end{aligned}
\tag{21}
$$

When $k=1$ (*viz.*, for the dividing polhode) the dn u and the cn u merge, by (12) and (18′), and equations (20) reduce to hyperbolic functions, illustrated in Figure 4:

$$
\begin{aligned}
W_1' &\to -W_2W_1, & W_1 &= \operatorname{cn} u \to \operatorname{sech} u; \\
W_2' &= +W_1^2, & W_2 &= \operatorname{sn} u \to \tanh u; \\
W_3' &\to 0, & W_3 &= \operatorname{dn} u \to \operatorname{sech} u.
\end{aligned}
\tag{22}
$$

In this sense the elliptic functions, as illustrated in Figure 7, may be viewed as a blend between the circular and the hyperbolic functions.

Multiplying the first two of equations (20) by cn u and sn u, adding, integrating and taking account of the initial conditions, we have

$$
\operatorname{cn}^2 u + \operatorname{sn}^2 u = 1. \tag{23}
$$

Hence the maximum magnitude of sn u is 1, and ω_{2M} of (15) or (15′)* is indeed the maximum magnitude of ω_2. In the classical notation, (23) is the only one in the theory of elliptic functions that is exactly the same as its analog in

* We are speaking here of the common case when $k<1$; *i.e.*, each loop on Figure 1 is considered with respect to its own center. For the "off-center" loops of the case $k>1$, the maximum value of sn u is $1/k$; formula (23) then, curiously, exchanges its physical meaning with (24).

trigonometry. All other analogs are more complicated.

Multiplying the second two of equations (20) by $k^2 \operatorname{sn} u$ and $\operatorname{dn} u$, adding, integrating and taking account of the initial conditions, we have the definition of $\operatorname{dn} u$ (which was absent—or rather was simply 1—in trigonometry):

$$\text{(24)} \qquad \operatorname{dn}^2 u = 1 - k^2 \operatorname{sn}^2 u = (1 - k^2) + k^2 \operatorname{cn}^2 u = \operatorname{cn}^2 u + (1 - k^2) \operatorname{sn}^2 u.$$

The quantity $k' = \sqrt{(1-k^2)}$ often recurring in this theory, is called the complementary modulus. Note that k' is not the same as $\bar{k}$ of (17'). Incidentally, with (23) and (24) the last equation of (1) can be recognized as the square of the second one of (20). Similar equations can easily be worked out for cn u and dn u.

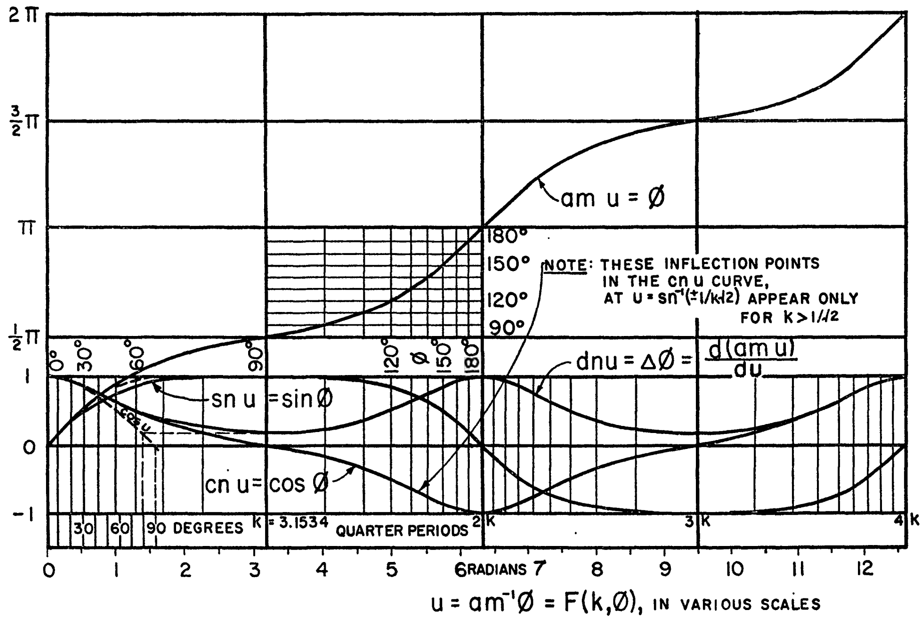

FIG. 7. Elliptic functions as stretched circular functions for $k=.9848=\sin 80°$.

Because of (23) an angle, say ϕ, exists and is very important, such that $\operatorname{cn} u = \cos\phi$ and $\operatorname{sn} u = \sin\phi$. This angle is called the "amplitude" of u:

$$\text{(25)} \qquad \phi = \operatorname{am} u = \cos^{-1}(\operatorname{cn} u), \qquad \operatorname{cn} u = \cos(\operatorname{am} u), \qquad \operatorname{sn} u = \sin(\operatorname{am} u).$$

The expression $\operatorname{dn} u = \sqrt{(1-k^2 \operatorname{sn}^2 u)} = \sqrt{(1-k^2 \sin^2\phi)}$ is often written $\Delta\phi$, and called "delta function" of ϕ, or delta function of amplitude (*viz.*, "delta-amplitude") of u. Obviously, the concept of such an angle is the main reason for the standard definition of elliptic functions.

Since the first two of equations (20) can be recognized as the formulas

$$d(\cos\phi)/du = -(\sin\phi)\cdot(d\phi/du), \qquad d(\sin\phi)/du = (\cos\phi)(d\phi/du),$$

it is obvious that

$$(26)\qquad \begin{aligned} &\operatorname{dn} u = d\phi/du = d(\operatorname{am} u)/du, \qquad \phi = \operatorname{am} u = \int_0^u \operatorname{dn} u\, du, \\ &u = \int_0^{\operatorname{am} u} \frac{d(\operatorname{am} u)}{\operatorname{dn} u} = \int_0^{\phi} \frac{d\phi}{\Delta\phi}. \end{aligned}$$

Thus the elliptic functions are simply the functions cos ϕ and sin ϕ, whose abscissa ϕ is stretched nonuniformly into the function $u(\phi)$ such that the rate $d\phi/du$ is $\Delta\phi$ called dn u. At zeros of sine and extrema of cosine, the rate of u is the same as that of ϕ (*viz.*, dn $u=1$); at extrema of sine and zeros of cosine, the rate of u is the maximum, and $d\phi/du$ has the minimum value, which is readily seen to be the complementary modulus k'. This is illustrated in Figure 7.

The value of u viewed as a function of am u and denoted by $F(k, \phi)$ is called the incomplete elliptic integral of the first kind. The term "integral" refers, of course, to the last expression of (26), and strongly connotes the pre-computer-era origin of this theory. The inverted function $u(\phi)$ is mathematically much more formidable than the more-relevant-to-all-sorts-of-physical-problems function $\phi(u)$: it is restricted to a certain interval of abscissa ϕ, is multivalued, and has in many places an infinite, or just very large, derivative. It seems amusing that precisely those classical methods (of inversion) which have been evolved to circumvent the labor of solving the differential equations are the ones which lead to difficulties with the computing machines. Thus in putting on the machines some practical problems whose skeleton involves the elliptic functions it may be well to think twice before proceeding with the refinement of these classical methods of inversion.

Commonly listed in the tables are the values of u for am $u=\frac{1}{2}\pi$, or the quarter-period of the motion, viewed as a function of an angle $\theta=\sin^{-1} k$ and called the complete elliptic integral of the first kind, $K(\theta)$. It varies from $\frac{1}{2}\pi$ for $\theta=k=0$, to ∞ for $k=1$, $\theta=\frac{1}{2}\pi$.

With their three reciprocals and the six ratios between them, the functions cn, sn, dn constitute the set of twelve basic, or Jacobian, elliptic functions. While six of them (such as tn $u=\tan\phi=\operatorname{sn} u/\operatorname{cn} u$ and ns $u=1/\operatorname{sn} u$) have obvious trigonometric analogs, the six involving dn (such as cd $u=\operatorname{cn} u/\operatorname{dn} u$) do not. There are many other elliptic functions; most of them, though, arose apparently either in sophisticated methods of computing the Jacobian functions or elliptic integrals, or as steps in the generalizations of the theory of doubly-periodic functions of complex variable. They are beyond the scope of this introduction.

Geometrical interpretation. The student may yet desire to have a simple geometric interpretation of elliptic functions, analogous to the classical trigonometric construction. It is perhaps the main pedagogical reason for the estrangement of the elliptic functions from trigonometry that no such construction

apparently exists. It may, perhaps, be sketched in principle as follows.

Let the unit circle AB on Figure 8a be shrunk vertically in the ratio $OC/OB = k' < 1$. The radius vector OD at an angle ϕ is shrunk in the ratio

$$\sqrt{\{\cos^2\phi + (k'\sin\phi)^2\}} = \sqrt{\{1 - (1 - k'^2)\sin^2\phi\}} = \sqrt{(1 - k^2\sin^2\phi)} = \Delta\phi,$$

so that the quantities $\operatorname{sn} u = \sin\phi$, $\operatorname{cn} u = \cos\phi$ and $\operatorname{dn} u = \Delta\phi$ can be recognized in the ratios FE/OC, OF/OA, OE/OA. Similarly, all other Jacobian elliptic functions can be recognized: thus, $\operatorname{cd} u = (OF/OA)/(OE/OA) = OF/OE$, etc. Given a value of k, we can compute the complementary modulus k', and construct the ellipse; and for each point E of this ellipse we can construct all Jacobian elliptic functions. However, they are still functions of the eccentric angle ϕ of the ellipse—that is, basically the trigonometric functions of the amplitude of u—rather than functions of u directly.

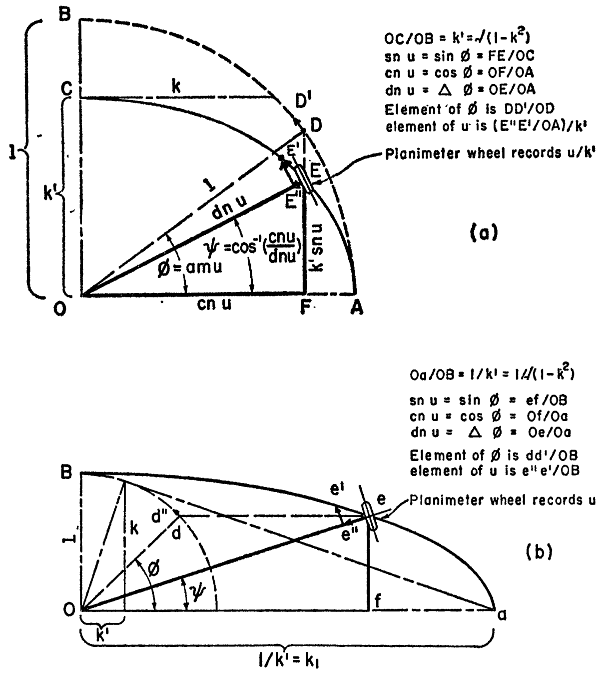

FIG. 8. Geometric interpretation of elliptic functions.

What, then, is—in this ellipse—the independent variable u? It is neither the arc AE (*that* can be expressed in terms of the more-commonly-met elliptic integral of the second kind), nor the angle AOE (*that* is a still different integral), nor the area of the elliptical sector (*that* is simply proportional to ϕ, or is a "circular integral").

This variable is the elliptic integral of the first kind,

$$u = F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\Delta\phi} = \int_0^{y} \frac{dy}{\sqrt{(1 - y^2)}\sqrt{(1 - k^2y^2)}},$$

where $y = \sin\phi$, which apparently is only met in the inversion of the problems stated as differential equations. While u cannot be readily recognized on Figure 8a, (nor can it be constructed directly with a ruler and a compass), its element—as shown by the integrand—can readily be constructed. It turns out that u is simply the circumferential travel of the tip of the radius vector, divided by k'. The travel DD' of D is always purely circumferential; but the displacement EE' of E consists of a radial displacement EE'' and a circumferential one $E''E'$. We may imagine a planimeter wheel, with its axis along the radius vector, which if attached to the radius vector at D would record the angle ϕ; if this planimeter now is constrained to slide upon the radius vector OE and be always located at E on the ellipse, its reading divided by k' will give u. The radial displacement of E on Figure 8a, of course, is simply $1 - \operatorname{dn} u$, which vanishes in trigonometry.

Indeed, if we let the angle AOE be ψ, differentiating $\sin\psi = (k'y)/\sqrt{(1-k^2y^2)}$ and noting that $\cos\psi = \sqrt{(1-y^2)}/\sqrt{(1-k^2y^2)}$, we get

$$d\psi = \frac{k'dy}{(1 - k^2y^2)\sqrt{(1 - y^2)}}.$$

Clearly, the element of u is $1/k'$ times $\sqrt{(1-k^2y^2)}d\psi$ or $(OE)d\psi$. But this is the "circumferential travel" of E.

The factor $1/k'$ can be avoided by the equivalent construction shown on Figure 8b, in which the possible infinite value for the quarter-period K of u (for $k=1$) appears more naturally.

Further properties of the elliptic functions—addition formulas. Even in this sort of introduction it is necessary to mention the addition formulas, *i.e.*, to evaluate the functions of $u+v$ in terms of the functions of u and v. While they are not often needed, these formulas would serve for checking the oddness and evenness of these functions, and their reflectivity about the instants of u represented by the integer number of quarter-periods K; for deriving the formulas for the functions of $u-v$, $2u$, $\frac{1}{2}u$, etc.; and for "connecting" our simplified theory with that of the functions of complex variable. All of these formulas do possess some similarity to their trigonometric analogs, but are much more complicated. The most essential addition formulas are:

$$\begin{aligned} \operatorname{cn}(u + v) &= (\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{sn}(u + v) &= (\operatorname{sn} u \operatorname{dn} v \operatorname{cn} v + \operatorname{cn} u \operatorname{sn} v \operatorname{dn} u)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v). \end{aligned} \tag{27}$$

Since

$$\operatorname{cn} u + i \operatorname{sn} u = \cos\phi + i \sin\phi = e^{i\phi} = e^{i \operatorname{am} u}, \tag{28}$$

it would have been preferable to have a simple formula for $\operatorname{am}(u+v)$; for then (27) could be easily derived by separating

$$e^{i\,\mathrm{am}\,(u+v)} = \mathrm{cn}\,(u+v) + i\,\mathrm{sn}\,(u+v) \tag{28'}$$

into real and imaginary parts. Unfortunately, no simple formula giving $\mathrm{am}(u+v)$ in terms of $\mathrm{am}\,u$ and $\mathrm{am}\,v$ seems to exist; but there is one in terms of $\mathrm{cn}\,u$, $\mathrm{cn}\,v$, $\mathrm{sn}\,u$, $\mathrm{sn}\,v$, $\mathrm{dn}\,u$, $\mathrm{dn}\,v$.

Formulas (27) may be derived on the basis of a very interesting analogy, which relates the elliptic functions to the spherical trigonometry. The amplitude of a sum $\mathrm{am}(u+v)$ is given by a side c of a spherical triangle whose other two sides are $a=\mathrm{am}\,u$ and $b=\mathrm{am}\,v$, provided that the angle C between these two sides is such that

$$\frac{\sin C}{\sin c} = k, \text{ and } C \text{ is obtuse.*} \tag{29}$$

This analogy is based on the fact that if in the sum

$$u + v = \int_0^{\mathrm{am}\,u} d\phi/\Delta\phi + \int_0^{\mathrm{am}\,v} d\phi/\Delta\phi = \int_0^{\mathrm{am}\,(u+v)} d\phi/\Delta\phi$$

the addends u and v are varied without affecting the sum, the differentiation yields $d(\mathrm{am}\,u)/\mathrm{dn}\,u+d(\mathrm{am}\,v)/\mathrm{dn}\,v=0$, which is analogous to the spherical trigonometric formula

$$\frac{da}{\cos A} + \frac{db}{\cos B} = 0 \tag{30}$$

for the spherical triangle whose sides a and b are varied in such a manner that the third side c and the angle C between a and b are kept as constants related by (29). That the ratio $\cos A/\cos B$ in that case is the same as $\mathrm{dn}\,u/\mathrm{dn}\,v$, follows from (29): for then $\sin A=k \sin a$ and $\sin B=k\sin b$, so that

$$\frac{\cos A}{\cos B} = \frac{\sqrt{(1-k^2\sin^2 a)}}{\sqrt{(1-k^2\sin^2 b)}} = \frac{\mathrm{dn}\,u}{\mathrm{dn}\,v}\,. \tag{29'}$$

Since C depends upon the unknown c, the derivation of (27) calls for some algebra, and the construction of such a spherical triangle cannot be undertaken directly. The possible variations of such triangle are illustrated, and certain details of the proof are spelled out, in Figure 9.

The convenient formula for the amplitude of a sum is

$$\mathrm{am}(u+v) = x_u + x_v, \tag{31}$$

where $x_u=\tan^{-1}(\mathrm{tn}\,u\,\mathrm{dn}\,v)$, $x_v=\tan^{-1}(\mathrm{tn}\,v\,\mathrm{dn}\,u)$ are the angles of the complex numbers

$$(\mathrm{cn}\,u + i\,\mathrm{sn}\,u\,\mathrm{dn}\,v) \quad\text{and}\quad (\mathrm{cn}\,v + i\,\mathrm{sn}\,v\,\mathrm{dn}\,u), \tag{32}$$

---

* With C acute, this procedure yields functions of $|u-v|$.

which are illustrated in Figure 9b and are certainly easy enough to remember: each one represents the radius vector OE of Figure 8a, reckoning the "shrinking factor" k' not as the complementary modulus, but as the dn of the other addend.

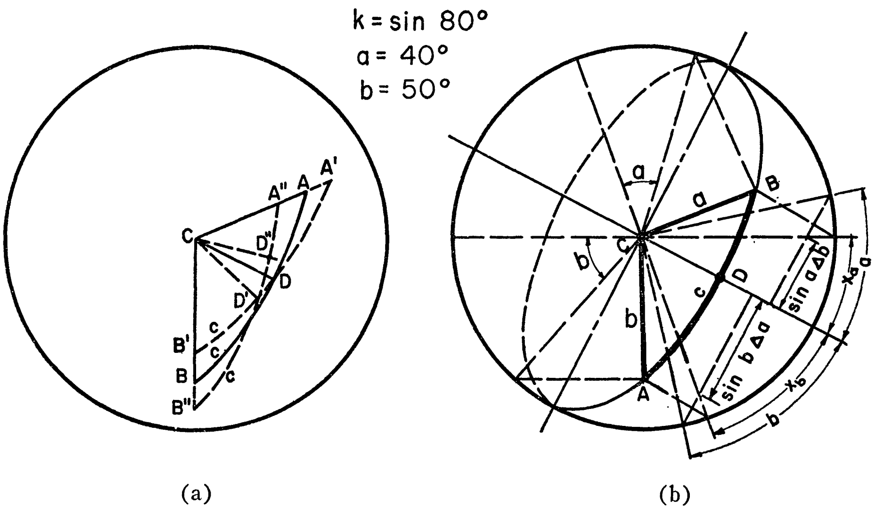

FIG. 9. Spherical-trigonometric interpretation of (31).

(a) Variation of shape of triangle for $c=$const. (b) Relations in the view of the triangle
(c) Derivation of (30): In $\cos c=\cos a \cos b-\sin a \sin b \cos C$, put c and C constant and differentiate. Thus:

$$(\sin a \cos b+\cos a \sin b \cos C)da+(\sin b \cos a+\cos b \sin a \cos C)db=0.$$

Note that

$$\cos B=-(\sin a \cos b+\cos a \sin b \cos C)/\sin c,$$
$$\cos A=-(\sin b \cos a+\cos b \sin a \cos C)/\sin c,$$

from which (30) follows.

Formula (31) can be derived by noting that the right-hand side of (28′), formed from (27), can be factored as

$$(33) \qquad \frac{(\operatorname{cn} u+i \operatorname{sn} u \operatorname{dn} v)(\operatorname{cn} v+i \operatorname{sn} v \operatorname{dn} u)}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

the magnitude of which, of course, must be 1; indeed, the magnitude of each factor in the numerator is the square root of the denominator. *E.g.*,

$$\operatorname{cn}^2 u+\operatorname{sn}^2 u \operatorname{dn}^2 v=1-(1-\operatorname{dn}^2 v)\operatorname{sn}^2 u=1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v.$$

Thus (33) has the form

$$e^{i \operatorname{am}(u+v)}=(\cos x_u+i \sin x_u)(\cos x_v+i \sin x_v)=e^{i(x_u+x_v)}$$

which is (31).

All addition formulas, such as (27), can readily be derived from (31). A more direct *a priori* derivation of (31) would be of interest. It is also interesting to note that the arcs x_u and x_v, into which $c=\mathrm{am}(u+v)$ is thereby broken, are the "spherical projections" of $a=\mathrm{am}\ u$ and $b=\mathrm{am}\ v$; that is, the point D in Figures 9a and 9b divides AB in such a way that the arc CD is normal to AB.

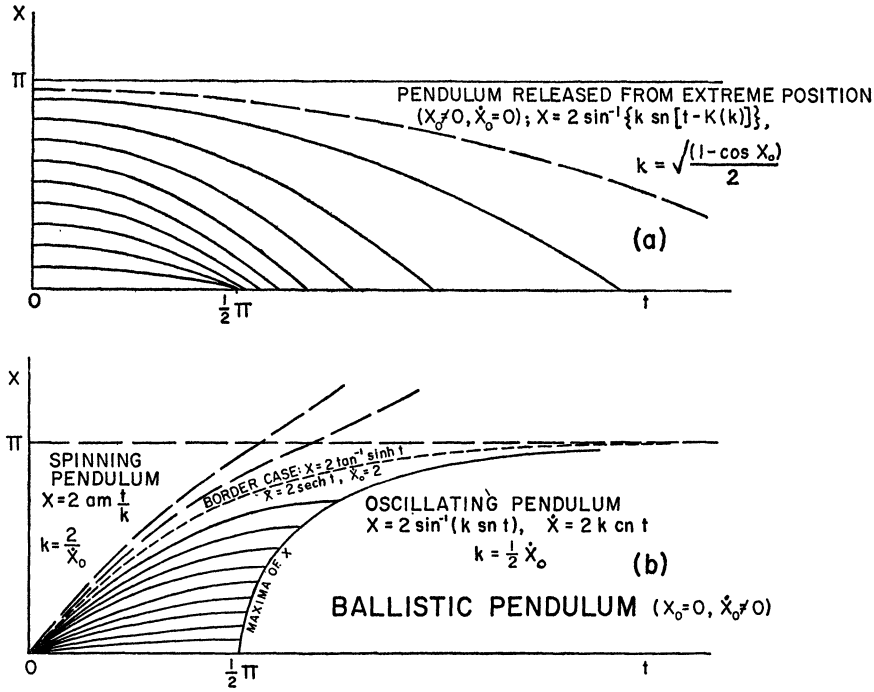

FIG. 10. A sample of a systematized set of solutions of (2). The remainder of the solution can be produced by the appropriate reflections. These "natural" curves are well known and are presented here merely as an example of the systematization of a practical problem treated with the aid of computing machines.

Remarks on the pendulum. Although the classical example (2) for the introduction of elliptic functions does not possess the symmetry of (4), it certainly describes the simpler physical phenomenon; for this reason, perhaps, this problem belongs to an introduction of this sort. It is particularly interesting for our purposes because it divides in two parts—oscillating and spinning pendulum—which correspond to the two groups of loops on Figure 1.

Faced with (2), our hypothetical student is apt to introduce some "elliptic functions" of his own: he would simply produce on a machine a systematized multiplicity of solutions $x(t)$, as in Figures 10a and 10b, and will be tempted to label these very "natural" functions the "elliptic cosine" and the "elliptic sine"; he would look into the theory of elliptic functions merely as a matter of con-

forming to the standard terminology. He will also note that the neater-looking plot of Figure 10a is limited (for obvious physical reason) to the oscillating pendulum, while the plot of Figure 10b can include also the case of the spinning pendulum; and would therefore concentrate on Figure 10b. After a heavy struggle with the theory (which would have to include, if only for lexicological reason, a nodding acquaintance with Mathieu functions) he will find that in the standard terminology his "natural" functions—of Figure 10b—have rather clumsy form:

(3′) $x = 2 \sin^{-1} (k \operatorname{sn} t)$ for the oscillating pendulum,* with $k = \frac{1}{2}\dot{x}_0 < 1$;

(3″) $x = 2 \operatorname{am}(t/k)$ for the spinning pendulum, with $k = 2/\dot{x}_0 < 1$;

(22′) $x = 2 \tan^{-1} (\sinh t)$, $\dot{x} = 2 \operatorname{sech} t$, with $\dot{x}_0 = 2$

for the borderline between these two cases (pendulum stopping asymptotically at the top). A far reaching socio-pedagogical reality is suggested here. Noting that his numerical approach (which merely reflects the physical reality) nicely blends the three cases which appear so distinct in the theory, our student is prone to develop a new appreciation of the machine, but to experience some corollary disenchantment toward the theory; the first is a healthy development, the second—in the long run—is certainly regrettable. In fact, (3′) and (3″) are equivalent, by (19″).

After all, the "theory" to him means no more than that "These sort of things have been done before, and are 'well known'; study the more general things, and you will be able to chop up these little problems quickly, without making a production of it." He would therefore be concerned with how, with a knowledge of (20) and (26), he could have recognized (2) as one leading to elliptic functions.†

Basically, he would now want to express the derivative in (2) as a product of two functions; it is therefore natural to put $x = 2y$ and write (2) as

(2′) $$\ddot{y} = - \sin y \cos y.$$

The solution is easiest for the spinning pendulum (indeed, physically it is the simpler case, as nearer to a steady rotation; and technologically it is rather the more important case, as a skeleton of the operation of an engine). In this case we may simply let t be u on a certain scale, *e.g.*, let $t = Uu$ as in (13) and identify $\sin y$ and $\cos y$ with $\operatorname{sn} u$ and $\operatorname{cn} u$, y with $\operatorname{am} u$; in fact, we could write ϕ for y. By (26), y' is identified with $\operatorname{dn} u$, and hence y'' with $(\operatorname{dn} u)'$. Thus (2′) is at once identified with the third equation of (20), and there only remains to evaluate the constants. Since

$$y'' = U^2\ddot{y} = - U^2 \operatorname{sn} u \operatorname{cn} u = (\operatorname{dn} u)' = - k^2 \operatorname{sn} u \operatorname{cn} u,$$

* For Figure 10a, we have simply $x = 2 \sin^{-1}(k \operatorname{sn}(t - K(k)))$; the concepts of cn *vs.* sn imply not only the change of phase, but also a change of the wave shape.

† The classical treatment (which proceeds by inverting the problem and introducing first the elliptic integral of the first kind), at least for the oscillating pendulum, is found in many texts, and need not be repeated here.

obviously, $U=k$. To evaluate it, we must consider the initial conditions:

$$x_0 = 0; \qquad \dot{x}_0 = (dx/dt)_0 = (dx/Udu)_0 = x_0'/U = 2y_0'/U = 2/U,$$

since $y_0' = \text{dn } 0 = 1$; hence $U = 2/\dot{x}_0$. Thus $x = 2y = 2 \text{ am } u = 2 \text{ am}(t/U)$; which is (3'').

With the oscillating pendulum the recognition is less easy (since the angle y is not monotonic, it cannot be identified with am u). Yet, it is intuitively obvious that the wave shape of sn u is relevant to the problem; we may expect it to represent x, y or sin y (note that it may not represent sin x, since $\ddot{x}$ must be small for x approaching π, while sn u is maximum). Let us try putting $\sin y = k \text{ sn } u$, where k is the maximum magnitude of sin y. Then

$$\cos y = \sqrt{(1 - (k \text{ sn } u)^2)} = \text{dn } u,$$

and (2') becomes $\ddot{y} = -(k \text{ sn } u)(\text{dn } u)$; which begins to resemble the first equation of (20). The question then is whether $\ddot{y}/k$, which is y''/U^2k, can be identified with $(\text{cn } u)'$; or, whether an identification $y'/U^2k = \text{cn } u$ is consistent with our assumption $\sin y = k \text{ sn } u$. Differentiating the latter expression,

$$y' \cos y = k \text{ cn } u \text{ dn } u,$$

and cancelling $\cos y = \text{dn } u$, we have indeed $y' = k \text{ cn } u$; hence $U = 1$. Our result, so far, is $\dot{x} = x' = 2y' = 2k \text{ cn } u$, with $k = \frac{1}{2}\dot{x}_0$ (since cn $0 = 1$); this is the only simple result in this case. To find x, we must know how to integrate cn t; but for our present purposes the simplest way of doing that is to check the result by differentiating (3').

THE INVARIANT FACTOR ALGORITHM

B. ABRAHAMSON, Rhodes University

R. V. Andree [1] gives a procedure for obtaining the inverse of a square matrix A. It amounts to reducing

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$$

by row and column operations to the form

$$\begin{pmatrix} P & I \\ 0 & Q \end{pmatrix}.$$

This implies that $PAQ = I$, whence $A^{-1} = QP$. In fact essentially the same procedure enables one to calculate the Smith canonical form of A when its entries

are drawn from a Euclidean ring (or even a principal ideal ring). Furthermore an analysis of the matrices P and Q which are by-products of this calculation provides one with an algorithm for many problems in module theory.

The invariant factor theorem states that if M is a free module over a principal ideal ring E, and if N is any submodule of M, then there exists a basis $x_1, \cdots, x_m$ of M, and scalars $e_1, \cdots, e_n$ in $E (n \leqq m)$ such that $e_1x_1, \cdots, e_nx_n$ form a basis of N, and e_i is a divisor of e_{i+1} for $i<n$. Furthermore the e_i, called invariant factors, are unique up to associates. The proof of this, together with a discussion of the applications, can be found in [2].

In proving the invariant factor theorem one begins with an arbitrary basis $x_1, \cdots, x_m$ of M. As N is a submodule of the free module M, N is free and so has a basis $y_1, \cdots, y_n$ (in practice the y's need only be a set of generators of N). Then

$$y_j = \sum_{i=1}^{m} a_{ij}x_i, \quad \text{where} \quad j = 1, \cdots, n, \quad \text{and} \quad a_{ij} \in E.$$

If the matrix (a_{ij}) is denoted by A, we can write this as $(y_1, \cdots, y_n) = (x_1, \cdots, x_m)A$. Observing that the operations (i) of multiplying x_i by a unit u of E, (ii) interchanging x_i and x_j, and (iii) adding any multiple of x_i to x_j have the effect of converting a basis into another basis, we apply these operations to the x- and y-bases and calculate the corresponding changes in A.

If $D(i, u)$ is the (unimodular) matrix obtained from the identity matrix I by multiplying u into the ith row of I, we have

$$(x_1, \cdots, x_i, \cdots, x_m) = (x_1, \cdots, ux_i, \cdots x_m)D(i, u^{-1}).$$

Thus multiplication of x_i by the unit u results in premultiplying A by the unimodular matrix $D(i, u^{-1})$. If the ith and jth rows of I are interchanged we get the unimodular matrix $R(i, j)$. Then

$$(\cdots, x_i, \cdots, x_j, \cdots) = (\cdots, x_j, \cdots, x_i, \cdots)R(i, j),$$

so that interchanging x_i and x_j amounts to premultiplying A by $R(i, j)$. Finally if $R(i+aj)$ denotes the matrix obtained from I by adding a times the jth row to the ith, we have

$$(\cdots, x_i, \cdots, x_j, \cdots) = (\cdots, x_i, \cdots, x_j + ax_i, \cdots)R(i - aj),$$

so adding ax_i to x_j amounts to premultiplying A by $R(i-aj)$. Note that if the matrices $D(i, u^{-1})$, $R(i, j)$, $R(i-aj)$ are premultiplied into A, they produce the corresponding row operations on $A: D(i, u^{-1})A$ has each element of the ith row of A multiplied by u^{-1}, $R(i, j)A$ has the ith and jth rows of A interchanged, $R(i-aj)A$ has a times the jth row of A subtracted from the ith. We also have

$$D(i, u^{-1})^{-1} = D(i, u), \quad R(i, j)^{-1} = R(i, j), \quad R(i - aj)^{-1} = R(i + aj).$$

If these matrices are postmultiplied into A, or the corresponding $n \times n$ ones

when A is an $m\times n$ matrix, we get the corresponding column operations performed on A in the cases of $D(i, u)$ and $R(i, j)$, but in the third case postmultiplication by $R(i+aj)$ has the effect of adding a times the ith column onto the jth.

Now it is proved that under the given hypotheses a succession of elementary row and column operations may be found to put A into diagonal form D with the invariant factors $e_1, \cdots, e_n$ down the main diagonal and all other entries zero. As each of these row (column) operations can be represented by pre(post)-multiplication by a unimodular matrix, and since the product of unimodular matrices is unimodular, we have the existence of unimodular matrices P and Q so that PAQ has the prescribed diagonal form D. To compute P and Q we write an $m\times m$ identity matrix I_m on the left of A and an $n\times n$ identity matrix I_n below:

$$\begin{pmatrix} I_m & A \\ & I_n \end{pmatrix}.$$

If a given row operation applied to A is also applied to I_m, it will convert I_m into a unimodular matrix R such that RA gives the result of the row operation on A. Similarly for column operations. If a sequence of row operations is carried out on A to produce B, then the same sequence will convert I_m into a unimodular matrix P such that $PA=B$. We have a similar statement for column operations. Applying both row and column operations to A to reduce it to D, we get the algorithm:

$$\begin{pmatrix} I_m & A \\ & I_n \end{pmatrix} \to \begin{pmatrix} P & D \\ & Q \end{pmatrix}.$$

A systematic procedure for carrying this out is to bring $e_1=\text{g.c.d.}(a_{ij})$ to the top left position by elementary transformations, annihilate the remaining elements of the first row and column and proceed by recursion. In a Euclidean ring one has the Euclidean algorithm available to find the g.c.d. so that the above procedure can always be carried out in this case.

If y is a column matrix consisting of m scalars, then Py may be found by submitting y to the same row operations as A. We can carry it along with the main computation:

$$(*) \qquad \begin{matrix} I_m & I_m & A & y \\ & & I_n & \end{matrix} \to \begin{matrix} P^{-1} & P & D & Py. \\ & & Q & \end{matrix}$$

To find P^{-1} observe that if P is obtained by the elementary row operations with matrices $R_1, \cdots, R_r$ in that order, then $P=R_rR_{r-1}\cdots R_2R_1$ so that $P^{-1}=R_1^{-1}\cdots R_r^{-1}$. It can be found by applying the inverse column operations to I_m while the row operations are being applied to A. Thus if the ith row of A is multiplied by u^{-1}, multiply the ith column of I_m (or its successor) by u; if a times the jth row of A is added to the ith, subtract a times the ith column of I_m

from the jth; if two rows of A are permuted, permute the corresponding columns of I_m, etc. This procedure is efficient but (it must be admitted) takes will-power.

In the following example E is taken to be the ring of integers, M the two-dimensional module of all ordered pairs of integers,

$$A = \begin{pmatrix} -18 & -22 \\ 12 & 14 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The form of the calculation is as in (*).

Set up the array:

$$\begin{array}{cccccccc} 1 & 0 & 1 & 0 & -18 & -22 & & 1 \\ 0 & 1 & 0 & 1 & 12 & 14 & & -1. \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{array}$$

Note that the g.c.d. of the entries of A is 2. We obtain 2 by applying the Euclidean algorithm to -22 and 14, the numbers in the second column of A. Hence we apply this algorithm in stages by means of row operations in the above array, using the inverse column operations on the leftmost matrix. To begin, add twice the second row to the first:

$$\begin{array}{cccccccc} 1 & -2 & 1 & 2 & 6 & 6 & & -1 \\ 0 & 1 & 0 & 1 & 12 & 14 & & -1. \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{array}$$

Then subtract twice the first row from the second:

$$\begin{array}{cccccccc} -3 & -2 & 1 & 2 & 6 & 6 & & -1 \\ 2 & 1 & -2 & -3 & 0 & 2 & & 1. \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{array}$$

Eliminate the 6 by subtracting three times the second row from the first:

$$\begin{array}{cccccccc} -3 & -11 & 7 & 11 & 6 & 0 & & -4 \\ 2 & 7 & -2 & -3 & 0 & 2 & & 1. \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{array}$$

Interchange rows and also columns:

$$\begin{array}{rrcrrcrrcr}
-11 & -3 & & -2 & -3 & & 2 & 0 & & 1 \\
7 & 2 & & 7 & 11 & & 0 & 6 & & -4. \\
 & & & & & & 0 & 1 & & \\
 & & & & & & 1 & 0 & &
\end{array}$$

Thus

$$P = \begin{pmatrix} -2 & -3 \\ 7 & 11 \end{pmatrix}, \qquad Py = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the actual applications one omits the parts which are not needed.

As to the significance of P and Q, if $PAQ=D$, then reverting to the x- and y-bases,

$$(y_1, \cdots, y_n) = (x_1, \cdots, x_m)A = (x_1, \cdots, x_m)P^{-1}DQ^{-1},$$

whence

$$(y_1, \cdots, y_n)Q = (x_1, \cdots, x_m)P^{-1}D.$$

If we set

$$(y_1', \cdots, y_n') = (y_1, \cdots, y_n)Q, \quad (x_1', \cdots, x_m') = (x_1, \cdots, x_m)P^{-1},$$

then we have that

$$(y_1', \cdots, y_n') = (x_1', \cdots, x_m')D,$$

so that $y_k' = e_k x_k'$ for $k \leqq n$ as required by the invariant factor theorem. P^{-1} is thus the unimodular matrix giving the change in the x-basis and Q that giving the change in the y-basis.

The first application, to the solution of a linear Diophantine system, illustrates a broad class of commonly occurring problems. If $Ax=y$ is to be solved for x, where A is an $m\times n$, x an $n\times 1$, y an $m\times 1$ matrix of scalars belonging to an Euclidean ring E, put A into diagonal form by means of $PAQ=D$. Then $PAQQ^{-1}x=Py$, or $Dx'=y'$, where $x'=Q^{-1}x$ and $y'=Py$. If a solution of $Dx'=y'$ exists it is easily found. Having found x', we have $x=Qx'$. Observe that in this calculation we need Py rather than P. It may be set out as follows:

$$\begin{array}{llll}
A & y \rightarrow & D & Py \rightarrow Qx'. \\
I & & Q & x'.
\end{array}$$

For example, to solve the system

$$\begin{aligned}
3p + 2q - 4r &= 14, \\
p + 4q - 2r &= 28,
\end{aligned}$$

in integers. Set up the array:

$$\begin{array}{rrrr} 3 & 2 & -4 & 14 \\ 1 & 4 & -2 & 28 \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} .$$

As the g.c.d. of the elements of A is 1, aim to isolate a 1 in a given row and column by making the remaining entries in that row and column zero. Here subtract three times the second row from the first:

$$\begin{array}{rrrr} 0 & -10 & 2 & -70 \\ 1 & 4 & -2 & 28 \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} .$$

Subtract 4 times the first column from the second and add twice the first column to the third:

$$\begin{array}{rrrr} 0 & -10 & 2 & -70 \\ 1 & 0 & 0 & 28 \\ 1 & -4 & 2 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} .$$

Add five times the third column to the second:

$$\begin{array}{rrrr} 0 & 0 & 2 & -70 \\ 1 & 0 & 0 & 28 \\ 1 & 6 & 2 & \\ 0 & 1 & 0 & \\ 0 & 5 & 1 & \end{array} .$$

Interchange rows and columns:

$$\begin{array}{rrrr} 1 & 0 & 0 & 28 \\ 0 & 2 & 0 & -70 \\ 1 & 2 & 6 & \\ 0 & 0 & 1 & \\ 0 & 1 & 5 & \end{array} .$$

Fill in the solution, by inspection, in the lower right hand column:

$$\begin{array}{ccccl}
1 & 0 & 0 & & 28 \\
0 & 2 & 0 & & -70 \\
\\
1 & 2 & 6 & & 28 = p' \\
0 & 0 & 1 & & -35 = q' \\
0 & 1 & 5 & & n = r' \ (n \text{ an arbitrary integer}).
\end{array}$$

The solution

$$Qx' = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} 28 \\ -35 \\ n \end{pmatrix} = \begin{pmatrix} -42 + 6n \\ n \\ -35 + 5n \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A second application is to the analysis of the null space and range of a linear transformation on a module. Let M and N be free modules with bases respectively $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$. Let $T: M \to N$ be a linear transformation with matrix A. Thus $Tx_i = \sum_{k=1}^{n} a_{ki} y_k$, or

$$(y_1, \cdots, y_n)A = (Tx_1, \cdots, Tx_m) = T(x_1, \cdots, x_m).$$

If $PAQ = D$ then

$$(y_1, \cdots, y_n)P^{-1}D = (Tx_1, \cdots, Tx_m)Q = T(x_1, \cdots, x_m)Q.$$

Setting $(y_1', \cdots, y_n') = (y_1, \cdots, y_n)P^{-1}$ and $(x_1', \cdots, x_m') = (x_1, \cdots, x_m)Q$, we have new bases for N and M respectively such that the matrix corresponding to T is D. If $D = \text{diag}(e_1, \cdots, e_r, 0, \cdots, 0)(n \times m$, not square!) then $Tx_k' = e_k y_k'$ for $k \leqq r$ and $Tx_k' = 0$ for $r < k$. Hence the null space of T is spanned by $x_{r+1}', \cdots, x_m'$ and its range by $e_1 y_1', \cdots, e_r y_r'$. Referring elements of M and N to their coordinates with respect to the original x- and y-bases respectively, the null space is spanned by the last n-r columns of Q and range by the first r columns of $P^{-1}D$. When E is a field and M and N are vector spaces, then $e_k = 1$ for $k \leqq r$ and the range is then spanned by the first r columns of P^{-1}.

A third application is the decomposition of a finitely generated module M into a direct sum of cyclic submodules. Let $g_1, \cdots, g_m$ be a set of generators for M and let $x_1, \cdots, x_m$ be a basis for a free module F over the same ring. There is a unique linear transformation $T: F \to M$ such that $Tx_i = g_i$. If N is the null space of T, a basis $x_1', \cdots, x_m'$ of F may be found such that $e_1 x_1', \cdots, e_n x_n'$ is a basis for N and e_i is a divisor of e_{i+1} for $i < n$. Putting $g_i' = Tx_i'$ the module M splits into a direct sum of cyclic submodules $(g_1') \oplus \cdots \oplus (g_m')$, where $(g_i') = E/e_i E$.

The computation starts with a set $y_1, \cdots, y_n$ of generators of N, the null space of T. A is defined by $(y_1, \cdots, y_n) = (x_1, \cdots, x_m)A$. As before, we have

$(y_1, \cdots, y_n)Q=(x_1, \cdots, x_m)P^{-1}D$ so that the required new basis of F is $(x_1', \cdots, x_m')=(x_1, \cdots, x_m)P^{-1}$. Thus the new generators required for the original module M are $(g_1', \cdots, g_m')=(g_1, \cdots, g_m)P^{-1}$. In the present application it is P^{-1} that we wish to find.

As a numerical example consider the abelian group G with generators g_1, g_2, g_3 and relations $53g_1-25g_2-9g_3=0$, $21g_1-10g_2-3g_3=0$, considered as a module over the integers. Let x_1, x_2, x_3 be a basis for the free module F. The null space of T is spanned by $y_1=53x_1-25x_2-9x_3$ and $y_2=21x_1-10x_2-3x_3$. Then

$$A = \begin{pmatrix} 53 & 21 \\ -25 & -10 \\ -9 & -3 \end{pmatrix}.$$

As we have to find P^{-1} only, write I to the left of A and as row operations are applied to A, apply the inverse column operations to I. Column operations may be freely applied to A without affecting the computation of P^{-1}. Thus

$$\begin{matrix} 1 & 0 & 0 & \quad 53 & 21 \\ 0 & 1 & 0 & \quad -25 & -10 \\ 0 & 0 & 1 & \quad -9 & -3 \end{matrix} \;\rightarrow\; \begin{matrix} 1 & -2 & 0 & \quad 3 & 1 \\ 0 & 1 & 0 & \quad -25 & -10 \\ 0 & 0 & 1 & \quad -9 & -3 \end{matrix}$$

on adding twice the second row to the first. We proceed as before to make the other entries in the row and column containing the 1 vanish, obtaining

$$\begin{matrix} 21 & -2 & 0 & \quad 0 & 1 \\ -10 & 1 & 0 & \quad 5 & 0. \\ -3 & 0 & 1 & \quad 0 & 0 \end{matrix}$$

Interchanging the last two columns:

$$\begin{matrix} 21 & -2 & 0 & \quad 1 & 0 \\ -10 & 1 & 0 & \quad 0 & 5. \\ -3 & 0 & 1 & \quad 0 & 0 \end{matrix}$$

Hence

$$P^{-1} = \begin{pmatrix} 21 & -2 & 0 \\ -10 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

so that $e_1=1$, $e_2=5$, $g_1'=21g_1-10g_2-3g_3$, $g_2'=-2g_1+g_2$, $g_3'=g_3$. So g_1' generates the zero subgroup, g_2' a cyclic subgroup of order 5, g_3' an infinite cyclic group. We have $G=(-2g_1+g_2)\oplus(g_3)$.

The same technique permits the reduction of an $n\times n$ matrix A over a field

Φ to canonical form, enabling one to calculate a nonsingular matrix N such that $N^{-1}AN$ has that form. Here A acts as a linear operator over the vector space M of all n-rowed column matrices with entries in Φ. Taking the ring E to be $\Phi[t]$, the ring of all polynomials in an indeterminate t with coefficients in Φ, we make M into a module over E by setting $p(t)g=p(A)g$, where $p(t)$ is a polynomial in t and $g\in M$. This defines the product of a scalar $p(t)$ in E and a vector g of M. It is easily verified that with this definition of multiplication, M is a module over the ring E. Clearly if $g_1, \cdots, g_n$ is a basis for the *vector space* M (over Φ), it will also be a set of generators for the *module* M (over E). Let F be the free module with basis $x_1, \cdots, x_n$ over E; let T be the module linear transformation defined from F into M by $Tx_k=g_k$. Then the null space of T is spanned by

$$y_i = \sum_{j=1}^{n} (\delta_{ji}t - a_{ji})x_j, \qquad i \leqq n.$$

Thus we apply the invariant factor algorithm to the matrix $tI-A$ to obtain unimodular (polynomial) matrices $P(t)$ and $Q(t)$ such that $P(t)(tI-A)Q(t)=D(t)$, where the invariant factors of A appear as the diagonal elements of $D(t)$. The vectors $(g_1', \cdots, g_n')=(g_1, \cdots, g_n)P(t)^{-1}$ will generate the cyclic subspaces of M with respect to A, the different invariant factors $e_1(t), \cdots, e_n(t)$ being their respective annihilating polynomials.

Consider the example

$$A = \begin{pmatrix} -6 & 0 & 2 \\ -15 & 0 & 5 \\ -18 & 0 & 6 \end{pmatrix}.$$

To find $P(t)^{-1}$ set up the array

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \qquad \begin{array}{ccc} t+6 & 0 & -2 \\ 15 & t & -5 \\ 18 & 0 & t-6 \end{array}$$

where the entries are elements of the ring E, *i.e.*, polynomials in t. The g.c.d. of the right-hand matrix is 1, which must be the first invariant factor. Applying the Euclidean algorithm to the first two rows we get a 1 in the third column. Thus subtract twice the first row from the second and then multiply through the second row by -1 to get

$$\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{array} \qquad \begin{array}{ccc} t+6 & 0 & -2 \\ 2t-3 & -t & 1. \\ 18 & 0 & t-6 \end{array}$$

The remaining elements in the second row are now annihilated by column operations:

$$\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{array} \qquad \begin{array}{ccc} 5t & -2t & -2 \\ 0 & 0 & 1. \\ -2t^2+15t & t^2-6t & t-6 \end{array}$$

Annihilating the remaining elements in the third column by row operations:

$$\begin{array}{ccc} 1 & -2 & 0 \\ 2 & -5 & 0 \\ 0 & t-6 & 1 \end{array} \qquad \begin{array}{ccc} 5t & -2t & 0 \\ 0 & 0 & 1. \\ -2t^2+15t & t^2-6t & 0 \end{array}$$

The g.c.d. of the remaining members is t. Add twice the second column to the first:

$$\begin{array}{ccc} 1 & -2 & 0 \\ 2 & -5 & 0 \\ 0 & t-6 & 1 \end{array} \qquad \begin{array}{ccc} t & -2t & 0 \\ 0 & 0 & 1. \\ 3t & t^2-6t & 0 \end{array}$$

Annihilate the remaining elements in the first row and column:

$$\begin{array}{ccc} 1 & -2 & 0 \\ 2 & -5 & 0 \\ 3 & t-6 & 1 \end{array} \qquad \begin{array}{ccc} t & 0 & 0 \\ 0 & 0 & 1. \\ 0 & t^2 & 0 \end{array}$$

Rearrange:

$$\begin{array}{ccc} -2 & 1 & 0 \\ -5 & 2 & 0 \\ t-6 & 3 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 0 & 0 \\ 0 & t & 0. \\ 0 & 0 & t^2 \end{array}$$

Thus

$$P(t)^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ -5 & 2 & 0 \\ t-6 & 3 & 1 \end{pmatrix}.$$

Then $g_1' = 0$, $g_2' = g_1 + 2g_2 + 3g_3$ is annihilated by A and $g_3' = g_3$ is annihilated by A^2. If the set g_2', Ag_3', g_3' is taken as a basis for the vector space M, A will have the matrix representation

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is the Jordan canonical form corresponding to the elementary divisors t and t^2. The analysis in the case where the invariant factors are products of

powers of distinct irreducible factors is more complex but runs on the same lines. As for N, its columns consist of the coordinates of the new basis of M with respect to the original basis. Thus

$$N = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 1 \end{pmatrix}.$$

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ARITHMETICAL FUNCTIONS OF GENERALIZED PRIMES

E. M. HORADAM, University of New England, Armidale, Australia

Introduction. Generalized primes were first introduced by Beurling [**1**], and later developed by Nyman [2]. In this paper they will be defined as follows:—

Suppose given a finite or infinite sequence $\{p\}$ *of real numbers (generalized primes) such that*

$$1 < p_1 < p_2 < p_3 < \cdots.$$

Form the set $\{l\}$ *of all possible p-products, i.e., products* $p_1^{v_1}p_2^{v_2}, \cdots$, *where* v_1, $v_2, \cdots$ *are integers* $\geqq 0$ *of which all but a finite number are* 0.

Call these numbers "l numbers" and suppose that no two l numbers are equal (if their v's are different). Then arrange $\{l\}$ *as an increasing sequence*:

$$1 = l_1 < l_2 < l_3 < \cdots.$$

Examples.

(a) $\{p\}: 6 < 35 < 143 < 323 < \cdots$ (infinite).

$\{l\}: 1 < 6 < 35 < 36 < 143 < 210 < 216 < \cdots.$

(b) $\{p\}: \dfrac{13}{11} < \dfrac{7}{5} < \dfrac{3}{2}.$ (finite).

$$\{l\}: \quad 1 < \frac{13}{11} < \frac{13^2}{11^2} < \frac{7}{5} < \frac{3}{2} < \frac{13^2}{11^3} < \frac{13}{11}\cdot\frac{7}{5} < \cdots.$$

THEOREM 1. *The power of a generalized prime in* $l_n! = l_n l_{n-1} \cdots l_2 l_1$ *is*

$$\left[\frac{l}{p}\right] + \left[\frac{l}{p^2}\right] + \left[\frac{l}{p^3}\right] + \cdots,$$

where $[m]$ *is the number of* l *numbers* $\leqq m$.

Proof. The number of factors of the product $l!$ which are multiples of p, is $[l/p]$; amongst these there are $[l/p^2]$ multiples of p^2, and so on. The sum of all these numbers gives the required power of p. The same reasoning also gives

THEOREM 2. $\phi(p^n) = [p^n] - [p^{n-1}]$, *where* $\phi(m)$ *is the number of* l *numbers less than* m *which are prime to* m.

The general formula for the Euler ϕ function does not hold as $\phi(m)$ is not multiplicative.

THEOREM 3. $\sum_{d|l} \phi(d) = [l]$.

(In this notation $d|l$ if $d \in \{l\}$ and there exists $l_d \in \{l\}$ such that $d \cdot l_d = l$. Call d the greatest common divisor of l_r and l_s, *i.e.*, $(l_r, l_s) = d$ if

(1) d is a common divisor of l_r and l_s

(2) every common divisor of l_r and l_s divides d.)

Proof. Consider the sequence

$$1 = l_1, l_2, \cdots, l. \tag{1}$$

If d is any divisor of l, there are in this sequence $[l/d]$ multiples of d, namely the numbers

$$1 \cdot d, l_2 \cdot d, \cdots l_r \cdot d, \cdots, (l/d) \cdot d. \tag{2}$$

Some of the numbers (2) have a g.c.d. d with l. The g.c.d. of $l_r \cdot d$ with l is d if and only if $(l_r, l/d) = 1$. There are $\phi(l/d)$ numbers $l_r d$ with this property. Since all numbers (1) have a g.c.d. with l,

$$\sum_{d|l} \phi(l/d) = [l].$$

But when d runs through all divisors of l, so does l/d, and the proof is complete.

Multiplicative functions. $\theta(l)$ is multiplicative if it is defined for all l and is not identically zero for any l and if $\theta(l_m l_n) = \theta(l_m)\theta(l_n)$ when $(l_m, l_n) = 1$. If $\theta(l)$ is multiplicative then $\theta(1) = 1$ and also

THEOREM 4. *If* $l = \prod_{j=1}^{k} p_j^{v_j}$, then

$$\sum_{d|l} \theta(d) = \prod_{j=1}^{k} \{1 + \theta(p_j) + \cdots + \theta(p_j^{v_j})\}.$$

Proof. If $l=1$, the right-hand side is taken to be 1. If $l>1$, the right-hand side is the sum of products of the form

$$\theta(p_1^{\beta_1}) \cdots \theta(p_k^{\beta_k}) = \theta(p_1^{\beta_1} \cdots p_k^{\beta_k})$$

in which all possible products of this form are contained and no product occurs more than once. This is what occurs on the left-hand side.

COROLLARY. *The function l^s is multiplicative and so*

$$\sum_{d|l} d^s = \prod_{j=1}^{k} (1 + p_j^s + \cdots + p_j^{v_j s}).$$

THEOREM 5. $\sum_{r=1}^{n} \tau(l_r) = \sum_{h=1}^{n} [l_n/l_h]$, *where $\tau(l)$ is the number of divisors of l.*

Proof. Let $f(l_n)$ denote the right-hand side. Now $[l_n/l_h] - [l_{n-1}/l_h] = 1$ or 0 according as l_h is a divisor of l_n or not. Hence

$$\begin{aligned} f(l_n) - f(l_{n-1}) &= \sum_{h=1}^{n}\left[\frac{l_n}{l_h}\right] - \sum_{h=1}^{n-1}\left[\frac{l_{n-1}}{l_h}\right] \\ &= \sum_{h=1}^{n}\left[\frac{l_n}{l_h}\right] - \sum_{h=1}^{n}\left[\frac{l_{n-1}}{l_h}\right] + \left[\frac{l_{n-1}}{l_n}\right] \\ &= \tau(l_n) + 0, \end{aligned}$$

$$\sum_{v=1}^{n} \tau(l_v) = \sum_{v=1}^{n} \{f(l_n) - f(l_{n-1})\} + 1.$$

(Putting $s=0$ and $s=1$ in the corollary to Theorem 4 gives the number of divisors of l and the sum of the divisors of l.)

The Möbius function. Define $\mu(l)=0$ if l has a square factor; $\mu(l)=(-1)^k$, where k denotes the number of prime divisors of l and l has no square factor; $\mu(1)=1$. Then $\mu(l)$ is multiplicative.

THEOREM 6.

$$\sum_{d|l} \mu(d) = \begin{cases} 0 & \textit{when} \quad l \neq 1, \\ 1 & \textit{when} \quad l = 1. \end{cases}$$

$$\sum_{d|l} \mu(d)/d = \begin{cases} \prod_{j=1}^{k} (1 - p_j^{-1}) & \textit{when} \quad l \neq 1, \\ 1 & \textit{when} \quad l = 1. \end{cases}$$

These results follow immediately from Theorem 4 by putting $\theta(l)=\mu(l)$ and $\theta(l)=\mu(l)/l$, respectively.

THEOREM 7. (An inversion formula). *If* $G(l_n) = \sum_{d|l_n} F(d)$, then $F(l_n) = \sum_{d|l_n} \mu(l_n/d)\, G(d)$.

Proof.

$$\sum_{d|l_n} \mu(l_n/d)\, G(d) = \sum_{d|l_n} \mu(d)\, G(l_n/d) = \sum_{d|l_n} \mu(d) \sum_{c|(l_n/d)} F(c)$$
$$= \sum_{(cd)|l_n} \mu(d)F(c) = \sum_{c|l_n} F(c) \sum_{d|(l_n/c)} \mu(d).$$

But, from Theorem 6,

$$\sum_{d|(l_n/c)} \mu(d) = 0$$

unless $c = l_n$ when the sum is 1. Hence the theorem is proved.

(Using this theorem with Theorem 3 shows that

$$\phi(l_n) = \sum_{d|l_n} \mu(l_n/d)[d].)$$

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THE 1960 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twenty-first William Lowell Putnam Mathematical Competition held on December 3, 1960, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of the University of California, Berkeley, California. The members of the team were George Bergman, Stefan Burr, and Jon Folkman; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Stephen Adler, M. Alan Landman, and Daniel Quillen; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Stephen Orszag, Frank Rubin, and John Wells; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Michigan State University, East Lansing, Michigan. The members of the team were Richard Freeman, Robert E. Greene, and Ted E. Petrie; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Harold G. Diamond, William B. Easton, and Jeffrey P. Rubens; to each of these a prize of ten dollars is awarded.

The five individuals ranking highest in the examination, named in alphabetical order, are William R. Emerson, California Institute of Technology; Jon Folkman, University of California at Berkeley; M. Hochster, Harvard University; Louis Jaeckel, University of California at Los Angeles; and Samuel Klein, College of the City of New York.

The five individuals ranking second highest in the examination, named in alphabetical order, are Robert Kilmoyer, Lebanon Valley College; John N. Mather, Harvard University; Daniel Quillen, Harvard University; Bill Waterhouse, Harvard University; and John F. Wilkinson, California Institute of Technology.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being Edward A. Bender, Harold M. Stark, and John F. Wilkinson; Polytechnic Institute of Brooklyn, Brooklyn, New York, the members of the team being Burton Fein, George Glauberman, and Errol Pomerance; Reed College, Portland, Oregon, the members of the team being Lynne Barnes, Loyd Hopper, and David Ragozin; University of California at Los Angeles, the members of the team being George Chapline, Louis Jaeckel, and Stanton Philipp; University of Notre Dame, Notre Dame, Indiana, the members of the team being Robert Burckel, William O'Connel, and James Wirth; and Yeshiva University, New York, New York, the members of the team being Stanley Boylan, Jonathan Ginsberg, and Benjamin Weiss.

Honorable mention is given to the following eighteen individuals, named in alphabetical order: Stephen Adler, Harvard University; (Mrs.) Rae H. Alderfer, Ursinus College; George Bergman, University of California at Berkeley; Stefan Burr, University of California at Berkeley; Rufus Clay, Yale University; Larry Dornhoff, University of Nebraska; Joseph Ferrar, Michigan State University; George Glauberman, Polytechnic Institute of Brooklyn; John A. Holbrook, Queen's University (Kingston, Ontario); Anthony Knapp, Dartmouth College; Martin Lampe, Harvard University; M. Alan Landman, Harvard University; John Lindsey, California Institute of Technology; Stephen Orszag, Massachusetts Institute of Technology; David Ragozin, Reed College; Lawrence C. Shepley, Swarthmore College; Stuart Sidney, Yale University; and Benjamin Weiss, Yeshiva University.

A total of eleven hundred and nine contestants from one hundred and sixty-six colleges and universities (one hundred and thirty-seven of these having teams)

entered the Competition. Eight hundred and sixty-seven contestants from one hundred and sixty colleges and universities (one hundred and twenty-eight having teams) participated in the examination on December 3, 1960.

The following is a list, in alphabetical order, of all colleges and universities which entered teams in the Competition: Adams State College, Agricultural and Mechanical College of Texas, American University, Amherst College, Anna Maria College, Arizona State College, Bethel College, Bowdoin College, Brandeis University, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Case Institute of Technology, Central Michigan University, Chatham College, College of the City of New York, College of the Holy Cross, Columbia University, Cornell University, Dakota Wesleyan University, Dana College, Dartmouth College, Delaware State College, Eastern New Mexico University, East Central State College, Fordham University, George Pepperdine College, Georgetown University, Georgia Institute of Technology, Grambling College, Grinnell College, Hamilton College (McMaster University), Harvard University, Haverford College, Humboldt State College, Illinois Institute of Technology, Iowa State University, Kalamazoo College, Kenyon College, King College, Knox College, Lebanon Valley College, Linfield College, Manhattan College, Mankato State College, Marquette University, Marymount College, Massachusetts Institute of Technology, McGill University, Memphis State University, Michigan State University, Monmouth College, Mount Allison University, Mundelein College, Newark College of Engineering, New Mexico State University, Northeastern University, Northern Montana College, North Texas State College, Oberlin College, Occidental College, Ohio State University, Ohio Wesleyan University, Oklahoma Baptist University, Oklahoma State University, Oregon State College, Otterbein College, Polytechnic Institute of Brooklyn, Pomona College, Purdue University, Queens College, Queen's University, Radcliffe College, Reed College, Rockford College, Rockhurst College, Rosary College, Rose Polytechnic Institute, Saint Ambrose College, Saint Francis Xavier University, Saint Martin's College, San Diego State College, San Jose State College, Seattle University, Shepherd College, South Dakota School of Mines, Southeastern State College, Southwestern at Memphis, Stanford University, Swarthmore College, Texas College, Tulane University, University of Alberta, University of Arizona, University of Buffalo, University of British Columbia, University of California (Berkeley), University of California (Los Angeles), University of Colorado, University of Detroit, University of Dallas, University of Illinois, University of Kansas, University of Manitoba, University of Michigan, University of Minnesota, University of Montreal, University of Nebraska, University of Notre Dame, University of Ottawa, University of Puerto Rico, University of Rochester, University of San Francisco, University of Santa Clara, University of Southeastern Louisiana, University of Texas, University of the South, University of Toronto, University of Washington, University of Waterloo, University of Western Ontario, United States Air Force Academy, United States

Naval Academy, Ursinus College, Valparaiso University, Villanova University, Wake Forest College, Washington State University, Washington University, Wayne State University, Wesleyan University, William Jewell College, William Marsh Rice University, Yale University, and Yeshiva University.

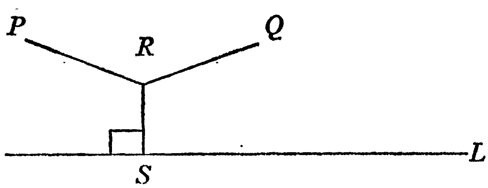
The following institutions, in alphabetical order, entered individual contestants only: Albright College, Antioch College, Bethany Nazarene College, Clemson College, College of Saint Catherine, College of Saint Thomas, Eastern Baptist College, Fresno State College, Hofstra College, Hunter College, Idaho State College, Kent State University, Macalester College, Madison College, Montana State College, North Carolina State College, Sacramento State College, Saint Olaf College, Southern University, State University of New York (Long Island Center), University of Arkansas, University of Houston, University of Iowa, University of Kentucky, University of Massachusetts, University of Wyoming, Westminster College, West Virginia University, and Xavier University.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

Those participating in the competition were given the following problems to solve:

Part I

1. Let n be a given positive integer. How many solutions are there in ordered positive integer pairs (x, y) to the equation
$$\frac{xy}{x+y} = n?$$
2. Show that if 3 points are inside a closed square of unit side, then some pair of them are within $\sqrt{6}-\sqrt{2}$ units apart.
3. Show that if t_1, t_2, t_3, t_4, t_5 are real numbers, then
$$\sum_{j=1}^{5} (1 - t_j)e^{\sum_{k=1}^{j} t_k} \leqq e^{e^{e^{e}}}.$$
4. Given two points in the plane, P and Q, at fixed distances from a line L, and on the same side of the line, as indicated, the problem is to find a third point R so that $PR+RQ+RS$ is a minimum, where RS is perpendicular to L. Consider all cases.



5. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.
6. A player throwing a die scores as many points as on the top face of the die, and is to play until his score reaches or passes a total n. Denote by $p(n)$ the probability of making exactly the total n, and find the value of $\lim_{n\to\infty} p(n)$.

7. Let $N(n)$ denote the smallest positive integer N such that $x^N = 1$ for every permutation x on n symbols, where 1 denotes the identity permutation. Prove that if $n > 1$,

$$\begin{aligned} \frac{N(n)}{N(n-1)} &= 1 \text{ if } n \text{ is divisible by 2 distinct primes,} \\ &= p \text{ if } n \text{ is a power of a prime } p. \end{aligned}$$

Part II

1. Find all solutions of $n^m = m^n$ in integers n and m $(n \neq m)$. Prove that you have obtained all of them.
2. Evaluate the double series

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-3k-j-(k+j)^2}.$$

3. The motion of the particles of a fluid in the plane is specified by the following components of velocity

$$\frac{dx}{dt} = y + 2x(1 - x^2 - y^2), \qquad \frac{dy}{dt} = -x.$$

Sketch the shape of the trajectories near the origin. Discuss what happens to an individual particle as $t \to +\infty$, and justify your conclusion.

4. Consider the arithmetic progression $a, a+d, a+2d, \cdots$, where a and d are positive integers. For any positive integer k, prove that the progression has either no exact kth powers, or infinitely many.
5. Define a sequence as follows:

$$a_0 = 0, \qquad a_1 = 1 + \sin(-1), \cdots, a_n = 1 + \sin(a_{n-1} - 1) \cdots.$$

Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k.$$

6. Any positive integer may be written in the form $n = 2^k(2l+1)$. Let $a_n = e^{-k}$ and $b_n = a_1 a_2 \cdots a_n$. Prove that $\sum b_n$ converges.
7. Let $g(t)$ and $h(t)$ be real, continuous functions for $t \geq 0$. Show that any function $v(t)$ satisfying the differential inequality

$$\frac{dv}{dt} + g(t)v \geqq h(t), \qquad v(0) = c,$$

satisfies the further inequality $v(t) \geqq u(t)$, where

$$\frac{du}{dt} + g(t)u = h(t), \qquad u(0) = c.$$

From this, conclude that for sufficiently small $t > 0$, the solution of

$$\frac{dv}{dt} + g(t)v = v^2, \qquad v(0) = c_1$$

may be written

$$v = \max_{w} \left[c_1 e^{-\int_0^t [g(s) - 2w(s)]ds} - \int_0^t e^{-\int_s^t [g(s_1) - 2w(s_1)]ds_1} w^2(s) ds \right],$$

where the maximization is over all continuous functions $w(t)$ defined over some t-interval $[0, t_0]$.

Solutions.* Part I

The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants.

1. $x=n$ does not furnish a solution and therefore $y=(xn)/(x-n)$ is an equivalent equation. (x, y) is a solution of the desired type of this equation if and only if

$$(x, y) = \left(n + i, \frac{n^2}{i} + n\right),$$

where i is a positive integral divisor of n^2.

2. If the points are collinear then two of them must be within $\frac{1}{2}\sqrt{2}$ units. If the points are not collinear a closed triangle with vertices on the boundary of the square and containing these points will have sides not less than the minimum distance between these points. If two vertices are on the same side of the square they will be within 1 unit. If two vertices are on opposite sides of the square a parallel motion of the side joining them until one vertex coincides with a vertex of the square yields a triangle not having smaller sides. Such a triangle will have its short side a maximum provided it is equilateral with side $\sqrt{6}-\sqrt{2}$ units.

3. Let

$$f(t_1, t_2, t_3, t_4, t_5) = \sum_{j=1}^{5} (1 - t_j) \exp\left[\sum_{k=1}^{j} t_k\right]$$

and let f_i denote the partial derivative of f with respect to t_i. The solution of the system $f_i=0$ $(i=1, \cdots, 5)$ yields

$$(*) \qquad f(e^{e^e}, e^e, e, 1, 0) = e^{e^{e^e}}$$

as the only possible extremum of f. Since $f\to-\infty$ as each $t_i\to\infty$ and $f\to 0$ as each $t_i\to-\infty$, it follows that $(*)$ is a maximum value of f.

4. Choose the coordinate system so that the x-axis coincides with L and the points P and Q have coordinates $(-a, p)$ and (a, q), respectively. Without loss of generality it can be assumed that $0<p\leqq q$. To solve the problem we need to find coordinates (x, y) of R such that the function

$$F(x, y) = |y| + \sqrt{\{(x+a)^2 + (y-p)^2\}} + \sqrt{\{(x-a)^2 + (y-q)^2\}}$$

is a minimum. Consideration of $F_x=0$, $F_y=0$ yields $x=\frac{1}{2}\sqrt{3}(p-q)$, $y=\frac{1}{2}(p+q) - \frac{1}{3}\sqrt{3}a$ as a solution provided that $(q-p)/(2a)\leqq\frac{1}{3}\sqrt{3}$. Otherwise $x=-a$, $y=p$ furnishes the minimum.

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper graders will enter into any correspondence concerning them.

5. Let $g(x)=x+h$ so that $f(x+h)=f(x)+h$ and $\{f(x+h)-f(x)\}/h=1$ for all $h\neq 0$. Therefore $f'(x)=1$ and $f(x)=x+c$. Letting $g(x)$ be the zero polynomial shows that it is necessary that $c=0$ and thus $f(x)=x$. It is easily seen that $f(x)=x$ is sufficient.

6. Let $p(n)=\frac{1}{6}\{p(n-1)+p(n-2)+p(n-3)+p(n-4)+p(n-5)+p(n-6)\}$ where $p(0)=1$ and $p(n)=0$ for negative n. Using this in the generating function equation $P(s)=p(0)+p(1)s+\cdots+p(n)s^n+\cdots$ after multiplication of both sides by $(6-s-s^2-s^3-s^4-s^5-s^6)$, we obtain $P(s)=6/(6-s-s^2-s^3-s^4-s^5-s^6)$. This can be expressed as

$$P(s)=\frac{2/7}{1-s}+\frac{2(15+10s+6s^2+3s^3+s^4)}{7(6+5s+4s^2+3s^3+2s^4+s^5)}\cdot$$

Expanding the two fractions into power series we obtain from the coefficients of $P(s)$ that $p(n)=2/7+f_n$ where the f_n are coefficients in the expansion of the second fraction. It can be shown that $f_n\to 0$ and therefore that $\lim p(n)=2/7$.

7. The symmetric group S_n has order $n!$ and contains elements of orders $1,\cdots,n$. Therefore $N(n)=LCM(1,\cdots,n)$. If n is divisible by two distinct primes, then $N(n)=\text{LCM}(1,\cdots,n-1)=N(n-1)$. If n is a power of a prime p, then $N(n)=p\,\text{LCM}(1,\cdots,n-1)=pN(n-1)$.

Solutions, Part II

1. It is clear that neither m nor n may be zero, and it is also readily verified that neither m nor n may equal $+1$ or -1 without having $m=n$. If m and n are both positive, the equation $n^m=m^n$ is equivalent to $(\log n)/n=(\log m)/m$. Now for $x>1$, $(\log x)/x$ is positive, increasing for $1<x<e$, and decreasing for $x>e$. Hence if $(\log n)/n=(\log m)/m$ is to be satisfied, with $m\neq n$, one of m and n must be between 1 and e, while the other is greater than e. Since 2 is the only integer between 1 and e, one of m and n must be 2, and it is immediately seen that this leads to a solution $2^4=4^2$, which must be the only solution if both m and n are positive and $m\neq n$. It is not possible for m and n to be of opposite sign, since the absolute value of one of m^n and n^m would be less than one while the other is greater than one. There is no solution with m and n both negative and of opposite parity since then m^n and n^m have opposite sign. If m and n are both negative and of the same parity, $m^n=n^m$ is equivalent to $(-m)^{-n}=(-n)^{-m}$, so that the only solution with m and n both negative is $(-2)^{-4}=(-4)^{-2}$, since there is only one positive solution. Thus the only solutions are $2^4=4^2$ and $(-2)^{-4}=(-4)^{-2}$.

2. If the given double series is written out it is seen that every term of the geometric progression $\sum_{n=0}^{\infty}2^{-2n}$ occurs once and only once. The double series is positive and convergent, hence absolutely convergent, so the rearrangement does not alter the sum. The sum of the double series is therefore $4/3$.

3. The last letter in the first equation is y although broken type made it appear to be v, and it was considered to be y by a majority of those attempting the problem. The behavior of the trajectories near the origin is determined from the linear terms, *i.e.* from the system $dx/dt=y+2x$, $dy/dt=-x$. This system can be solved explicitly. The trajectories are the curves

$$\frac{x}{x+y}=\log|x+y|+C,$$

all of which go to the origin tangent to the line $x+y=0$. The direction of increasing t is away from the origin. To find the behavior as $t\to+\infty$, introduce polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$. It is found that $(d/dt)r^2=4r^2\cos^2\theta(1-r^2)$ so that r^2 is nondecreasing when $r^2<1$ and r^2 is nonincreasing when $r^2>1$. Thus for any solution, the polar coordinate r approaches a limit which must be 1, since dr/dt is different from zero somewhere on every circle $r=$constant, where $r\neq 1$. Thus the solution curves all spiral outward or inward to the circle $r=1$.

If the last letter in the first equation is taken to be a constant v the same procedure will hold at the origin. The linear system is $dx/dt=y+2x(1-v^2)$ $dy/dt=-x$, and so long as $v^2\neq 1$, the complete system corresponds to the linear system. It is found that for $v^2>1$, all trajectories go toward the origin, while if $v^2<1$, all trajectories go away from the origin. When $v^2=1$, the original system is $dx/dt=-2x^3+y$, $dy/dt=-x$, and we find $(d/dt)r^2=-4x^4$, which implies that all trajectories go to the origin. As $t\to+\infty$, we have all trajectories going to the origin if $v^2\geqq 1$. If $v^2<1$, it can be shown that there is a closed curve surrounding the origin to which all trajectories converge.

4. Suppose that an integer x is such that $x^k\equiv a \pmod d$, *i.e.* x^k is a member of the given progression. Then for any positive integer n, $(x+nd)^k\equiv x^k \pmod d$ $\equiv a \pmod d$, so that $(x+nd)^k$ is also in the given progression. Thus, if the progression contains one kth power it contains infinitely many. There are arithmetic progressions which contain no kth powers. *E.g.*, the progression $2, 5, 8, \cdots$ contains no squares (or any even powers).

5. Let $b_n=a_n-1$. Then $b_0=-1$ and $b_n=\sin b_{n-1}$. By induction it follows that every b_n is negative, and that the b_n form an increasing sequence. Hence they have a limit L which must satisfy $L=\sin L$. Thus $L=0$, $b_n\to 0$, and $a_n\to 1$. It is well known that if a sequence $\{a_n\}$ has a limit, the sequence of averages $\{n^{-1}\sum_1^n a_k\}$ has the same limit. Hence the required limit is 1.

6. The problem means that every positive integer can be written uniquely in the form $2^k(2l+1)$, where k and l are nonnegative integers. Thus $k=0$ when n is odd, and k is a positive integer when n is even. Hence $a_n=1$ when n is odd, and $a_n\leqq e^{-1}$ when n is even. Therefore $b_{2n}=b_{2n+1}$ and the series is equivalent to $b_1+2b_2+2b_4+\cdots$. But since $b_{2n+2}/b_{2n}=a_{2n+1}a_{2n+2}\leqq e^{-1}$, the latter series converges. Since the terms approach zero, the grouping of the given series does not affect convergence and the given series converges.

7. Let $p(t)=v(t)-u(t)$. At $t=0$, $p(0)=v(0)-u(0)=c-c=0$. Moreover, subtracting the relation for u from that for v, we find that $(dp/dt)+g(t)p\geqq 0$ or, multiplying by $\exp\{\int_0^t g(s)ds\}$ that

$$d/dt\left[p(t)\exp\left\{\int_0^t g(s)ds\right\}\right]=0.$$

Integrating and noting that $p(0)=0$, we have for $t\geqq 0$, $p(t)\exp\{\int_0^t g(s)ds\}\geqq 0$. But, since the exponential is positive, we have $p(t)\geqq 0$ for $t\geqq 0$, so that $v(t)\geqq u(t)$ for $t\geqq 0$.

Now let $w(t)$ be any continuous function on $t\geqq 0$. Let $g_1(t)=g(t)-2w(t)$ and let $h(t)=-\{w(t)\}^2$. Then, if v satisfies $(dv/dt)+g(t)v=v^2$ with $v(0)=c_1$,

$$(dv/dt)+g_1(t)v=v^2-2wv=(v-w)^2-w^2\geqq -w^2=h(t)$$

with equality if and only if w is chosen to be exactly the solution v. But, if $u'+g_1(t)u=h(t)$ with $u(0)=c_1$, we can solve for u and obtain

$$u(t)=c_1\exp\left\{-\int_0^t[g(s)-2w(s)]ds\right\}$$
$$-\int_0^t\exp\left\{-\int_s^t[g(s_1)-2w(s_1)]ds_1\right\}w^2(s)ds.$$

Now by the first part, $v(t)\geqq u(t)$ with equality if and only if $w(t)$ is chosen equal to $v(t)$. Thus $v(t)$ is the maximum of all possible $u(t)$ as $w(t)$ ranges over all continuous functions.

CORRECTIONS

In the paper *The wedge product* by Gerald Berman (this MONTHLY, vol. 68, 1961, pp. 112–119), there is an omission following (2.2) on page 112. This should read "where the summation is taken over all values of the μ's and γ's such that $\mu_i+\gamma_i\equiv\epsilon_i \pmod 2$, $i=1,\cdots,n$, and $\cdots$." Again, in (2.5), the sums $\mu_i+\gamma_i$ are taken mod 2. Finally, the * mapping in Section 3 is *not* an automorphism, but it is useful in describing the algebra.

In the paper *A generalized Fibonacci sequence* by A. S. Horadam (this MONTHLY, vol. 68, 1961, pp. 455–459), equation (17) on page 457 is incorrect. It should read

$$\frac{H_{n+r}+(-1)^rH_{n-r}}{H_n}=F_{r+1}+F_{r-1}=a^r+b^r. \tag{17}$$

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, University of Washington

Material for this department should be sent to Roy Dubisch, Department of Mathematics, University of Washington, Seattle 5, Washington.

A FUNCTIONAL INEQUALITY*

GEORGE BRAUER, University of Minnesota

Ewing and Utz [2] determined all real-valued continuous functions which satisfy the functional equation

$$f^n(x) = f(x) \qquad -\infty < x < \infty, \tag{1}$$

where f^n denotes the nth iterate of f and n is a given positive integer. Donald I. Kurth and I [1] solved the following functional inequality

$$f^n(x) \geqq x, \qquad -\infty < x < \infty. \tag{2}$$

In this note I shall study a more general functional inequality.

Throughout the paper $\phi(x)$ denotes a continuous strictly increasing function such that $\lim_{x\to\infty} \phi(x) = \infty$, $\lim_{x\to-\infty} \phi(x) = -\infty$ and $f(x)$ denotes a continuous function of x. I shall obtain some properties of the solutions of the functional inequality

$$f^n(x) \geqq \phi(x) \qquad -\infty < x < \infty. \tag{3}$$

THEOREM 1. *If $f(x)$ is a solution of* (3), *then* $\lim_{x\to\infty} f(x) = \infty$ *or* $\lim_{x\to\infty} f(x) = -\infty$.

Proof. If the theorem is false, there exists a number a, and a sequence $\{x_j\}$ tending to infinity such that $f(x_j) \to a$. For each value of n, $\lim_{j\to\infty} f^n(x_j) = f^{n-1}(a)$, while x_j tends to infinity. This situation is impossible if $f(x)$ is a solution of (3).

THEOREM 2. *If $f(x)$ is a solution of* (3) *and* $\lim_{x\to\infty} f(x) = -\infty$, *then* $\lim_{x\to-\infty} f(x) = \infty$.

Proof. If not, there exists a number $b \geqq -\infty$ and a sequence $\{y_j\}$ tending to minus infinity such that $f(y_j)$ tends to b. Since $f(x)$ tends to minus infinity as x tends to infinity, for sufficiently large values of j there exists a point x_j such that $f(x_j) = y_j$ and x_j tends to infinity. Then $f^2(x_j)$ tends to b and for $n > 2, f^n(x_j)$ tends to $f^{n-2}(b)$ as j tends to infinity. On the other hand $\phi(x_j)$ tends to infinity and we have the same type of contradiction as in Theorem 1.

THEOREM 3. *If $f(x)$ is a solution of* (3) *for an odd value of n, then* $\lim_{x\to\infty} f(x) = \infty$.

Proof. If not, $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$. There exists a fixed-point c of the mapping f. The interval $[c, \infty)$ is mapped onto an interval con-

* This work was supported by Nonr 710(16).

taining $(-\infty, c]$ by f, and the interval $(-\infty, c]$ is mapped onto an interval containing $[c, \infty)$. It is easy to see that all odd iterates of f map the interval $[c, \infty)$ onto an interval containing $(-\infty, c]$. Since $\phi(x)$ tends to infinity as x tends to infinity, f cannot be a solution of (3) for an odd value of n.

LEMMA. *If* $\limsup_{x\to\infty} f(x) = \infty$ *and* x_0 *is a point such that* $f(x_0) < x_0$, *then, for each integer* r, *there is a point* $x_r \geqq x_0$ *such that* $f^r(x_r) = x_0$.

This lemma is proved rather easily by use of finite induction and the intermediate value theorem.

THEOREM 4. *If* $f(x)$ *is a solution of* (3) *such that* $f(x)$ *tends to infinity as* x *tends to infinity, then* $f(x) \geqq \phi(x)$ *whenever* $f(x) < x$. *In particular this holds if* $f(x)$ *is a solution of* (3) *for an odd value of* n.

Proof. Suppose x_0 is such that $f(x_0) < x_0$, $f(x_0) < \phi(x_0)$. By the preceding lemma, there exists a point x_{n-1} such that $x_{n-1} > x_0$, $f^{n-1}(x_{n-1}) = x_0$. Thus $f^n(x_{n-1}) = f(x_0) < \phi(x_0) < \phi(x_{n-1})$, and f is not a solution of (3).

COROLLARY. *If* $f(x)$ *is a solution of*

$$f^n(x) \geqq x \tag{4}$$

for an odd value of n, *then* $f(x) \geqq x$.

LEMMA. *If* $f(x)$ *satisfies a Lipschitz condition, with Lipschitz constant* L,

$$|f^n(x) - x| \leqq \left(\sum_{k=1}^{n} L^k\right)|f(x) - x|.$$

Proof. We have $|f^2(x) - f(x)| = |f(f(x)) - f(x)| \leqq L|f(x) - x|$ and $|f^3(x) - f^2(x)| = |f(f^2(x)) - f(f(x))| \leqq L^2|f(x) - x|$. More generally $|f^k(x) - f^{k-1}(x)| \leqq L^{k-1}|f(x) - x|$. Hence

$$|f^n(x) - x| \leqq \sum_{k=1}^{n} |f^k(x) - f^{k-1}(x)| \leqq \left(\sum_{k=1}^{n} L^k\right)|f(x) - x|.$$

THEOREM 5. *If* $f(x)$ *is a solution of* (3), *for an odd value of* n, *and it satisfies a Lipschitz condition with Lipschitz constant* L, *and*

$$m = \inf_{\phi(x) > x} [(f(x) - x)]/[(\phi(x) - x)],$$

then

$$n > \log[(1 + (L-1)/m)]\log L \qquad \textit{for } L \neq 1 \quad \textit{and} \quad n > 1/m \qquad \textit{for } L = 1.$$

In particular if $m = 0$, *then* $f(x)$ *is not a solution of* (3) *for any odd value of* n.

Note: It follows from Theorem 1 that $m \geqq 0$.

Proof. If ϵ is a positive number and x is a point such that $\phi(x) > x$, then

$0<f(x)-x<(m+\epsilon)(\phi(x)-x)$. From the preceding lemma we have

$$f^n(x) - x \leqq \sum_{k=1}^{n} L^k[f(x) - x] \leqq \sum_{k=1}^{n} L^k(m + \epsilon)(\phi(x) - x).$$

If $L\neq 1$,

$$f^n(x) - x \leqq \left(\sum_{k=1}^{n} L^k \right) (m + \epsilon)(\phi(x) - x).$$

If $f(x)$ is a solution of (3) then, since ϵ is arbitrary,

$$m \sum_{k=1}^{n} L^k = m(L^n - L)/(L - 1) < 1,$$

and thus $n>[\log[1+(L-1)/m]]/[\log L]$. If $L=1$, we show in a similar manner that $n>1/m$.

If the requirement that $f(x)$ satisfy a Lipschitz condition is removed, then the theorem is no longer true. For example let $\phi(x)=x^{1/5}$ if $0\leqq x\leqq 1$ and $\phi(x)=x$ if $x<0$, or $x>1$. Then, let $f(x)=x^{1/3}$ if $0\leqq x\leqq 1$ and $f(x)=x$ if $x<0$ or $x>1$. Then $f^n(x)\geqq\phi(x)$ for $n\geqq 2$ despite the fact that $(f(x)-x)/(\phi(x)-x)$ approaches zero as x approaches zero through positive values.

THEOREM 6. *If $\phi(x)-x$ is nondecreasing on the set $\{x|\phi(x)>x\}$, then the following conditions ensure that $f(x)$ is a solution of* (3) *for $n>[1/m]$*:

(i) $$f(x) \geqq \phi(\beta)$$

whenever $f(x)<x$, where $\beta=\sup\{x|f(x)<x\}$,

(ii) $$\inf [f(x) - x)]/[\phi(x) - x] \geqq m > 0,$$

where the infinum is taken over all x such that $\phi(x)>x$.

Proof. We note first that if x' is a point such that $\phi(x')\leqq x'$ then our assumptions insure that $f(x')\geqq\phi(x')$. Let n_0 denote the least positive integer such that $f^{n_0}(x')<\phi(x')$. Then $f^{n_0-1}(x')\geqq\phi(x')$, so that $f(f^{n_0-1}(x'))<f^{n_0-1}(x')$; hence $f^{n_0-1}(x')<\beta$. By (i) we have $f^{n_0}(x')=f[f^{n_0-1}(x')]>\phi(\beta)$. If $\beta<x'$, $f(x')>x'$ and it is easy to show that $f^n(x')\geqq x'\geqq\phi(x')$ for all n. If $\beta\geqq x'$, then $f^{n_0}(x')>\phi(\beta)>\phi(x')$, since ϕ is increasing; we thus have a contradiction and we may conclude that $f^n(x')\geqq\phi(x')$ for all positive integers n and all points x' such that $\phi(x')\leqq x'$.

It remains to deal with points x'' such that $\phi(x'')>x''$.

If for some integer n_0 we have $f^{n_0}(x'')>\phi(f^{n_0}(x''))$, then, by the same kind of induction as was used earlier in the proof, we may show that $f^n(x'')\geqq\phi(x'')$ for all $n\geqq n_0$. Thus we need only consider points x'' such that $f^k(x'')<\phi(f^k(x''))$ ($k\leqq[1/m]$), that is, points x'' such that for $k\leqq[1/m]$ the point $f^k(x'')$ belongs to the set $\{y|\phi(y)>y\}$. For such points we have $f^k(x'')-f^{k-1}(x'')\geqq m[\phi(f^{k-1}(x'')-f^{k-1}(x'')]$. The point x'' belongs to the set $\{y|\phi(y)>y\}$, and

since $\phi(y)-y$ is nondecreasing on this set, all points to the right of x'' belong to it. In particular, since $f^{k-1}(x'')>x''$ for $0<k<[1/m]$, by our assumption on ϕ and the above equation, $f^k(x'')-f^{k-1}(x'')\geqq m[\phi(x'')-x'']$. If $n>[1/m]$,

$$f^n(x) - x'' = \sum_{k=1}^{n} f^k(x'') - f^{k-1}(x'') \geqq nm[\phi(x'') - x''] \geqq \phi(x'') - x'',$$

so that $f^n(x'')\geqq\phi x''$. This completes the proof of the theorem.

We now deal with the case where $f(x)$ is a solution of (3) for an even value of n.

THEOREM 7. *If $f(x)$ is a solution of* (3) *for an even value of n, then $f^2(x)\geqq\phi(x)$ whenever $f^2(x)<x$.*

Proof. It follows from Theorems 1 and 2 that $f^2(x)$ tends to infinity as x tends to infinity. Theorem 7 now follows from Theorem 5.

For the inequality

$$f^n(x) \geqq x \tag{4}$$

we have more specific information.

THEOREM 8. *If $f(x)$ is a solution of* (4) *for an even value of n then either $f(x)\geqq x$ or $f^2(x)=x$.*

Proof. If there exists a point x such that $f(x)<x$, then $\lim_{x\to-\infty}f(x)=\infty$, $\lim_{x\to\infty}f(x)=-\infty$. There exists a fixed point c of the mapping f. If $x>c$, then $f(x)\neq c$ for otherwise there would exist a point $x_1>c$ such that $f^k(x_1)=c<x_1$ for $k\geqq 2$. Hence for $x>c$ we have either $f(x)>c$ or $f(x)<c$. In the former case $\lim_{x\to\infty}f(x)=\infty$ and $f(x)>x$ for all values of x. In the latter case it is easy to see that $f(x)>c$ for $x<c$. If this were not true, there would exist a point $y_2<c$, such that $f(y_2)\leqq c$. Since $\lim_{x\to\infty}f(x)=-\infty$ in the case we are considering, there exists a point $x_2>c$ such that $f(x_2)=y_2$. Hence $f^2(x_2)\leqq c<x_2$. This is impossible by Theorem 7. Hence $f(x)>c$ for $x<c$.

Now let M_1 be a number greater than c. The function f maps the interval $[c, M_1]$ into an interval $[m_1, c]$ (note that $m_1<c$). This interval is mapped by f onto an interval $[c, M_2]$ with $M_2>c$ and the latter interval is mapped by f onto an interval $[m_2, c]$ with $m_2<c$. We will show that $M_1=M_2$.

We have

$$M_2 = \max_{c\leqq x\leqq M_1} f^2(x) \geqq f^2(M_1) \geqq M_1.$$

If $M_1<M_2$, then $[c, M_1]\subset[c, M_2]$ and $[m_1, c]\subseteq[m_2, c]$; that is $m_1\geqq m_2$. But since

$$m_2 = \min_{m_1\leqq x\leqq c} f^2(x),$$

there exists a point x_1 such that $m_1\leqq x_1<c$ and $f^2(x_1)=m_2\leqq m_1\leqq x_1$. Since $f(x)$ is a solution of (4) we must have $m_1=m_2$ and consequently $M_1=M_2$ or $f^2(M_1)=M_1$.

Since M_1 was chosen arbitrarily we have $f^2(x)=x$ for $x>c$. Similarly one can show that $f^2(x)=x$ for $x<c$. Hence $f^2(x)$ is identically equal to x. This completes the proof.

I wish to thank Professor Seymour Schuster and my colleague Mr. William A. Dolid for suggesting the problem.

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A BOUND FOR THE SOLUTION OF A LINEAR EQUATION

A. B. FARNELL, Convair Scientific Research Laboratory

The object of the present note is to establish a bound for solutions of the equation $\ddot{y}+f(t)y=0$, assuming that $f(t)$ is continuous and that $0<f_1\leqq f(t)\leqq f_2$.

The case $y_0>0$, $\dot{y}_0>0$ is considered first. For $y>0$, $\dot{y}>0$, it follows that

$$-2f_2y\dot{y} \leqq 2\dot{y}\ddot{y} \leqq -2f_1y\dot{y}. \tag{1}$$

Integrating from 0 to t, we obtain $\dot{y}_0^2-f_2(y^2-y_0^2)\leqq\dot{y}^2\leqq\dot{y}_0^2-f_1(y^2-y_0^2)$. Noting the zeros of the left- and right-hand sides of this inequality, and that $\ddot{y}$ is bounded away from zero if y is bounded away from zero, we find the following inequality for the first maximum M occurring to the right of $t=0$:

$$M_1 \equiv \sqrt{(y_0^2+\dot{y}_0^2/f_2)} \leqq M \leqq \sqrt{(y_0^2+\dot{y}_0^2/f_1)} \equiv M_2. \tag{2}$$

Supposing this maximum occurs at $t=t'$, integration of (1) from t to t', where $0\leqq t\leqq t'$, yields $-f_2(M^2-y^2)\leqq-\dot{y}^2\leqq-f_1(M^2-y^2)$, from which we obtain

$$\int_0^{t'}\sqrt{f_2}dt \geqq \int_{y_0}^{M}\frac{dy}{\sqrt{(M^2-y^2)}} \geqq \int_0^{t'}\sqrt{f_1}dt,$$

and thus that

$$[\tfrac{1}{2}\pi - \arcsin(y_0/M)]/\sqrt{f_2} \leqq t' \leqq [\tfrac{1}{2}\pi - \arcsin(y_0/M)]/\sqrt{f_1}.$$

To investigate the behavior of y between $t=t'$ and the first zero of y to the right of t', occurring at $t=T$, we begin with the inequality $-2f_1y\dot{y}\leqq 2\dot{y}\ddot{y}\leqq -2f_2y\dot{y}$, and integrate from t' to t, where $t'\leqq t\leqq T$, to obtain

$$-f_1(y^2-M^2) \leqq \dot{y}^2 \leqq -f_2(y^2-M^2). \tag{3}$$

Hence

$$\int_{t'}^{T}-\sqrt{f_2}\,dt \leqq \int_M^0\frac{dy}{\sqrt{(M^2-y^2)}} \leqq \int_{t'}^{T}-\sqrt{f_1}\,dt,$$

implying in turn that

$$(\tfrac{1}{2}\pi)/\sqrt{f_2} \leqq T - t' \leqq (\tfrac{1}{2}\pi)/\sqrt{f_1},$$

and thus that

$$[\pi - \text{arc}\,\sin(y_0/M)]/\sqrt{f_2} \leqq T \leqq [\pi - \text{arc}\,\sin(y_0/M)]/\sqrt{f_1}.$$

Since $\ddot{y}<0$ for $y>0$, the maximum of $|\dot{y}|$ over $[0, T]$ occurs at either $t=0$ or $t=T$. Inequality (2) implies $|\dot{y}_0| \leqq \sqrt{(f_2)}M$, while (3) yields

$$\sqrt{f_1}\,M \leqq |\dot{y}_T| \leqq \sqrt{f_2}\,M.$$

Thus $\sqrt{(f_2)}M$ furnishes an upper bound for $|\dot{y}|$ over $[0, T]$.

If $y_0>0$ and $\dot{y}_0<0$, we obtain, proceeding similarly,

$$[\text{arc}\,\sin\,(y_0/M_1)]/\sqrt{f_2} \leqq T \leqq [\text{arc}\,\sin\,(y_0/M_2)]/\sqrt{f_1},$$

and,

$$\sqrt{(f_1)}\,M_2 \leqq |\dot{y}_T| \leqq \sqrt{(f_2)}\,M_1.$$

Since $M_1 \leqq M \leqq M_2$, we can alter these inequalities to resemble the ones obtained previously:

$$[\text{arc}\,\sin\,(y_0/M)]/\sqrt{f_2} \leqq T \leqq [\text{arc}\,\sin\,(y_0/M)]/\sqrt{f_1},\quad \sqrt{(f_1)}\,M \leqq |\dot{y}_T| \leqq \sqrt{(f_2)}\,M.$$

Exactly similar results are obtained for $y_0<0$.

The simplest results follow for $y_0=0$. In this case, beginning the first interval with a slope of $|\dot{y}_0|$, we have an upper bound of $|\dot{y}_0|/\sqrt{f_1} \equiv a$ for $|y|$ over the first interval, an upper bound of $\sqrt{(f_2)}M \leqq |\dot{y}_0|\sqrt{f_2/f_1} \equiv |\dot{y}_0|r$ for $|\dot{y}|$, and a lower bound of $\pi/\sqrt{f_2} \equiv b$ for the interval width. These conditions give rise to an upper bound for the greatest possible variation of y. Thus, for the second interval, we begin with a slope in absolute value not greater than $|\dot{y}_0|r$, obtain an upper bound for $|y|$ of $|\dot{y}_0|r/\sqrt{f_1}=ar$, and end with a slope in absolute value not greater than $\sqrt{f_2}(|\dot{y}_0|r/\sqrt{f_1}) = |\dot{y}_0|r^2$, etc. Thus $|y|$ will be bounded by the expression

$$ar^{t/b} = [|\dot{y}_0|/\sqrt{f_1}][\sqrt{(f_2/f_1)}]^{\sqrt{(f_2)}t/\pi} = Be^{k't},$$

which has values of $a, ar, \cdots$, at $t = 0, b, \cdots$, where $B = |\dot{y}_0|/\sqrt{f_1}$, $k' = (\sqrt{f_2}/2\pi)\ln\,(f_2/f_1)$.

If we begin with $y_0 \neq 0$, then the first interval must be treated separately. It would seem simplest to state a bound over the first interval of width b', where $b' = [\pi - \text{arc}\,\sin\,(y_0/M)]/\sqrt{f_2}$, corresponding to $y_0>0$, $\dot{y}_0>0$, or $b' = [\text{arc}\,\sin\,(y_0/M)]/\sqrt{f_2}$, corresponding to $y_0>0$, $\dot{y}_0<0$. For these cases, we take $\sqrt{(y_0^2+\dot{y}_0^2/f_1)}$ as an upper bound of M, and $\sqrt{f_2}$ times this quantity as an upper bound for $|\dot{y}|$, over the first interval. The only modification then of the previous bound, beyond an axis shift, for $t \geqq b'$ is the replacement of $|\dot{y}_0|$ by $\sqrt{(\dot{y}_0^2+f_1y_0^2)}$.

Loud [1] obtained a bound on $|y|$ of the form Ae^{kt}, where k is equal to $(f_2-f_1)/[2\sqrt{2}\sqrt{(f_2+f_1)}]$, established assuming only that $3f_1+f_2 \geqq 0$. For

certain values of the constants, the result just obtained is somewhat better; *e.g.*, for $f_1=9$, $f_2=16$, we find that $k=0.495$ and $k'=0.366$, approximately, or for $f_2/f_1=1+\epsilon$, where $\epsilon>0$ is small, k/k' is approximately $\pi/\sqrt{2}$.

To explicitly illustrate the type of unbounded behavior inferred by the previous discussion, it is simplest to consider an example in which $f(t)$ is piecewise continuous. Let $f(t)$ be of period $\frac{3}{4}\pi$, and

$$f(t) = 1,\ 0 \leqq t < \tfrac{1}{2}\pi; \qquad f(t) = 4,\ \tfrac{1}{2}\pi \leqq t < \tfrac{3}{4}\pi.$$

The basic equation has a unique continuous solution for functions of this type, and the previously established bound is valid.

Assuming $y(0)=0$, $\dot{y}(0)=\dot{y}_0>0$, we obtain

$$\begin{aligned} y(t) &= \dot{y}_0 \sin t, & 0 &\leqq t < \tfrac{1}{2}\pi, \\ y(t) &= \dot{y}_0 \cos 2(t - \tfrac{1}{2}\pi), & \tfrac{1}{2}\pi &\leqq t < \tfrac{3}{4}\pi, \\ y(t) &= -2\dot{y}_0 \sin (t - \pi), & \tfrac{3}{4}\pi &\leqq t < \tfrac{5}{4}\pi, \end{aligned}$$

etc. Thus, beginning with a slope of $\dot{y}_0$, $y(t)$ attains a maximum of $\dot{y}_0$ on the first interval of width $\frac{3}{4}\pi$ and its slope approaches $-2\dot{y}_0$ as t approaches $\frac{3}{4}\pi$. These numbers will be multiplied by -2 for the second interval, etc. A bounding curve for this example could be $|\dot{y}_0|2^{t/b}$, where $b=\frac{3}{4}\pi$, and, if we shift this curve $\frac{1}{2}\pi$ units to the right, the maxima of $|y(t)|$ actually lie on the resulting curve.

Perhaps the best-known example is furnished by the Mathieu equation: $\ddot{y}+(a+b\cos 2t)y=0$. For $a>|b|>0$, it is of the type under consideration here. Letting y_1 and y_2 represent a pair of independent solutions, Ince ([2], pp. 175–178) shows that for certain values of a and b, y_1 may be chosen as a periodic function. In such a case y_2 is then aperiodic and of the form $c_1tp_1(t)+c_2p_2(t)$, where $p_1(t)$ and $p_2(t)$ are periodic functions representable by series of Fourier type. Thus a bound for $|y(t)|$ would be of the form $At+B$.

The derivation of the inequalities for M, T, and $|\dot{y}_T|$ was suggested by similar derivations established by Taam [3] for the equation $\ddot{x}+p(t)x+2q(t)x^3=F(t)$. Assuming that p, q, and F have a common period L, p and q are even while F is odd, and $0<p_1\leqq p\leqq p_2$, $0<q_1\leqq q\leqq q_2$ for all t, and $0\leqq F_1\leqq F\leqq F_2$ over $(0, \frac{1}{2}L)$, he established a variety of facts concerning the existence and behavior of harmonic and subharmonic solutions. A careful study of his derivations indicates that, apart from the corollaries, all the arguments and conclusions seem to be valid if $p_1=0$, or even if $p(t)\equiv 0$. Thus his paper seems to yield results which complement those obtained by Morris [4, 5] for the equation $\ddot{x}+2x^3=e(t)$ where it was assumed that the forcing term was even and of period 2π and it was established that this equation yields a profusion of subharmonic solutions.

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SOME SPECIAL INTEGRALS

M. G. BEUMER, Abadan Institute of Technology, Abadan, Iran

1. In 1957 the following problem was published by Bremekamp*: "Show that

$$\int_0^{\pi} (\log \sin \phi)^2 d\phi = \frac{1}{12}\pi^3 + \pi(\log 2)^2."$$

Some solutions were published in 1958†. In this note I will give not only a solution of this problem but also suggest some methods which lead to integrals of the more general type $\int_0^{\pi}$ $(\log \sin \phi)^n d\phi$ and point out that there are relations between these integrals and some Dirichlet series.

2. The starting point is the function

$$D(s) = \sum_{n=0}^{\infty} \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n} \cdot \frac{1}{(2n+1)^s} \qquad (s = \sigma + \tau i). \tag{1}$$

The right-hand series is a Dirichlet series which converges absolutely for $\sigma > \frac{1}{2}$. From $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$ follows that

$$\frac{1}{(2n+1)^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-(2n-1)x} x^{s-1} dx. \tag{2}$$

Considering, on the other hand, the uniform convergence of the series

$$\sum_{n=0}^{\infty} \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n} (e^{-x})^{2n+1} \qquad \text{for } x > 0,$$

we easily derive that

$$D(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\sqrt{(e^{2x} - 1)}} dx \qquad \text{for } \sigma > \tfrac{1}{2}. \tag{3}$$

The substitutions $e^{-x} = \sin \phi$ and $s = n$ (where n is an integer) give that

$$D(n) = \frac{(-1)^{n-1}}{(n-1)!} \int_0^{\frac{1}{2}\pi} (\log \sin \phi)^{n-1} d\phi. \tag{4}$$

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* Nieuw Arch. Wisk. (ser. 3), vol. 5, 1957, p. 48.
† Wisk. Opgaven, vol. 20, 1958, pp. 9–10.

As the series

$$\sum_{n=0}^{\infty} \frac{(\log \sin t)^n}{n!} x^n = (\sin t)^x$$

converges uniformly for each $0<x<\frac{1}{2}\pi$, a term-by-term integration may be applied, which gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x^n}{n!}\int_0^{\frac{1}{2}\pi} (\log \sin t)^n dt &= \sum_{n=0}^{\infty} (-1)^n D(n+1)x^n \\ &= \int_0^{\frac{1}{2}\pi} (\sin t)^x dt = \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(\frac{1}{2}x+\frac{1}{2})}{\Gamma(\frac{1}{2}x+1)}, \quad |x|<1.\end{aligned} \tag{5}$$

In a similar way we find that

$$\sum_{n=0}^{\infty} D(n+1)x^n = \sqrt{\pi}\,\frac{1}{x}\,tg\,\tfrac{1}{2}x\pi\,\frac{\Gamma(\frac{1}{2}x+1)}{\Gamma(\frac{1}{2}x+\frac{1}{2})}\cdot \tag{6}$$

By multiplication of (5) and (6) we find that

$$\sum_{n=0}^{\infty} (-1)^n D(n+1)x^n \sum_{n=0}^{\infty} D(n+1)x^n = \sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} B_n \pi^{2n} x^{2n-2}, \tag{7}$$

where the Bernoulli numbers B_n are defined by

$$\frac{1}{x}\,tg\,\tfrac{1}{2}x\pi = 2\sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} B_n \pi^{2n-1} x^{2n-2}.$$

By means of comparison of the coefficients of x^{2n-2} in both members of (7) we find that

$$D(1)D(2n-1) - D(2)D(2n-2) + \cdots + D(2n-1)D(1) = \frac{2^{2n}-1}{(2n)!}\pi^{2n}B_n, \tag{8}$$

which is a recurrence relation for the D's. From Dirichlet's series follows:

$$D(1) = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{1}{2n+1} = \arcsin 1 = \tfrac{1}{2}\pi,$$

and from (3)

$$D(2) = -\int_0^{\frac{1}{2}\pi} \log \sin t dt = \tfrac{1}{2}\pi \log 2.$$

From (8) we find for $n=2$ that $D(1)D(3)-D(2)D(2)+D(3)D(1)=(1/48)\pi^4$; so that

$$D(3) = \frac{1}{48}\pi^3 + \tfrac{1}{4}\pi(\log 2)^2 = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\log \sin x)^2 dx,$$

which is Bremekamp's formula.

3. Another recurrence relation can be derived from (5) by differentiation; the result is:

$$(9)\quad \sum_{n=1}^{\infty}(-1)^n nD(n+1)x^{n-1} = \tfrac{1}{2}[\psi(\tfrac{1}{2}x+\tfrac{1}{2}) - \psi(\tfrac{1}{2}x+1)]\sum_{n=0}^{\infty}(-1)^n D(n+1)x^n,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$.

We now give the following formula:

$$(10)\quad \psi(\tfrac{1}{2}x+\tfrac{1}{2}) - \psi(\tfrac{1}{2}x+1) = -\log 4 + 2\sum_{n=1}^{\infty}(-1)^{n+1}\left(1-\frac{1}{2^n}\right)\zeta(n+1)x^n,$$

where $\zeta(n+1) = \sum_{k=1}^{\infty} k^{-n-1}$ denotes Riemann's zeta function. The formula (10) can be derived by expanding the function $f(x) = \psi(\frac{1}{2}x+\frac{1}{2}) - \psi(\frac{1}{2}x+1)$ in a series, with the help of Maclaurin's formula, and with the following result, known from the theory of gamma functions,

$$\psi(x) = \int_0^{\infty}\left(e^{-a} - \frac{ae^{-ax}}{1-e^{-a}}\right)\frac{1}{a}\,da.$$

The substitution of (10) in (9) gives

$$\sum_{n=1}^{\infty}(-1)^n nD(n+1)x^{n-1}$$
$$= \left[-\log 2 + \sum_{n=1}^{\infty}(-1)^{n+1}\left(1-\frac{1}{2^n}\right)\zeta(n+1)x^n\right]\cdot\sum_{n=0}^{\infty}(-1)^n D(n+1)x^n,$$

from which, through comparison of the coefficients of x^{n-2}, it follows that

$$(11)\quad (n-1)D(n) = D(n-1)\cdot\log 2 + \sum_{k=1}^{n-2}\left(1-\frac{1}{2^k}\right)\zeta(k+1)D(n-k-1).$$

From the theory of zeta functions we know that

$$\zeta(2m) = \frac{(2\pi)^{2m}}{4m(2m-1)!}B_m.$$

It is not possible to continue to express all D's, one after the other, in terms of "known" constants since no such expression is known for $\zeta(2m+1)$; however, all D's can be expressed in terms of the zeta function of positive integers. Thus

$$D(4) = \frac{1}{12}\pi(\log 2)^3 + \frac{1}{48}\pi^3\cdot\log 2 + \frac{1}{8}\pi\cdot\zeta(3),$$

$$D(5) = \frac{19}{11520}\pi^5 + \frac{1}{96}\pi^3(\log 2)^2 + \frac{1}{48}\pi(\log 2)^4 + \frac{1}{8}\pi\cdot\log 2\cdot\zeta(3), \quad \text{etc.}$$

REMARKS ON THE FIBONACCI SERIES MODULO m

S. E. MAMANGAKIS, International Business Machines Corporation
Research Center, Yorktown Heights, N. Y.

In a recent paper D. D. Wall [1] was concerned with determining the length of the period of the recurring series obtained by reducing a Fibonacci series modulo m. In this paper we will use the same notation as in [1].

Thus, u_n denotes the Fibonacci series with $u_0=0$ and $u_1=1$, where $u_{n+1}=u_n+u_{n-1}$. Also, $k(m)$ denotes the length of the period of u_n mod m and as in [1] we let $m=p^e$, where p is a prime number. In [1] Wall poses a question that has so far remained unanswered: "The most perplexing problem we have met in this study concerns the hypothesis $k(p^2)\neq k(p)$. We have run a test on a digital computer which shows that $k(p^2)\neq k(p)$ for all p up to 10,000; however, we cannot yet prove that $k(p^2)=k(p)$ is impossible." This paper furnishes a proof of the hypothesis $k(p^2)\neq k(p)$ under certain mild conditions.

THEOREM 1. *If c and p are relatively prime and cp occurs in u_n, then $k(p^2)\neq k(p)$.*

Proof. Let $u_j=cp$ and consider the sequences

(1) $$u_n, \ \bmod p \text{ which we will denote by } {}_1u_n;$$

(2) $$u_n, \ \bmod p^2 \text{ which we will denote by } {}_2u_n$$

which begin, respectively, 0, 1, 1, 2, $\cdots$, ${}_1u_{j-1}=Rp+Q$, ${}_1u_j=cp\equiv 0$, ${}_1u_{j+1}\equiv Q$, $\cdots$ (mod p) and 0, 1, 1, 2, $\cdots$, ${}_2u_{j-1}=Rp+Q$, ${}_2u_j=cp\equiv cp$, ${}_2u_{j+1}\equiv(c+R)p+Q$, $\cdots$ (mod p^2), where $0<Q<p$. It can be shown by mathematical induction that

(3) $${}_1u_{tj-1}\equiv Q^t, \qquad {}_1u_{tj}\equiv 0 \pmod p;$$

(4) $${}_2u_{tj-1}\equiv tRpQ^{t-1}+Q^t, \qquad {}_2u_{tj}\equiv ctpQ^{t-1} \pmod{p^2}.$$

To see that (4) holds, we note that for $t=1$, the formulas hold. Next, assume the formulas hold for $t\leqq i$, then ${}_2u_{ij-1}\equiv iRpQ^{i-1}+Q^i$ and ${}_2u_{ij}\equiv cipQ^{i-1}$ (mod p^2). Now consider the new sequence U_n with $U_0=iRpQ^{i-1}+Q^i\equiv{}_2u_{ij-1}$ and $U_1=cipQ^{i-1}\equiv{}_2u_{ij}$ (mod p^2). But, by the well-known formula for a Fibonacci series in [1], $f_n=u_nb+u_{n-1}a$, where $f_1=a$, $f_2=b$ and $f_{n+1}=f_n+f_{n-1}$, we have $U_j=u_j(cipQ^{i-1})+u_{j-1}(iRpQ^{i-1}+Q^i)$ or $U_j\equiv(i+1)RpQ^i+Q^{i+1}\equiv{}_2u_{(i+1)j-1}$ (mod p^2), and $U_{j+1}=u_{j+1}(cipQ^{i-1})+u_j(iRpQ^{i-1}+Q^i)$ or $U_{j+1}\equiv(i+1)cpQ^i\equiv{}_2u_{(i+1)j}$ (mod p^2). Hence (4) holds; and (3) is implied by (4).

We will, therefore, obtain in the series (1) $\cdots$, ${}_1u_{tj-1}\equiv Q^t$, ${}_1u_{tj}\equiv 0$ (mod p), and in (2) $\cdots$, ${}_2u_{tj-1}\equiv tRpQ^{t-1}+Q^t$, ${}_2u_{tj}\equiv ctpQ^{t-1}$ (mod p^2).

Now series (1) will first repeat when Q belongs to t mod p. (In other words t is the smallest number satisfying $Q^t\equiv 1$ mod p.) In this case, p^2 does not divide ${}_2u_{tj}\equiv ctpQ^{t-1}$ since t divides $p-1$, which means series (2) does not repeat with sequence (1). This proves our theorem.

THEOREM 2. *Let c and p be relatively prime, $e \leqq d$, and $u_j = cp^d$ be the first multiple of p to occur in u_n. Then $k(p^e) = k(p)$ if and only if u_{j-1} has the same order* mod p *and* mod p^e.

Proof. Since u_j is the first multiple of p to occur in u_n, the period of u_n mod p will be jt where u_{j-1} belongs to t mod p. But u_j is also the first multiple of p^e to occur in u_n and so its period is equal to js where u_{j-1} belongs to s mod p^e. Therefore if $k(p) = k(p^e)$ we have $k(p) = jt = js = k(p^e)$ which implies $t = s$. Conversely, under the same hypotheses, if u_{j-1} has the same order mod p and mod p^e it is obvious that $k(p^e) = k(p)$.

Remarks. In [2] Kraitchik has a table of u_n in their prime factorization for odd n up to $n = 129$, with a few missing entries, and none of the u_n listed satisfy the hypothesis of Theorem 2 for $1 < e \leqq d$ in this paper. Furthermore, I have computed all of the u_n up to $n = 50$ using Lehmer's prime and factor tables plus the tables in [2] as a check and, again, none of the u_n satisfy the hypothesis of our Theorem 2 for $1 < e \leqq d$. Could it be that the mild conditions of Theorem 1 are strong enough to apply to all prime numbers? That is to say, one would like to make Theorem 1 read: *If c and p are relatively prime, then cp occurs in u_n and $k(p^2) \neq k(p)$.*

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NOTE ON THE DISTRIBUTIVE LAWS (Supplement)

TÔRU SAITÔ, Tokyo Gakugei University, Japan

J. L. Kelley [1] defines a ring to be a system which we called a c-ring in [2]. Then he defines a field to be a system which is a ring (in his sense) such that the set of all nonzero elements forms a multiplicative commutative group. In this supplementary note, we show that the definition of a field in Kelley's sense coincides with that in the ordinary sense.

We use terminologies and results in [2] freely.

We define a *w-division ring* (*c-division ring*) to be a w-ring (c-ring) F such that F has at least two elements and $F - \{0\}$ forms a multiplicative group. When the multiplicative group is commutative, it is called a *w-field* (*c-field*). Hence a field in Kelley's sense is a c-field in our definition.

THEOREM 1. *If a w-division ring F contains an element which is neither zero nor the defining element, then F is a division ring* (*in the ordinary sense*).

Proof. It suffices to show that F is a ring. By [2] Lemma 3, we have, in F, $e^2 = e$, where e is the defining element of F. If e were not 0, e would be an idempotent element of the multiplicative group $F^* = F - \{0\}$, and so e would be the identity element of F^*. Then for any $x \in F^*$, we have, using [2] Lemma 3 again,

$x = xe = e$. Hence F^* would consist of solely the defining element e, contradicting the assumption. Hence $e = 0$, and therefore F is a ring.

THEOREM 2. *Every c-division ring is a division ring. Conversely, every division ring is a c-division ring.*

Proof. Let F be a c-division ring. By [2] Theorem 3, F can be considered as a w-division ring of order 3 or 1 with the defining element $e = 00$. If F were of order 3, the element $2e$ would be neither zero nor the element e. Hence by Theorem 1, F would be a division ring and so $e = 0$, which is absurd. Hence F is of order 1 and therefore is a division ring. The converse part of the theorem is evident.

COROLLARY. *Every c-field is a field and conversely every field is a c-field.*

References

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CORRECTION

In the note *On some equations involving functions $\phi(n)$ and $\sigma(n)$* by A. Makowski (this MONTHLY, vol. 67, 1960, pp. 668–670), the first number in (b) on page 670 should be $2^2 \cdot 3 \cdot 5 \cdot 13$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

SO-CALLED "EXCEPTIONAL" EXTREMUM PROBLEMS

HUGH A. THURSTON, University of British Columbia

Although the invitation "rejoinders to earlier notes are encouraged" is no longer printed in the MONTHLY, I should like to make a rejoinder to Ogilvy's Classroom Note [1]. I shall refer also to Thurston [2], Oakley [3], and Walsh [4].

Ogilvy maintains that the "elementary calculus method" of finding maxima and minima may fail for two reasons, and he gives an example of each. The second failure he imputes to an "unsuitable independent variable" and remarks: "what we should like is a criterion for judging in advance the suitability of an independent variable. Unfortunately this seems rather too much to ask."

I maintain that the elementary method does not fail in either case and that therefore no criterion is needed. By the elementary method I mean the method

suggested by common sense: we examine the stationary values of the function being investigated, the values where the derivative fails to exist, and the endpoint values. The greatest of these is the maximum and the least the minimum, provided of course the function does have a maximum and a minimum. A similar method occurs in Walsh's paper (the case where there is just one extremum in the range under consideration) and is sufficient to exemplify the principle that we should realize precisely what we are looking for, rather than mechanically apply a learnt test.

Ogilvy's first problem is essentially to maximize $(50-x)(100+x)$ under the conditions $0<x<50$. He writes that if we proceed "in the usual fashion" then "x comes out to be -25, which is not permitted by the conditions of the problem." But to get this result, the standard (and, I would have hoped, usual) method must be replaced by something like "set the derivative equal to zero and solve."

The second problem is essentially:—

$$\text{Minimize } (x-1)^2+y^2 \text{ under the condition } y^2=4x.$$

This is clearly equivalent to: Minimize $(x-1)^2+4x$ under the condition $x\geqq 0$. Because the derivative is always positive throughout the range considered, the minimum occurs when $x=0$. Ogilvy states that his method of solution "yields $x=-1$, a point not on the parabola."

In both cases, the incorrect results can be imputed to *neglect of the domain of the function under investigation*. (*E.g.*, the minimum of $(x-1)^2+4x$ with no restriction on x does occur when $x=-1$; the maximum of $(50-x)\cdot(100+x)$ does occur when $x=-25$.) In my note, I said regretfully that the concept of domain is apt to be pushed into the background, and gave an example (from a well-known textbook) of a problem whose solution goes wrong if domains are neglected. There are no such things as "unsuitable" variables: the use of this term is an attempt to put on the variable the blame which rightly belongs to an illogical method of solution.

Why are both common sense and Walsh's thirteen-year-old paper neglected? It must be because teaching is too mechanical: finding stationary points is a convenient way for finding certain local maxima and minima; and the *technique* of setting the derivative equal to zero is emphasized at the expense of *understanding* what the problem is and why the technique is being used. Why else would a student expect to find the least value of $a-ex$ as x varies from $-a$ to a by differentiation? And yet Oakley found that "even an able student" might do this.

The fact is that as one goes through life most maximum and minimum problems are endpoint problems. How fast do I run to get to the telephone as quickly as possible? Answer: as fast as I can—an endpoint solution. Of course, most of these problems are trivial; the interesting ones are where there is some balancing factor: in maximizing $x(1-x)$ the increase in x balances the decrease in $(1-x)$.

However, we teachers of calculus seem to have so lost our sense of proportion that we regard these as normal and the endpoint solutions as wicked exceptions.

To show how far our sense of proportion has been lost, let us consider the following problem. Choose P between P_0 and the vertex P_m on the parabola $y=9x^2-28x+24$ (Fig. 1), complete the shaded region and rotate it about the y-axis. Where should P be taken to give a maximum volume? Because P is restricted to a finite closed interval, the endpoint maximum (at P_m) and minimum (at P_0) stare us in the face; it would certainly be possible to have extrema in between, but at any rate the endpoint values are the most obvious suspects.

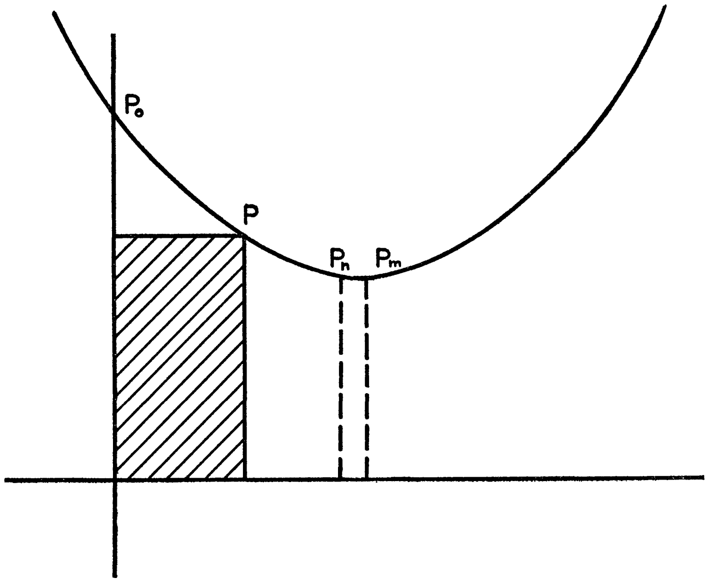

FIG. 1

Indeed, no one would doubt that the value given by P_m is greater than that given by P_n. And yet the problem is described [5] as "A *well-concealed* endpoint maximum"! (My italics.) However, we are not alone in making this mistake. I heard that, until recently, geographers, when asked for the highest point in Florida, would cite the highest of Florida's few hills. Now, however, someone has noticed that the highest point in Florida is on the Alabama border: an endpoint maximum.

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THE AREA OF THE ELLIPSE

ROGER BURR KIRCHNER, Harvard University

Consider the ellipse E whose equation is $x^2/a^2+y^2/b^2=1$, where $a, b>0$. Suppose that we make the transformation $(x, y)\to(ax, by)$. The x- and y-axes are stretched (or compressed) into new axes which we call the x' and y' axes. If the x-y coordinates of a point P are (x, y) and its x'-y' coordinates are (x', y'), then clearly

$$(1)\qquad x = ax', \qquad y = by'.$$

Thus, in the x'-y' system, E has the equation $x'^2+y'^2=1$.

Let Δ be a triangle with one side parallel to the x-axis. Suppose that the x-y coordinates of its vertices are (x_1, y_1), (x_2, y_1), and (x_3, y_3), and the corresponding x'-y' coordinates are (x_1', y_1'), (x_2', y_1'), and (x_3', y_3'). If the area of Δ in the x-y system is denoted by A, and in the x'-y' system by A', then

$$(2)\qquad A = \tfrac{1}{2}\left| x_2 - x_1\right| \cdot \left| y_3 - y_1\right|, \qquad A' = \tfrac{1}{2}\left| x_2' - x_1'\right| \cdot \left| y_3' - y_1'\right|.$$

Hence, from the equations (1), we find that $A=abA'$.

Since any region bounded by a polygonal curve can be broken up into triangles of the type Δ, we see that its area in the x-y system is just ab times its area in the x'-y' system. The same must then be true for regions whose boundaries are limits of polygonal curves. Such a region is the ellipse E. Since E is just the unit circle in the x'-y' system, its x'-y' area is π. Hence the area of the ellipse with equation $x^2/a^2+y^2/b^2=1$ is πab.

SIMPLE CONSTRUCTIONS OF NONDIFFERENTIABLE FUNCTIONS AND SPACE-FILLING CURVES

WILLIAM C. SWIFT, Rutgers, The State University

Although the beginning calculus student is commonly informed of the existence of continuous functions which are nowhere differentiable, the demonstration of the celebrated examples of such functions is beyond his competence. We exhibit here a function of this class which should be meaningful to anyone comprehending the binary (base 2) representation of real numbers. In a similar manner we define functions $u(t)$ and $v(t)$, each continuous and nowhere-differentiable, with the added property that the image of $\{t \mid 0\leqq t\leqq 1\}$ completely fills the square $\{(u, v) \mid 0\leqq u\leqq 1,\ 0\leqq v\leqq 1\}$.

1. The value of the function $f(x)$ for x in the interval $[0, 1]$ is based on the ternary (base 3) decimal expansion of x. Let

$$x = .x_1 \cdots x_n \cdots \quad \text{(base 3)},$$

in which of course $x_n=0$, 1 or 2. Then $f(x)$ is given by the binary decimal

$$f(x) = .y_1 \cdots y_n \cdots \quad \text{(base 2)},$$

where $u_n=0$ or 1 and $v_n=0$ or 1 as given by the following rule:

First, $u_1=0$ if and only if $t_1=0$, 1 or 2, and $v_1=0$ if and only if $t_1=0$, 1 or 3.

To determine u_{n+1} and v_{n+1} we distinguish six cases:

1. if $t_{n+1}=0$ or 5 and $t_{n+1}=t_n$, then $u_{n+1}=u_n$ and $v_{n+1}=v_n$;
2. if $t_{n+1}=0$ or 5 and $t_{n+1}\neq t_n$, then $u_{n+1}\neq u_n$ and $v_{n+1}\neq v_n$;
3. if $t_{n+1}=1$, then $u_{n+1}=0$ and $v_{n+1}=0$;
4. if $t_{n+1}=2$, then $u_{n+1}=0$ and $v_{n+1}=1$;
5. if $t_{n+1}=3$, then $u_{n+1}=1$ and $v_{n+1}=0$;
6. if $t_{n+1}=4$, then $u_{n+1}=1$ and $v_{n+1}=1$.

That $u(t)$ and $v(t)$ are continuous functions which are nowhere differentiable may be demonstrated following the procedure accorded $f(x)$ above. Moreover, considering the functions simultaneously, it is immediate from their definition that every point of the square $\{(u, v)\,|\,0\leqq u\leqq 1, 0\leqq v\leqq 1\}$ is the image of some t in the interval $[0, 1]$.

ON THE USE OF TABLES TO OBTAIN CONFIDENCE INTERVALS

W. C. GUENTHER, University of Wyoming

In some statistical problems it may appear that one needs tables that are not readily available. Sometimes tables possessed by nearly all statisticians are better adapted to the situation. Two examples of this follow.

When the general method of obtaining confidence intervals is applied to the binomial case, it leads to the equations

$$\sum_{y=k}^{n}\binom{n}{y}p^y(1-p)^{n-y}=\tfrac{1}{2}\alpha, \tag{1}$$

$$\sum_{y=0}^{k}\binom{n}{y}p^y(1-p)^{n-y}=\tfrac{1}{2}\alpha, \tag{2}$$

where n, k, and α are all known numbers. If the solution of (1) is p_1 and that of (2) is p_2, then (p_1, p_2) is a confidence interval for p with coefficient $1-\alpha$. That is, $\Pr[p_1<p<p_2]=1-\alpha$.

The above equations can be solved with the aid of binomial tables. However, frequently tables are not readily available and, even if they are, some kind of interpolation is necessary. It is well known that

$$\frac{n!}{(k-1)!(n-k)!}\int_0^p w^{k-1}(1-w)^{n-k}dw=\sum_{y=k}^{n}\binom{n}{y}p^y(1-p)^{n-y}.$$

This is easily demonstrated by repeated parts integration. Thus tables of the incomplete Beta function could be used to solve (1). The difficulties are the same as with the binomial tables.

Good F tables are generally available and adapted to this kind of problem. Letting

$$w = \frac{\dfrac{kF}{n-k+1}}{1+\dfrac{kF}{n-k+1}},$$

it is found that

$$\Pr[0 < w < p] = \Pr\left[0 < F < \frac{n-k+1}{p}\,\frac{p}{1-p}\right],$$

where F has $2k$ and $2(n-k+1)$ degrees of freedom. Thus to solve (1), solve instead

$$\Pr\left[0 < F < \frac{n-k+1}{k}\,\frac{p}{1-p}\right] = \tfrac{1}{2}\alpha.$$

When $\{(n-k+1)/k\}\{p/(1-p)\}$ is equated to the tabulated F value one has a linear equation in p to solve for p_1. Similarly (2) may be replaced by

$$\Pr\left[F > \frac{n-k}{k+1}\,\frac{p}{1-p}\right] = \tfrac{1}{2}\alpha,$$

where F has $2(k+1)$ and $2(n-k)$ degrees of freedom and the solution is p_2.

In the Poisson case where

$$f(x;m) = e^{-m}\frac{m^x}{x!}, \qquad x = 0, 1, \cdots,$$

one is led to the equations

$$\sum_{y=y'}^{\infty} e^{-nm}\frac{(nm)^y}{y!} = \tfrac{1}{2}\alpha, \tag{3}$$

$$\sum_{y=0}^{y'} e^{-nm}\frac{(nm)^y}{y!} = \tfrac{1}{2}\alpha, \tag{4}$$

where n, α and $y' = \sum_{i=1}^{n} x_i$ are known.

The solutions of (3) and (4) yield m_1 and m_2 such that

$$\Pr[m_1 < m < m_2] = 1 - \alpha.$$

Repeated parts integration gives

$$\int_m^{\infty} \frac{1}{\Gamma(n+1)}\,x^n e^{-x}dx = \sum_{y=0}^{n} e^{-m}\frac{m^y}{y!},$$

but both the Poisson tables and the incomplete Gamma tables possess the same objections mentioned previously. Since χ^2/degrees of freedom is extensively

tabulated and readily available (see Dixon and Massey) let $x=\frac{1}{2}w$. Then w is χ^2 with $2n+2$ degrees of freedom so that

$$\sum_{y=0}^{n} e^{-m}\frac{m^y}{y!} = \Pr[w > 2m] = \Pr\left[\frac{w}{2n+2} > \frac{m}{n+1}\right],$$

$$\sum_{y=0}^{y'} e^{-nm}\frac{(nm)^y}{y!} = \Pr\left[z > \frac{nm}{y'+1}\right],$$

where z is χ^2 divided by degrees of freedom $2y'+2$. If the tabular value is b, then $m_2=b(y'+1)/n$. Similarly

$$\sum_{y=y'}^{\infty} e^{-nm}\frac{(nm)^y}{y!} = \Pr\left[z < \frac{nm}{y'}\right] = \tfrac{1}{2}\alpha$$

gives $m_1=ay'/n$ where z is χ^2 divided by degrees of freedom $2y'$ and a is the tabulated value.

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A CHARACTERIZATION OF COMPACT METRIC SPACES

NORMAN LEVINE, The Ohio State University

1. Introduction. It is well known that a compact metric space is bounded; the converse of course is not true, for example, $M: 0<x<1$ with the usual metric. It is possible to remetrize M so that M becomes unbounded relative to the new metric. We shall see in the next section that every noncompact metric space can be remetrized so that it becomes unbounded relative to the new metric.

2. A characterization of compact metric spaces.

LEMMA. *If M is a metric space, then M is compact if and only if every f: $M\to R$ is bounded where f is continuous and R is the set of all real numbers with the usual topology.*

A proof of this lemma is given in [1], page 119.

THEOREM. *Let M be a metric space with metric d. Then M is compact if and only if M is bounded relative to d^* where d^* is any metric for M equivalent to d.*

Proof. The necessity is well known. To show the sufficiency let M be bounded relative to d^* where d^* is any metric equivalent to d. Assert then that M is compact. Deny. Then by the above lemma there exists an f^*: $M\to R$ such that f^* is continuous and unbounded, R being the space of reals. We now define a new metric for M as follows: for x, y in M let $d^\#(x, y)=d(x, y)+|f^*(x)-f^*(y)|$. The theorem is proved when we show that (a) $d^\#$ is a metric for M equivalent to

d and (b) M is unbounded relative to $d^\#$. To show (a), we note that (1) $d^\#(x, y) \geqq 0$, (2) $d^\#(x, y) = 0$ if and only if $x = y$, (3) $d^\#(x, y) = d^\#(y, x)$, (4) $d^\#(x, z) = d(x, z) + |f^*(x) - f^*(z)| \leqq d(x, y) + d(y, z) + |f^*(x) - f^*(y)| + |f^*(y) - f^*(z)| = d^\#(x, y) + d^\#(y, z)$, and finally (5) let $y \in M$ and $y_i \in M$ for $i = 1, 2, \cdots$ and suppose $\lim d^\#(y, y_i) = 0$. Then $\lim d(y, y_i) = 0$ since $d(y, y_i) \leqq d^\#(y, y_i)$. Conversely let $\lim d(y, y_i) = 0$. Then $\lim f^*(y_i) = f^*(y)$ since f^* is continuous and thus $\lim d^\#(y, y_i) = \lim d(y, y_i) + \lim |f^*(y) - f^*(y_i)| = 0$. To show (b) choose x_i in M so that $\lim |f^*(x_i)| = \infty$. Then

$$\begin{aligned} d^\#(x_i, x_1) &= d(x_i, x_1) + |f^*(x_i) - f^*(x_1)| \\ &\geqq |f^*(x_i) - f^*(x_1)| \\ &\geqq \big||f^*(x_i)| - |f^*(x_1)|\big|. \end{aligned}$$

Thus $d^\#(x_i, x_1) \to \infty$.

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MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE TEACHING OF MATHEMATICS*

Professor J. P. Mathieu of the French delegation presented an introduction to the teaching of mathematics to physicists:

I have tried to find out the opinion of my French colleagues, but unfortunately without success, and I must therefore give you a personal view rather than a national one.

First we must decide on the level of education we are concerned with. In secondary education a number of important problems arise in the teaching of mathematics. At this level education is intended to give a general culture, and the teaching of mathematics can be considered as being an end in itself, giving the pupils a certain attitude of mind and a way of reasoning of which a mathematician is very proud and jealous.

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Problems do arise, however, at this level between the teaching of mathematics and physics—problems of syllabus and problems of teaching method. In France from a very early stage we give importance to abstract education, more, probably, than in many other countries, and this raises the question of the relation between mathematics and physics.

At the university level the student has made his choice and is going to specialize. He knows the road he is taking, and what he wants from mathematics is an instrument, rather than a formative discipline. One, of course, does not exclude the other, and the statement in the report of the (British) Institute of Physics* is quite correct when it says that if an instrument must be used, let it be used with intelligence and not mechanically.

A special problem is the mathematics that is essential in the study of physics by biologists and medical students, for whom physics is an accessory discipline.

I will not, however, dwell on this question, as what interests us above all is the training of the physicists. I will not treat the training of the chemist as a separate problem, as Professor Aigrain did. In my opinion, the same mathematical education can be given to the physicist and the chemist. This does not happen in my country, but I do not consider it wrong that it does happen in other countries.

Here again, one must be careful to stipulate at what level the teaching of mathematics is to be. As the British report I have quoted so rightly states, it is the degree of emphasis on generalization or rigor that must be decided in teaching of mathematics for the earlier ages.

People often suppose that there is a difference between the spirit in which mathematics should be taught and the spirit that should animate the teaching of physics, but if one looks at the theoretical form which the concepts of physics take when they have matured, there is no difference of spirit. There is in fact no difference of spirit in the scientific disciplines generally.

What then do we consider the minimum of mathematics that must be taught to physicists, and what need not be taught? This depends on the level considered and is a question of utility.

In France, the mathematician wishes to teach very seriously in order to educate the pupil in a general way. What are the advantages of so doing? First, economy of thought. If we create concepts that have a very wide general applicability, then we teach at one and the same time a number of ideas that can appear to be unconnected. The physicst knows this well. What are the disadvantages of this tendency to discuss ideas in their most general form?

In the first place, it is illusory to abstract oneself from reality. An axiomatic system does not constitute a science, and this may be missed by a student to whom only general notions are taught.

Further, it will take much more time and effort to teach mathematics in

* The Teaching of Mathematics to Physicists. The Institute of Physics and The Physical Society, London, 1960.

this way than in a more practical manner.

Let us look at the situation in practice.

The physicist will require from the mathematician a suitable syllabus, and, as part of this suggested syllabus will not be of great use to him, it will be deleted. The Institute of Physics' report, in fact, gives a syllabus of this type which I find interesting.

Second, the physicist will expect from the mathematician a method, and this point has been very much discussed in France. The mathematician needs more generalized proofs than we physicists do. In other words, the physicist is ready to believe the mathematician when he gives him details of existence theorems. He will believe, because he has every reason to believe.

There is a third point. It is of the greatest importance to the physicist to be able to interpret the solutions of equations, and on this the mathematician—at least in our country—gives no help to the physicist.

What is a practical way out of these difficulties? I believe that there must be close collaboration between the teacher of mathematics and the physicist who is an expert in mathematics. In France the situation is very unsatisfactory. We never seem able to arrange for a physicist expert in mathematics to teach mathematics to future physicists.

We must, of course, remember that, although the mathematician uses his physical concepts drawn from reality, it is also true that certain abstract and disinterested mathematical studies have been found eventually to be of use to physicists. We must, therefore, leave mathematicians to make their own researches.

The important thing is that the mathematics which is taught should be of some use.

The discussion after Professor Mathieu's address followed two main lines: whether or not mathematics for physicists should be taught by mathematicians or physicists, and detailed discussions on what subject matter of mathematics should be included, and in what form.

On the first question, as to whether or not mathematics for physicists should be taught by mathematicians or not, no agreement was achieved, and strong opinions were stated on both sides.

During the discussion of techniques and the content of a mathematics course, several interesting comments were made by Professor A-K. M. Kinani of the United Arab Republic:

We have been impressed that in courses of mathematics for physicists, and also in the books on mathematics especially for physicists, all examples for giving the fundamental concepts of mathematics are, in general, abstract and dull. We have tried giving physical examples to the students to learn the applications of mathematics to physics, both at secondary-school and university levels. For example, in secondary school when we study fractions and variables, we use the physical example of the length of a bar as a function of temperature.

As examples of derivatives we use the connection between electric current and the quantity of electric charge.

At the university we take as an example of the exponential function the discharge of a capacitor or the fundamentals of radioactive disintegration. We have tried this system in our country, and we have seen that this method interests the students very much.

Dr. J. Topping of the United Kingdom delegation pointed out the following:

One of the most important developments of mathematics in recent years is the development of numerical mathematics. It is a new and tremendously important branch of mathematics which has meant that problems which could not be tackled before can now be solved. This has led to the use of electronic computers and all sorts of devices of this kind. One of the developments in England in teaching mathematics to physicists in the Diploma in Technology courses is that they are introduced to these machine methods and the general methods of numerical analysis.

At the more advanced level, Professor A. Borsellino of Italy had this to say:

I am a theoretical physicst, and I am very much intersted in the mathematical background for physicists. I think it is one of the greatest dangers in the relationship between physicists and mathematicians at the present time that we do not understand each other on the research plane. Naturally we uncerstand each other on the plane of elementary mathematics, on the plane of mathematics of one century ago. But now we do not understand each other. Essentially mathematicians are now working on subjects that we as physicists know very little about. We should make contact with them at some points to permit real communication from modern mathematics to the physical knowledge of the average physicist, and not let the only contact be through the specialist in the most advanced theoretical physics.

MATHEMATICS AND STATISTICS DEGREES DURING THE DECADE OF THE FIFTIES

Clarence B. Lindquist, U. S. Office of Education

During the decade of the 1950's the number of doctorates in mathematics and statistics conferred by institutions of higher education in the United States experienced a steady gain, while the number at the master's and bachelor's level decreased during the first half of the decade but rose during the last half, with an especially sharp increase in the late 1950's.

The numbers of degrees that were conferred annually at each level from 1949–50 through 1958–59, together with the percents that these numbers were of all degrees at each level, are shown in Table 1. The data in this table were obtained from the annual series of *Earned Degrees Conferred by Higher Educational Institutions*, published by the U. S. Office of Education. Each annual report covers a period from July 1 of one year to June 30 of the next. No distinction is made

between degrees earned in liberal arts or in preparation for teaching, or in any other category. The criterion that is used in classification is that there be a substantive major in mathematics or statistics. Double majors are counted as one-half in each discipline. Thus, a double major by a student in mathematics and physics is allocated one-half to mathematics and one-half to physics.

TABLE 1

NUMBER OF DEGREES CONFERRED ANNUALLY IN MATHEMATICS AND STATISTICS* BY LEVEL OF DEGREES

Year ending June 30	Bachelor's degrees		Master's degrees		Doctorates (Ph.D., Ed.D., etc.)	
	Number of degrees conferred	Percent† of all degrees at this level	Number of degrees conferred	Percent of all degrees at this level	Number of degrees conferred	Percent of all degrees at this level
1950	6,392	1.47	974	1.67	160	2.41
1951	5,753	1.49	1,109	1.70	184	2.51
1952	4,721	1.42	802	1.26	206	2.68
1953	4,396	1.44	677	1.11	241	2.90
1954	4,090	1.40	706	1.24	227	2.52
1955	4,034	1.40	761	1.31	250	2.83
1956	4,660	1.50	898	1.51	235	2.64
1957	5,546	1.63	965	1.56	249	2.84
1958	6,924	1.89	1,234	1.88	247	2.76
1959	9,019	2.34	1,499	2.16	282	3.01

* Includes actuarial science.

† The percent at the bachelor's level is based upon all bachelor's and first-professional degrees conferred that year.

Facts for the ten-year period. For the ten-year period as a whole, the 55,535 degrees conferred in mathematics and statistics at the bachelor's level constituted 1.62 percent of the total number of bachelor's and first-professional degrees; the 9,625 master's degrees, 1.55 percent of the total number of master's degrees; and 2,281 doctor's degrees, 2.72 percent of the total number of earned doctorates.

Beginning with the 1955–56 survey, the category "mathematics" was replaced by "mathematical subjects," which consisted of two subcategories, "mathematics" and "statistics." Presumably, prior to 1955–56 degrees in statistics were classified by respondents as mathematics. The average number of degrees in statistics at the bachelor's level for the four years, 1955–56 through 1958–59, was 67; at the master's level, 100; and at the doctoral level, 32.

Also beginning with the 1955–56 survey, the Office of Education published the numbers of institutions, by level of degrees, that conferred degrees in the various disciplines for each year. The number of institutions conferring degrees

in mathematics at the bachelor's level rose steadily from 751 in 1955–56 to 871 in 1958–59, and at the master's level from 146 to 177 for the same years. However, at the doctorate level the number of institutions did not change much during any of the four years, averaging about 60 per year. The number of institutions granting bachelor's degrees in statistics averaged about 20 per year; the number conferring master's degrees, about 23 per year; and the number awarding doctorates, about 10 per year. It must be remembered that the number of institutions granting degrees in a specific discipline at a certain level in a given year will usually not represent the universe of institutions authorized to grant these degrees because in some years some of the institutions fail to grant such a degree.

Degrees awarded to women. The percent of mathematics and statistics degrees awarded to women, by level, for the ten-year period, is shown in Table 2, along with the comparable percents for the biological and physical sciences as well as for all fields combined. A fact not shown in the table is that the percent of women receiving bachelor's degrees in mathematics and statistics rose steadily from 22.6 in 1949–50 to 32.7 in 1955–56 and then decreased to 27.9 by 1958–59. The percent for women at the master's level declined from 19.5 in 1949–50 to 16.5 in 1952–53 and then rose to 20.7 by 1958–59. There were no significant trends at the doctorate level.

TABLE 2

THE PERCENT OF DEGREES AWARDED TO WOMEN OVER THE TEN-YEAR PERIOD 1949–50 THROUGH 1958–59

Level	Percent of degrees awarded to women in			
	Mathematics & statistics	Physical sciences	Biological sciences	All disciplines
Bachelor's and first-professional	28.6	12.2	22.2	32.3
Master's	18.8	8.3	20.1	32.0
Doctorate (Ph.D., Ed.D., etc.)	5.3	3.9	10.9	9.8

Junior-year majors in mathematics. In 1957 the Office of Education commenced an annual series of reports, *Junior-Year Science and Mathematics Students*. In the fall of 1957, 9,133 students were reported as junior-year majors in mathematics and statistics. These students would normally be expected to graduate in 1958–59. The number of bachelor's degrees conferred in mathematics and statistics in 1958–59 was 9,019, which gives a ratio of 0.988 for completion of degrees.

The fall 1958 report lists 11,961 junior-year majors in mathematics and

statistics, and the fall 1959 report lists 14,065. If the ratio of the number of graduates in 1958–59 to the number of junior-year majors in the fall of 1957 prevails, we can expect slightly less than 12,000 bachelor degrees in mathematics and statistics in 1959–60, and nearly 14,000 degrees in 1960–61. If these data materialize the number of degrees that will be awarded in 1960–61 will be more than 50 percent greater than the number awarded in 1958–59. It is interesting to note that, based upon similar calculations, the expected increase in the biological sciences for 1960–61 over 1958–59 is expected to be about 7 percent, and in the physical sciences about 1 percent.

Number of doctorates awarded by individual institutions over the ten-year period. Table 3 shows the total number of doctorates in mathematics and statistics awarded by individual institutions for the ten-year period, 1949–50 through 1958–59. The 75 different institutions which awarded the degrees are ranked in decreasing order of the total number each institution awarded during the ten-year period. The first 13 institutions granted more degrees than the remaining 62 did.

The University of California heads the list with 170. It must be remembered, however, that this figure is the total for all campuses, including both Berkeley and Los Angeles. The University of California chooses to report to the Office of Education in this fashion. On the other hand, the University of North Carolina sends separate data to the Office for Chapel Hill, the State College, and the Women's College.

During the 1960's there are certain to be new institutions which have never conferred the doctorate in mathematics and statistics before. Under Title IV of the National Defense Education Act of 1958 alone, fellowships have been awarded, up to the present time, for approved new doctoral programs in mathematics at twelve institutions. These are:

University of Alabama
University of Arizona
Florida State University
University of Idaho
Brandeis University
Montana State College
New Mexico State University
Polytechnic Institute of Brooklyn
Yeshiva University
University of South Carolina
Agricultural and Mechanical College of Texas
Washington State University

Outlook ahead. With the growing recognition of mathematics, not only as an indispensable tool for the scientist and engineer but as a worthwhile discipline in itself, the indications are that the trends in evidence since the middle 1950's will continue. The increasing opportunities for employment for mathematically trained persons in industry, government, and teaching at all levels will attract even larger numbers of students. Expanding graduate programs will produce larger numbers of degree holders at the master's and doctorate levels. It is quite apparent that there will be a real problem to staff adequately in mathematics and statistics the institutions which are expanding not only in number but in size as well.

TABLE 3

TOTAL NUMBER OF DOCTORATES IN MATHEMATICS AND STATISTICS AWARDED BY INDIVIDUAL INSTITUTION: 1949–50 THROUGH 1958–59

Rank Order	Institution	10-yr. total	Rank Order	Institution	10-yr. total
1	University of California (all campuses)	170	36	Syracuse University	18
			36	Virginia Polytechnic Inst.	18
2	New York University	120	39	University of Notre Dame	16
3	Princeton University	114	39	University of Kentucky	16
4	University of Chicago	101	39	University of Tennessee	16
4	University of Michigan	101	39	George Washington University	16
6	University of Illinois	92	43	Rice University	15
7	Columbia University	82	43	Catholic University of America	15
8	Harvard University	71	45	Illinois Institute of Technology	14
8	Massachusetts Inst. of Tech.	71	45	Northwestern University	14
10	Brown University	70	47	University of Georgia	13
11	University of North Carolina, Chapel Hill	68	48	University of Missouri	12
			48	University of So. California	12
12	University of Wisconsin	57	48	University of Oklahoma	12
13	Iowa State University	55	48	Oregon State College	12
14	Purdue University	54	52	George Peabody College for Teachers	11
15	Yale University	50			
16	University of Minnesota (all campuses)	48	52	University of Virginia	11
			54	University of Colorado	10
17	Stanford University	47	55	Boston University	9
18	University of Pittsburgh	43	55	University of Cincinnati	9
18	University of Texas	43	55	Washington University	9
20	Carnegie Inst. of Tech.	40	58	University of Buffalo	8
21	University of Florida	34	59	Louisiana State University	7
21	University of Pennsylvania	34	59	Johns Hopkins University	7
23	University of Washington	33	59	University of Rochester	7
24	Calif. Inst. of Tech.	32	59	Case Institute of Technology	7
25	University of North Carolina, State College	31	63	Auburn University	6
			63	Oklahoma State University	6
26	Indiana University	29	63	Pennsylvania State University	6
26	Ohio State University	29	63	Vanderbilt University	6
28	Cornell University	26	67	Rutgers, The State University	5
29	Duke University	24	67	Lehigh University	5
30	Tulane University	23	69	University of Utah	4
30	University of Oregon	23	69	American University	4
32	University of Maryland	22	71	Wayne State University	3
33	State University of Iowa	21	71	University of Nebraska	3
34	University of Kansas	19	71	Radcliffe College	3
34	Michigan State University	19	74	Rensselaer Polytechnic Institute	1
36	St. Louis University	18	74	Bryn Mawr College	1

CONTEMPORARY MATHEMATICS*

The following report presents the uses of the two courses presented by Continental Classroom in the Contemporary Mathematics series—*Modern Algebra* in the first semester and *Probability and Statistics* in the second semester—during the academic year 1960–61.

Both courses were utilized by a variety of groups, including: (1) colleges and universities, (2) high schools, (3) local boards of education, (4) industrial organizations, (5) vocational organizations, and (6) the general public.

Colleges and universities. Approximately 300 colleges and universities across the country offered *Modern Algebra* (and/or *Teaching of Modern Algebra*) for credit during the first semester. In most cases, three semester hours of undergraduate credit was awarded for Dr. John L. Kelley's Monday, Wednesday, Friday lessons; similarly, graduate credit was granted for students following these lessons in addition to Dr. Julius H. Hlavaty's Tuesday, Thursday lessons—*Teaching of Modern Algebra.* The credit students numbered close to 5,000.

In the second semester, the number of participating institutions increased to 325. The same amount of credit was given, *i.e.*, three semester hours to undergraduate students following Dr. Frederick Mosteller's lectures in *Probability and Statistics* on Monday, Wednesday, and Friday, and graduate credit to those following these lessons in addition to Professor Paul Clifford's Tuesday and Thursday sessions—*Teaching of Probability and Statistics.* The number of credit students for these courses seems to be more than 5,000.

High schools. More than 100 high schools utilized *Modern Algebra,* for either credit or audit students (or, in some cases, both.) In the second semester, 66 high schools or school districts used *Probability and Statistics.* The number of students per school varies from several all the way to 250; approximately 1200 high school students are participating in the program.

Local boards of education. Many city boards of education have been offering inservice credit to their teachers for the mathematics courses. Among these are: New York City; Reading, Pennsylvania; Kansas City, Missouri; Newton, Massachusetts; Farmingdale, New York; Norwalk, Connecticut; Rochester, Minnesota; Cleveland Heights, Ohio; and, Ithaca, New York.

Industrial organizations. Industrial organizations have evinced particular interest in the *Probability and Statistics* series since it is so pertinent to their quality control and computer employees. Some of these organizations have set up formal classes in their plants with one of the employees (often a training officer) acting as teacher; others have merely encouraged their employees to watch the TV lessons. Among the formalized groups are: Western Electric Company, Winston-Salem, North Carolina and Allentown, Pennsylvania; Otarion Listener Corporation, Ossining, New York; New York Telephone Company, Brooklyn, New York; Market Research Corporation of America, New York City; *et al.*

Vocational organizations. Among the vocational organizations actively interested in the second semester offerings are the American Society for Quality Control and the American Orthopsychiatric Association. The former has done much promotion in its Journal, both in news articles and in printing of the TV Lesson Schedule. The latter, through its Committee on Research, invited LRI to participate in a joint mailing to its membership of 1,800 persons. The mailing was organized and sent out early in January.

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* See also p. 688 of this issue.

Another joint mailing—LRI and Addison-Wesley Publishing Company—was sent out early in January to the membership lists of the American Statistical Association and the National Council of Teachers of Mathematics. College teachers of mathematics were also included in this mailing.

General public. Based on letters and requests for TV Lesson Schedules that we have received, we may conclude that there is a large, noncredit audience among the general public in all parts of the country. This group includes: mathematicians, people engaged in scientific and other vocations related to mathematics, teachers and laymen who studied mathematics prior to the developments contained in the *Modern Algebra* and *Probability and Statistics* courses, "students" not seeking a degree, housewives, businessmen, etc.

These are the uses of the Contemporary Mathematics series to date. There will be future use and lasting value due to the availability of kinescopes made from the video tapes of the lectures. Also, it is likely that the courses will be re-run over network and educational facilities.

(Report from Learning Resources Institute. The courses are sponsored by the Conference Board of the Mathematical Sciences.)

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1476. *Proposed by M. T. Salhab, Illinois Institute of Technology*

In triangle ABC, $AB=AC$, D is the midpoint of BC, E is the foot of the perpendicular from D on AC, and F is the midpoint of DE. Prove that AF is perpendicular to BE.

E 1477. *Proposed by Sidney Kravitz, Dover, New Jersey*

It is known that $f(n)=n^2-n+41$ yields prime numbers for $n=1, \cdots, 40$. Prove that (a) $f(n)$ is never divisible by a number <41, (b) $f(n)$ is never a perfect square except for $n=41$, (c) for each n there exists an m such that $f(m)=f(n)f(n+1)$, (d) $f(1722)$ is the smallest f with four, not necessarily distinct, prime factors.

E 1478. *Proposed by Jonathan Sondow, University of Wisconsin*

Prove that the digital root of every even perfect number except 6 is 1.

E 1479. *Proposed by Morton Abramson, McGill University*

Find the number of ways of choosing k elements from n elements $x_1, \cdots, x_n$ so that no three consecutive elements appear in each choice.

E 1480. *Proposed by Leo Flatto and A. G. Konheim, IBM, Yorktown Heights, New York*

Let the line segment $[0, 1]$ be divided into $n+1$ segments by n points $P_1, \cdots, P_n$. Assume the $\{P_i\}$ are independent random variables each uniformly distributed on $[0, 1]$. What is the probability p that the $n+1$ segments can be joined to form an $(n+1)$-sided polygon?

SOLUTIONS

E 1434, Correction and Addition

E 1434 [1960, 802; 1961, 381]. *Proposed by Anatole Beck, University of Wisconsin*

Show that every open set in the plane can be represented as a disjoint union of closed straight line segments.

Editorial Note. Solutions I and II [1961, 381] are faulty. Solution I states that an open interval can be represented as the union of its closed "middle thirds," whereas there is actually a set of uncovered points of the open interval having the power of the continuum (namely the Cantor set minus a countable set). Also, the statement in Solution I that an open linear segment is the union of disjoint closed nondegenerate segments can be shown to be false. Solution II is incorrect since the unfilled part of R is *not* again a half-open rectangle, but a rectangle open on *three* sides. A correct solution along the lines of Solution II can easily be formulated, and such a solution appears below. Robert Spira has made the interesting conjecture that every open set in the plane can be represented as a disjoint union of closed straight line segments, *no two having the same direction.*

IV. *Solution by R. G. Kayel, Michigan State University.* The following construction shows how to cover the half-open square $S=\{(x, y): 0 \leqq x<1, 0 \leqq y<1\}$ with a union of disjoint line segments. (1) Construct the step function $S(x)=\frac{1}{2}$ if $x \in [0, \frac{1}{2}]$, $\frac{3}{4}$ if $x \in (\frac{1}{2}, \frac{3}{4}]$, $\frac{7}{8}$ if $x \in (\frac{3}{4}, \frac{7}{8}]$, $\cdots$. (2) Let the vertical segment (VS) $[0, \frac{1}{2}]$ move on the horizontal segment (HS) $[0, \frac{1}{2}]$; VS $[0, \frac{3}{4}]$ on HS $(\frac{1}{2}, \frac{3}{4}]$; VS $[\frac{3}{4}, \frac{7}{8}]$ on HS $(\frac{3}{4}, \frac{7}{8}]$; $\cdots$. (3) Let HS $[0, \frac{1}{2}]$ move on VS $(\frac{1}{2}, \frac{3}{4}]$; HS $[0, \frac{3}{4}]$ on VS $(\frac{3}{4}, \frac{7}{8}]$; $\cdots$.

The Unique and Mystic Order of Blushing Beauties

E 1446 [1961, 62]. *Proposed by David Bickerstaff, University of Mississippi*

"How about telling me confidentially the secret order of the five beauties to be featured in this year's Annual?" I proposed to the editor. She, of course, refused, but agreed to pass judgment on my guess. "Is it *A-B-C-D-E*?" I asked.

"You are most skillful at being wrong," she chided. "You not only got each person out of her true position but, furthermore, not one in your ranking followed correctly her immediate predecessor."

"Well, then, is it *D-A-E-C-B*?" I asked.

"Now you are improving," she encouraged cautiously. "You have two in proper position and you have two following correctly their immediate predecessors."

After a little figuring I then told her the correct order, and she swore me to secrecy. What is the correct order?

Solution by J. F. Leetch, Ohio State University. In the ordering *D-A-E-C-B*, the correct pair must be adjacent, for otherwise we would have a proper follower after a correct position, implying three correct positions. This means that *D-A*, *A-E*, *E-C*, or *C-B* are properly located. The pairs *A-E* and *E-C* must be eliminated since a second proper follower could not occur. The pair *D-A* can be eliminated by checking two cases. This leaves only *E-D-A-C-B* meeting all requirements.

Also solved by Philip Anderson, R. H. Anglin, Merrill Barnebey, R. E. Beals, Jeanette Bickley, A. M. Broshi, Brother Joseph Heisler, Brother T. C. Wesselkamper, Mike Brown, W. E. Buker, F. P. Callahan, Jr., Robert Carlos, Virginia Christian, Curt Gilchrist, D. I. A. Cohen, James Cooley, R. W. Cottle, Monte Dernham, R. D. Dickson, Jane Evans, D. P. Giesy, Michael Goldberg, L. D. Goldstone, S. H. Greene, W. G. Griggs, F. C. Hall, J. E. Homer, Jr., J. T. Humphrey, A. R. Hyde, Lawrence Isenecker, V. F. Ivanoff, J. L. Johnson, P. B. Johnson, Sidney Kravitz, C. W. Kreke, Betty Levine, Adolph Lu, Glen Luchau, Frank McGee, Robert Maas, D. C. B. Marsh, J. M. Mettler, Otto Mond, D. A. Moran, J. B. Muskat, Herbert Nadler, R. W. Neufeld, J. C. Nichols, C. S. Ogilvy, J. M. Pasachoff, Jon Petersen, J. L. Pietenpol, C. F. Pinzka, Stephen Porcari, E. H. Primoff, M. B. Richins, Jonathan Robinson, O. J. Roman, David Sachs, Sister Mary Denis, E. L. Spitznagel, Jr., W. B. Stovall, Jr., R. S. Strichartz, D. V. Susco, Fred Suvorov, Guy Torchinelli, J. D. Vineyard, R. M. Warten, W. C. Waterhouse, M. J. Wiedel, R. J. Wisner, and Bell Yung. Late solution by Eric Sturley.

Construction of a Triangle

E 1447 [1961, 62]. *Proposed by Walter Bluger, Dominion Bureau of Statistics, Ottawa, Canada*

Construct a triangle given R, r, h_a.

Solution by L. D. Goldstone, Watervliet, New York. The datum (h_a, r, r_a) yields r_a, then the datum (R, r_a-r, a) yields a, and then the triangle is easily constructed.

For the relations yielding the above data see Altshiller-Court, *College Geometry* (2nd ed.), Sec. 139 and Sec. 144 (a). These sections, along with Sec. 140, Sec. 152, and some elementary geometry, show that the solution is essentially unique and that for compatibility we must have

$$(2R - h_a)(h_a - 2r) \geqq r^2, \qquad 2R > h_a > 2r, \qquad R \geqq 2r.$$

Also solved by Carole Colebob, Michael Goldberg, D. C. B. Marsh, Beckham Martin, Peter Ploch, O. J. Roman, Guy Torchinelli, and the proposer.

Goldberg showed that the distance from vertex A to the incenter is the mean proportional between $2R$ and h_a-2r. The construction of the triangle now easily follows.

Marsh found constructible expressions for the sides a, b, c in terms of R, r, h_a.

Section of an Oblate Spheroid

E 1448 [1961, 62]. *Proposed by C. B. Grosch, General Mills, Inc., Minneapolis, Minnesota*

Show that any plane section of an oblate spheroid, not perpendicular to the axis of the spheroid, is an ellipse with major axis parallel to the equatorial plane of the spheroid and with minor axis (or minor axis extended) intersecting the axis of the spheroid.

I. *Solution by Amos Nannini, University of Manitoba.* With no loss of generality on account of the symmetry involved, we may assume that, the xy-plane being the equatorial plane and the z-axis the axis of revolution, the cutting plane is parallel to the y-axis. The intersection, necessarily an ellipse, again by reasons of symmetry, has one axis parallel to the y-axis and the other axis coplanar with the z-axis. To prove that the former must be the major axis, let us assume it is instead the minor. Then, by making the inclination of the cutting plane closer and closer to 90°, by reasons of continuity the intersection ought to be a circle for a definite value of the angle of inclination, because an exactly vertical plane would cut out a "squat" ellipse. But this is impossible, because an oblate spheroid has no umbilics except its poles.

The problem (and proof) can be generalized to any surface of revolution under some broad conditions of continuity and differentiability. The following might be a suitable generalization: "Any plane section of a surface of revolution has an axis of symmetry coplanar with the axis of revolution and therefore either parallel to or intersecting it, naturally or extended. Needless to say, if the curve of intersection has an axis of symmetry normal to the previous one, it must be orthogonal to the axis of revolution."

The statement of the problem as presented should be completed by mentioning the possible parallelism of the minor axis to the axis of revolution.

II. *Solution by Michael Goldberg, Washington, D. C.* A plane section of a sphere is a circle. If the sphere is deformed parallel to a diameter by a reduction of scale, the sphere becomes an oblate spheroid. The plane section remains plane and the circular section becomes an ellipse. The diameter of the circle along the line joining the center of the circle to the axis of contraction is reduced, while the diameter of the circle normal to this line remains unchanged. Hence, by symmetry, these are the minor and major axes of the ellipse.

Conversely, any oblate spheroid can be converted into a sphere by an increase in scale parallel to the axis. A plane section of the spheroid is transformed into a plane section of the sphere.

Also solved by A. P. Boblétt, Jane Evans, D. C. B. Marsh, and the proposer. All these solutions were analytical.

A Pseudo Mean

E 1449 [1961, 62]. *Proposed by C. S. Patlak, Department of Health, Education, and Welfare, Bethesda, Maryland*

Assume that (1) A_i, B_i, C_i, D_i $(i=1, \cdots, n)$ are all positive, (2) $\sum A_i \geqq \sum C_i$, (3) $A_i - C_i = B_i - D_i$ $(i = 1, \cdots, n)$. Set $P_i = A_iB_i/C_iD_i$ and $P=\max(P_1, \cdots, P_n)$. Prove that $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i) \leqq P$.

Solution by D. C. B. Marsh, Colorado School of Mines. The proposition is obviously true for $n=1$.

Assume that $n=2$ and set $A=A_1+A_2$, $B=B_1+B_2$, $C=C_1+C_2$, $D=D_1+D_2$. Consider $P-AB/CD$, which may be written in the form

$$(1/4CD)\{4(P-1)CD + (C+D)^2 - (A+B)^2\}.$$

Since $P \geqq A_iB_i/C_iD_i$ $(i=1, 2)$, we have

$$\{4(P-1)C_iD_i + (C_i+D_i)^2\}^{1/2} \geqq A_i + B_i > 0.$$

Denoting the radical by R_i we have

$$\begin{aligned} P-AB/CD &\geqq (1/4CD)\{4(P-1)CD+(C+D)^2-(R_1+R_2)^2\} \\ &\geqq (1/4CD)\{4(P-1)(C_1D_2+C_2D_1)+2(C_1+D_1)(C_2+D_2)-2R_1R_2\}, \end{aligned} \tag{1}$$

when we expand $(R_1+R_2)^2$ and simplify algebraically. Since $A \geqq C$ it follows that, for some i, $A_i \geqq C_i$. For this i we also have $B_i \geqq D_i$, whence $P_i \geqq 1$, or $P \geqq 1$. Thus

$$4(P-1)(C_1D_2 + C_2D_1) + 2(C_1+D_1)(C_2+D_2) + 2R_1R_2 > 0. \tag{2}$$

Multiplying both sides of inequality (1) by the positive expression (2), we find that the right-hand side reduces to $(1/4CD)\{16P(P-1)(C_1D_2-C_2D_1)^2\}$, which is nonnegative. Thus the left-hand side must also be nonnegative, whence $P-AB/CD \geqq 0$, proving the proposition for $n=2$.

Now assume as an induction hypothesis that the proposition is true for $n=k>2$, and set

$$A^* = \sum_{i=1}^{k} A_i, \qquad B^* = \sum_{i=1}^{k} B_i, \qquad C^* = \sum_{i=1}^{k} C_i, \qquad D^* = \sum_{i=1}^{k} D_i.$$

Then, applying the proposition for $n=2$, we have

$$\begin{aligned} &(A^* + A_{k+1})(B^* + B_{k+1})/(C^* + C_{k+1})(D^* + D_{k+1}) \\ &\qquad \leqq \max[A^*B^*/C^*D^*, A_{k+1}B_{k+1}/C_{k+1}D_{k+1}] \\ &\qquad \leqq \max[\max(P_1, \cdots, P_k), P_{k+1}] = \max[P_1, \cdots, P_{k+1}], \end{aligned}$$

and the proposition holds for arbitrary positive integral n by mathematical induction.

Also solved by the proposer, who gave examples to show that all three conditions are necessary for the theorem to be true in general. He also gave an example illustrating that the quotient $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i)$ is not a true mean of the expressions A_iB_i/C_iD_i; in order that also $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i) \geqq \min(P_1, \cdots, P_n)$, condition (2) would have to be that $\sum A_i = \sum C_i$.

Application of Abel's Partial-Summation Theorem

E 1450 [1961, 62]. *Proposed by Lawrence Shepp, Princeton University*

If $a_n, b_n > 0$, $a_n \downarrow 0$, then $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j = \sum_{n=1}^{\infty} a_n b_n$.

I. *Solution by W. C. Waterhouse, Harvard University.* We note, by Abel's partial-summation theorem,

$$S_k \equiv \sum_{n=1}^{k} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j = \sum_{n=1}^{k} a_n b_n - a_{k+1} \sum_{n=1}^{k} b_n = \sum_{n=1}^{k} (a_n - a_{k+1}) b_n.$$

Thus $\sum_{n=1}^{k} a_n b_n > S_k$, so if $S_k \to \infty$ then $\sum_{n=1}^{k} a_n b_n \to \infty$.

Suppose $\lim S_k = A$. Then $\sum_{n=1}^{\infty} a_n b_n \geqq A$. But for any k and any $\epsilon > 0$ we can pick an $m > k$ such that $a_m/a_k < \epsilon$. Then

$$A \geqq S_{m-1} \geqq \sum_{n=1}^{k} (a_n - a_m) b_n > (1 - \epsilon) \sum_{n=1}^{k} a_n b_n,$$

whence $A \geqq \sum_{n=1}^{\infty} a_n b_n$. It follows that $A = \sum_{n=1}^{\infty} a_n b_n$.

II. *Solution by Michael Goldberg, Washington, D. C.* Consider the step function of vertices $(0, a_1)$, (b_1, a_1), (b_1, a_2), $(b_1 + b_2, a_2)$, $\cdots$, $(\sum_{j=1}^{k} b_j, a_j)$, $(\sum_{j=1}^{k} b_j, a_{j+1})$, $\cdots$. Then, if the area A under this function is divided into vertical strips, it is given by $A = \sum_{n=1}^{\infty} a_n b_n$, while if it is divided into horizontal strips, it is given by $A = \sum_{n=1}^{\infty} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j$. Hence the desired result follows.

Also solved by J. L. Brown, Jr., P. R. Chernoff, Gus DiAntonio, S. H. Greene, Betty Levine, Jiang Luh, D. C. B. Marsh, J. M. Pasachoff, J. T. Rosenbaum, O. E. Stanaitis, Fred Suvorov, and the proposer. Late solution by Eric Sturley.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4977. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let S be a system with a finite number of elements with two operations, addition and multiplication, satisfying the following axioms:

1) Under addition the elements of S form an abelian group.
2) For a, b, c in S, $a \neq b$, the equation $xa = xb + c$ has a unique solution.
Prove that also the equation $ay = by + c$ has a unique solution.

4978. *Proposed by R. Nathan, University of Idaho*

Let $n > 0$ and $k \geqq 2$ be integers. Show

$$\sum_{\mu=0}^{n} \frac{(-1)^{\mu}}{\mu![(\mu+1)k-1](n-\mu)!} = \frac{k^n}{(k-1)(2k-1)\cdots([n+1]k-1)}.$$

4979. *Proposed by H. S. Shapiro, New York University*

If z_1, z_2, z_3 are distinct numbers of modulus 1, and

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1{}^m & z_2{}^m & z_3{}^m \\ z_1{}^n & z_2{}^n & z_3{}^n \end{vmatrix} = 0,$$

then either two rows or two columns of the determinant are identical.

4980. *Proposed by G. H. Meisters, RIAS, Baltimore, Md.*

If P is a topological property, we call a topological space X *locally-P* if and only if every neighborhood N of every point x contains a neighborhood N^* of x which has the property P in its relative topology. Otherwise our terminology is that of J. L. Kelley: *General Topology*. Prove (or disprove) the following statement: If X is a compact, locally-Hausdorff topological space, then X is locally-compact.

4981. *Proposed by Lawrence Shepp, Princeton University*

Show

$$\int_0^1 \left(\sum_{n=1}^{\infty} \{2^{-n}(1 - \cos 2^{2^n} z)\} \right)^{-1} dz < \infty.$$

4982. *Proposed by G. Di Antonio, Duquesne University*

On any of the five regular solids, let two points be given, not both on the same face. Determine the geodesic between the two points.

SOLUTIONS

A Corollary of a Theorem of Schwartz

4906 [1960, 479]. *Proposed by P. L. Butzer, Technical University, Aachen, Germany*

If the function $f(x)$ is continuous on an interval (a, b) and, as $h \to 0$, $h^{-3}\int_0^h [f(x+u) + f(x-u) - 2f(x)]du \to 0$ for all x in (a, b), then $f(x)$ is a linear function.

II. *Solution by Joshua Barlaz, Rutgers—The State University.*

Let $[\alpha, \beta]$ be a closed interval in (a, b). With x in $[\alpha, \beta]$, k real, let

$$\psi(x) = f(x) + \frac{\beta - x}{\alpha - \beta} f(\alpha) + \frac{x - \alpha}{\alpha - \beta} f(\beta) + k^2(x - \alpha)(x - \beta).$$

Then $\psi(x)$ is continuous on $[\alpha, \beta]$ and, as $h \to 0$,

$$(1/h^3) \int_0^h [\psi(x + u) + \psi(x - u) - 2\psi(x)] du \to 2k^2/3 > 0.$$

Suppose now that the maximum of $\psi(x)$ occurred at $x = \xi$ and $\alpha < \xi < \beta$. Then, for sufficiently small u,

$$\psi(\xi + u) + \psi(\xi - u) - 2\psi(\xi) \leqq 0$$

and the limit above could not be positive. Therefore the maximum of $\psi(x)$ is either at $x = \alpha$ or at $x = \beta$. But $\psi(\alpha) = \psi(\beta) = 0$. Consequently $\psi(x) \leqq 0$. Letting $k \to 0$, it follows that

$$f(x) + \frac{\beta - x}{\alpha - \beta} f(\alpha) + \frac{x - \alpha}{\alpha - \beta} f(\beta) \leqq 0, \qquad \alpha \leqq x \leqq \beta.$$

A parallel argument also proves

$$f(x) + \frac{\beta - x}{\alpha - \beta} f(\alpha) + \frac{x - \alpha}{\alpha - \beta} f(\beta) \geqq 0$$

and therefore $f(x)$ is linear on $[\alpha, \beta]$. But α, β are arbitrary in (a, b). Therefore $f(x)$ is linear on (a, b).

This proof parallels that given for Schwartz's theorem in Chaundy, *The Differential Calculus.*

Note by H. W. Brinkmann, Swarthmore College. The solution previously given [1961, 385] is incorrect since it uses, not l'Hospital's rule, but a converse of it which is notoriously not true. The presumed theorem (*If $f(x)$ and $g(x)$ are of class C' and $f(x)/g(x) \to L$ as $x \to a$, then $f'(x)/g'(x) \to L$*) is easily disproved by the counter example: $f(x) = x^4 \sin(1/x)$, $g(x) = x^3$, $(a = 0)$.

Arguments of Two Sets of Complex Numbers

4918 [1960, 699]. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let α_k, β_k be complex numbers, $k = 1, \cdots, n$, with $|\alpha_k| = |\beta_k| = 1$. Let $0 \leqq \arg \alpha_1 < \cdots < \arg \alpha_n < 2\pi$, $0 \leqq \arg \beta_1 < \cdots < \arg \beta_n < 2\pi$, where $0 \leqq \arg \gamma < 2\pi$ for any nonzero complex number γ. Also let $\sum_1^n \alpha_k = \sum_1^n \beta_k = 0$. Prove that $\sum_1^n \alpha_k \bar{\beta}_k \neq 0$.

Editorial Note. A solution by P. J. van Albada, G. Laman, and J. H. van Lint appears in the *Michigan Journal of Mathematics*, vol. 7, 1961.

A Definite Integral Property

4919 [1960, 699]. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let $\phi(\theta)$ be real valued for $0 \leqq \theta \leqq 2\pi$, let $\phi'(\theta)$ be continuous and positive, let $\phi(0)=0$, $\phi(2\pi)=2\pi$ and let $\int_0^{2\pi} e^{i\phi(\theta)}d\theta=0$. Prove that

$$\left|\int_0^{2\pi} e^{i(\phi(\theta)-\theta)}d\theta\right| > 4.$$

Editorial Note. No proof of the problem as stated has been received. However, the proposers have solved it with an additional hypothesis on $\phi(\theta)$. This will appear in an early issue of the *Michigan Journal of Mathematics.*

Number of Games Won in a Major League Baseball Season

4921 [1960, 700]. *Proposed by David Gale, Brown University*

Prove that the numbers $w_1, \cdots, w_8$ can be the numbers of games won by the eight teams at the end of a major league baseball season if and only if

$$\sum_{i=1}^{8} w_i = 616, \qquad \sum_{i=1}^{k} w_i \leqq 11k(15-k), \qquad k = 1, \cdots, 7.$$

(During a major league season every two teams play each other 22 times.)

Solution by D. C. B. Marsh, Colorado School of Mines. Assuming no ties and no incompleted games, the necessity of the given conditions is easily shown. Putting w_{ij} for the number of times i won from j; $w_{ii}=0$ and $w_{ij}+w_{ji}=22$ for all i, j with $i\neq j$, we have

$$\sum_{i=1}^{k} w_i = \sum_{j=1}^{8}\sum_{i=1}^{k} w_{ij} = \sum_{j=1}^{k}\sum_{i=1}^{k} w_{ij} + \sum_{j=k+1}^{8}\sum_{i=1}^{k} w_{ij}.$$

Now $\sum_{i=1}^{k}\sum_{j=1}^{k} w_{ij}=22C(k,2)=11k(k-1)$ and $0\leqq \sum_{j=k+1}^{8}\sum_{i=1}^{k} w_{ij}\leqq 22(8-k)k$, whence

$$(3) \qquad 11k(k-1) \leqq \sum_{i=1}^{k} w_i \leqq 11k(k-1) + 22k(8-k) = 11k(15-k),$$

for $k=1, \cdots, 8$, which implies the given conditions.

The conditions as stated are not sufficient, for they are satisfied by $w_1=154$, $w_2= \cdots =w_7=0$, $w_8=462$, which nevertheless contradict $w_{23}+w_{32}=22$, etc.

Also solved by John E. Freund, David Greenstein, Sidney Kravitz; and also (necessity only) J. R. Blum, J. B. Bohac, G. S. Cunningham, G. DiAntonio, Jane Evans, D. L. Muench, Martha M. Pennell, D. C. Stevens, and Eric Sturley.

Editorial Note. If we relabel the w's so that $w_1 \geqq w_2 \cdots \geqq w_8$, the left member of (3) can be replaced by $77k$. It is then not difficult to show that the modified inequalities provide also sufficient conditions. One way of establishing this result is to consider n teams each playing 22 games with each of the others. Then (3) becomes $11k(n-1)\leqq \sum_{i=1}^{k} w_i \leqq 11k(2n-1-k)$, $k=1, \cdots, n$. The sufficiency is obvious for $n=2$, and we then show that sufficiency for n follows from sufficiency for $n-1$.

Abelian Group without Maximal Subgroups

4922 [1960, 700]. *Proposed by Peter Crawley, California Institute of Technology*

In the solution of a recent problem (4761, [1959, 67]) it was shown that the additive group of rationals has no maximal subgroups. Prove the generalization: an abelian group G has no maximal subgroups if and only if G is divisible (*i.e.*, $NG=G$ for all integers n).

Solution by Paul Hill, Institute for Advanced Study. If H is a proper subgroup of a divisible abelian group G, then G/H is divisible and therefore cannot be cyclic. Thus H is not maximal in G. To prove the converse, use is made of the following obvious propositions: (1) if H is a subgroup of an abelian group G having no maximal subgroups, then G/H has no maximal subgroups, (2) a group G having a cyclic summand has a maximal subgroup unless G is prime order cyclic. Now suppose that the abelian group G is not divisible. The quotient group G/nG is nontrivial for some positive integer n. If G/nG is prime cyclic, then nG is maximal in G; otherwise, G/nG contains a maximal subgroup since from the well-known first theorem of Prufer it follows that G/nG is a direct sum of cyclic groups. Thus G contains a maximal subgroup.

Also solved by T. N. Delmer, C. Franke, E. R. Gentile, Nathaniel Grossman, J. M. Irwin, S. Lajos, R. A. McHaffey, Fred Suvorov, and the proposer.

Editorial Note. The stated proposition is equivalent to: *An Abelian group G is divisible if and only if $\phi(G)=G$, where $\phi(G)$ is the Frattini subgroup of G*, part of an exercise in L. Fuchs, *Abelian Groups*, 1958, p. 67.

Gentile proves the following generalization: *Let R be a Dedekind ring. Then an R-module is divisible if and only if it has no maximal submodules.*

The Divisor Set of a Set of Real Numbers

4923 [1960, 808]. *Proposed by D. J. Newman, Yeshiva University*

For any set of real numbers S we define its divisor set $\hat{S}$ to be the set of all numbers in (0, 1) for which some integral multiple lies in S. Are there sets of arbitrarily large measure for which $\hat{S}$ has arbitrarily small measure?

Solution by J. B. Kelly, Michigan State University. The following example provides an affirmative answer. Let

$$S = \sum_{k=1}^{[n/\epsilon]} \left(\frac{n}{k}, \frac{n+\epsilon}{k}\right).$$

Here n is a positive integer and ϵ is a positive number less than unity whose dependence upon n will be given later. The intervals comprising S are disjoint since $n/k<(n+\epsilon)/(k+1)$ implies $k>n/\epsilon$. Hence, denoting the measure of S by $m(S)$, we have

$$m(S) = \sum_{k=1}^{[n/\epsilon]} \epsilon/k > \epsilon \log (n/\epsilon).$$

The divisor set $\hat{S}$ is the same as the divisor set for the open interval $(n/k, (n+\epsilon)/k)$. The divisor set for the latter consists of intervals $(n/k, (n+\epsilon)/k)$ where k runs from $n+1$ to $[n/\epsilon]$, plus the interval from 0 to ϵ. Hence, using the fact that $\sum_{k=m}^{rm} 1/k \sim \log r$, we have

$$m(\hat{S}) = \epsilon + \sum_{k=m+1}^{[n/\epsilon]} \epsilon/k < \epsilon + 2\epsilon \log(1/\epsilon)$$

for n sufficiently large. Now, setting $\epsilon = (\log n)^{-1/2}$ and letting $n \to \infty$, we see that $m(S) \to \infty$ and $m(\hat{S}) \to 0$.

Also solved by Robert Breusch, Fred Suvorov, and the proposer.

A Pseudo-integral

4924 [1960, 808]. *Proposed by J. H. Blau, Antioch College*

Let P denote a partition of $[0, 1]$ into n disjoint measurable sets E_i. Denote $\sup m(E_i)$ by $|P|$, and let $x_i \in E_i$. For which functions f on $[0, 1]$ does the pseudo-integral exist:

$$\lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i) m(E_i)?$$

Solution by the proposer. The integral exists and equals J if and only if those values of x (if any) for which $f(x) \neq J$ form a sequence $\{a_i\}$, and $f(a_i) \to J$. For the proof, let $J=0$. The extension to general J is clear.

Let the integral exist. Suppose $f(x) \geqq \epsilon > 0$ for infinitely many distinct values b_i of x. Then, for any $\delta > 0$, there is a partition P with $|P| < \delta$, and with $b_i \in E_i$. Thus $\sum_{i=1}^{n} f(b_i) m(E_i) \geqq \epsilon$, contradicting the integral's existence. Hence $f(x) < \epsilon$ with a finite number of exceptions. The same is true for $f(x) > -\epsilon$, and hence for $|f(x)| < \epsilon$. It follows that the set where $f(x) \neq 0$ is denumerable. Enumerating it as $\{a_i\}$, we have $f(a_i) \to 0$.

Conversely, let the condition be satisfied. Then $M = \sup |f|$ is finite. If $\epsilon > 0$, then $2|f(x)| \geqq \epsilon$ for only a finite number, say r, of values of x. Let P be such that $2rM|P| < \epsilon$, let $x_i \in E_i$, and renumber the E_i so that $2|f(x_i)| < \epsilon$ for $i > r$. Now

$$\left| \sum_{i=1}^{n} f(x_i) m(E_i) \right| \leqq \sum_{i=1}^{n} |f(x_i)| m(E_i).$$

The sum of the first r terms on the right is at most $rM|P|$, and the remaining sum is less than $\epsilon/2$. Hence the integral exists.

Also solved by G. A. Heuer, and Fred Suvorov.

RECENT PUBLICATIONS

Edited by Richard V. Andree, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Zbornik Matematičkih Problema, III. D. S. Mitrinović (Ed.). Univerzitet u Beogradu, Beograd, 1960. xvi+334 pp. Purchasable through La Faculté d'Électrotechnique, Département Mathématique, Université de Belgrade. Zbornik I, $7.00; Zbornik II, $8.00; Zbornik III, $8.00.

With a revised edition of Volume II appearing last year and Volume I currently undergoing its third revision, this paper-bound work by the well-known European problemist presents a new set of some 870 problems to supplement the earlier pair. The main sources of material are *Jahresbericht der Deutschen Mathematiker Vereinigung*, *The American Mathematical Monthly*, *The Mathematical Gazette*, *Mathematics Magazine*, and *Mathesis*, as well as original results obtained by Mitrinović and his associates.

The chapter headings give some indication of the scope of subject matter. After an introduction and a generous table of symbols and notation, we have complex numbers and functions, special functions (Legendre, Bessel, Laguerre), abstract algebra, projective geometry, miscellaneous problems, and an appendix touching on such topics as stereographic projection and generalized Jensen formulas. The work is concluded with twenty-six pages of tables (Kelvin functions, binomial coefficients, zeros of $J(x)$ and others).

The section on abstract algebra, for example, is a veritable handbook, developing semi-group and group concepts through exercises on properties and examples (matrices, permutations) up to rings and fields, and including isomorphism, ordering, *et al.*

With or without the assistance of a bilingual dictionary, one may rely upon the universal nature of the symbolism and technical terms to be able to interpret most of the content. Still there is enough expository material to warrant an English translation and one may hope that the National Science Foundation will include these volumes among those Yugoslavian works to be translated.

The printing and quality of the materials used are simple but adequate. All three volumes are recommended particularly to those interested in modern problems *per se.*

D. C. B. Marsh
Colorado School of Mines

Introduction to Analytic Geometry and Linear Algebra. By Arno Jaeger. Holt, Rinehart and Winston, New York, 1960. xii+305 pp. $6.00

This is an attempt to "present a modern . . . and a somewhat novel treatment of fundamental analytic geometry based on groups, vector spaces, and

Euclidean vector spaces together with an introduction to these algebraic structures at a level suitable for freshmen and sophomores."

The 23 chapters are divided into four major sections: I. *Foundations* introduces set theory, groups, vector spaces, and establishes the connection between algebra and geometry in terms of 1-1 onto maps between point-space $\mathcal{R}^n$ and vector-space of translations $\mathbf{R}^n$, between $\mathbf{R}^n$ and coordinate space of n-tuples R^n, and between $\mathcal{R}^n$ and R^n, where n is 1 or 2 or 3. II. *Linear Geometry and Algebra* concerns systems of linear equations, bases and dimension of vector spaces, linear maps, matrices. There is a nice digression on linear programming. III. *Multilinear Geometry and Algebra* introduces the inner product and deals with metric notions. Euclidean vector space is defined and put in context. Basic properties of determinants are developed. IV. *Quadratic Geometry and Algebra* deals with conics, quadrics, and canonical forms.

Demands on verbal ability and ability to handle symbols will make this book hard reading for freshmen and sophomores. There is some tendency to define concepts in greater generality than the main theme of the text requires, but in any event, the deliberately high level of abstraction requires introduction of an extensive technical vocabulary. The exposition is often quite condensed. An excessive use of italics for emphasis becomes quite tiresome, and what is more serious, reduces the effectiveness of italics in all uses.

The reviewer feels that for undergraduates having some prior college mathematics this book can be the basis for a very useful course. The many connections with geometry are welcome in an introductory algebra course. The organization is good. Theorems are well stated, definitions accurate. Abundant illustrative examples are well chosen. Lists of exercises are generally excellent. There is an adequate index supplemented by a useful index of symbols and another of examples.

ROBERT M. EXNER
Syracuse University

Complex Variable and the Laplace Transform for Engineers. By Wilbur R. LePage. McGraw-Hill, New York, 1961. xvii+475 pp. $12.50.

This is an excellent book for the purpose for which it is intended and rather than dwell on the technical qualities of the book (which are really good), the reviewer feels it is the spirit of the book which is important and should be discussed. The author is a professor of engineering and obviously feels that a properly trained engineering student is one who does his work creatively and forgets the quest for the "magic formula." This means that the student must have a real understanding of the basic area from which he draws his tools (*i.e.*, mathematics), not only from the point of view of selecting tools (theorems), when possible, to solve a problem, but also having sufficient understanding in order to create additional tools (when they are needed) with a feeling of confidence. The author must feel that the best way for the student to get to this position is to understand what definitions, theorems, and proofs are. It might

be noted at this point that he does an outstanding job of motivating his point of view. For example, in discussing the Fourier and Laplace integral, he essentially makes a good point concerning the mathematical idealization of system for analysis and the transforming of the ideal system into the construction of a practical engineering image. Hence, in the first situation the Fourier integral is effective (since energy considerations are important here) and in the second case (which is the important case for the practitioner) the Laplace integral is effective since the physical synthesis and realization of the idealized system inevitably requires the use of complex variables.

Insofar as the text itself is concerned, there are fifteen chapters. Most of the important topics found in the usual books on engineering mathematics are developed from scratch in a manner that is intellectually honest. Particular emphasis is given to complex function theory inasmuch as this plays a central and unifying role in the text. A large number of problems appear at the end of each chapter. Also, the author makes good use of physical problems to motivate the type of theorem he needs to develop and to illustrate that a hasty use of formal calculus doesn't always lead to the right result (if any at all).

A publicist for the book company writes on the front flap of the jacket that the text was written for the serious student, "probably" at the graduate level. He is really suffering from a case of commercial temerity since the contents and presentation of the book is in no way out of line with the recommendations made a few years ago by a select committee of the American Society for Engineering Education.

PASQUALE PORCELLI
Louisiana State University

Introductory Algebra. A College Approach. By M. D. Eulenberg and T. S. Sunko. Wiley, New York, 1961. xi+290 pp. $4.95.

The preface of this book states that it is intended for students in college who have not had sufficient preparation for college algebra or an equivalent course. It is supposed to be finished in one semester, since such students are more mature mentally than those in preparatory school. The introduction is well done. The book contains a six-page index, answers to odd-numbered problems and tables (1) of squares and cubes, square roots and cube roots and (2) of common logarithms of numbers.

It is a question as to whether a "set" is defined here or not. The authors say that we will regard a set as "any distinct collection of objects," but that it is an undefined word. Certainly the definition of a finite set is inadequate. It is not the *elements* that are less than a given fixed number, but the *number* of the elements.

The authors are very good in their explanation of division by zero, in many sets of problems and on functional relationships. The reviewer is glad to see the material on sets in the book. It is developed in an interesting manner.

It is distressing to find frequent use of a singular subject with a plural verb, as well as a split infinitive, the term "consecutive numbers" and a painful collection of numerical errors.

The student who uses this book will certainly know how to deal with fractions, though he may be halted temporarily when told (on p. 128) that $18-6=24$ and (at the top of p. 134) that one root is 3/4 when it really is 4/3. The book seems to have been somewhat carelessly proofread.

MARION E. STARK
Wellesley College

Intermediate Algebra. By F. J. Mueller. Prentice-Hall, Englewood Cliffs, N. J., 1960. 374 pp. $5.95.

Written in traditional style, this book is developed by means of rules. Over seventy of these rules are included in the book, each of which is illustrated by several step-by-step examples, and followed by a large number of problems.

There is occasional mention of a term used in modern mathematics, although little effort is made to incorporate this language into the development. In the first chapter (From Arithmetic to Algebra) the principles of closure, commutativity, and associativity for addition and multiplication of a system of numbers are stated, along with the distributive principle for multiplication with respect to addition. These are illustrated with arithmetic examples but *are not used for anything.* The distributive principle is the only one of them mentioned in later chapters.

Included in the book are techniques for dealing with all the traditional problems of elementary algebra. Mr. Mueller is so clear in his statements of "how to do it" that an average student could learn how to solve the problems of this book with little extra help from a teacher. Whether he would gain any understanding of algebraic proof is questionable.

VIRGINIA CARLTON
Centenary College

Theoretical Physics in the Twentieth Century; A Memorial Volume to Wolfgang Pauli. By M. Fierz and F. F. Weisskopf (eds.). Interscience, New York, 1960. x+328 pp. $10.00.

Fourteen papers, including a foreword by Niels Bohr and a bibliography of Pauli's works by C. P. Enz, range over topics and fundamental problems close to Pauli's interests in quantum theory, relativity, and mathematical physics. Important events in research during the 1930's are also among the subjects discussed by such contributors from Europe, Russia, and the United States as W. Heisenberg, L. D. Landau, B. L. van der Waerden, G. Wentzel, C. S. Wu, and others.

TOM SMITH
University of Oklahoma

Silhouette Mathematics. By R. S. Underwood. Texas Technological College Bookstore, Lubbock, Texas, 1961. iv+148 pp. $2.00.

This text, which is paper bound and photographically reproduced from typewritten pages, largely contains the present status of the author's researches of the important and rather neglected problem of picturing the equations of three, and higher, dimensional analytic geometry on a plane. The material is elementary and merits examination. With some give and take, the text can be used in a beginning course in analytic geometry, though it would better serve a three-semester-hour tailor-made course. There are ample exercise sets, and answers are furnished in the rear of the book. There is no index, but there is a good table of contents. In the last chapter various "loose ends" and possible areas of application (such as to the solution of some Diophantine equations) are pointed out. The theory is not complete and there is room for improvements and extensions. Indeed, a dozen or so master's theses have originated with the ideas of this text.

HOWARD EVES
University of Maine

The Simplex Method of Linear Programming. By F. A. Ficken. Holt, Rinehart and Winston, New York, 1961. 64 pp. $1.50.

In this excellent little book, the author gives a clear exposition directed primarily to the mathematical theory of the simplex method of linear programming which falls within the field of algebra for the consideration of the general solution of a system of linear equations having certain constraints associated with them. The purpose appears to be an introduction to the subject in that, as is indicated by the author, there are many omissions. These, in the reviewer's opinion, are not at all detrimental. Also, the omissions are to a large extent neutralized by a rather well selected Bibliography. The reviewer also opines that one additional reference should have been mentioned, namely: *Mathematical Methods of Operations Research*, by T. L. Saaty, New York, 1959.

Ficken's book will give a reader with moderate maturity in mathematics a very intelligent idea of the basis, nature and uses of the simplex method of linear programming. The professional engineer and the business specialist as well as others who have need for linear programming will find this book very useful and informative. The topics treated have been wisely selected and their treatment is orderly and clear.

It is very gratifying to see such a lot of useful information put in a clear and understandable fashion in such a concise way. In my opinion, the author is to be congratulated and commended for a piece of work well done.

FRANK M. WEIDA
The George Washington University

Modern Fundamentals of Algebra and Trigonometry. By Henry Sharp, Jr. Prentice-Hall, Englewood Cliffs, N. J., 1961. ix+340 pp. $6.50.

This book gives an excellent treatment of basic topics in algebra and trigonometry. It emphasizes an axiomatic approach in the development of the number system and introduces set terminology before treating the concept of a function. Proofs of the binomial theorem, DeMoivre's theorem and important results in the theory of equations demonstrate the usefulness of the induction axiom. Trigonometry proceeds from the solution of right and oblique triangles to a careful treatment of trigonometric and inverse trigonometric functions. Throughout the book graphical methods are stressed—in the study of inequalities, in the analytic geometry of the straight line, in graphs of polynomials and transcendental functions, in presenting statistical data. In a chapter on theory of equations the author defines the slope of a polynomial function and thus paves the way to the study of calculus. Answers to most odd-numbered exercises are given. Appendices on inequalities, roots of polynomials and binomial series add to the worth of the book, as do various tables. The latter include trigonometric values for angles in degrees and for numbers, 4-place common logarithms and logarithms of trigonometric values. The book admirably combines a modern approach to underlying principles of mathematics with a detailed and rigorous study of algebra and trigonometry.

HELEN G. RUSSELL
Wellesley College

Strukturtheorie der Wahrscheinlichkeitsfelder und -Räume. By Demetrios A. Kappos. Springer-Verlag, Berlin, Göttingen, and Heidelberg, 1960. 136 pp. About $5.50.

The author points out that since there already exist systematic treatises on probability theory from the measure-theoretic point of view, this book does not attempt such a presentation. The book thus confines itself, as the title implies, to the systematic presentation of the mathematical theory of the structure of probability fields and, in terms of these, to that of the structure of probability spaces. The former are defined in terms of a probability function and the latter in terms of measure, but since it is shown that every probability space may be represented as a probability field, the former approach permits a more direct formulation of problems that have been treated by the latter.

The latter part of the book includes material on cartesian products and concepts of independence that play a dominant role in the characterization of the structure of probability fields. A theory of the nonseparable invariant extensions of Lebesque measure related to concepts of independence of Kakutani and Oxtoby, and some treatment of a generalization of the concept of probability space due to Renyi, appear at the end.

A. A. GRAU
Oak Ridge National Laboratory

Elementary Algebra for College Students. By Irving Drooyan and William Wooton, Wiley, New York, 1961. 272 pp. $4.95.

This book, which is traditional in its approach, assumes a knowledge of arithmetic only. It would be helpful to a student with no previous experience with algebra or to a student engaged in self-instruction in algebra. Algebraic concepts are developed intuitively but logically, and algebraic terminology is made to contribute to understanding rather than to confusion. Each new concept, including "word problems," is carefully illustrated with sample problems. Word problems early become an integral part of the text. The entire text should be covered in a first course in elementary algebra.

TRUMAN WESTER
Federal Aviation Agency Training Center, Oklahoma City

Integral Quadratic Forms, Cambridge Tracts in Mathematics and Mathematical Physics no. 51. By G. L. Watson. Cambridge University Press, New York, 1960. 143 pp. $5.00.

The author has brought together in a well-organized and well-written fashion the basic theory of quadratic forms with integral coefficients and variables. Three classes of problems are identified and studied. First there is the problem of the equivalence of quadratic forms under linear integral transformations of determinant ± 1. Proper equivalence, where the determinant is restricted to $+1$, is not introduced; this is just as well because it is strange to have x^2+2y^2 not equivalent to $2x^2+y^2$. Second, the author is concerned with decomposition; a form f in n variables is said to be decomposed into $g+h$, where g and h are forms in r and $n-r$ variables, if f is equivalent to $g+h$. The third problem is the representation of integers by forms; an integer k is represented by $f(x_1, \cdots, x_n)$ if the equation $f=k$ can be solved in integers $x_1, \cdots, x_n$. The background material needed for the study of the book is not great, nothing more than elementary number theory and an acquaintance with Dirichlet's theorem on the infinitude of primes in an arithmetic progression. However, the pace is fairly fast and sophisticated, so that typical undergraduates in universities in the U. S. would not find it easy going.

While the book for the most part constitutes a compilation and synthesis of known results, it contains some theorems by Watson not previously in the literature. The book is a felicitous addition to the small collection of such works available. The Carus monograph on the subject, *The Arithmetic Theory of Quadratic Forms* by Burton W. Jones, uses a different approach on many questions, and the emphasis is not the same. About the only other book in English on the subject is L. E. Dickson's *Studies in the Theory of Numbers*, and it is much more specialized in content than the works by Watson and Jones.

IVAN NIVEN
University of Oregon

Boolean Algebra and its Applications. By J. Eldon Whitesitt. Addison-Wesley, Reading, Mass., 1961. x+182 pp. $6.75.

One of the trying times in the life of a mathematics teacher occurs when an eager young electrical engineering student steps up at the end of a calculus lecture to ask "what do you know about this new subject called Boolean algebra?" Safe answers range from "nothing" to "a little, but it's not really in my field." Whether we like it or not, the elementary theory of Boolean algebras pops up in a remarkably wide variety of applications. It is comforting to know that there is now a book—Whitesitt's *Boolean Algebra and its Applications*—which treats the subject with clarity, care and honesty.

A look at the chapter headings in Whitesitt's book shows the range of material which it covers. In order, these are: "The algebra of sets, Boolean algebra, symbolic logic and the algebra of propositions, switching algebra, relay circuits and control problems, circuits for arithmetic computation, and introduction to probability in finite sample spaces." None of these topics are treated in detail. Nevertheless, a beginning student will find each chapter an attractive introduction to the ideas and tools of the subject with which it deals.

In the smooth, logical development of subject matter, the careful statement and explanation of definitions, the detailed presentation of examples, and the extensive offering of exercises (many of them equipped with solutions), Whitesitt's book rates a solid A. As a textbook, it should provide help and solace to students and teachers, as well as the great mass of engineers bent on self-improvement.

R. S. PIERCE
University of Washington

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Mr. R. E. Barnhill, University of Kansas, has been awarded a Woodrow Wilson National Fellowship for graduate study in the academic year 1961–62.

Professor S. R. Beyma, Hampton Institute, was named a recipient of the Christian R. and Mary F. Lindback Award for Distinguished Teaching. This award carries a $500 stipend.

Professor J. A. Brown, University of Delaware, represented the Association at the

inauguration of Dr. L. I. Mishoe as President of Delaware State College on April 16, 1961.

Professor J. M. Calloway, Kalamazoo College, represented the Association at the inauguration of Dr. J. W. Miller as President of Western Michigan University on May 20, 1961.

Professors Shiing-Shen Chern, University of California, and J. W. Tukey, Princeton University, have been elected members of the National Academy of Sciences.

Professor M. S. Hendrickson, University of New Mexico, represented the Association at the inauguration of Dr. D. C. Moyer as President of Eastern New Mexico University on May 30, 1961.

Professor Emeritus Norman Miller, Queen's University, was awarded the honorary degree of Doctor of Laws by Queen's University on May 20, 1961.

Professor Deane Montgomery, Institute for Advanced Study, received the honorary degree of Doctor of Humane Letters from Yeshiva University on June 15, 1961.

Professor A. W. Tucker, Princeton University, was awarded the honorary degree of Doctor of Science by Dartmouth College on June 11, 1960.

Agnes Scott College: Mr. R. E. R. Nelson, University of Virginia, has been appointed Instructor; Assistant Professor Sara Ripy has been promoted to Associate Professor.

Massachusetts Institute of Technology: Drs. A. E. Hurd, Norman Lebovitz and J. H. Simons have been appointed C. L. E. Moore Instructors; Drs. F. P. Bretherton, R. E. Briney, T. J. Eisler, R. L. Finney, III, R. A. Gangolli, J. P. Levine, G. J. Maltese, R. M. Moroney, M. M. Robertson, and H. C. Rumsey, Jr., have been appointed Instructors; Drs. F. M. Leslie, J. R. McCord, and Hironori Onishi have been appointed Research Associates; Assistant Professors Sigurdur Helgason, D. M. Kan, A. P. Mattuck, M. L. Minsky, and F. P. Peterson have been promoted to Associate Professors; Associate Professor Bertram Kostant, University of California and Dr. R. H. Lüst, Max Planck Institute, Munich, Gemany, have been appointed Visiting Professors; Associate Professor Felix Browder, Yale University, and Dr. Ekkehart Kröner, Stuttgart Institute of Technology, Stuttgart, Germany, have been appointed Visiting Associate Professors; Assistant Professor H. J. Weinitschke, University of California, Los Angeles, has been appointed Visiting Assistant Professor; Professor I. M. Singer is on leave on an Alfred P. Sloan Research Fellowship; Professor G. G. Whitham is on leave and will spend the year at California Institute of Technology; Associate Professor L. M. Howard is on leave at Cambridge University on a Guggenheim Fellowship; Assistant Professor K. M. Hoffman is on leave at the University of California, Los Angeles, for one year.

Rensselaer Polytechnic Institute: Associate Professors B. A. Fleishman and J. W. Hollingsworth have been promoted to Professors; Assistant Professor T. Y. Chow has been promoted to Associate Professor.

University of Arizona: Dr. J. A. Dyer, University of Texas, has been appointed Assistant Professor; Professors L. M. Milne-Thomson, Mathematics Research Center, University of Wisconsin, and L. J. Mordell, Cambridge University, Cambridge, England, have been appointed Visiting Professors; Dr. M. S. Cheema, University of California, and Dr. N. C. Giri, Stanford University, have been appointed Visiting Assistant Professors.

University of Rochester: Drs. A. H. Stone and Dorothy M. Stone, University of Manchester, Manchester, England, and Professor J. H. B. Kemperman, Purdue University, have been appointed Professors; Drs. W. W. Comfort, Harvard University, K. A. Ross, University of Washington, and C. E. Watts, Institute for Advanced Study, have been appointed Assistant Professors; Assistant Professor Yuzo Utumi, McGill University, has been appointed Visiting Assistant Professor.

Mr. E. Z. Andalafte, University of Missouri, has been appointed Instructor at Southwest Missouri State College.

Dr. J. M. Anderson, University of Nebraska, has accepted a position as Mathematician at the Radio Corporation of America Laboratories, Princeton, New Jersey.

Mr. Dov Avishalom, Bar Ilan University, Tel Aviv, Israel, has been appointed Instructor at the University of Minnesota.

Mr. S. D. Beck, Battelle Memorial Institute, has been appointed Nuclear Engineering Specialist at Alco Products, Schnectady, New York.

Associate Professor E. G. Begle, Yale University, has been appointed Professor of Mathematics Education at Stanford University's School of Education.

Mr. R. J. Benice, University of Buffalo, has been appointed Senior Mathematician at the Sylvania Electronic Systems, Buffalo, New York.

Mr. W. H. Benson, University of California, Berkeley, has accepted a position at the Lawrence Radiation Laboratory, Berkeley, California.

Mr. B. D. Biegun, University of Minnesota, has been appointed Teacher at Napa Senior High School, Napa, California.

Mr. J. H. Braun, Chrysler Corporation, Detroit, Michigan, has accepted a position as Systems Engineer with International Business Machines, Columbus, Ohio.

Brother H. Columban, Xavier High School, Appleton, Wisconsin, has been appointed Teacher at St. George High School, Evanston, Illinois.

Dr. S. D. Conte, Space Technology Laboratories, Los Angeles, California, has accepted a position as Manager of the Computing Sciences Department, Systems Research and Development Division, Aerospace Corporation, Los Angeles, California.

Professor F. L. Kiokemeister has been appointed Chairman of the Department of Mathematics at Mount Holyoke College.

Mr. S. J. Einhorn, University of Pennsylvania, has been appointed to the Technical Staff of Averbach Electronics as Mathematical Analyst.

Mr. Samuel Feder, Bulova Research & Development Laboratories, Woodside, New York, has accepted a position at System Development Corporation, Paramus, New Jersey.

Dr. E. H. Hanson, Land-Air, Point Mugu, California, has accepted a position as Manager of the Advanced Analysis Department, Earth Sciences Division, United ElectroDynamics, Pasadena, California.

Associate Professor M. C. Hartley, University of Illinois, has been appointed Visiting Professor at the University of Puerto Rico, Mayaguez, Puerto Rico.

Mr. N. W. Johnson, Technical Operations, Washington, D. C., has accepted a position at Computer Associate, Woburn, Massachusetts.

Professor Edgar Karst, Brigham Young University, has been appointed Professor at Evangel College.

Dr. H. C. Kennedy, St. Louis University, has been appointed Assistant Professor at Providence College.

Assistant Professor L. H. Lange has been appointed Acting Head of theDepartment of Mathematics at San Jose State College.

Professor Joseph Lehner, Michigan State University, will be on leave during the academic year of 1961–62 and will attend the Number Theory Institute at the University of Pennsylvania.

Associate Professor V. O. McBrien, College of the Holy Cross, has been appointed Professor and Chairman of the Department of Mathematics.

Mr. G. E. Murine, Head of the Mathematics Department, Solon School System, Cleveland, Ohio, has been appointed Assistant Professor at John Carroll University.

Dr. J. B. Muskat, Massachusetts Institute of Technology, has been appointed Assistant Professor and Research Associate in Computing, University of Pittsburgh.

Professor J. H. Neelley, Carnegie Institute of Technology, has been appointed Visiting Professor at Ball State Teachers College.

Mr. W. K. Rapp, University of Missouri, has accepted a position as Senior Engineer with Motorola Military Electronic Center, Scottsdale, Arizona.

Associate Professor N. C. Severo, University of Buffalo, has been promoted to Professor of Mathematical Statistics.

Mr. Lawrence Sokoloff, Sylvania Electric Products, Needham, Massachusetts, has accepted a position with Auerbach Electronics, Philadelphia, Pennsylvania.

Dr. J. W. Summers, University of California, Berkeley, has been appointed Assistant Professor at Alameda State College.

Dr. G. H. Swift, Jr., International Business Machines, Poughkeepsie, New York, has been appointed Manager, Technical Plans for Large Computers, International Business Machines, Federal Systems Division, Rockville, Maryland.

Dr. T. T. Tanimoto, International Business Machines, has been appointed Head of the Pattern Recognition Laboratory, Melpar, Watertown, Massachusetts.

Professor Emeritus W. B. Carver, Cornell University, died on July 4, 1961. He was a charter member of the Association. Professor Carver served the Association as Editor of the MONTHLY (1932–1936), President (1939–1940), and Secretary-Treasurer (1943–1947). He had the distinction of being the only person to hold these three offices. From 1947 Professor Carver continued to serve as a member of the Finance Committee until January, 1961, when he found it necessary to resign because his health was failing.

Professor E. S. Ashcraft, Stetson University, died December 17, 1960. He was a member of the Association for 10 years.

Professor Emeritus W. W. Denton, University of Arizona, died January 22, 1961. He was a Charter Member of the Association.

Assistant Professor Aaron Herschfield, Pennsylvania State University, died February 19, 1961. He was a member of the Association for 4 years.

Professor Emeritus A. J. Hoare, University of Wichita, died April 23, 1961. He was a Charter Member of the Association.

Dr. R. P. Johnson, Louisa, Virginia, died March 16, 1961. He was a member for 40 years.

Associate Professor R. B. Pinson, Stephen F. Austin State College, died February 19, 1961. He was a member of the Association for 11 years.

It has recently been pointed out that, in the notice of the death of Professor E. J. Finan, Catholic University of America (this MONTHLY, vol. 67, 1960, p. 713), the name was misspelled. The Editor regrets this error.

CONTINENTAL CLASSROOM, 1961–62

The National Broadcasting Company announced on June 28 that last season's Continental Classroom course in Contemporary Mathematics will be repeated on color tape recordings from 6 to 6:30 a.m. local time beginning September 25, 1961. Professor John L. Kelley, University of California, Berkeley, teaches *Modern Algebra* during the first semester; Professor Frederick Mosteller teaches *Probability and Statistics* during the second.

The new course on Continental Classroom will be a course in American Government This course will be televised in color and carried by approximately 170 stations in every part of the country, Monday through Friday, from 6:30 to 7:00 a.m. local time beginning September 25, 1961.

The Conference Board of the Mathematical Sciences is one of the sponsors of Contemporary Mathematics; the others are Learning Resources Institute and the National Broadcasting Company.

DOCTORAL PROGRAM FOR COLLEGE TEACHERS OF MATHEMATICS

The Graduate School of Science and the Graduate School of Education of Yeshiva University announce a new doctoral program designed primarily for college teachers of mathematics. The program is aimed toward the training of mathematicians interested primarily in college teaching, rather than in research.

The requirements of the new program include the same number of credits in graduate mathematics as is required in the Ph.D. program of the Graduate School of Science. However, the student is encouraged to select courses with an eye toward breadth of coverage, rather than intensive specialization.

In addition to the content courses, there is a sequence of courses of special pertinence to the needs of the college teacher. In particular, the course "Readings in the Masterworks of Mathematics" is designed to acquaint students with original writings in the field (using the original language when possible), and it is valuable both in itself and as a prelude to the writing of the dissertation.

For more information write to: Director of Admissions, Yeshiva University, Amsterdam Avenue & 186th Street, New York 33, New York.

OPPORTUNITIES FOR STUDY IN U.S.S.R.

The Inter-University Committee on Travel Grants, representing American colleges and universities, wishes to announce that it is soliciting inquiries and applications from graduate students and scholars who wish to spend all or part of the academic year 1962–63 engaged in study and research in the Soviet Union as participants in the academic exchange between the United States and the U.S.S.R.

American citizens under forty years of age are eligible if they are *graduate students*, *post-doctoral researchers* or *faculty members* at the time of application. A knowledge of Russian adequate to the needs of study and research is required. Other criteria for selection include intellectual ability, maturity, emotional stability, proven scholarly competance or indication of future professional promise, and substantial knowledge of both American and Russian history and culture.

Periods of study and research between one semester and fifteen months can be arranged. Funds are available to cover all or part of the exchange participant's expenses, including maintenance of family, depending on the participant's own financial needs and resources.

For further information and applications write to: Stephen Viederman, Deputy Chairman, Inter-University Committee on Travel Grants, 719 Ballantine Hall, Indiana University, Bloomington, Indiana. Applications be must received no later than December 15, 1961, to be considered for the 1962–63 exchange.

HUME MATHEMATICS HONOR GALLERY REOPENED AT MISSISSIPPI

In autumn of 1890 a young man named Alfred Hume was chosen for the Chair of Mathematics in the University of Mississippi. A few years later he was a charter member of the Mathematical Association of America and retained his membership throughout his long and active lifetime. At the very beginning of his teaching, Dr. Hume instituted a practice which turned out to be perhaps as unique as anything of its kind in the world. At the end of each year he would give critical examination of the performance of his students and honor those rated at the top by placing their pictures in the classroom. This discriminating practice resulted in groups ranging from one to perhaps seven or eight. The gallery grew and in 1924 it had gone completely around the spacious classroom and it contained names that became outstanding in many walks of life. In this year, Dr. Hume became Chancellor of the University and the practice was discontinued.

The sequel to this account is the decision made at this institution to re-activate this gallery and choose annually a recipient for the award designated as the Alfred Hume Memorial Award in Mathematics. Among the provisions of the award are membership in the Association and the privilege of having the recipient's photograph entered into this Honor Gallery which started 70 years earlier. The first recipient of this second phase is James R. Price. He will, of course, continue graduate study in mathematics in a leading mid-western University.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1961 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	Paul Eberhart, Washburn University
Missouri	R. J. Michel, Southeast Missouri State College
New Jersey	H. O. Pollak, Bell Telephone Laboratories
Northeastern	D. E. Richmond, Williams College
Ohio	R. R. Stoll, Oberlin College
Pacific Northwest	D. C. Murdoch, University of British Columbia
Southeastern	H. S. Thurston, University of Alabama
Southwestern	J. B. Giever, New Mexico State University
Upper New York State	Harriet F. Montague, University of Buffalo

The greatest number of votes (230) was cast by the members of the Northeastern Section. The highest percentage of votes was 47% in the Missouri Section.

H. M. GEHMAN, *Executive Director*

THE 1961 HIGH SCHOOL MATHEMATICS CONTEST

The Annual High School Mathematics Contest Examination, sponsored by the Mathematical Association of America and the Society of Actuaries, was administered March 9, 1961 to approximately 160,000 students in 5300 schools throughout the United States and Canada, and including 14 APO (Army) and FPO (Navy) schools, scattered widely around the world. These figures compare with approximately 152,000 students in 5200 schools in 1960.

A stronger international flavor was added to the contest examination this year by having it administered on an unofficial basis in two English schools and two Dutch schools, the latter with provision for translation. All four schools performed well, and, in communications with the teachers in the Netherlands schools, we learned that their students were very much interested in the test and that extended participation is planned for 1962. Also arrangements are being made to have the test printed in the Dutch journal *Euclides*.

On the reasonable assumption that the body of participants over the years constitute a pedagogical constant, we must conclude that this year's examination was more difficult than that of 1960. Limited statistics describing the national and regional perform-

ances are found in the summary, sent to all participating high schools and available by request from the contest chairman.

In team performance, Brooklyn Technical High School ranks first with a score of 366.75 points out of 450, and Abraham Lincoln High School (Brooklyn), second with a score of 358. The best individual performance is credited to Aviad Broshi, Yeshiva of Flatbush High School (Brooklyn) with a score of 143.75 points out of 150. Close behind is Robert Rosenstein, Abraham Lincoln High School (Brooklyn) with a score of 141.25.

The discriminatory quality of the examination on all levels of achievement continues to be satisfactory, and no changes are contemplated in basic pattern. For next year, however, we may use a dual wording, one classic and one modern, for problems based on "modern mathematics." The few sprinkled through this year's examination appeared in essentially classic garb.

To date, our most ambitious effort to interest high school students in mathematical careers is this year's three-page brochure, *How About a Career in Mathematics*, distributed to all participating students. Although the expense of printing and distributing these brochures is heavy, we are of the opinion that the educational value of the brochures warrants their continuance.

CHARLES T. SALKIND, *Chairman,*
Committee on High School Contests

THE 1961 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-second annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 2, 1961. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed about October 1 to the regular mailing list. If an application blank is not received by October 15, one may be secured by writing the director, Professor L. E. Bush, 308 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 6, 1961. For further details of the examination and the list of prizes (including the $3,000.00 scholarship to Harvard), see the announcement which will accompany the application blank.

Reports of the previous competitions and the examinations may be found in this Monthly for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947; December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; August–September 1958; August–September 1959; January (questions and solutions for eighteenth, nineteenth and twentieth competitions) 1961; and in this issue, pages 629–637.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 25, 1961, at Wayne State University, Detroit, in conjunction with the annual meeting of the Michigan Academy of Science, Arts and Letters. Professor E. D. Rainville of the University of Michigan, Chairman of the Section, presided at all sessions of the meeting. The attendance at the morning and afternoon sessions was about 80 persons and 67 attended the luncheon and business meeting.

The nominating committee consisting of Professor J. G. Hocking, Michigan State University, Chairman, Professor L. G. Woodby, Central Michigan University, and Sister M. Ignatia, Marygrove College proposed the following slate of officers: Professor

F. L. Celauro of Central Michigan University as Chairman; Professor R. H. Oehmke of Michigan State University as Vice-Chairman; Professor L. E. Mehlenbacher of the University of Detroit as Secretary-Treasurer. This slate of officers was elected unanimously.

The report of the Governor of the Michigan Section, Professor R. M. Thrall, was read by Professor Frank Harary. Professor R. K. Ritt of the University of Michigan reported for the Committee on the Michigan Mathematics Prize Competition that there were over nine thousand participants from Michigan high schools, and that the general results show improvement over previous years. Professor J. S. Frame of Michigan State University reported on the Visiting Scientist Program sponsored by the Michigan Academy and on the resolution in support of certain changes in the proposed teacher certification code for Michigan.

The morning program included the presentation of three papers and a report of the work of the Panel on Teacher Training of the CUPM. The latter report was presented by Professor R. J. Wisner, Executive Director of the CUPM. The afternoon program included the presentation of one paper and a panel discussion on the topic "The School Mathematics Study Group Programs for Elementary and Secondary School Mathematics." The Panel consisted of Professor P. S. Jones of the University of Michigan, Moderator; Miss Irene Sauble, Supervisor of Elementary Mathematics, Division of Instruction, Detroit Public Schools; Professor Charles Brumfiel, University of Michigan; Miss Hope Chipman, University of Michigan High School.

The following papers were presented:

1. *A very independent axiom system*, by Professor Frank Harary, University of Michigan.

This paper was published in this MONTHLY, vol. 68, 1961, pp. 159–162.

2. *Fitting an exponential curve to the frequencies of the lengths of precipitation periods at South Haven, Michigan*, by Professor W. D. Baten, Michigan State University.

This article contains an application of fitting an exponential curve to observed frequencies of the lengths of precipitation periods at South Haven, Michigan, for the past 30 years. The length of a precipitation period is the number of consecutive days during which there is measurable precipitation.

3. *On the set of maximum points of a regular function*, by Professor Fritz Herzog, Michigan State University.

Let F be the class of nonconstant functions $f(z)$, regular for $|z|<1$ and continuous for $|z|\leqq 1$. Let F_1 be the subclass of F, consisting of those $f\in F$ which are regular for $|z|\leqq 1$. The point set on the unit circle C where $|f(z)|$ assumes its maximum value will be called the *set of maximum points*. It is shown that, (a) for the class F, the most general set of maximum points is any closed nonempty subset of C, and (b) for the subclass F_1, the most general set of maximum points is either all of C or a finite nonempty subset of C.

4. *Invertible spaces*, by Professor J. G. Hocking, Michigan State University.

This paper by P. H. Doyle and J. G. Hocking is scheduled for publication in this MONTHLY, November, 1961.

L. E. MEHLENBACHER, *Secretary*

THE MARCH MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of Arizona, Tucson, Arizona, March 17–18, 1961. Professor Harvey Cohn, Chairman of the Section, presided at the afternoon session on March 17, and also at the morning session on March 18. There were 87 persons in attendance, including 40 members of the Association.

The following officers were elected: Chairman, Professor I. I. Kolodner, University

of New Mexico; Vice-Chairman, Professor Deonisie Trifan, University of Arizona; Secretary-Treasurer, Professor George Baldwin, New Mexico State University.

Dr. S. M. Ulam, Los Alamos Scientific Laboratory, gave the invited address, "Study of Combinatorial Problems on Computers."

The following papers were presented:

1. *Some computations of class numbers in quadratic integral domains*, by Professor Harvey Cohn, University of Arizona.

The author considers $h(f^2d)$ the class number for the quadratic integers of type $x+yf(d+\sqrt{d})/2$ where d is the field discriminant. Dirichlet showed that for fixed d if $f=p^aq^b\cdots$ (for a *fixed* set of primes and variable exponents), $h(f^2d)$ is bounded; also for certain sets of primes in f, $h(f^2d)=h(d)$. The survey made on GEORGE tends to suggest certain possible conjectures on the class numbers. For example: $h(12f)=1$ exactly when $f=3^a$ or $2\cdot 3^a$; $h(24f)=1$ only when $f=3$, $h(24f)=3$ only when $3<f=3^a$, and $h(24f)$ is even otherwise. When d is the sum of two squares, the values of $h(f^2d)/h(d)$ tend to be odd more often and smaller more often (as f varies) than for other d.

2. *Rings with group*, by Mr. E. L. Walter, New Mexico State University.

Let R be a ring with $1\neq 0$, and R^{**} be the set of nondivisors of zero. Define R to be a ring with group if R^{**} is a group. Several nontrivial examples are given. Ring R is a ring with group if and only if it has no proper left (right) ideals which intersect R^{**}. If the left (right) ideals of R satisfy the descending chain condition, then R is a ring with group. If X is a separable Banach space, then $B(X, X)$ is a ring with group if and only if X is finite.

3. *Closed linear operations with closed range*, by Mr. B. A. Benn, New Mexico State University' introduced by the Secretary.

Let T be a closed linear operator with domain $D(T)$ dense in Banach space X and range $R(T$ in Banach space Y. It is well known that $R(T)$ is closed in Y if and only if $R(T')$ is closed in X'. The purpose of this paper is to show that the theorem follows readily from the "state diagram." (S. Goldberg, *Linear operators and their conjugates*, Pacific J. Math., vol. 9, 1959, pp. 69–79.)

4. *A theorem on power dissipation in circuits with periodically varying elements*, by Professor I. I. Kolodner, University of New Mexico.

The response of a damped linear circuit (say in RLC circuit, $R>0$) to a sinusoidal electromotive force of circular frequency ω can be split into a harmonic "steady state" and a decaying transient. The Ohmic energy dissipated in the circuit over a period is a continuous function of ω on $(0, \infty)$; the energy stored in the reactive components averages to zero over a period. Under the same circumstances, if the circuit elements vary periodically with period T, the response still can be split into an almost-periodic steady state and a decaying transient. An average power P_s dissipated or stored in a set s of circuit elements can still be defined by $P_s=\lim_{t\to\infty} E_s(t)/t$, where E_s is the energy dissipated or stored in s.

THEOREM. *P_s has removable singularities on $A=\{\omega \mid \omega=K\pi/T,\ K\in I^+\}$, and is continuous on $(0, \infty)-A$.*

5. *Some remarks on associative functions*, by Professor Berthold Schweizer, University of Arizona.

The history of the functional equation of associativity, $F(x, F(y, z))=F(F(x, y), z)$ is discussed and a number of recent results, as well as a conjecture, concerning properties of its solutions are presented.

6. *Geometric progressions modulo 2^k*, by Mr. Milton Levy, White Sands Missile Range, New Mexico.

Elementary proofs are given of three theorems which are applicable to the generation of pseudo-random numbers in a binary digital computer. In all the following theorems, $k\geqq 3$. (1) If $r\equiv\pm 1 \pmod 8$, then $r^{2^{k-3}}\equiv 1 \pmod{2^k}$. (2) If $r\equiv\pm 3 \pmod 8$, then $r^{2^{k-3}}\not\equiv 1 \pmod{2^k}$ and $r^{2^{k-2}}\equiv 1 \pmod{2^k}$. (3) If $r\equiv -3\equiv 5 \pmod 8$, then the sequence of integers generated by $x_0\equiv a$

(mod 2^k), $x_n+1 \equiv rx_n$ (mod 2^k) for $0 \leqq n < 2^{k-2}$ is a permutation of $(1, 5, \cdots, 2^k-3)$ if $a \equiv 1$ (mod 4) and is a permutation of $(3, 7, \cdots, 2^k-1)$ if $a \equiv 3$ (mod 4).

7. *Differentials on a topological space*, by Professor E. D. Gaughan, New Mexico State University.

The purpose of this paper is to present the notion of a differential for a function from a topological space X into R^n, R a nondiscrete Hausdorf topological field, and to show that this differential possesses some of the properties of the usual differential as defined on a vector space.

8. *Some remarks about moment sequences*, by Professor J. W. P. Mayer-Kalkschmidt, University of New Mexico.

The following theorems are proved: (1) Given the moment sequence $u_n = \int_0^1 t^n dg(t)$, where tot. $\text{var.}_{[0,1]}(g) = K$, if $\sum_{\nu=0}^{\infty} |a_\nu| K^\nu < \infty$, then $\delta_n = \lim_{k\to\infty} \sum_{\nu=0}^{k} a_\nu \mu_n^\nu = \delta_n{}^k$ is a moment sequence, and $\{\delta_n^{(k)}\}$ converges to $\{\delta_n\}$ in the norm topology. (2) Under the conditions of (1), the Hausdorff method (H, δ) is regular if (H, μ) is regular and if $\sum_{\nu=0}^{\infty} a_\nu = 1$, where the a_ν are real. The case: $\mu_n = 1/(n+1)$, $a_\nu = \alpha^\nu$ is studied for $1 \leqq \alpha$, and $\alpha < 1$.

9. *General block decompositions of a space and the general computation of homology groups*, by Professor J. B. Giever, New Mexico State University.

Although the homology groups of, for example, a polytope are usually defined using simplicial decompositions, they are usually computed using more general "block decompositions" of one type or another. By using a much greater generality in the type of blocks, one can have the homology groups of the blocks determine that of the space without having a very effective means of computing by means of the blocks. Nevertheless, such a procedure can be shown to have quite useful applications.

10. *Discriminants and determinants*, by Professor Gordon Pall (Illinois Institute of Technology), Visiting Professor, University of Arizona.

The theory of quadratic forms with coefficients in a ring R in which division by 2 is either impossible or not unique has been largely neglected, principally because such forms lack symmetric matrices and determinants. By a simple device, it is shown how to associate with a quadratic form having coefficients in any commutative ring a polynomial in its coefficients, called the discriminant, which behaves much like the usual determinant and makes possible extensions of the classical theories. If R is not commutative, a similar method is given for Hermitian quadratic forms.

11. *A condition for convexity in non-Euclidean geometry*, by Professor John Irwin, New Mexico State University.

Poincare's hyperbolic model of non-Euclidean geometry consists of the points in the interior Ω of the unit circle. The lines in this geometry are arcs of circles orthogonal to the circumference of the unit circle. A set $S \subset \Omega$ is convex if and only if for each pair of points w_1 and w_2 in S, the (non-Euclidean) line segment joining w_1 and w_2 is contained in S. A condition for the convexity of regions in the complex plane is given and it is shown that a corresponding theorem for Ω holds

12. *Some unsolved problems concerning extensions of Abelian groups*, by Professor E. A. Walker, New Mexico State University.

This paper is a discussion of some of the unsolved problems concerning $\text{Ext}(A, B)$. Recent progress on some of these problems is indicated, including some recent theorems of the author. Whitehead's problem concerning $\text{Ext}(A, Z)$ when Z is the additive group of integers is given particular attention.

13. *Set functions and their determining collections*, by Dr. H. H. Wicke, Sandia Corporation, Albuquerque, New Mexico.

It is shown (as a special case of a more general theorem) that if M is a set, then every function whose domain and range is the power set of M corresponds to a mapping of M into collections of subsets of M. Certain interesting properties of set functions such as isotonicity are characterized by properties of the corresponding collections. The assignment of neighborhoods to each point of

M and the determination of a topology is an application. Another application is a characterization of the class of all isotone set functions f satisfying $f=cfc$, where c is the complement function.

14. *Banach spaces isomorphic to a conjugate space*, by Professor Seymour Goldberg, New Mexico State University.

Let X be a Banach space. New and simple proofs are given of theorems concerning necessary and sufficient conditions that X be linearly homeomorphic to a conjugate space of a Banach space.

15. *Some issues in mathematics education today*, by Professor Charles Wexler, Arizona State University.

A report on the Board of Governors' meeting, the CUPM meeting, and some of the problems in Mathematics Education discussed by these groups.

16. *Discussion*: *Engineering curriculum in mathematics*, Chairman: Professor J. W. P. Mayer-Kalkschmidt, University of New Mexico.

17. *Discussion*: *Undergraduate program in mathematics*, Chairman: Dr. J. R. Foote, University of New Mexico and Holloman Air Force Base.

G. L. BALDWIN, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The 48th regular meeting of the Iowa Section of the Mathematical Association of America was held at Simpson College, Indianola, on the afternoon of April 14 and the morning of April 15, 1961. Professor Irvin Brune presided as Acting Chairman in behalf of Professor H. C. Trimble, Chairman of the Section, who could not be present. Total attendance was 81, including 35 members of the Association. Routine business was considered during the afternoon meeting of April 14.

The following officers were elected: Chairman, Professor Hazel M. Rothlisberger, University of Dubuque; Vice-Chairman, Professor W. L. Waltmann, Wartburg College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program.

1. *Proof of the remainder theorem*, by Professor H. A. Heckart, Simpson College.

2. *Integration of* $\cot x \sin 2mx \ln (\sin x/\sin a)dx$, by Professor Don Kirkham, Iowa State University.

For $m=1, 2, \cdots$, the title integral and some related ones are evaluated. The results are obtained in terms of psi-functions.

3. *A mixed boundary value problem for an infinite elastic cylinder*, by Professor Harry Weiss, Iowa State University.

4. *A thin cylindrical shell problem*, by Mr. Thomas Rogge, Iowa State University, introduced by the Acting Chairman.

The problem of a thin cylindrical shell sector clamped on two curved edges and one straight edge, free on the remaining straight edge and loaded by a load similar to a hydrostatic load is considered. The method of solution is that of superimposing the solutions of three separate problems with the appropriate boundary conditions.

5. *Number theoretic densities for the Gaussian integers*, by Mr. W. H. Richardson, Iowa State University.

Some of the standard number-theoretic densities are defined for the Gaussian integers; and for these are established results analogous to corresponding cases for the rational integers. For example, one possible asymptotic density is given by $\lim\inf A(n)/N(n)$, where $A(n)$ is the number of elements of A in the half-open square determined by 0, n, and ni, and $N(n)$ is the norm of the Gaussian integer n.

6. *A modified Runge-Kutta solution of ordinary differential equations*, by Mr. G. D. Byrne, Cyclone Computer Laboratory and Professor R. J. Lambert, both of Iowa State University.

Suppose it is required that a particular solution to the differential equation $dy/dx = F(x, y)$ be found. Suppose further that the points (x_{n-1}, y_{n-1}) and (x_n, y_n) lie on the particular solution curve and that they are given. Let $x_n + h = x_{n+1}$, $y_n + k = y(x_{n+1}) = y_{n+1}$. Here h, the step-size, is fixed. Therefore, k, the change in y, must be evaluated to find the next point, (x_{n+1}, y_{n+1}), on the particular solution curve. A set of equations is given describing a modified Runge-Kutta method of numerical integration, which has an accuracy of the order of h^3 and which requires only two substitutions into the differential equation for each step of integration.

7. *A note on Boolean algebras*, by Professor M. F. Ruchte, Iowa State University.

8. *Integral transforms and boundary value problems*, by Mr. Gary Anderson, Iowa State University, introduced by the Acting Chairman.

9. *Lilliputian dynamics—the physics of extreme size change*, by Mr. Robert Gordon, Bettendorf High School, introduced by the Acting Chairman. (By invitation)

Whenever the size of an object or animal is changed, the scale factor must be considered. It was discovered experimentally that because strength is proportional to cross-sectional area, while mass is proportional to volume, a large animal or object, built similar to a smaller one, will be weaker, proportionately, by a factor of scale. Also, the smaller an object or animal becomes, the more surface area, relative to mass, it has. This explains why very small animals seem extremely hungry and are easily water-logged. Scaling is also important when the behaviorial properties of various size ships are considered. These are but a few of the many aspects of scaling.

10. *Recommendations of CUPM Panel on Teacher Training*, by Professor W. R. Orton, University of Arkansas.

11. *Iowa college reactions to CUPM recommendations*, by staff members of representative Iowa Colleges: Professor J. O. Chellevold and Professor O. C. Kreider, Iowa State University, Mr. N. L. Jacobson, Graceland College, Professor H. V. Price, State University of Iowa. Questions and comments from the floor.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-sixth annual meeting of the Kansas Section of the Mathematical Association of America was held at Ottawa University, Ottawa, Kansas, on April 15, 1961, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. Of the 195 people registered, 72 were members of the Association. Professor W. D. Bemmels, Chairman, presided at the sessions.

The following officers were elected for one-year terms: Chairman, Professor L. E. Fuller, Kansas State University; Vice-Chairman, Professor A. M. Wedel, Bethel College; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State College.

At the joint session, held in the morning, Professor J. G. Kemeny, Dartmouth College, addressed the group on *Recommendations for the Training of Teachers of Mathematics*.

The following short papers were presented at the afternoon session:

1. *Decision procedures for a propositional calculus*, by Professor B. J. Thorne, Kansas State University.

The calculus of material implication has the remarkable property of being decidable. The decision procedure in this case is the well-known system of truth tables. This paper presents a generalization of this method. An abstract definition of "propositional calculus" is given. "Finite model of a propositional calculus" is then defined and shown to have some very useful properties as a decision procedure for propositional calculi in general.

2. *Theorems on congruence*, by Professor A. A. Nafoosi, Kansas State College.

THEOREM 1. *Every odd number is the sum of at most nine odd squares, and every even number is the sum of at most ten odd squares. The proof depends on Gauss' theorem that every number is the sum of three triangular numbers.* THEOREM 2: *If* $s \geqq 4$, $N=s \pmod{B}$ *and* $h \geqq 1$, *then* $\sum_{i=1}^{s}(Bx_i+1)^2 \equiv N \pmod{p^h}$ *is always solvable for* $p \neq 2$. The proof is carried out by induction on h.

3. *Axiomatic set theory*, by Professor John Johnston, University of Kansas.

Axiomatic set theory is considered to be a portion of a general first order predicate calculus whose object language contains only bound formulae. The primitive symbols are indicated and the concept of formula defined in such a manner that no individual variable lies within the scope of two quantifiers acting on that variable. Rosser's six axiom schemata and rule of inference are stated in a manner that produces only bound formulae. After machinery for logical manipulation is developed, axioms governing the binary predicate constant "ϵ" are stated, essentially as in Gödel.

4. *Orthogonality conditions for polynomial solutions of a class of fourth order linear differential equations*, by Professor J. W. Meux, Kansas State University.

A class of fourth-order linear differential equations, each equation having a single set of polynomial solutions, is considered. A set of five conditions, sufficient to insure orthogonality of the solution set with respect to a weight function, over the fundamental interval (a, b), is derived. Under these conditions, analogues of the classical orthogonal polynomials, as well as other sets, may be obtained as solutions of this class of fourth-order differential equations.

5. *Lipschitzian homeomorphisms of bounded convex sets*, by Professor Robert Adams, University of Kansas.

Let R_1 and R_2 be bounded open convex sets in a normed linear space H, and $\rho_1(x)$, $\rho_2(x)$ their respective support functions. It is shown that the mapping $Tx=[\rho_1(x)/\rho_2(x)]x$ is a Lipschitzian homeomorphism of H onto H such that $T(R_1)=R_2$. The Lipschitzian constant (for T and T^{-1}) is bounded by $(1+r_2/r_1)^2$, where r_1 and r_2 are the radii of a pair of concentric spheres contained in and containing, respectively, both R_1 and R_2.

HELEN F. KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Annual Spring Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Aberdeen Proving Ground, Aberdeen, Maryland, on Saturday, April 29, 1961. Professor D. B. Lloyd, Chairman of the Section, presided. Sixty-three persons were present, including 60 members of the Association.

The following officers were elected to serve during the year 1961–62; Professor W. K. Morrill, Johns Hopkins University, Chairman; Professor J. C. Abbott, U. S. Naval Academy, and Mr. L. K. Meals, David Taylor Model Basin, Vice-Chairmen; Professor S. B. Jackson, University of Maryland, Treasurer, and Professor Herta T. Freitag, Hollins College, Secretary. Professor D. B. Lloyd and Dr. Michael Goldberg will represent the Section on the Joint Board on Science Education of the Washington, D. C. area.

The following papers were presented:

1. *A new characterization of Boolean algebras*, by Professor J. C. Abbott and Mr. P. R. Kleindorfer, U. S. Naval Academy, presented by Mr. Kleindorfer.

Since most mathematical statements are in the form of implications, a system of logic based on such a concept may be analytically valuable. This paper defines an implication algebra by an existence and closure postulate and five axioms. This unrestricted implication is equivalent to con-

ventional "material implication." Defining a "contained-in" operation for implication algebras, "formal implication" results, and partial ordering under this operation is established. Join and meet operations are defined and discussed, distributive and absorptive laws are proved. Upon introduction of a null element, implication algebras are shown to be Boolean algebras. Theorems and illustrations are given.

2. *On the converging factor for the asymptotic series connected with the exponential integral*, by Dr. F. D. Murnaghan, and Dr. J. W. Wrench, Jr., Applied Mathematics Laboratory, David Taylor Model Basin, Washington, D. C.

The converging factor for the asymptotic series $1+1!/x+2!/x^2+\cdots$ is discussed. When x is negative, the leading term of the series which furnishes the converging factor is $\frac{1}{2}$, while when x is positive, the leading term is $\frac{2}{3}$. The problem of determining the converging factor is more difficult in the latter case than in the former, and heretofore only four terms of the series which furnishes it have been given. The present paper gives twenty-one terms of this series, and these enable us to calculate, for example, $Ei(10)$ correct to within a unit in the twenty-third decimal place. As a by-product, we obtain twenty-one terms of the Stirling asymptotic series for $\Gamma(x)$.

3. *Two-dimensional slide rule—reduced cubic equation*, by Mr. C. R. White, Ballistic Research Laboratories, Aberdeen Proving Grounds, Maryland.

This slide rule is designed to solve not only the reduced cubic equation, $w^3+3pw+2q=0$, but also the general quadratic equation, $x^2+2ax+b=0$. It gives simultaneously for either equation; (1) the roots real and/or complex with no ambiguity in signs, (2) the complex roots in both rectangular and polar forms, (3) all numerical answers on ten-inch logarithmic scales. These results make possible approximate solutions to the general cubic and quartic equations and thus lead to direct computations for root-gain loci problems corresponding to transfer functions of the types $G_{n,m}(s)$ for $n=2, 3, 4$ and $m=1, 2, \cdots, n-1$.

4. *A procedure for determining a family of minimum-cost network flow patterns*, by Mr. R. G. Busacker, Johns Hopkins University.

Given a finite network (linear graph) joining nodes A and B, with a "unit cost" (nonnegative real number) and "capacity" (nonnegative integer) associated with each directed link, a procedure is presented for solving jointly the following set of linear programming problems: For every positive integer k, produce (when one exists) a feasible flow pattern from A to B minimizing total cost subject to "delivering" k units. A feasible flow pattern delivering k units from A to B is an assignment of "link flows" (integers) which do not exceed the corresponding link capacities, such that output equals input at nodes other than A and B and output minus input equals k at A.

5. *Asymptotic behavior in a problem of subsidence*, by Mr. J. H. Giese, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

T. Leser and A. Jenike have proposed a mathematical model of the subsidence of material in overstope mining that can be reduced to the consideration of $v(t)=(1+t)^{-2}+a\int_0^t(1+r)^{-2}v(t-r)dr$. Here $v(t)$ is the velocity of the disturbance wave advancing upward into undisturbed overburden, t is time, and a is a positive constant. By elementary methods it can be proved that as t becomes infinite $v(t)$ approaches infinity or zero accordingly as a is greater or less than unity. In the latter case the disturbance rises only a finite distance. Similar results can be obtained by Laplace transforms.

6. *On relation of distribution functions to the one-step functions*, by Dr. Ceslovas Masaitis, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Let M be a normed linear space of real valued bounded functions defined on the space T of reals with the norm $\|x\| = \sup_{t\in T}|x(t)|$. Let K be the subset of M consisting of all functions K of the type $K(p, t)=0$ if $t<p$, 1 if $t\geqq p$. Then the set D of all distribution functions is identical with the closure of the convex hull of K.

HERTA T. FREITAG, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held on April 22, 1961, at the University of Missouri, Columbia, in conjunction with the meeting of the Missouri Council of Teachers of Mathematics. Professor J. L. Zemmer, Chairman of the Section, presided at the morning session and Professor J. J. Andrews, Vice-Chairman of the Section, presided at the afternoon session. A total of 89 persons attended the meetings, including 39 members of the Association.

The officers elected for 1961–62 are: Professor R. M. Rankin, Missouri School of Mines as Chairman, Professor C. V. Fronabarger, Southwest Missouri State College as Vice-Chairman, and Professor C. A. Johnson, Missouri School of Mines as Secretary-Treasurer. A report of the Annual High School Mathematics Contest sponsored by the Section was given. A motion that the Section continue to administer the advanced placement examination in algebra and trigonometry for another year was passed.

The following papers were presented at the morning session:

1. *Multiplicative functions with special reference to Ramanujan's trigonometrical function* $C_m(n)$, by Professor M. V. S. Rao, Department of Mathematics, Sri Venkateswara University, Triupati, India; Visiting Professor, University of Missouri.

Using the fact that multiplicative arithmetic functions of r arguments form an abelian group under suitable operations like composition and compounding, various properties of these functions have been derived. This approach has been used to study Ramanujan's trigonometric function (representing the sum of the nth powers of the mth order primitive roots of unity) and its various generalizations, together with their applications to various problems involving relative partitions.

2. *The Advanced Placement Tests for Missouri high school seniors*, by Professor J. J. Andrews, St. Louis University.

The Misouri Section of the M.A.A. is sponsoring a state wide advanced placement examination in algebra and trigonometry. Successful completion of this examination qualifies a student to enter analytic geometry and calculus when enrolling at a cooperating university in Missouri. College credit is decided by the individual university. Thirty-six universities and junior colleges are participating in the program, where 998 students from 153 high schools have signed up for the examination. The concern of the high schools and the interest of the students is most encouraging.

At the joint afternoon session Professor W. R. Orton, Jr., University of Arkansas, addressed the group on *Recommendations for the training of teachers of mathematics*. This was a discussion of the report of CUPM.

NOLA A. HAYNES, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-seventh annual meeting of the Nebraska Section of the Mathematica Association of America was held on April 14 and 15, 1961, at the University of Nebraska, Lincoln, Nebraska, in conjunction with the seventy-first annual meeting of the Nebraska Academy of Sciences. Professor R. L. Moenter, Chairman of the Section, presided. There were 75 persons present at the sessions, including 30 members of the Association.

The following officers were elected for 1961–62: Chairman, Professor W. E. Mientka, University of Nebraska; Vice-Chairman, Professor R. L. Moenter, Midland College; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska. Professor J. M. Earl was continued as Chairman of the Committee on Mathematics Contests. The Committee consists of representatives of the Nebraska Section of the Mathematical Association of America, the Nebraska Section of the National Council of Teachers of Mathematics, the Nebraska Actuaries Club, and the Nebraska Academy of Sciences.

The following papers were presented:

1. *Properties of solutions of* $u''+g(t)u^{2n-1}=0$, by Professor W. R. Utz, University of Missouri. (By invitation)

Let $g(t)$, defined for all real t, be an oscillation coefficient provided there exists a real number K such that for all k, $K>k>0$, the solutions of $u''+kg(t)u=0$ are oscillatory. Several known theorems give sufficient conditions that $g(t)$ be an oscillation coefficient. The principal theorem of the paper asserts that if $x=x(t)$, $x(t)\not\equiv 0$, is a solution of $x''+g(t)x^{2n-1}=0$ valid for all large t and if $g(t)$ is continuous, $g(t)>0$ and $g(t)$ is an oscillation coefficient, then $x(t)$ oscillates as $t\rightarrow\infty$.

2. *Bases and infinite series in Banach spaces*, by Professor Gerald Heuer, Visiting Assistant Professor, University of Nebraska. (By invitation)

Several definitions of bases for infinite dimensional topological vector spaces are possible. A Hamel basis always exists (and does not require a topology). In a Banach space E, the following has been considered by several authors. A sequence $\{x_n\}$ in E is a B-basis if for every x in E, there exists a unique set of scalars $\{a_n(x)\}$ such that $\sum_{n=1}^{\infty} a_n(x)=x$. If the convergence is unconditional, it is an absolute B-basis. Existence of a B-basis requires E to be separable. Whether separability implies existence of a B-basis is unknown.

3. *Some theorems and problems in combinatorial geometry*, by Professor W. R. Utz, University of Missouri. (By invitation)

Beginning with a 1935 paper of Vincensini, suggested by the well-known Helly theorem for convex sets, the speaker considered the problem of restricting families of plane sets, generally convex, in such a way that one can determine a positive integer n such that if each n of the sets are intersected by a line, then all sets of the family are intersected by some line. Attention was confined to finite families and results of Santalo, Hadwiger, Danzer, and Grünbaum were emphasized in which the families are parallelograms, line segments, or circles.

4. *The Fourth Nebraska (Twelfth National) Mathematics Contest*, by Professors J. M. Earl, University of Omaha, and H. M. Cox, University of Nebraska, presented by Professor Earl.

A total of 242 schools (31 new this year) have participated in one or more of the four Nebraska Contests. There have been 1635, 2428, 2616, and 2828 students from 127, 133, 140, and 140 schools in the first, second, third, and fourth contests, respectively. Some 580 students participated in both the third and fourth contests; their median scores were 11 and 13, respectively ($r=0.53$). The name of one Nebraska contestant appears on the National Honor Roll.

5. *A physical study of hail*, by Professor L. V. Andrews, Nebraska State Teachers College, Chadron, introduced by the secretary.

Measurements of the temperature of rainfall, nucleating properties of silver iodide as determined by cooling curves, silver iodide concentration in precipitation, and physical characteristics of hail have been made in an effort to distinguish between thunder storms which have been seeded with silver iodide and those which proceed with natural nucleating agents.

6. *Recommendations for the training of teachers of mathematics*, by Professor Henry Van Engen, University of Wisconsin. (By invitation)

7. *The Computing Center of the University of Nebraska*, by Professor John Christopher, University of Nebraska.

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE SOUTHEASTERN SECTION

The fortieth annual meeting of the Southeastern Section of the Mathematical Association of America was held April 7–8, 1961, at Wofford College, Spartanburg, South Carolina. Professors E. B. Shanks, Vice-Chairman of the Section, D. E. South, J. W. Lasley, J. V. Hancock, and Nathaniel Macon presided over the general and divisional sessions. There were 139 members and guests of the Association in attendance.

The following officers were elected for the coming year: Chairman, Professor E. B.

Shanks, Vanderbilt University; Vice-Chairman, Professor Anne L. Lewis, Woman's College, University of North Carolina. A motion was passed that the term of office of the Secretary-Treasurer be changed from one to three years and that no holder of this office be re-elected for more than one term. Professor C. L. Seebeck, Jr. was elected to this position for the first three-year term. Professor R. M. Thrall, University of Michigan and Visiting Lecturer for the MAA, was the featured lecturer at the meeting. A motion that the Section maintain its present boundaries was approved. The invitations of the Woman's College of North Carolina for the 1962 meeting and of the University of Chattanooga for the 1963 meeting were confirmed and of the Citadel for 1964 and of North Carolina State University for 1965 were accepted.

The following program was presented:

1. *Set-valued classes and the axiom of exchange in set theory*, by Professor E. B. Shanks, Vanderbilt University.

A class A is set-valued if and only if for each set v there exists a set x such that for each set u, if the ordered pair $\langle uv\rangle$ belongs to A then u belongs to x. The Exchange Axiom is then stated: $(\alpha, A)\{\text{Set-val}(A).\rightarrow.(\exists y)(u)[u\in y.\equiv.(\exists v)(v\in\alpha.\langle uv\rangle\in A)]\}$. In this statement, α is an ordinal set. If α is replaced by a general set x, the axiom is called the Generalized Exchange Axiom, which is provable in the author's axiom system. In Godel's system, it will replace the sum and substitution axioms.

2. *On polynomial elimination theory*, by Professor L. H. Williams, Duke University.

Polynomial elimination theory is often eliminated from senior and graduate courses in algebra probably for the reason that use of the theory is not practical for pencil and paper calculations. The IBM 650 computer has been taught to do the necessary formal manipulations on polynomials in several variables, thus making classical elimination theory practical. The theory and applications are discussed.

3. *A method for the computation of the greatest root of a positive matrix*, by Professor Alfred Brauer, University of North Carolina.

4. *Computer short circuits*, by Professor W. G. Miller, Clemson College.

The title refers to circuitry which may simplify certain computer operations. Developments stem from "human" reasoning. (1) Extraordinary advances resulting from intuition and flashes of inspiration in processes of human thinking overshadow the fact that much of the reasoning involves merely the elimination of conditions conflicting with explicitly defined rules. This is translated into circuits which include more sophisticated binary concepts such as IF-THEN, IF-AND-ONLY-IF, OR-ELSE, etc. (2) The capacity of the human mind for manipulating "weighted" inputs is interpreted by two "multi-valued" logic techniques: (i) combinations of analogue-type inputs (potentiometers) with digital-type operations; and (ii) use of ac-current and transformer-flux relations.

5. *The use of personal equations in teaching numerical analysis*, by Professor E. P. Miles, Jr., Florida State University.

When learning to find real roots of polynomial equations by the Newton's and False Position methods, each student forms a polynomial with coefficients determined from his birth date and year. Although the polynomials formed differ for each student, all have a zero between 0 and 1. Students locate the smallest such zero to successive tenths by a preliminary analysis and then program the IBM 650 to approximate this root to six decimal places using the two methods studied. The machine thus does most of the grading which would otherwise be quite tedious with each student doing a different problem.

6. *On the existence of a nonzero form*, by Professor E. H. Hadlock, University of Florida.

The purpose of this paper is to show the existence of a nonzero ternary quadratic form $f=ax^2+by^2+cz^2+2ryz+2sxz$ with arbitrary a and of determinant d, when the invariants Ω and Δ of f are given. Formulas or conditions involving Legendre's and Jacobi's symbols were derived which

when satisfied by a given set of conditions show the existence of a form f. Next formulas compatible with the previously mentioned formulas were derived and when satisfied show the existence of a nonzero indefinite form f. These formulas are readily applicable to the construction of a positive, or a nonzero indefinite form f.

7. *Linear fractional Tschirnhaus transformations in algebraic fields*, by Professor H. S. Thurston, University of Alabama.

If ρ is a root of an irreducible equation of degree n over a field F, and η is a primitive number of $F(\rho)$, necessary and sufficient conditions are found such that ρ shall be expressible in the form $(a\eta+b)/(c\eta+d)$ where a, b, c, and d are in F.

8. *On ternary rings*, by Professor J. R. Wesson, Vanderbilt University.

The definition of a ternary ring requires that there exists an element e such that $ea0=ae0=a$ for all a. Let S be the system obtained by dropping the postulate above. Then let xyz indicate the ternary operation in S, and define a new ternary operation $\langle xyz\rangle$ on the elements of S by selecting $e\neq 0$ and defining $\langle xyz\rangle=xaz$, where $ea0=y$. The new system satisfies the conditions required of S, and furthermore $\langle ey0\rangle=y$ for all y. Similarly, a system S satisfying $ey0=y$ for all y can be transformed into a system satisfying $\langle ey0\rangle=\langle ye0\rangle=y$ for all y.

9. *A method for constructing involutory matrices*, by Professors Jack Levine and H. M. Nahikian, North Carolina State College, presented by Professor Nahikian.

A method for constructing all involutory matrices over a field $\mathfrak{F}$ is developed in this paper. The essential theorem proven may be stated: A necessary and sufficient condition that an $n\times n$ matrix H with elements in a field $\mathfrak{F}$ be involutory is that there exist column vectors X_α, $Y_\alpha\in V_n(\mathfrak{F})$ $(\alpha=1,\cdots,\ s,\ s\leqq[\frac{1}{2}n])$, such that $H=I+\sum_{\alpha=1}^{s}Z_\alpha$, $Z_\alpha=X_\alpha Y_\alpha'$, where $X_\alpha' Y_\beta=-2\delta_\alpha^\beta(\alpha,\beta=1,\cdots,s)$.

10. *On diophantine equations of the form $x^n+y^n=hp^m$*, by Mr. J. E. Shockley, University of North Carolina.

11. *Stress distribution of a rotating limaçon*, by Mr. J. L. Tilley, University of Florida.

Consider a thin plate of isotropic material in the shape of a limaçon given by the equation $r=2\alpha+2\beta\cos\theta$, $\alpha>\beta$, rotating in its plane about an axis through its centroid. The method of attack is to consider the equivalent problem of the limaçon in a fixed position acted on by appropriate body forces and is further simplified by considering the body forces in the x-direction and the y-directed separately. The final solution is thus found by superposition of these two results.

12. *The stress distribution due to triangular discontinuities in plates under edge forces*, by Professor C. B. Smith, University of Florida.

A large rectangular plate lying in the xy-plane is subjected to a uniform tension in the y-direction. If the plate is homogeneous, the stress distribution is quite simple. However, if a small rigid triangular region is assumed to lie in the center of the plate, the stress distribution is, of course, considerably changed. The stress distribution near various triangular shaped regions are discussed and the results contrasted with the stresses arising when open holes of similar shape occur in the plate.

13. *On particular values of the derivatives of the gamma function*, by Professor R. W. Cowan, University of Florida.

By successively differentiating the relation $\Gamma(x+1)=x\Gamma(x)$, employing the duplication and other formulas from the theory of the gamma function, values are obtained for the first two derivatives of the gamma function for a positive integer and half of an odd positive integer. These results are established by mathematical induction.

14. *A method to obtain conformal mapping functions by a direct approach and its application*, by Professor S. F. Yeung, University of Florida.

By means of a direct consideration of the polar form of the boundary of a certain type of

simply connected domain D, a conformal mapping which will map the unit circle onto domain D can be obtained. The mappings thus obtained are applied to solve torsion problems in elasticity.

15. *Wifeless tournaments for mixed doubles*, by Professor C. W. Huff, Winthrop College.

This paper supplements a couple of recent papers in this MONTHLY on designing mixed doubles tournaments by giving an arrangement of a schedule of play for couples' party bridge. The restrictions are that there shall be at least six matches (or rounds), and no two players shall play at the same table more than once. An additional restriction is easily supplied; namely, that husband and wife shall not play at the same table any time during the tournament.

16. *Comments on honors courses*, by Professor P. K. Smith, University of South Carolina.

The approach to the question of honors courses was that of setting forth the major inherent difficulties involved with the intent of evoking statements of views that might be helpful to those present with an interest in the subject of the paper. The different plans for challenging the superior student were outlined. The problem of setting up classes for superior students, as well as a general honors program, for a university was discussed. The problem of selecting the students to be invited into classes for superior students and the selection of the most capable teachers for these classes was considered. The need of reducing the load of the teacher handling honors classes was especially emphasized.

17. *A discussion of some of the steps taken by the Department of Mathematics at North Carolina State College to improve the mathematical competence and achievement of its students*, by Professor H. V. Park, North Carolina State College.

In this paper specific attempts to improve the mathematical competence and achievement of students in two areas is discussed. First, significant progress has been made, relative to high school students and students from junior colleges planning to transfer to North Carolina State, through the following programs: (a) participating in NSF sponsored institutes for high school mathematics teachers, (b) conducting a special summer program for a select group of rising high school seniors, and (c) participating in a joint university-small college conference. Secondly, capable students, upon entering State College, are given an opportunity to advance academically and to pursue more depth in mathematics by (a) being certified for credit by examination, and (b) participating in the Superior Student Program in Mathematics.

18. *Are sets omnipotent?* by Professor C. G. Phipps, Tennessee Polytechnic Institute.

Carried away by their enthusiasm, the proponents of "modern mathematics" are claiming too much for it. In many cases the word *modern* could be replaced by the word *fashionable*. Set theory is an example. The axiomatic approach to such theories is similar to the postulational approach to geometry of past years now discarded as pedagogically harmful. Another questionable practice is the inclusion of assumptions in proposed definitions.

19. *Plane geometry and complex numbers*, by Professor R. G. Blake, University of Florida.

The complex number z and its complex conjugate $\bar{z}$ can be used instead of x and y in the equations of plane figures. Such equations are useful in studying the effect of conformal maps from the z-plane to the w-plane. Equations are developed for the straight line and the conic sections.

20. *A note on Baire functions of the first class*, by Professor R. D. McWilliams, Florida State University.

If a bounded Baire function f is the pointwise limit of a uniformly bounded sequence $\{f_n\}$ of continuous real functions on a closed interval, then f is the pointwise limit of a sequence $\{g_n\}$ of finite linear combinations of $f_1, f_2, \cdots$ such that for each n the least upper bound of $|g_n(t)|$ on the interval is the same as that of $|f(t)|$. If the sequence $\{f_n\}$ is not unformly bounded, the conclusion need not be true.

21. *Exterior products in analytic geometry*, by Professor Johann Sonner, University of South Carolina.

In elementary vector algebra an oriented area (resp. volume) is described by a product $x\wedge y$ (resp. $x\wedge y\wedge z$) which unfortunately is identified with an element of R^3 (resp. R). One should not make this identification, but regard $x\wedge y$ (resp. $x\wedge y\wedge z$) as a new entity called bivector (resp. trivector). Let E be a vector space over R of finite dimension n. Extending the previous method an oriented p-dimensional volume in E may be described by a product $x_1\wedge \cdot \cdot \cdot \wedge x_k$. This suggests the construction of an exterior algebra $\wedge E$ whose elements are linear combinations of multivectors $x_1\wedge \cdot \cdot \cdot \wedge x_p$ and obey the rules: $x\wedge(y+z)=x\wedge y+x\wedge z$; $x\wedge x=0$. Exterior algebras are useful in finding determinants, volumes, distances.

22. *Groups having every subgroup as a direct summand*, by Miss Wai-Kit Leung, University of South Carolina.

If a group G is finite and every subgroup is a direct summand, then G is the direct sum of cyclic subgroups each having order some prime, and the converse is true.

23. *A probability distribution function*, by Professor D. E. South, University of Florida.

For the problem of sampling, without replacement, from a finite population, a general expression is obtained for the probability density function of x successes. Defining $P_{a,b}(n, x)$ as the probability of x successes in n trials, beginning with the ath trial, having had b successes, and α_{ij} as a mapping function from $P_{a,b}(n, x)$ to $P_{a+i,b+j}(n, x)$, the density function for $P_{1,0}(n, x)$ is shown to be $P_{1,0}(n, x)=\sum_\pi(p_{1,0}\alpha_{1,1})^x(q_{1,0}\alpha_{1,0})^{n-x}P_{1,0}(0, 0)$. The summation extends over the products of all possible permutations of n operator factors, x of which are $p_{1,0}\alpha_{1,1}$ and $n-x$ are $q_{1,0}\alpha_{1,0}$.

C. L. SEEBECK, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the Mathematical Association of America was held at the Stephen F. Austin State College, Nacogdoches, Texas, on April 14–15, 1961. There were 175 persons present, including 122 members of the Association. Professor H. S. Vandiver of the University of Texas was the invited speaker. Officers for the next year are: Chairman, Professor W. I. Layton, Stephen F. Austin State College; Vice-Chairman, Professor G. R. MacLane, Rice University; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The following papers were presented Friday afternoon and Saturday morning:

1. *A lemma for maximum and minimum values of functions of many variables*, by Professor H. A. Luther, Agricultural and Mechanical College of Texas.

Let the function to be studied be $f(x_1, \cdot \cdot \cdot, x_n)$. Let $n-r$ of the first-order partial derivatives when equated to zero determine uniquely the functions $x_k=g_k(x_1, \cdot \cdot \cdot, x_r)$, where $k=r+1, \cdot \cdot \cdot, n$. Then under suitable circumstances one may study instead $f(x_1, \cdot \cdot \cdot, x_r, g_{r+1}, \cdot \cdot \cdot, g_n)$.

2. *Invariants in extended analytic geometry*, by Professor R. S. Underwood, Texas Technological College.

A locus on the XY-plane for an equation in n variables, as obtained by the rules of extended analytic geometry, is called a *silhouette* if any part of the plane, with the exception of lines and points, remains uncovered. It has been proved for various general cases that all silhouettes of any given quadratic equation, as obtained from different linear plotting rules, are invariant in a topological sense. That is, one such silhouette infallibly shows the nature of all others. Furthermore, the silhouette reveals certain intrinsic algebraic properties of the equation.

3. *Wiener's smoothing and prediction technique as an extension of the least squares technique*, by Professor E. R. Keown, Agricultural and Mechanical College of Texas.

This paper is a heuristic discussion of Wiener's root-mean-square technique of prediction and filtering. The integral equation of the theory is replaced by an infinite series and the observation made that for stationary time series the technique becomes that old time favorite "the method of least squares."

4. *On developments in an arithmetic theory of the Bernoulli numbers*, by Professor H. S. Vandiver, University of Texas.

Consider the recursion formula $(b+1)^k=b_k$ for $k>1$. If the left-hand member of this equation is expanded by the use of the binomial theorem, and b_i is substituted for b^i, $i=1, \cdots, k$, a relation is obtained such that if the values $k=2$, $k=3$, etc. are taken in turn, the values of b_a, $a=1, 2, \cdots$, can be calculated. Thus after defining b_0 as 1, it is found that $b_1=-\frac{1}{2}$, $b_2=\frac{1}{6}$, $b_3=0$, etc. When the fractions b_{2n}, $n=1, 2, \cdots$, are expressed in their lowest terms, then the properties of the numerators and the denominators are considered mainly with the use of congruences. The methods used are quite elementary.

5. *The use of a second order correct boundary condition in the numerical solution of parabolic differential equations*, by Mr. G. W. Batten, Jr., Rice University.

The solution of the parabolic partial differential equation $\partial^2u/\partial x^2=F(x, t, \partial u/\partial x, \partial u/\partial t)$, subject to specified linear boundary conditions can be approximated by the solution of a difference equation subject to corresponding boundary conditions. In this paper the form of the boundary difference operator is given such that the solution of the difference equation converges to the solution of the differential equation like $O((\Delta x)^2+\Delta t)$.

6. *The discrete harmonic kernel function*, by Professor C. R. Deeter, Texas Christian University.

The discrete Laplace operator is defined by replacing the partial derivatives of the Laplacian by their corresponding difference quotients with mesh width h. For a bounded, simply connected region R, and a corresponding discrete region R_h, discrete counterparts of the Green's and Neumann functions, $G^h(z, \zeta)$ and $N^h(z, \zeta)$, are defined. The discrete harmonic kernel function of the region is defined as $K^h(z, \zeta)=N^h(z, \zeta)-G^h(z, \zeta)$. It is shown that $K^h(z, \zeta)$ is discrete harmonic, symmetric in its arguments, reproduces discrete harmonic functions on the region R_h with respect to a certain inner product, and that these properties characterize it completely.

7. *On the geometry of functions holomorphic in the unit circle, of arbitrarily slow growth, which tend to infinity on a sequence of curves approaching the circumference*, by Professor G. R. MacLane, Rice University.

It is well known that there exist functions $f(z)$, holomorphic in $|z|<1$, with $M(r)<\mu(r)$, where $\mu(r)$ is a given positive function which $\to\infty$ as $r\to 1$, and such that $\min_{|z|=r_n}|f(z)|$ approaches ∞ as $n\to\infty$. Here $r_n\to 1$ is an appropriately chosen sequence. Such functions may be constructed by the use of gap series or via an infinite product. The object of the present note is to construct such a function geometrically by starting with the Riemann surface S onto which $w=f(z)$ maps $|z|<1$.

8. *Velocities in two-dimensional potential flow*, by Professor George Copp, North Texas State College.

By using the growth of circulation about an airfoil computed by Herbert Wagner, a method is developed for computing velocities in unsteady flow in the vicinity of an airfoil in a wind tunnel. The computed velocities agree closely with measured velocities given by P. B. Walker in *Experiments on the Growth of Circulation about a Wing.*

9. *The summability of some Newton series*, by Professor Louis Brand, University of Houston.

The Newton series (in factorial powers) which are the analogues of the power series for e^x, $\cos x$, $\sin x$; e^{-x}, $\cosh x$, $\sinh x$ represent respectively the functions 2^x, $2^{\frac{1}{2}x}\cos\frac{1}{4}x$, $2^{\frac{1}{2}x}\sin\frac{1}{4}x$; 0, 2^{x-1}, 2^{x-1}. If λ and μ denote the abscissas of convergence and absolute convergence of these series, $\lambda=-1$, $\mu=0$ for the first three, $\lambda=0$, $\mu=0$ for the last three. The first three series are Cesàro-summable of order n when $-(n+1)<x\leqq -n$, $n=1, 2, \cdots$. More generally, the series $\sum_{k=0}^{\infty}x^{(k)}/k!$ is Cesàro-summable of order p (not necessarily an integer) when $x>-(p+1)$. When $x=0, 1, 2, \cdots$ the convergence of this series is of an especially simple character for it terminates after $n+1$ terms. When $x=-1, -2, -3, \cdots$ the Euler transform of this divergent series terminates after n terms

and represents 2^x. Thus the series is Euler-summable to 2^x by means of a *finite* series of n terms when $x=-n$, a negative integer.

10. *Estimation of parameters in a translated log-normal distribution with incomplete data*, by Professor P. D. Minton and Mr. Frederick Backer, Jr., Southern Methodist University.

It is assumed that survival times of patients treated for certain diseases are distributed as a translated log-normal random variable. Upon analysis of the results of the treatment, it is often found that some patients have withdrawn from treatment and others are still living when the results are being analyzed. A method originating with Lea is generalized and applied to cases where patients are still living at the time the experiment is stopped. This method gives maximum likelihood estimates for the parameters to be estimated in the assumed translated log-normal distribution.

11. *Some decomposition theorems for some classes of matrix summability operators*, by Professor E. P. Kelly, Jr., Stephen F. Austin State College.

Certain subclasses of matrix summability operators are considered as subspaces of the vector space of matrix summability operators on the set of bounded sequences of real numbers. Let T_c denote the subspace of all conservative matrix summability operators, T_b the subspace of matrix summability operators which map bounded sequences into convergence sequences, T_0 the subspace of matrix summability operators which map bounded sequences into null sequences, and $T_z \subset T_c$ the subspace of matrix summability operators which map null sequences into null sequences. A proof was given that T_c has the coset decomposition $T_c/T_0=T_b/T_0 \oplus T_z/T_0$. Other theorems of this type are stated.

12. *Regularized set operations*, by Professor Arlen Brown, Rice University.

Define $A^{\sim}=A^{0-}$ (in a topological space) and new set operations by applying the operation "$\sim$" to the old set operations. The sets A such that $A=A^{\sim}$ ("regularly closed" sets) form a Boolean ring with respect to these operations.

13. *Holmgen-Riesz ($H=R$) transform equations of Riemannian type*, by Professor M. A. Al-Bassam, Texas Technological College.

Let α_i, a_i be numbers, $R(n-w)>0$ and $z(x)\in C^2$ on $[a, b]$, and

$$E\colon I^{-w}\prod_{i=1}^{3}(x-a_i)^{\alpha_i}I^{-1}(x-a_i)^{1-\alpha_i}I^{w-1}z=0,$$

where the transform $I^\alpha f=D_x^n/\Gamma(\alpha+n)\int_a^x(x-t)^{\alpha+n-1}f(t)dt$, $R\alpha+n>0$ $(n=0, 1, \cdots)$, and $f\in C^n$ on $[a, b]$. Then it is shown that: (1) E is a differential equation of Riemannian type if and only if $F\colon w-\sum_{i=1}^{s}\alpha_i+I=0$, a condition which is satisfied by the indices of Riemann P-function; otherwise E is a differential-integral equation of Riemann-Volterra type. (2) E is reduced to the Gauss's equation if for fixed i (say $i=2$) $a_2\to\infty$, and by the operational properties of the transform the twenty-four Kummer's solutions have been obtained for this case.

14. *Ratio estimators in the balanced incomplete block design*, by Professor V. Seshadri, Southern Methodist University.

This paper proposes an estimator for the ratio of the block variance to the error variance σ_b^2/σ_e^2 in a balanced incomplete block design. This estimator is shown to be unbiased and then its variance is compared with the variance of another unbiased estimator. The difference between the variances is expressed as a quadratic function of the ratio σ_b^2/σ_e^2 and the roots of this quadratic have been examined. It is proved that the proposed estimator is uniformly better than the existing estimator for all designs that are possible.

15. *A sufficient condition that the topological space of a topological group be a Moore space*, by Mr. L. R. Carry, North Texas State College.

This paper presents a notion of point and region in a topological group; then defines a sequence of collections of regions which satisfy Professor Moore's Axiom 0 and Axiom 1, conditions 1), 2), and 3). The following theorem is then presented and proved.

THEOREM. *Let G denote a topological group. If the identity element in G admits a countable basis in the topological space G, then the topological space G is a Moore space.*

16. *An imbedding of a ring in a ring with unity*, by Professor D. E. Edmondson, University of Texas.

If R is a ring, a ring with unity is defined, R', and a homomorphism of R into R' is defined, with the properties that 1) the mapping is an isomorphism if R has a unity or no divisors of zero and 2) R' has no divisors of zero if R has no divisors of zero, and 3) concept of characteristic does not enter into the construction.

17. *Trace preserving isomorphism of operator algebras*, by Mr. Carl Pearcy, Rice University.

The author discussed the question of when a trace preserving isomorphism between two operator algebras on a finite-dimensional (real, complex) vector space is implemented by a nonsingular (orthogonal, unitary) operator.

18. *Cluster sets and pseudoanalytic functions*, by Mr. A. A. Armendarez, Rice University.

A function is said to be pseudoanalytic in a domain D if in D it is (a) an interior mapping in the sense of Stoilow, (b) it has continuous first partial derivatives, and (c) its Jacobian is positive except on at most a countable set. For a function pseudoanalytic in the unit circle the notions of cluster set $C(f, e^{i\theta})$ and of radial boundary cluster set modulo a set $E(C_{R-E}(f, e^{i\theta}))$ were discussed.

THEOREM. *Let $f(z)$ be pseudoanalytic on $|z|<1$ and let its modulus have radial limit 1 everywhere on $|z|=1$ except possibly on a set of capacity zero. If E is an arbitrary set of capacity zero on $|z|=1$, then for every $e^{i\theta}$, $f(z)$ takes on in every neighborhood of $e^{i\theta}$, $C(f, e^{i\theta})-C_{R-E}(f, e^{i\theta})$* except for at most a set of capacity zero.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 35th meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at West Virginia University on May 6, 1961. There were 118 persons present, including 63 members of the Association.

At the business meeting several items were acted upon. (1) The Committee on High School Contest Examinations indicated that approximately $750 had been accumulated over the past few years. It was voted that these funds be used to pay expenses of visiting lecturers to high schools in the area. (2) It was reported that the Association of Teachers of Collegiate Mathematics in West Virginia had passed a resolution and requested the endorsement of this resolution by the Allegheny Mountain Section. In essence, the resolution endorses the spirit of the CUPM recommendations and offers specific modifications for the state of West Virginia. The resolution contains detailed proposals concerning training of teachers and requirements for admission to West Virginia colleges, and recommends consideration of a mathematics course in the general education program. The resolution was endorsed by the Section, without dissent. (3) Officers for the next two years were elected: Chairman, Professor Evan Johnson, Jr., Pennsylvania State University; Secretary-Treasurer, Professor W. A. Beck, Chatham College; members of the Executive Committee, Dr. B. H. Mount, Westinghouse Corporation, Pittsburgh, and Professor I. D. Peters, West Virginia University.

Professor R. C. Buck, Chairman of the Committee on the Undergraduate Program in Mathematics, delivered the invited address. Professor Buck discussed the organization and the various areas of interest of the CUPM panels and summed up the major recommendations of the Committee. The floor was open for general discussion at the close of the address.

The following papers were presented:

1. *An elementary operator solution of the heat equation*, by Professor L. R. Bragg, West Virginia University.

Consider the Cauchy problem $u_t(x, t) = u_{xx}(x, t)$, $u(x, 0) = \phi(x)$. The author uses the series representation of the operator $\exp(tD^2)$ to give an elementary introduction to this problem. Through the linear and multiplicative properties of this operator, a class Ω of analytic initial functions $\phi(x)$ is built up that gives rise to closed analytic solution of the above Cauchy problem. One such elementary result is: Let $\phi(x) \in \Omega$. Then $P(x)\phi(x) \in \Omega$ for $P(x)$ a polynomial. It is also observed that this operator yields the usual results when formally applied to the Fourier series representation of a function.

2. *Operational formulas connected with generalized Hermite polynomials*, by Mr. H. W. Gould, West Virginia University.

This paper develops the two operational relations $(x + hrd^{r-1})^n f(x) = \exp(hD^r)x^n \exp(-hD^r)f(x)$ and $(D + hrx^{r-1})^n f(x) = \exp(-hx^r)D^n \exp(hx^r)f(x)$ which include well-known results involving the Hermite polynomials when $r = 2$. A further generalization is indicated in terms of the Bell polynomials.

3. *A class of additive arithmetical functions*, by Mr. R. L. Duncan, Pennsylvania State University.

A discussion and proof of this result will appear in the Mathematical Notes Section of this MONTHLY.

4. *A generation of high school calculus*, by Professor Emeritus J. H. Neelley, Carnegie Institute of Technology.

This paper will appear in Mathematical Education Notes of this MONTHLY.

5. *The teaching of mathematics for management careers*, by Mr. Carlos Fallon, Radio Corporation of America, Moorestown, New Jersey.

The classical undergraduate curriculum leading to the calculus is now competing with special courses in modern mathematics for students of the biological and social sciences. Similarly, the mathematics of finance is facing rivals in the areas of decision theory and of strategy. Modern mathematics has become a most useful part of the decision-making process. It is suggested, therefore, that a program of modern mathematics be taught, not as *the* mathematics of this or that field of concentration, but *as mathematics*, having for its principal supporters, science, engineering and business administration undergraduates, but offered to all students on the campus.

6. *Assertion vs consideration in mathematical exposition*, by Professor W. A. Beck, Chatham College.

Attention is drawn to the dual role which statements play in mathematical exposition—on the one hand as merely considered, on the other as asserted. More extensive use of some sign of assertion is encouraged in symbolic logic in order to reflect this distinction in the character of statements. Illustrations are drawn from standard usage of proofs by induction, proofs by contradiction, statements of equality, and statements involving quantifiers.

7. *Mixed strategies: a geometric approach*, by Professor F. H. Steen, Allegheny College.

A geometric solution for zero-sum two person games was discussed.

EVAN JOHNSON, JR., *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The fortieth annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, Urbana, Illinois, on May 12–13, 1961. Professor Douglas Daly, Chairman of the Section, presided at all sessions. There were 84 persons in attendance, including 67 members of the Association.

The following officers were elected to serve for the coming year: Chairman, Professor T. E. Rine, Illinois State Normal University; Vice-Chairman, Professor Anice Seybold,

North Central College; Secretary-Treasurer, Professor Wayne McGaughey, Bradley University.

The Friday evening banquet speaker was Professor J. L. Doob, University of Illinois. He showed slides which were taken in Moscow and Leningrad and told of his experiences during the three weeks he spent in Russia as an exchange professor of mathematics, sponsored by the Academy of Science.

Following a brief welcome by Professor M. M. Day, Chairman of the Department of Mathematics, University of Illinois, the following program was presented.

1. *A note on Vandermonde's convolution*, by Professor Michael Skalsky, Southern Illinois University.

By use of Lagrange's formula for the inversion of power series, the following combinatorial identity was proved:

$$\sum_{k=0}^{n} A_k(a, b)\cdot A_{n-k}(c, b) = A_n(a + c, b),$$

where

$$A_n(a, b) = \binom{a + bn}{n}\frac{a}{a + bn}.$$

This identity is valid for any numbers a, b, and c. If $b=0$, it reduces to the well-known Vandermonde's convolution

$$\sum_{k=0}^{n}\binom{a}{k}\binom{c}{n-k} = \binom{a+c}{n}.$$

2. *The psychological appeal of deductive proof*, by Professor Gertrude Hendrix, University of Illinois.

The primitive fascination of power to foretell provides a strong motive for first experience with formal proof. Again and again one finds that he could have derived results obtained previously by much labor. This repeated success in prediction—this finding that things learned by experiment could have been foretold by deduction—pays off as confidence in deductive proof and taste for proving. Thus do students acquire an *inductive* basis for faith in *deduction*. In the beginning *everyone* needs to prove things he already knows.

3. *The CUPM recommendations for the training of teachers*, by Professor Rothwell Stephens, Knox College.

The underlying philosophy of the CUPM recommendations was discussed and progress made in implementing the recommendations was reported.

4. *The coconut problem*, by Mr. Clyde Bridger, Illinois Department of Public Health, Springfield.

During the day, p men and a monkey gathered T coconuts. That night each man in turn went to the pile, divided it into p equal parts, tossed the residue of k coconuts to the monkey, and hid his share. Require that $0\leq k<p$ and let R be the remainder after the last man took his share. Then $p^p[R+k(p-1)]=(p-1)^p[T+k(p-1)]$. The general solution is $T=p^pt-k(p-1)$ and $R=(p-1)^pt-k(p-1)$, where t is a positive integer. For the classical solution, let $p=5$, $t=1$, $k=1$. Then $T=3121$ and $R=1020$.

5. *Similarity transformations on symmetric matrices*, by Professor John Christiano, Northern Illinois University.

The purpose of the paper was to demonstrate how to construct similarity transformations $R=R^{-1}$ $(RR^{-1}=1)$ for certain types of symmetric matrices so that a given matrix A when transformed into RAR^{-1} is either a diagonal matrix or one whose elements appear in a form suitable for evaluating the characteristic equation $|A-I\lambda|=0$. Symmetries and "combination" of symmetries

of four types were considered.

6. *Simple connectivity*, by Professor M. H. Heins, University of Illinois. (By invitation)

Definitions of simple-connectivity employed in the elements of the theory of functions of a complex variable were examined and related.

7. *The differential equation of a vibrating beam*, by Professor Earl McKinney, Northern Illinois University.

The vibration of a single span beam subjected to constant end and uniformly distributed axial load was considered. The equation of motion can be developed through moment analysis or, in the case of deflections about the equilibrium position, by the use of energy methods. This paper considered the determination of the equations of motion through the use of Rayleigh's principle and techniques of the variational calculus.

8. *Creativity and the search for beauty*, by Professor Rose Lariviere, University of Illinois, Navy Pier.

An awareness of the continuity of mathematical ideas contributes to the feeling of competence necessary for creative activity. Instances where continuity can be emphasized and elegance improved were given, and the retention in or restoration to the curriculum of certain topics was urged for their asethetic value regardless of their practical importance or unimportance.

9. *The fragilities of logic*, by Professor Rubin Gotesky, Department of Philosophy, Northern Illinois University, introduced by R. J. Cormier. (By invitation)

A "fragility of logic" is described by the author as consisting of two characteristics (a) a breakdown in logical use and (b) an attitude of disbelief that the "breakdown" is serious. The author discusses a number of important fragilities: (1) the impossibility-possibility fragility, (2) the disagreement fragility, (3) the familiarity fragility and (4) the rules fragility. Examples are given. The essential point of the author is that logic is a tool, an instrument. Its function and structure are determined by the specific problems, objectives, goals which arise in the life of man. There is no ultimate logic. There are only logics suited to and appropriate for given cultural and theoretical needs.

A. W. McGaughey, *Secretary*

THE MAY MEETING OF THE OHIO SECTION

The forty-fifth annual meeting of the Ohio Section of the Mathematical Association of America was held at Ohio Wesleyan University, Delaware, Ohio, on Saturday, May 6, 1961. Professor Wade Ellis, Chairman of the Section, presided at the morning and afternoon sessions. There were 82 persons registered in attendance, including 71 members of the Association.

Officers selected for the coming year are: Chairman, Professor R. L. Wilson, Ohio Wesleyan University; Secretary-Treasurer, Professor Foster Brooks, Kent State University; third member of the Executive Committee, Professor Charles Saltzer, University of Cincinnati; Program Committee, Professor W. E. Restemeyer, University of Cincinnati, Chairman, Professor Clarence Heinke, Capital University, and Mr. J. W. Warner, College of Wooster.

The following papers were presented:

1. *On Boolean algebras*, by Professor Wade Ellis, Oberlin College. (Chairman's Address)

Current procedures reviewing and reconstructing mathematics curricula have apparently given inadequate consideration to the desirability of including some topics from Boolean algebras. For such algebras axiomatic bases can be brief without requiring more than elementary mathematical experience for logical development. Systems constructed on such bases have in them, at a very early stage, features which permit interesting interpretations and applications. Later developments in such systems stimulate the student's curiosity about certain modern applications of mathematics in much the same way that a study of classical geometry stimulates interest in its

classical application and interpretation.

2. *The reducibility of polynomials*, by Professor R. L. Wilson, Ohio Wesleyan University.

Let $p(x)$ be a polynomial with coefficients in the field F. This paper presents a finite method for determining whether $p(x)$ is reducible over F. If $q(x) \mid p(x)$, the product of the zeros of $q(x)$ must be in F. If $p(x)$ is of degree n and $q(x)$ of degree m, the product of the zeros of $q(x)$ is one of the C_m^n products $Q_i\,[i=1, \cdots, C_m^n]$, of m of the n zeros of $p(x)$. An induced polynomial is formed for which the Q_i are zeros, and the zeros of this polynomial are investigated. This investigation for all $m \leqq [\frac{1}{2}n]$ provides the factors of $p(x)$ or indicates irreducibility.

3. *The Committee on the Undergraduate Program in Mathematics*, by Professor J. H. McKay, Michigan State University Oakland.

The activities, recommendations, and current areas of concern to CUPM during the past year were briefly summarized.

4. *Some conjectures associated with the Goldbach conjecture*, by Professor I. A. Barnett and Mr. Ted Cook, University of Cincinnati, presented by Professor Barnett.

The first conjecture, a stronger form of the Goldbach conjecture for odd numbers, says that every odd number $2k-1=x+2y$, where x and y are primes, $k \geqq 4$. The second conjecture is that it is possible to find a representation of every odd number of the form $6k+1$ or $6k+5$ as $2x'+3y'$ (x', y' prime) $k \geqq 3$, for which either the x' or y' appears as one of the primes in *some* representation of $2k-1$ as the sum of a prime plus the double of a prime. Both conjectures have been verified to about 15,000. If $6k+3$ is such that $2k-1$ is prime, then $6k+3$ may be written $6+3y'$ (y' prime). If, however, $2k-1$ is composite, then there is no representation of $6k+2$ as the double of a prime plus the triple of a prime.

5. *Computers and automata*, by Professor Charles Saltzer, University of Cincinnati. (By invitation).

Turing machines are defined. Some of the results of the theory concerning noncomputability and universal Turing machines are derived. Programming of Turing machines is described and the interpretation of computers and computer programs as Turing machines is given.

6. *Classroom note on an approach to Ceva's theorem*, by Professor C. H. Heinke, Capital University.

The theorem of Ceva affords an excellent opportunity, in relatively elementary mathematics, for students to experience discovery of a general proposition through a careful study of some of its special cases that are known to them. The "medians concurrent" theorem of high school geometry suggests the question; the "interior angle bisectors concurrent" theorem suggests an answer; the "altitudes concurrent" theorem suggests a proof of the general theorem. Although now well known and seemingly elementary, this theorem, requiring nothing not known to Euclid, was not published until 1678!

7. *A general theory of sums and products*, by Professor A. A. Johnson, Ohio Wesleyan University.

Category-theoretic definitions of sums and products are given in which the product is the dual of the sum. Examples from set theory include sums, (Cartesian) products, unions, and intersections. Other examples are topological sums and Cartesian products of topological spaces, logical sums and products of sentences, least upper bounds and greatest lower bounds, direct sums and Cartesian products of R-modules, tensor products, free R-modules, etc. Associativity and commutativity of such sums and products is proved. It is shown that distributivity does not always hold. Quotients and differences also arise in the theory.

8. *Some generalizations of the Fibonacci sequence*, by Professor S. E. Ganis, Ohio Wesleyan University.

The sequence $\{f_n\}$ is defined by $f_1=f_2=1$, $f_n+f_{n+1}=f_{n+2}$, $n \in J^+$. The well-known relations

(1) $f_{n-1}f_{n+1}-f_n^2=(-1)^n$ and (2) $f_{n-2}f_{n+2}-f_n^2=(-1)^{n+1}$ generalize for $s\in J^+$ as (3) $f_{n-s}f_{n+s}-f_n^2=(-1)^{n+s-1}f_s^2$. (4) $f_{n-2}f_{n+2}-f_{n-1}f_{n+1}=2(-1)^{n+1}$ generalizes as (5) $f_{n-s-1}f_{n+s+1}-f_{n-s}f_{n+s}=(-1)^{n+s}f_{2s+1}$. For $s_1, s_2\in J^+$ (3) and (5) further generalize as (6) $f_{n-s_1}f_{n+s_1}-f_{n-s_2}f_{n+s_2}=(-1)^{n-1}[(-1)^{s_1}f_{s_1}^2-(-1)^{s_2}f_{s_2}^2]$.

FOSTER BROOKS, *Secretary*

THE MAY MEETING OF THE OKLAHOMA SECTION

The Spring Meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma State University, Stillwater, Oklahoma, on Friday, May 12–13, 1961. Professor J. A. Nickel, Chairman of the Section, presided. There were 72 persons registered, including 57 members of the Association.

The following research papers were presented on Friday:

1. *Conformally differentiable points of arcs*, by Mr. Louis De Noya, Oklahoma State University.

2. *Nonoptical illusion*, by Mr. J. W. Schestedt, Stigler, Oklahoma.

3. *A cross and scalar product in function spaces*, by Mr. E. E. Slaughter, University of Oklahoma.

4. *A theorem related to a theorem of E. Helly*, by Mr. A. G. Haddock, Oklahoma State University.

5. *Indexed systems of neighborhoods for general topological spaces*, by Mr. Allen Davis, University of Oklahoma.

6. *Combinatorial equivalence of matrices*, (invited hour address) by Professor R. A. Good, University of Oklahoma and University of Maryland.

7. *An invariant subspace of a differential operator*, by Dr. J. E. Scroggs, University of Arkansas.

8. *Some applications of the Ramanujan function*, by Professor C. A. Nicol, University of Oklahoma.

9. *Nondegenerate convex cones*, by Mr. R. G. Laatsch, Oklahoma State University.

10. *Equivalence relations and equals relations*, by Professor W. A. Rutledge, University of Tulsa.

11. *Harmonic transformations*, by Professor N. A. Court, University of Oklahoma.

Friday night there was an evening meeting of Oklahoma college teachers to review and discuss the teacher training recommendations of the CUPM. Dr. Robert Wisner, Executive Director of CUPM, was the principal speaker.

Saturday, May 13, was also devoted to an investigation and discussion of the CUPM recommendations. Representatives of the mathematics departments of sixteen of the seventeen Oklahoma institutions of higher education that give teacher training, and representatives of several private institutions, were present and recommended that a three-step program for Oklahoma colleges in meeting CUPM suggestions for teaching training at the secondary level be adopted on a state-wide basis. This program is scheduled for completion by June, 1965. They also recommended that intermediate algebra, business mathematics, general mathematics, solid geometry, and methods of teaching mathematics be dropped for college credit by June, 1964. These courses, according to the proposed program, will not apply towards certification and accreditation of mathematics teachers after June, 1962, even if kept in the mathematics program.

R. V. ANDREE, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-ninth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin, Madison, Wisconsin, on May 13, 1961. Professor Henry Van Engen, Chairman of the Section, presided. This meeting was held jointly with the May meeting of the Wisconsin Mathematics Council and there were 129 present, including 59 members of the Association and 70 members of the Wisconsin Mathematics Council.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Earl Swokowski, Marquette University; Vice-Chairman, Professor G. L. Bullis, Wisconsin State College and Institute of Technology, Platteville; Secretary-Treasurer, Professor E. F. Wilde, Beloit College.

The following papers were presented:

Morning Session for members of the Association

1. *A problem in matrix theory*, by Professor C. B. Hanneken, Marquette University.

Let $F=GF(p)$ and $\mathfrak{M}_n$ be the ring of $n\times n$ matrices over F. A nonderogatory matrix H with irreducible characteristic equation generates a subfield $F_H=\{\sum_{i=0}^{n-1} a_iH^i | a_i, \text{ scalars}\}$ of $\mathfrak{M}_n$ of order p^n. There exists a nonderogatory matrix $A\in\mathfrak{M}_n$ that generates the automorphism group $\{A\}$ of F_H. Furthermore $A^n=I$ and $f(x)=x^n-1$ is the minimal polynomial of A. Let $N_A(p^n)$ be the number of distinct subfields having the same automorphism group $\{A\}$. It follows that $N_A(p^n)=\omega_n(p)/n(p-1)$, where $\omega_n(p)$ is the order of the multiplicative group of the ring $\Omega=\{\sum_{i=0}^{n-1} a_iA^i | a_i \text{ scalars}\}$. Clearly Ω is a commutative ring with I with nonzero divisors and is isomorphic to $F[x]/(x^n-1)$. If $\alpha=\sum_{i=0}^{n-1} a_iA^i\in\Omega$, then $|\alpha|=\prod_{j=0}^{n-1}(\sum_{i=0}^{n-1} a_i\rho^{ij})$, where ρ generates the cyclic group of roots of $f(x)$. If $n|p-1$, the order $\omega_n(p)$ is found to be $(p-1)^n$.

2. *Some decompositions of a Euclidean space*, by Professor L. F. McAuley, University of Wisconsin.

Suppose that $f: X\rightarrow Y$ is continuous. Given conditions on either X or f, what can be said about Y? In particular, when is Y homeomorphic to X? An approach to such problems is that of studying various subsets of X which map onto points. What upper semicontinuous decompositions of a metric space X (in particular, X is Euclidean n-space E^n) define continuous mappings onto a space homeomorphic to X? Simple examples of u.s.c. decompositions of E^2 are given which lead up to a theorem due to R. L. Moore that an upper semicontinuous decomposition of E^2 yields a space homeomorphic to E^2 if the elements of the decomposition are compact continua which do not separate E^2. Results are mentioned concerning u.s.c. decompositions of E^3 which yield spaces homeomorphic to E^3.

3. *Some aspects of general quadratic transformation*, by Professor H. P. Pettit, Marquette University.

The general quadratic transformation falls naturally into four cases: (a) double pole at $z=\infty$, (b) simple poles at $z=\infty$, $z=\xi$, (c) simple poles at $z=\xi_1$, $z=\xi_2$, (d) double pole at $z=\xi$. Analysis of the transformation of an algebraic curve of order m, with circular points of multiplicity r and the poles of multiplicity k_i is made through use of the algebraic theory of plane curves. (a) A branch with ordinary asymptote transforms into a parabolic branch. (b) Transformed curve of order $3m-2r-2k$, circular points of multiplicity $m-k$, origin of multiplicity $m-2r$. (c) Transformed curve of order $4m-2r-(k_1-k_2)$, circular points of multiplicity $2m-2r-k_1-k_2$, origin of multiplicity $2m-4r$. Case (d) was not included in this report.

Morning Session—Wisconsin Mathematics Council

4. *What will we be teaching when the shouting and tumult dies down?*, by Professor C. T. Brumfiel, University of Michigan.

Afternoon Session—joint session of the Wisconsin Section M.A.A. and the Wisconsin Mathematics Council.

5. *Crises: past and present*, by Professor R. C. Buck, University of Wisconsin.

A brief summary of the changes (and lack of changes) in mathematics education at the college level, starting at 1900, and ending with a summary of the work of the Committee on the Undergraduate Program in Mathematics through June, 1961. Emphasis was placed upon the work of the panel on Teacher Training, the panel on Mathematics for Physical and Engineering Sciences, and upon the work directed toward a general statement of the objectives of college mathematics.

6. *The status of mathematics teaching in Wisconsin secondary schools*, by Mr. A. M. Chandler, Department of Public Instruction, Madison.

Research shows that little mathematics talent can be awakened later if it has not had opportunity to be nurtured during the high school years. In Wisconsin we are beginning to see some worthwhile trends. Many schools are including more mathematics courses in their offering; revising present course offerings; developing inservice training in mathematics; increasing enrollment in mathematics classes through guidance and counseling programs; and more effectively assigning mathematical personnel. As in other states, Wisconsin is facing a shortage of well-qualified mathematics teachers. In an effort to improve the quality of mathematics teachers, the state department of public instruction has raised its certification requirements.

E. F. WILDE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, Spring, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962

KENTUCKY, University of Kentucky, Lexington, Spring, 1962.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 2, 1961.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan Ann Arbor, March 24, 1962.

MINNESOTA, Moorhead State College, November 4, 1961.

MISSOURI, Missouri School of Mines, Rolla, Spring, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14 1962.

NEW JERSEY, St. Peter's College, Jersey City, November 4, 1961.

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO

OKLAHOMA, Oklahoma City University, October 27, 1961.

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, South Dakota School of Mines and Technology, Rapid City, Spring, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

SOUTHERN CALIFORNIA, Long Beach State College, March 9, 1962.

SOUTHWESTERN

TEXAS, Rice University, Houston, April, 1962.

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, Spring, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.

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THE REVOLUTION IN MATHEMATICS*

MARSHALL STONE, University of Chicago

We are in the midst of an intellectual crisis which has profound implications for education everywhere in the world. This crisis has arisen in conjunction with the development of science, and is the direct consequence of man's adoption of scientific ways of thinking and of acting. We shall not resolve its tensions until we accept science as an integral and all-pervasive part of our culture, not only at the material level but also in the tangent spheres of the intellectual life and of education. So suddenly has this crisis developed and so far do its effects extend that we are forced to recognize in it the symptoms characteristic of a major mutation in human culture. Even though imagination is inadequate to paint the transformed society which will at length emerge from changes but recently begun, we can nevertheless see somewhat more than dimly that only a few generations hence every aspect of man's life will have been radically altered. Already we can recognize how importantly the progressive formation of scientific habits of thought and action is influencing man's relation to his universe. Everything—his relation to his physical environment, his relation to time and space, his relations with himself and with the society in which he lives, even, we must add, his relation to the spiritual realm—is being viewed and treated under new aspects because man has found in science a new instrument for perceiving and understanding the conditions of his existence.

Already the changes thus wrought by science have outstripped the slow evolution of our systems of education. As a new world struggles to be born, we realize with more than a little concern that we are suddenly called upon to make a very determined effort toward bringing education up to the level of our times and orienting it as best we can toward a future vastly different from anything familiar to us from the past. The situation demands that we rethink our entire conception of education, in a spirit of seeking the truth about ourselves and the universe in which we live, and with the aim of embodying that truth in what we teach and in our manner of teaching it.

Twentieth-century developments in mathematics. As part of this task of reviewing and revising the ideas underlying our educational practices, it is essential to examine the quiet revolution which has taken place in mathematics in our own times and to appraise its now enormous potentialities for bringing fundamental advances in every domain where reason and scientific thinking have roles to play. Here it is not so important for the mathematician to recall that for something like two and a half millennia mathematics has held an exalted

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place in education—and among the humanities, be it noted—as it is for him to delineate clearly and strikingly the remarkable developments which have taken place during the twentieth century in mathematics and in its connections with other disciplines—and to explain why they now put such a tremendous pressure upon us to modernize our teaching of mathematics and even to give it a new and more central place in the educational scheme. What has to be grasped by all who are interested in education is that our conception of the nature of mathematics has been revolutionized, our technical knowledge of the subject vastly enlarged, and our dependence upon it for scientific and technological progress enormously increased. Indeed, it is becoming clearer and clearer every day that mathematics has to be regarded as the corner-stone of all scientific thinking and hence of the intricately articulated technological society we are busily engaged in building. We can foresee a time in the not very distant future when a complete identification of science, logic, and mathematics will be achieved: what has already happened in the case of mathematical physics, particularly the quantum field theory, foreshadows what will happen in the other branches of science as they probe more deeply into the complex relationships they seek to understand.

While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world. To put this just a little more precisely, mathematics is now seen to have no necessary connections with the physical world beyond the vague and mystifying one implicit in the statement that thinking takes place in the brain. The discovery that this is so may be said without exaggeration to mark one of the most significant intellectual advances in the history of mankind, comparable so far as mathematics is concerned with only one other great discovery—the recognition by the Greeks that the empirical facts of geometry fall into logical patterns which can be so amalgamated that the whole subject appears as a coherent logical structure based on a limited number of axioms. Not unrelated to this profound modern insight into the nature of mathematics are certain other achievements of the twentieth century. Thus it has been possible to determine quite precisely the connections between mathematics and logic, and to define the scope of mathematics in terms so broad and so simple that they appear to provide a nearly final answer to the question "What is mathematics?" It has been possible to see how the different branches of mathematics are united into one imposing whole and to reinforce the technical bonds which tie them together within the framework of a general definition. It has been possible to win our way to a new point of vantage from which we are able to appreciate the remarkable technical advances achieved in mathematics over the last sixty years as merely the prelude to greater accomplishments in the future. When we stop to compare the mathematics of today with mathematics as it was at the close of the nineteenth century we may well be amazed to note how rapidly our mathematical knowledge has grown in quantity and in complexity, but we should also not fail to observe how closely this development has been

involved with an emphasis upon abstraction and an increasing concern with the perception and analysis of broad mathematical patterns. Indeed, upon closer examination we see that this new orientation, made possible only by the divorce of mathematics from its applications, has been the true source of its tremendous vitality and growth during the present century. We realize, too, that the trend toward abstraction must inevitably continue, reinforced by the successes which are already to be credited to it. In following this trend and directing their attention more and more to the discernment and study of abstract patterns, mathematicians have become increasingly aware of the fundamental antithesis between the structural aspect of mathematics and the strictly manipulative aspect which so often appears to have paramount importance for the applications and so often is the principal preoccupation of the mathematics teacher. A deeper understanding and appreciation of this antithesis has recently been achieved through some remarkable technical studies in modern logic which have disclosed inherent limitations upon what can be accomplished by the purely mechanical manipulation of a symbolic calculus. It is entirely obvious that these new insights and advances, which in sum constitute a genuine revolution in mathematics, pose difficult practical problems for the educator. Merely to incorporate into the mathematical curriculum the essential elements of our new mathematical knowledge is a formidable enough task, but the necessity for presenting mathematics as the abstract subject it has become and reconciling its antithetical aspects greatly increases the difficulties involved in bringing mathematical instruction up to the level demanded by our times. Consequently we need to examine in considerable detail the several factors which could be mentioned only briefly in these introductory remarks.

The abstract nature of mathematics. The philosopher Bertrand Russell has described the abstract nature of mathematics in the epigram: "Mathematics is the subject in which we do not know what we are talking about or whether what we say is true." This is his way of asserting that mathematics is abstract —"we do not know what we are talking about"—and that the notion of mathematical truth is purely formal—we do not know whether what we say is true, in any factual sense. While Russell's description underscores the independence of mathematics from the phenomenal world it fails to define the content of mathematics. A modern mathematician would prefer the positive characterization of his subject as the study of general abstract systems, each one of which is an edifice built of specified abstract elements and structured by the presence of arbitrary but unambiguously specified relations among them. He would mean by the study of such mathematical systems not only the examination of the intrinsic properties of individual systems but also the comparison of the structures of different systems. He would maintain that neither these systems nor the means provided by logic for studying their structural properties have any direct, immediate, or necessary connection with the physical world. At the same time he would recognize that such mathematical systems can often usefully serve as

models for portions of reality, thus providing the basis for a theoretical mathematical analysis of relations observed in the phenomenal world. He would also acknowledge that this kind of accidental and to a certain extent arbitrary connection between some part of reality and a certain mathematical system has often led to the discovery of abstract features of the latter which could eventually be made the subject of abstract mathematical proof.

Naturally this view of mathematics, which differs so radically from the one held by the ancient Greeks or for that matter by all mathematicians prior to the nineteenth century, was not formulated at a single stroke but has evolved little by little over a fairly long period of time. The discovery from which our current view of mathematics as a totally abstract, strictly logical, and entirely independent discipline has emerged was the discovery of non-euclidean plane geometry by the two Bolyais, Gauss, and Lobachevski early in the nineteenth century. Gauss quickly sought to decide by experiment whether the geometry of space requires a Euclidean or a Lobachevskian model, but concluded that he could not decide the question without making measurements of a greater accuracy than he could attain. Thus two internally consistent but mutually incompatible mathematical systems were available for describing physical geometry. It was impossible to imagine that either of them could have at the expense of the other any necessary connection with the physical world. It was likewise impossible, at least for the time being, to choose between them on physical grounds. The subsequent development of many other kinds of abstract geometry hastened the emergence of the modern view which we have described here. Further impetus in the same direction came also from the side of algebra, where it gradually became evident that mathematics must deal not with a single number system but rather with an infinity of such systems, sharing many common features but at the same time having their individual peculiarities. A third contributing factor was the development of new techniques and concepts in logic during the last half of the nineteenth century and the early part of the twentieth, as a result of which it became possible to clarify the relation between mathematics and logic and to justify the modern definition of mathematics which has been given here.

Mathematics and logic. Indeed, we have now reached the point where we can actually identify mathematics with logic. The central facts on which this identification is based were established in minute detail by Russell and Whitehead in their monumental treatise, "Principia Mathematica," published in 1911. This remarkable work may be regarded as the culmination of a series of brilliant nineteenth century contributions to the study of the connections between mathematics and logic. We refer particularly to the symbolic treatment of logic by Boole, Schroeder, and Peano, the creation of set theory and transfinite number theory by Cantor, and the logical analysis of the real number system by Dedekind and Frege. Just as the Greeks had to develop logic in order to explore the implications which connect the empirical facts of geometry, so the mathe-

maticians of the nineteenth century had to go still more deeply into logic in order to treat the new mathematics of their time with sufficient accuracy and precision. They were thus led to make many technical improvements and extensions in the earlier logic and to understand the scope and nature of the function concept. As a matter of fact, from a mathematical point of view it is the growing preoccupation with this concept under one or another of the many names attached to it (*e.g.*, function, operator, transformation, mapping) which is the key to certain of the trends characteristic of modern mathematics. With only a little oversimplification it may be said that what had to be added to the logic of propositions and the set theory of Aristotle and the scholastic logicians in order to make logic equal to the new mathematical demands upon it was the analysis of this concept, as worked out by the mathematicians and logicians mentioned above. Russell and Whitehead first showed how all of mathematics could be expressed in terms of a formal logic embracing the concepts of set and function. In this sense mathematics may be considered a part of logic. On the other hand, the full realization of Leibnitz's dream of reducing logic to a symbolic calculus was achieved in a succession of steps associated with the names of Boole, Peano, Russell, Whitehead and some later workers. In consequence we can now recognize that formal logic, being the study of operations upon appropriate symbols, must take its place as a special chapter in algebra and hence appears as a part of mathematics. The fusion thereby achieved between mathematics and logic reinforces the conclusion that mathematics is a completely abstract formal discipline, and raises interesting questions as to how much can be accomplished by the mechanical manipulation of symbolic systems and how much must depend upon direct or intuitive insights into their structural patterns. Indeed, it is clear that mathematics may be likened to a game—or, rather an infinite variety of games—in which the pieces and moves are intrinsically meaningless and the absorbing interest lies in perceiving and utilizing the patterns of play allowed under the rules. When mathematics is viewed in this light, the questions just noted pose the problem of determining whether or not it is possible to reduce the play of one or another of these games to a prescribed automatic procedure, leaving no room for the exercise of judgment and inspiration. Thus the distinction between the manipulative and structural aspects of mathematics acquires a sharper meaning and a greater significance by virtue of the identification of mathematics and logic.

The unity of mathematics. The characterization of mathematics as the study of systems comprising certain abstract elements and certain abstract relations prescribed among them shows very clearly the essential unity of mathematics. Nevertheless it cannot adequately suggest the intimate structural connections which have actually been found among the different branches of mathematics, as a result of modern researches. During the last fifty or sixty years much has been done to identify and compare the mathematical systems dealt with in algebra, number theory, geometry, and analysis. The outcome has been surpris-

ing in two respects. On the one hand, the possibility of analyzing the mathematical systems known to us in detail has been exploited to the point where they are all seen to be derived from systems of three simple types: algebras, ordered systems, topological spaces. On the other hand, there have been discovered numerous processes, some of them very complex, for synthesizing new systems from given systems and using them to obtain information about the latter. Furthermore, many of the familiar systems—as for example, the number systems of analysis—can be characterized by the conjunction of simple algebraic and topological or ordinal properties. In a sense, therefore, the parts of mathematics about which we know a great deal are those lying fairly close to certain very familiar landmarks; and the parts which are not yet explored may very likely be of quite a different nature from those where we now feel more or less at home.

Even though, as a result, we have no warrant for supposing that the mathematical investigations of the future will necessarily use techniques or disclose phenomena generally like those known at the present time, we may still expect the new mathematics, however diversified, to be embraced in the unity laid down by the definition offered here. Indeed, this definition is so broad and admits so high a degree of abstraction that the limits it sets to mathematics can hardly be pushed any further back. We thus have good reason to believe that eventual modifications in our basic conception of the nature of mathematics must depend upon new developments in logic—developments involving new techniques and new points of view, which will grow out of our widening experience as mathematicians.

Abstraction and application. It may seem to be a stark paradox that, just when mathematics has been brought close to the ultimate in abstractness, its applications have begun to multiply and proliferate in an extraordinary fashion. There is no doubt that one of the most exciting features of intellectual life in the twentieth century is the penetration of mathematics into an ever widening circle of scientific disciplines, not only the natural sciences but also those devoted to the study of human behavior. As this penetration gradually leads to the beginnings of mastery over situations which had previously defied theoretical treatment, we have our first clear intimations of how far and how deep the writ of mathematics is destined to run. A whole new world of thought and understanding opens out before us to which mathematics alone is the key. Far from being paradoxical, however, this conjunction of two apparently opposite trends in the development of mathematics may rightly be viewed as the sign of an essential truth about mathematics itself. For it is only to the extent that mathematics is freed from the bonds which have attached it in the past to particular aspects of reality that it can become the extremely flexible and powerful instrument we need to break paths into areas now beyond our ken. The examples which buttress this argument are already numerous, and there is no doubt that they will be reinforced by many others during the second half of this century. We may mention the modern development of an elaborate mathematical theory of genetics,

the recent creation of game theory with applications to economics and to a variety of other situations characterized by competition, and the still more recent formulation of a mathematical theory of communication with applications in engineering and linguistics. Compared with the mathematics required in the more difficult parts of theoretical physics, such as the quantum field theory, that used in some of the newer applications often seems simple and even unsophisticated. Very likely this appearance will change as more powerful mathematical techniques can be developed and the scope of the applications broadened out. Indeed it would be altogether realistic to predict that the general trend will be towards increasingly intricate mathematical theorizing in all parts of science, even in those branches where we cannot yet discern anything beyond the first tentative formulations of a few rudimentary mathematical principles.

If we are to paint an accurate picture of what is taking place in mathematics in our time, we cannot confine ourselves to these generalities, but must try also to give some account of the important technical advances which have been made during the twentieth century. No period in history has seen such intense and fruitful mathematical activity as has the first half of the present century. Moreover this activity is now increasing very sharply indeed and will certainly continue to do so. In the last three or four years the number of active mathematicians, as measured in terms of publications, seems to have doubled. While this simple statistic might easily be twisted to give a distorted view of the situation, there is other strong evidence that mathematics is experiencing a literally explosive growth. More significant than anything else is the increasing rate at which important new ideas and techniques are being introduced into mathematics. While it is a difficult task to do more than hint at the nature of the tremendous technical progress currently being made, it is one which we now propose to undertake, albeit with an inevitable sense of inadequacy.

Modern algebra. Let us first look at algebra. By an algebra, or an algebraic system, we today mean a mathematical system comprising certain abstract elements and certain specified finitary operations applicable to them. In essence, an operation is identifiable with a functional relation; it is finitary if it is a relation among only a finite number of elements. Thus the various number systems studied in elementary mathematics—the integers, the rational numbers, the real numbers, and the complex numbers—are algebras with two basic operations, addition and multiplication. Our modern conception of algebra grew out of the study of these particular systems by a process of abstraction and generalization, which has now reached its natural limits: if we proceed to consider relations which are not finitary operations, we find ourselves dealing with mathematical systems in the most general sense of the term and we lose contact with the guiding features suggested by the examples from which we started. During the twentieth century great technical strides have been made in the study of algebraic systems, so that even at the level of elementary instruction we need to revise and reorient the presentation of our algebraic knowledge. Needless to

say, courses in algebra beyond the elementary ones are already vastly different from those offered fifty or even twenty-five years ago.

Equally striking has been the extensive formation of fruitful contacts between algebra and the other branches of mathematics. Today it is easy for us to understand why algebra has such an important role to play in other parts of mathematics, but history shows that mathematicians have been very slow in understanding this relationship and in learning how to exploit it successfully. In fact, every part of mathematics obviously involves the behavior of objects of interest to it under appropriate operations—that is to say, the mathematical systems which are to be studied are, in the nature of things, linked to certain algebraic systems. When these related systems are wisely selected and analyzed in the light of general algebraic principles, important insights and information can usually be gained. In geometry, for example, the ancient Greeks were aware that they needed to study the properties of certain operations and, as a perusal of Euclid's "Elements" shows, they devoted a great deal of effort to solving certain algebraic problems in a geometrical manner. Lacking simple algebraic techniques and being unready for the abstraction required, they encountered difficulties and complications which disappeared once the role of algebra was explicitly stated by Descartes in the seventeenth century. The extent to which algebra is able to contribute to analysis was not realized until much more recently, in our own century. Again difficult problems, once they are viewed in terms of algebraic concepts, may become much more perspicuous and much easier to solve. Even formal logic, as we have already had occasion to observe, appears as a part of algebra by virtue of the fact that it deals with operations on symbols. While algebra has been very suggestive in this connection, its role in logic is not yet so important as in either geometry or analysis. Today it is unthinkable that a mathematician should pretend to a mastery of either geometry or analysis without a thorough grounding in the elements of algebra, especially group theory and linear algebra. Geometry, indeed, relies so heavily on these parts of algebra that the two subjects must be viewed as inextricably bound up together. Nowhere is this more evident than in the field of topology where such brilliant progress is now going on. It was no doubt inevitable that the close association of these two branches of mathematics should have reactions in algebra itself; and, as a matter of fact, procedures which first proved themselves valuable in combinatorial topology have now been taken over into algebra and have led to the creation of a new discipline known as homological algebra.

It is also inevitable that the increasing importance of algebra in the other parts of mathematics should be reflected in many fields where mathematics is applied. However, the contacts between algebra and applied mathematics are increasingly more direct than that, because there are so many situations in which the problems of applied mathematics have to be formulated from the start in algebraic terms. This is true, for instance, in the case of quantum field theory, as well as in those of circuit analysis, linear programming, and game theory, to cite a few of the more important examples. Consequently, it is not

only the mathematician but also the applied mathematician who today needs a good grounding in algebra, especially group theory or linear algebra or both.

A special branch of mathematics which has always had the most intimate associations with algebra and might indeed even be considered a part of it, had not methods drawn from analysis been so extensively used there, is number theory. The study of the additive, multiplicative, and other algebraic properties of the natural numbers—that is, of the finite cardinal numbers—has always had a tremendous fascination. Many of the problems of number theory can be very simply formulated in the mathematical terminology of common speech and are thus easily understood without much mathematical preparation. Among them are some of the most difficult unsolved problems in the whole of mathematics. Such problems attract the attention not only of serious mathematicians but also of amateurs and even of mere publicity seekers. Who would not like to solve Goldbach's problem: to show that every natural number is the sum of a limited number of primes (perhaps at most three)? The problems of number theory are not restricted to those concerning the natural numbers, as they have generalizations or analogues in other algebraic systems. Indeed, this fact has led historically to the development of a good many very useful algebraic concepts and techniques. In the same way the reduction of certain problems of number theory to problems of mathematical analysis has stimulated very deep work in the latter field. The success of analytical methods, while incomplete in the case of some of the most interesting and difficult problems, has nevertheless been sufficiently marked that in recent years a good deal of effort has been devoted to devising more elementary attacks. Some remarkable achievements, such as the elementary proof of the so-called prime number theorem, have been made along this line in the last few decades. Number theory is not totally without interest for applied mathematics, but it remains largely the preserve of the pure mathematician. Some of the analytical problems, and perhaps also some of the algebraic applications, associated with number theory do indeed have a certain intrinsic interest for fields of application, but this would hardly justify teaching number theory in an applied mathematics program. On the other hand, the inclusion of elementary courses on number theory in the mathematical curriculum, whether on technical or cultural grounds, certainly needs no defense.

Developments in geometry. In our brief discussion of algebra, we have already noted how deeply geometry has been penetrated by algebraic concepts and techniques. There are some parts of geometry which have been completely taken over by algebra. For example the study of the sets defined by algebraic equations, originally taken up in relation to the real and complex number systems as a part of higher analytic geometry, is today purged of all analytical tendencies and is carried on by purely algebraic methods, applicable to much more general algebraic systems than the two classical number systems. Thus algebraic geometry is quite literally a part of algebra. Similarly, combinatorial topology, though basically concerned with continuity considerations, lent itself

readily to algebraic treatment and as a result has been nearly completely swallowed up by algebra—though not without influencing algebra in the process, as we have already noted. In spite of these examples, the fact that geometry is concerned with continuity properties not very amenable to discussion in an algebraic spirit has protected that ancient branch of mathematics against complete domination by modern algebra. Consequently some parts of geometry remain largely untouched by algebraic techniques. Since these parts are those where continuity considerations are prominent, and the outstanding problems often have an analytical aspect, there has been a tendency for them to gravitate towards analysis. Both differential geometry and general (or set-theoretical) topology illustrate this tendency. In differential geometry, of course, the ties with classical analysis are extremely close, most of the important problems leading directly to problems in the theory of differential equations. Geometry therefore has been strongly pulled in two apparently opposite directions and has seemed at times to be under the threat of being torn asunder. More recently, however, a closer study of the fundamental concepts of differential geometry has begun to produce a new synthesis of the algebraic and analytic points of view, made somewhat easier by the fact that the roles of algebra and topology in analysis have come to be better understood and appreciated. For some time now mathematicians have been groping for a satisfactory way of effecting such a synthesis, and there are now rather plentiful indications that their search has been at least moderately successful. In any case, there is no doubt that difficult problems involving algebraic, analytic, and topological features can now be clearly formulated and neatly solved. The prospects for a brilliant development of these complex aspects of geometry now seem to be assured. This circumstance in itself creates a difficult problem in mathematical instruction because we must evidently provide suitable introductory material to enable future mathematicians to make or at least to understand progress in these promising directions—and we do not yet quite know how to do this.

Mathematical analysis. Since mathematical analysis, like geometry, has to deal with situations in which continuity considerations and limit processes play a dominant role, there has been a double impetus toward the abstract treatment of such processes. Thus there has emerged the concept of a topological space as a mathematical system consisting of abstract elements called points and a single limit relation among them, specified in any one of several essentially equivalent ways. Historically the introduction of this concept was associated with the calculus of variations and thus had an analytical rather than a geometrical background. The subsequent development of general topology—that is, of the theory of topological spaces—and of the related parts of analysis known as functional analysis or abstract analysis has been a particularly characteristic and important achievement of twentieth century mathematics. While it has thrown much light on the foundations of both geometry and analysis, we must today regard general topology as a part of geometry rather than of analysis

because its major problems seem to have greater significance for geometry. As a matter of fact, the mathematical systems which the analyst finds genuinely interesting have linked algebraic and topological properties and cannot be adequately discussed in terms of general topology alone. Indeed, it would be possible to define modern abstract analysis as the study of topological algebras—that is to say, of mathematical systems which are simultaneously topological spaces and algebras with operations continuous relative to the underlying limit processes. A great deal of classical analysis has recently been brought into close relation with the theory of topological algebras, and many classical problems have thereby been shown in a new light and brought nearer to solution. Thus the study of differential equations has been formulated in terms of topological linear spaces (or vector algebras), and harmonic analysis brought under the theory of commutative topological groups. Many of the noteworthy advances being made currently in the theory of partial differential equations are actually based on the application of methods drawn from the theory of topological linear spaces. In the other direction, it has been possible to show that certain topological algebras are describable in terms of classical analysis. The most spectacular result of this kind was the recent solution of Hilbert's fifth problem—to express the operations of a topological group in terms of analytic functions. Although analysis has thus been very strongly influenced by the abstract tendencies of modern mathematics, it retains a very lively interest in a variety of special problems, special functions, and special functional equations, particularly some of those related to applications. As a result this branch of mathematics must be taught with particular care so as to give due emphasis to both its traditional and its modern aspects.

Computation. In connection with the applications of analysis great importance has always attached to the execution of extensive computations without which the passage from theory to practice would be impossible. As the other branches of mathematics begin to play a bigger part in the applications, the scope of computation needs to be correspondingly enlarged. Under this pressure for more powerful and efficient computing instruments, extraordinary progress has been made since the mid-thirties in developing machines of astounding speed, capacity, and versatility. The result has been a revolution in the art of computation and a complete alteration of the relations between mathematics and that art. Computations which a few years ago would have been considered quite beyond the limits set by practical considerations can now be carried out in a few days or weeks, if not hours. There is no finite sequence of mathematical or symbolic manipulations which cannot be effected, at least in theory, by a general-purpose computer. The practical limitations upon the length and complexity of the sequences which can be handled by today's machines will almost certainly be pushed back as the electronic and other physical techniques applicable to computer design are brought to perfection. With the introduction of these powerful aids, it is no longer the mathematician's task to strive for a

reduction of his solutions to a form readily computable by hand or with the aid of the simplest calculating machines; it is his role to translate them into programs for execution by high-speed electronic computers. In consequence, there is being created a new specialized branch of mathematics devoted to the theory and practice of setting up computer programs. It requires a knowledge of the mathematical techniques involved in computation and of their logical combination, and has obvious contacts with formal logic. Indeed, one can see—after the event—that the entire development of modern machine methods in mathematical computation has been made possible by two basic advances—Russell and Whitehead's discovery of the reduction of mathematics to formal logic and the invention of the vacuum tube. Accordingly, the emerging ties between the computing art and logic are in no sense accidental and will undoubtedly become closer and more intricate with the passage of time.

The versatility of today's computing machines not only makes possible mathematical and logical analyses which one would hardly have dared to imagine a few years ago, but has also focused our attention sharply upon that antithesis between manipulation and pattern-perception to which we have already referred. As we realize how much can be done by machines, we have to ask whether it would be possible to design one which could do everything done by the brain. This amounts, on the theoretical side, to asking whether the brain itself is a machine. Another way of putting the question is to ask whether all of mathematics and logic can be reduced to the execution of a routine program of symbol-manipulation, eliminating the need for the direct discovery and utilization of patterns. Although, as will presently be seen, the modern study of logic suggests that our instinctive negative reply to this question may be correct, it is sobering to realize how much can be accomplished by purely mechanical procedures. Already there are machines which can state and prove simple mathematical theorems or outplay a human opponent at a simple game like checkers, even learning from experience to avoid the repetition of past mistakes. It is no wonder then that one of the surprising uses of the new computers is to assist the sociologist in analyzing human behavior, by simulating the reactions of individuals to one another in accordance with the hypothetical principles which are to be tested. Here, as in many other applications of the new machines, the capacity for carrying out at high speed an incredibly large number of operations is essential, because only thus can the consequences of varying a relatively large number of variables and parameters be explored. This is the reason why the modern computers are especially useful whenever the number of variables and parameters entering into a problem is too large for the application of simple formulae and not large enough for a resort to asymptotic estimates. For instance, the practical solution of many problems in mathematical economics or linear programming would be impossible without the help of modern computing machines, not because the mathematics employed is difficult or complicated but because the variables involved are usually so numerous.

Modern developments in logic. In view of the foregoing remarks upon the relation of computing theory to logic, we shall not be making an abrupt change of subject if we turn at this point to examine some modern developments in logic. Having already dwelt upon the fundamental contribution of Russell and Whitehead, we must now consider a line of inquiry laid down by Hilbert in 1900, when he included in his eventually celebrated list of important problems that of demonstrating the consistency and completeness of at least one substantial portion of mathematics. Thirty years later Gödel shocked the mathematical world by showing in a brilliant paper that if a system comprising the so-called first order functional calculus of logic and the axioms of arithmetic is consistent then it must be incomplete in the sense that there is a proposition of arithmetic which can be formulated within it but which can neither be proved nor disproved in it. A related and equally intriguing phenonenom was discovered by Church and Turing who showed that it is impossible to prescribe a systematic procedure capable of deciding whether each particular proposition in this system is provable or not. In order to express this result in a particularly precise and vivid way, Turing chose to describe a certain imaginary machine and to demonstrate its limitations. He was thus among the first to point out the connection between a fundamental problem of logic and the theory of computing machines. It is, in fact, this result of Church and Turing which suggests that logic and mathematics cannot be reduced entirely to a routine program of mechanical manipulations with symbols, that there may be some things possible for the human mind which any particular machine cannot do. Thus the antithesis we have emphasized between manipulating symbols and perceiving patterns may perhaps be related to the logical phenomenon discovered by Church and Turing, and may be genuinely rooted in the distinction between mind and the machine.

Probability and statistics. Just as logic arose from the study of deductive reasoning, so the theory of probability and statistics originated in the attempt to formalize the processes of inductive inference. The theory dates from the sixteenth century, when the first mathematical analyses of games of chance were made, and it was quite highly developed by the end of the nineteenth. Successful applications had by then been made to the theory of errors of measurement, the theory of gases, and a good many biological and actuarial situations. The twentieth century has witnessed extensive technical developments in probability theory and statistics and the rapid multiplication of their applications. As purely mathematical theories they can now be identified as a part of measure theory or, equivalently, of integration theory. On the other hand, as branches of applied mathematics they have tremendously increased importance and scope. Full understanding of their epistemological significance was not gained until the present century, but we can now recognize their unique and fundamental role as tools for knowing the phenomenal world. Increasingly the

models employed in our attempts to analyze man and nature are statistical models. Even the deterministic models, such as those which have survived from classical physics, are admittedly asymptotic simplifications of more realistic statistical models. In consequence the student of any of the sciences, physical, biological or behavioral, now needs a good grasp of the elements of probability theory and statistics. Thus there is posed the problem of how and in what form modern courses in probability theory and statistics should be introduced into the mathematical curriculum. There is increasing support for putting the first introductory treatment at a very early stage so that our students may become acquainted as soon as possible with the elementary principles of statistical reasoning, now so prominent in a wide range of human thought.

No survey of the recent history of mathematics so brief and so topical as the one we have attempted here can possibly give an adequate picture of the scientific development of modern mathematics. It may, however, serve to persuade the general reader of the truly revolutionary and intellectually significant character of that development and to give him a useful background for the discussion of some of its educational implications. By reviewing, even if much too summarily, the important trends in modern mathematics and some of the advances which are in progress in its various branches—algebra, number theory, geometry, analysis, computation, logic, and probability theory and statistics—we have perhaps been able to suggest the spirit and the substance of the extensive changes which have become necessary by now in our teaching of mathematics.

The central problem of mathematics in education. As we have put it already, mathematics does not seek an essentially new place in the curriculum. It has always held a place of honor and importance there, both as a humanistic study and as a useful scientific discipline. What it now seeks is to adjust its place in education in accordance with its growing importance in the intellectual life of our times. In such an adjustment the central problem is to define a new core of mathematical knowledge embodying the leading ideas and the principal techniques of modern mathematics and to organize the teaching of this core as a well-articulated program, taking advantage of whatever light modern psychology can throw on intellectual development, concept-formation, and the learning process. Subordinate to this problem, but nonetheless of first importance, is the problem of grouping as satellites around this mathematical core a number of more specialized bodies of fundamental knowledge, each basic to some special purpose such as preparation for teaching, for the pursuit of higher mathematics, or for work in some field of applied mathematics. Another difficult and challenging problem is that of coordinating the teaching of pure mathematics, as represented by the core and its various associated bodies of more specialized mathematics, with the teaching of the widely different fields in which mathematics is being applied. This problem, already a critical one so far as physics is concerned, seems certain to command a great deal of attention

during the coming decade. It is made all the more pressing because of the abstract tendencies apparent in present-day mathematics and its involvement with the ever more sharply defined antithesis between the manipulative and the structural aspects of mathematics.

To open the discussion of the central problem, let me offer three poems—all scientific poems—which together occupy but a single line:

$$E = mc^2, \qquad E = h\nu, \qquad \alpha(Aa) = (\alpha A)a.$$

Like Chinese poems they must be taken in by the eye as well as by the ear. The first will be recognized at once as the famous equation between energy and mass, epitomizing a whole revolution in physics; it was discovered by Einstein early in this century as an essential principle of the theory of relativity. The second likewise states a revolutionary principle of modern physics, proposed by Planck shortly after the turn of the century and made the cornerstone of the quantum theory; it expresses a relation between energy and the frequency of vibration of a wave of light. An understanding of these two poems is impossible without a reasonably good grasp of the core of modern physics. Indeed, if one were to set out with the single aim of teaching their meaning and their implications, he would end by giving his students a very satisfactory introductory course in physics. In the same way, the last of these three poems is an epitome of a large part of what should be taught as an introduction to modern mathematics. Even for a mathematician this statement might remain obscure, unless he were given a hint to interpret the equation as symbolizing Stokes's theorem; and he would then understand at once just how much is needed from algebra, geometry, topology, and analysis before the significance of this equation can be fully grasped. As it stands, the equation merely expresses a kind of abstract associative law, analogous to the one which is verified in case the three symbols α, A, and a designate cardinal numbers. It is when we give altogether different interpretations to these same symbols that our equation becomes an expression of Stokes's theorem. Specifically, we should take a as designating a topological chain with numerical coefficients and boundary chain Aa (usually denoted as ∂a), α as designating an exterior differential form with differential αA (usually denoted as $d\alpha$), and βb as designating the integral of the exterior differential form β over the chain b (with respect to the appropriate geometric measure on the latter). The symbol A must then be thought of as designating a double operator which sends a into Aa and α into αA.

Whether we actually start in this or in some other way to define the core of the mathematical curriculum is of no great importance. Our goal is the same in any case: to list those topics from the various branches of mathematics which we deem significant enough and central enough to be taught to everyone seriously interested in gaining a mastery of the essential elements and techniques of modern mathematics. If we were to examine the current discussions of what these topics might be, we would find a good deal of evidence for believing that agreement might be reached on a list made somewhat as follows:

from algebra—the topics of linear algebra, group theory (with an elaboration of some of its ramifications in the theory of rings and fields), and elementary number theory; from geometry—the treatment of euclidean, analytic (properly, algebraic), projective, and differential geometry together with some kind of introduction to elementary topology; from analysis—the calculus, an introduction to the theory of differential equations, the elements of the theory of analytic functions of one complex variable, and an introduction to functional analysis; from the field of computation—elements of the theory and practice of numerical analysis, the central role of approximate numerical solution of problems in linear algebra; from logic—the elements of set theory including cardinal and ordinal numbers, the algebra of logic (Boolean algebra) and an introduction to symbolic (mathematical) logic; from probability theory and, statistics—the fundamental principles of inductive or statistical reasoning, probability distributions and their relation to measure theory, the principal statistics and their applications, and an introduction to the study of stochastic processes. The arrangement of such topics in a suitably articulated pattern of courses would require a detailed examination of their interrelations and could not be satisfactorily accomplished without a good deal of study and debate. It would be against the spirit of modern mathematics to obscure or conceal the fundamental unity of the subject by too much compartmentalization of the listed topics in some kind of logical scheme. The aim should be to bring out that unity by showing how the main branches of mathematics touch and penetrate one another even in their most elementary aspects. Naturally, many topics mentioned here have different degrees of importance for different classes of students—those interested in general knowledge, the future teachers, the future professional mathematicians, and the future specialized scientists. Accordingly some of the courses offered as belonging to the central core of the curriculum should not be uniformly stressed under all circumstances or for all students. The special interests of special groups of students must be catered to in special courses giving adequate introductions to the various fields of specialized mathematical knowledge which cluster around the core we have tried to describe. This is hardly the place to analyze more closely either the core or these associated special fields, and we shall not attempt to do so.

Problems for the college educator. Instead we may now turn to a brief consideration of a few specific problems which the college educator seems certain to face in the rather immediate future. We shall touch only upon such problems as arise in one way or another from the progress of mathematics, strictly excluding all those which have other origins. From what has preceded it is evident that among the most important of these problems are the following: first, to decide what part of the central core should be taught by the college and how that part shall be elaborated into a suitable course program; next, to decide what additional specialized courses should be offered by a particular college mathematics department; and finally to bring about adequate coordination be-

tween the instruction offered by the college mathematics department and the courses involving mathematics but falling under the jurisdiction of other departments.

In spite of the fact that mathematicians concerned with the general development and major trends in mathematics do have a reasonably good idea of how the central core is constituted at the present moment, that idea is not at all adequately expressed in the programs in mathematics currently offered by schools, colleges, and graduate departments. In recent years, however, the need for a general reform leading to the modernization of the mathematics program at all three levels has been widely felt, and active steps have been taken to meet this need. While our colleges have not neglected this problem, it seems clear that more substantial progress has been made towards solving it at the high school and graduate school levels. So far as the teaching of mathematics is concerned, our colleges therefore find themselves under the influence of the changes contemplated or already initiated both above and below. The practical effect of these changes is to shift a good deal of mathematical material downward to a somewhat earlier point in the student's experience. In fact, the graduate school presses on the college to offer somewhat more advanced courses, including some now generally offered at the graduate level, so that their students may begin graduate study better prepared and arrive more quickly at the more advanced points of our present mathematical knowledge. On the other hand, the high school is starting to eye some of the subjects traditionally taught in college as possible material for courses in the junior and senior secondary school years. For example, some high schools have been teaching the elements of the calculus for many years, and there are many indications that before ten years have passed a majority of our larger high schools will do likewise. Thus the college educator is faced with the problem of up-grading the program of his mathematics department and staffing the department with more highly qualified teachers. Assuming for the sake of our discussion that it will not be long before the standard first year college course in mathematics will be a thorough and rigorous treatment of the integral and differential calculus taught to students who already have had a more or less intuitive introduction in high school, we can reasonably expect that a great many of our larger colleges will be offering to juniors and seniors good introductory courses in the theory of analytic functions of a complex variable, modern differential geometry, and functional analysis, of the kind now given in graduate schools. If the upper and lower limits of the college mathematics curriculum should be fixed in some such manner, then the main problem could be narrowed down to that of elaborating the essential courses which are to be taught in this range. Here a great deal needs to be done; so much, indeed, that a concerted attack may very well be called for along the lines suggested by what is being done in the field of high school mathematics. It may be mentioned in particular that by and large the problem of teaching enough modern algebra at the college level remains open—and demands careful study by college mathematics teachers. Even the complete suc-

cess of the current efforts to modernize the teaching of algebra in the high schools will not mean that very much modern algebra will be taught there, for the simple reason that the main part of high school algebra as it is conceived of at the present moment is hardly algebra in the modern sense, being concerned mainly with the elementary analysis of real or complex polynomial functions. The determination of what should be taught at the college level in geometry is another problem which is far from being satisfactorily solved, as we have already mentioned in our earlier remarks. Indeed, we must expect that our ideas about this problem will shift a good deal over the next decade, in response to the progress being made by research in the field.

Clearly, the college can hardly undertake a richer and more advanced program in mathematics unless it is willing to eliminate a considerable amount of deadwood from the traditional college curriculum. There are a good many obvious candidates for elimination, subjects which should either be taught in high school or not be taught at all. Let us cite college algebra, solid geometry, most of numerical trigonometry, descriptive geometry, and some topics in the calculus. Since a great many—perhaps a majority—of our colleges are still teaching these subjects, the work of reform has to begin by banishing them from the college curriculum. Only when this has been accomplished can the college mathematics program be given its proper scope and brought to the proper level of quality.

So far as the courses offered as a specialized adjunct to the core program are concerned, it must be expected that there will be great variation from one college to another. The professional interests of the student body and the professional qualifications of the teaching staff will both have an influence on this offering. It is easy to mention a number of special subjects which could quite appropriately be taught for college undergraduates, such as numerical analysis and computational techniques, mechanics, introductory mathematical physics, logic, game theory, linear programming, and so on. Few colleges can afford to offer a complete range of the possible choices, and there is certainly no reason why any college should even attempt to do so. On the other hand, the interests of the students certainly demand that some courses of this nature be made available, either by the mathematics department or by some other departments in which applications of mathematics are discussed. Thus the college educator who is mindful of the modern developments in mathematics and of the proliferation of its applications will actively encourage his faculty to study what can be done in terms of the combined resources of the appropriate departments in order to provide adequate instruction in some of the more specialized parts of mathematics.

In any case, this problem is closely allied to the problem of coordination which, as we have already pointed out, can hardly be avoided by the college educator of today. Something more than encouragement may be required to bring about an adequate contact between the departments concerned. Certainly our experience suggests that close and continued cooperation between

two departments is difficult to maintain without some outside compulsion or supervision. At the present time, there are some fields where the circumstances seem unusually favorable to initiating a serious cooperative effort to correlate the curricula and teaching programs of certain departments. This is undoubtedly the case so far as mathematics and physics are concerned. If these opportunities are to be seized upon and exploited, the college president must therefore not hesitate to exert strong leadership, it seems to me. It is quite evident that any success he may be able to achieve in making coordination a reality will mark a real educational advance for his institution. The nature of the problem of coordination is not hard to define and the means to its solution lie close at hand. The greatest obstacle to achieving a solution is the human difficulty of maintaining cooperation in a situation which manifestly demands it despite marked differences in points of view. Some attempt to reconcile these differences should therefore be made as an initial step in establishing cooperation between the mathematics department and any other department which may happen to be concerned. Instead of going directly to a consideration of the details of how specific courses should be interlocked, it would be better to undertake first a joint examination of the general state of the two subjects and of the points of view held concerning them by the members of the two departments. In most cases the basic difference of opinion is very likely to rest upon value-judgments about the antithesis between the concrete and the abstract or the related mathematical antithesis between manipulation and pattern-discernment. Since abstraction and the discernment of patterns are demonstrably playing more important roles in the scientific study of nature—as is very strikingly plain in the case of quantum field theory—the mathematician's desire to emphasize the abstract and the structural aspects of mathematics both because of their intrinsic interest and because of the firmer grasp they give of concrete situations and manipulative techniques, may today begin to be more sympathetically received than was formerly the case. On the other hand, it is important that the mathematician should not neglect the manipulative aspects of his subject nor insist that his own concern for generality and precision should at all times and under all circumstances be shared by those who engage in making applications of mathematics. Granted that some mutual understanding can be reached on these points, the actual coordination of the teaching of mathematics with that of other subjects can be undertaken in a favorable atmosphere. The mathematics department should do its best to offer the right topics at the right time, with some orientation toward the applications which may be envisaged; and, on the other hand, the other department concerned should endeavor to use and use correctly the preparatory materials furnished by the mathematics department and should assume some obligation to give its students further practical training in the mathematical principles and techniques involved. Obviously, such coordination as we have thus described demands much hard and patient work for its realization. It cannot be achieved overnight or without the maximum of good will and of dedication to what should be one of the most important

educational purposes of our time—the purpose of bringing out the essential unity of human thought.

Summary. To summarize, then, we have seen how mathematics has undergone revolutionary changes and an explosive development during the twentieth century; we have seen how it has simultaneously multiplied its contacts with other fields of investigation, assuming in many of them a key role and in others at least an indispensable auxiliary one; and we have identified many serious educational problems which stem from the resulting urgent need for modernizing our teaching of mathematics. These problems are particularly acute at the college level and constitute a great contemporary challenge to the wisdom and the energy of our college faculties and college presidents.

CONVERGENCE REGIONS FOR CONTINUED FRACTIONS AND OTHER INFINITE PROCESSES*

W. J. THRON, University of Colorado

1. Introduction. A continued fraction is a sequence $\{A_n/B_n\}$ formed from two given sequences of complex numbers $\{a_n\}$, $\{b_n\}$ by the rule

$$\frac{A_n}{B_n}=\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cdots+\cfrac{a_n}{b_n}}}}\,.$$

To simplify notation one writes this in one line with depressed plus signs as follows

$$\frac{A_n}{B_n}=\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+\cdots+\frac{a_n}{b_n}\,.$$

The abbreviation

$$\frac{A_n}{B_n}=\mathop{\mathrm{K}}_{k=1}^{n}(a_k/b_k)$$

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is also used. The sequence $\{A_n/B_n\}$ as a whole is also denoted by

$$\mathop{\mathrm{K}}_{n=1}^{\infty} (a_n/b_n) \quad \text{or} \quad \frac{a_1}{b_1} + \frac{b_1}{b_2} + \cdots .$$

The fraction A_n/B_n is called the nth *approximant* of the continued fraction. The quantities A_n and B_n, which so far are determined only up to multiplicative constants, are called the nth partial numerator and denominator, respectively. By suitable choice of the multiplicative constants one can arrange it so that the A_n and B_n satisfy the following system of difference equations

$$\text{(1.1)} \qquad \text{(a) } A_n = b_nA_{n-1} + a_nA_{n-2}, \qquad B_n = b_nB_{n-1} + a_nB_{n-2},$$

with the initial conditions

$$\text{(1.1)} \qquad \text{(b) } A_0 = 0, \qquad A_1 = a_1, \qquad B_0 = 1, \qquad B_1 = b_1.$$

Until recently the study of continued fractions has been based almost exclusively on this system of difference equations or on a system of an infinite number of linear equations in an infinite number of unknowns which can be associated with a continued fraction.

However there is another way of looking at continued fractions, which is very suitable for deriving certain types of convergence criteria. One arrives at it in the following way: Let a sequence $\{t_n(z)\}$ of linear fractional transformations be given, where

$$t_n(z) = \frac{a_n}{b_n + z} \cdot$$

Now form from it a new sequence $\{T_n(z)\}$, defined inductively as follows:

$$\text{(1.2)} \qquad T_1(z) = t_1(z); \qquad T_n(z) = T_{n-1}(t_n(z)), \qquad n > 1.$$

Then one sees immediately that $T_n(0) = A_n/B_n$, so that a continued fraction can also be considered as a sequence $\{T_n(0)\}$.

That this was a possible interpretation of continued fractions, has been known for some time. References to it can be found in Isenkrahe [11] and Netto [15]. In 1917 Schur [20], in his famous paper on power series bounded in the interior of the unit circle, uses a sequence $\{T_n(z)\}$ built up according to (1.2) from general linear fractional transformations

$$t_n(z) = \frac{\alpha_n z + \beta_n}{\gamma_n z + \delta_n}$$

and calls it a "continued-fraction-like algorithm." Hamel, a year later, in two articles ([6], [7]) actually employs this approach in solving certain problems in continued fraction theory. It is not until the early 1940's that we find essential use made of this approach again in two papers by Scott and Wall [23] and by Paydon and Wall [16]. Since then the principle has been widely applied in the

study of the analytic theory of continued fractions.

The point of view which considers a continued fraction as being generated by generalized iteration (we have ordinary iteration only if all $t_n(z)$ are equal to each other) is not used to the exclusion of the difference equations approach, so (1.1) will still be used and we shall now establish them taking the "iteration" definition of continued fractions as a starting point.

Since the $t_n(z)$ are all linear fractional transformations, it is clear that their product $T_n(z)$ must also be a linear fractional transformation. However, we can prove the even more specific result that

$$T_n(z) = \frac{A_{n-1}z + A_n}{B_{n-1}z + B_n}, \tag{1.3}$$

where the A_n and B_n are exactly the quantities defined in (1.1). We proceed by induction. For $n=1$ we have

$$T_1(z) = \frac{A_0z + A_1}{B_0z + B_1} = t_1(z) = \frac{0z + a_1}{z + b_1}\cdot$$

Now assume the assertion has been proved for $n=k$. Then

$$\begin{aligned} T_{k+1}(z) = T_k(t_{k+1}(z)) &= \frac{A_{k-1}a_{k+1} + A_k(z + b_{k+1})}{B_{k-1}a_{k+1} + B_k(z + b_{k+1})} \\ &= \frac{A_kz + (A_kb_{k+1} + A_{k-1}a_{k+1})}{B_kz + (B_kb_{k+1} + B_{k-1}a_{k+1})}, \end{aligned}$$

and the result follows. Formula (1.3) also provides an easy proof for the formula

$$T_n(0) = \frac{A_n}{B_n} = \frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n},$$

whose validity we had already asserted. A very useful consequence of (1.1) is

$$A_nB_{n-1} - B_nA_{n-1} = (-1)^{n+1}\prod_{\nu=1}^{n} a_\nu, \tag{1.4}$$

which one establishes readily by induction.

Other infinite processes can also be generated by generalized iteration. Instead of starting with a special type of linear fractional transformation, as we did for the continued fractions, we now begin with an arbitrary complex valued *generating function*

$$f(a_n^{(1)}, \cdots, a_n^{(k)}, z)$$

of $k+1$ complex variables and call it $t_n(z)$. The definition of $T_n(z)$ is then the same as in (1.2). Through suitable choice of $z=c$, many well-known infinite processes can be obtained. Some of these are listed in the following table:

f	c	$\{T_n(c)\}$
$z + a_n$	0	$\sum_{n=1}^{\infty} a_n$
$a_n z$	1	$\prod_{n=1}^{\infty} a_n$
$\sqrt{z + a_n}$	0	$\sqrt{a_1 + \sqrt{a_2 + \cdots}}$
$e^{a_n z}$	1	$b_1^{b_2^{\cdot^{\cdot^{\cdot}}}}$ $(b_n = e^{a_n})$
$\dfrac{a_n}{1 + z}$	0	$\underset{n=1}{\overset{\infty}{\mathrm{K}}} (a_n/1)$
$\dfrac{1}{b_n + z}$	0	$\underset{n=1}{\overset{\infty}{\mathrm{K}}} (1/b_n)$

The infinite processes given are, in order: infinite series, infinite products, infinite radicals, infinite exponentials, two different kinds of continued fractions.

In this article we are concerned mainly with describing methods, based on the iterative definition of infinite processes, for obtaining convergence-region criteria for certain of these infinite processes. The methods have been successfully applied to all except the first two types enumerated in our table. That no nontrivial convergence regions exist for infinite series and infinite products will become clear after we have defined what a convergence region is.

We shall now restrict ourselves to generating functions involving only one parameter, that is functions of the form $f(a_n, z)$. Then a set of regions $C_1, \cdots, C_k$ will be called a set of *convergence regions* for the infinite process $\{T_n(c)\}$ if the conditions

$$a_{kn+i} \in C_i \qquad \text{for all } i = 1, \cdots, k \text{ and all } n \geqq 0$$

insures the convergence of the sequence $\{T_n(c)\}$.

Of particular interest have been the cases $k = 1$, 2 which are called *simple* and *twin* convergence regions, respectively. A remark about the use of the term region may be in order. We could have talked here of convergence sets and later of element sets and value sets, but since the sets that have proved useful in this

connection have all been open connected point sets including part or all of their boundary, that is, regions, we use this more restricted term.

It should now be clear that the only convergence sets for infinite series and products are the sets consisting of the point 0 or 1, respectively.

To avoid confusion it may be well to point out that the term convergence region as used here has nothing in common with such statements as "the region of convergence of a power series is a circular disk" or "the region of convergence of a Dirichlet series is a half-plane." There one deals with one sequence of functions and "region of convergence" refers to the variable z. Here we are concerned with a large family of sequences of constants and "convergence region" refers to sets in which the elements a_n are to lie.

Before proceeding we might call attention to a generalization of decimal and regular continued fraction expansions of real numbers x of the form

$$x = a_0 + \phi(a_1 + \phi(a_2 + \cdots) \cdots),$$

where ϕ is a fairly general montone function. These algorithms were introduced by Kakeya [**12**], and their ergodic properties were investigated by Bissinger [**2**], Everett [**4**], and Renyi [**19**]. In his article [**9**] on infinite radicals, Herschfeld also suggested the following generalized radical $\{u_n\}$

$$u_n = (a_1 + (a_2 + \cdots (a_n)^{\lambda_n} \cdots)^{\lambda_2})^{\lambda_1}$$

without investigating it. Hoffman [**10**] and Isenkrahe [**11**] had already used this iterative process with $a_n = a$, $\lambda_n = \lambda$, to obtain approximate solutions of trinomial equations $x^n + \alpha x + \beta = 0$.

2. Element regions and value regions. For all $n \geq 1$ let the values of the parameter a_n in the generating function f lie in some region E. Then it would be useful to know regions Z such that $T_n(c) \in Z$ for all $n \geqq 1$. Such a region Z is called a *value region* corresponding to the element region E. Some value regions corresponding to given element regions have been known for some time. Thus the Pringsheim convergence criterion ([**18**], p. 58) also gives information about the location of the approximants of the continued fraction. In 1941 Scott and Wall [**23**] devote an article to the study of the relation between element regions and value regions, and determine the value region corresponding to the original parabolic convergence region (see our Theorem 3.2). Paydon and Wall [**16**] in their derivation make essential use of value regions in arriving at their results. The same is true in the proof of the same result given by Leighton and Thron [**14**]. However, these authors recognized that, while it is in general exceedingly difficult to go from a given element region to a corresponding value region, the opposite approach is much less complicated. This consists in starting with a value region Z and then constructing for it an element region $E(Z)$ such that $a_n \in E(Z)$ for all $n \geqq 1$ insures that $T_n(c) \in Z$ for all $n \geqq 1$ (provided $c \in Z$). We shall now show that in many cases one can give an explicit formula for E in terms of the generating function f and the region Z.

The only restriction to be imposed on $w=f(a_n, z)$ is that it have a unique inverse function $a_n=g(w, z)$ for z, w in a suitable region G. That is, we require that f and g satisfy the identity

$$w = f(g(w, z), z). \tag{2.1}$$

For those generating functions f which have a unique inverse and for a value region $Z \subset G$ the following definition then is meaningful.

DEFINITION 2.1. $E(Z)=\bigcap_{z\in Z} g(Z, z)$.

A word of explanation may be in order. If U is a set in the complex plane and $h(z)$ is a complex-valued function of a complex variable then $h(U)$ is understood to be the set of all complex numbers $h(z)$, $z\in U$. $E(Z)$ is thus the intersection of all sets $g(Z, z)$ as z varies over Z.

We now prove that Z is a value region corresponding to the element region $E(Z)$. This is done in two steps. The first is the following theorem.

THEOREM 2.1. *If $a_n \in E(Z)$ then $t_n(Z)\subset Z$.*

Proof. Let z' be an arbitrary element of Z and let a_n be the parameter value in $t_n(z)=f(a_n, z)$. Then there exists a $w'\in Z$ such that $a_n=g(w', z')$. This follows from the fact that $a_n\in E(Z)$ and hence in particular $a_n\in g(Z, z')$. We thus obtain

$$t_n(z') = f(a_n, z') = f(g(w', z'), z') = w' \in Z,$$

and the theorem is proved.

THEOREM 2.2. *If $a_n\in E(Z)$ for all $n\geqq 1$ and if $c\in Z$, then $T_n(c)\in Z$ for all $n\geqq 1$.*

Proof. Repeated application of Theorem 2.1 yields $T_n(Z)\subset Z$; hence the theorem follows from the fact that $c\in Z$.

It should be noted that certain choices of Z, for some generating functions f, lead to empty sets $E(Z)$. Nor is there in general any assurance that Z will be the smallest value region corresponding to the element region $E(Z)$. The explanation for this is that for most regions Z there does not exist a region E such that every value in Z has a representation of the form $T_n(c)$, where the parameters $a_1, \cdots, a_n$ all lie in E. A region Z satisfying these conditions we shall call a *proper value region.*

It thus is of some importance to have some criteria for proper value regions. For continued fractions some results along this line have been obtained [**25**]. For continued fractions of the form $K(a_n/1)$ a necessary condition for a region Z to be a proper value region and for $E(Z)$ to be a convergence region is that Z and $-1-Z$ have no points in common. This suggests choosing for Z half-planes with the point $-\frac{1}{2}$ on the boundary and containing $z=0$ in their interior. (The corresponding $E(Z)$ are the parabolas of Theorem 3.2). The simplest of these regions is the half-plane defined by $\Re(z)\geqq -\frac{1}{2}$. For it we shall now carry through the derivation of $E(Z)$.

THEOREM 2.3. *For continued fractions $K(a_n/1)$ and for Z defined by $z \in Z$ if and only if $\mathcal{R}(z) \geqq -\frac{1}{2}$, $E(Z)$ is the parabolic region whose elements a satisfy the inequality*

$$|a| \leqq \tfrac{1}{2}(1 - \cos \arg a)^{-1}$$

or the equivalent one $(\mathcal{I}(a))^2 \leqq \mathcal{R}(a) + \frac{1}{4}$.

Proof. Since $f(a_n, z) = a_n/(z+1)$, $g(w, z)$ is uniquely defined and given by $g(w, z) = (1+z)w$. Thus

$$E(Z) = \bigcap_{z \in Z} (1 + z)Z.$$

This is the intersection of certain half-planes which are obtained from the half-plane Z by multiplication by $1+z$, which causes a rotation by an angle equal to $\arg(1+z)$ and a stretching or shrinking by an amount equal to $|1+z|$. Instead of letting z in $1+z$ range over all of Z, it is clearly sufficient to consider only the intersection of those half-planes $(1+z)Z$ for which z lies on the boundary of Z. The boundary of the region $E(Z)$, in polar coordinates, is then given by

$$\begin{aligned} r(\theta) &= \min_{\phi + \psi = \theta} \tfrac{1}{2} \sec \phi \cdot \tfrac{1}{2} \sec (\psi - \pi) \\ &= \tfrac{1}{4}\Big\{ \max_{|\phi| < \frac{1}{2}\pi} \cos \phi \cos (\theta - \phi - \pi) \Big\}^{-1}. \end{aligned}$$

An argument from elementary calculus then shows that the maximum of $\cos \phi \cos (\theta - \phi - \pi)$ is attained for $\phi = \frac{1}{2}(\theta - \pi)$. We thus arrive at

$$r(\theta) = \tfrac{1}{4}\{\cos \tfrac{1}{2}(\theta - \pi)\}^{-2} = \tfrac{1}{2}(1 - \cos \theta)^{-1}.$$

To give the boundary in rectangular coordinates we note that from the above relation follows

$$r^2 = \mathcal{R}^2(a) + \mathcal{I}^2(a) = (\tfrac{1}{2} + r \cos \theta)^2 = \tfrac{1}{4} + \mathcal{R}(a) + \mathcal{R}^2(a).$$

We give two further illustrations of the process of obtaining element regions from value regions.

THEOREM 2.4. *For continued fractions $K(1/b_n)$ and for Z defined by $\mathcal{R}(z) > 0$, $E(Z)$ is defined by $\mathcal{R}(b) > 0$.*

Proof. In this case $f(b_n, z) = 1/(b_n + z)$ and hence $g(w, z) = (1/w) - z$. For $E(Z)$ one thus obtains

$$E(Z) = \bigcap_{z \in Z} \left(\frac{1}{Z} - z\right).$$

Since Z is the open right half-plane the same is true for $1/Z$. For every $z \in Z$ the set $(1/Z) - z$ is therefore the right half-plane translated by the amount $-z$, that is translated to the left. All sets $(1/Z) - z$ contain the right half-plane and this is the only set common to all of these half-planes. This establishes the theorem.

THEOREM 2.5. *For infinite exponentials, and for Z defined by $|\log z| \leqq 1$, $E(Z)$ is the set of all a such that $|a| \leqq e^{-1}$.*

Proof. We now have $f(a_n, z) = e^{a_n z}$ and hence $g(w, z) = (1/z) \log w$, which is well defined if we take the principal branch of the logarithm and restrict, as we have done, Z to exclude the negative real axis. We note that Z as defined here is a kidney-shaped region contained in the region

$$-1 \leqq \Re(\log z) = \ln |z| \leqq 1, \qquad -1 \leqq \mathscr{I}(\log z) = \arg z \leqq 1,$$

that is, $e^{-1} \leqq |z| \leqq e$, $|\arg z| \leqq 1$. For $E(Z)$ we obtain

$$\bigcap_{z \in Z} \frac{1}{z} \log Z.$$

The set $\log Z$ is the circular disk with center at 0 radius 1. The sets $(\log Z)/z$ are thus circular disks with center at 0 and radius $1/|z|$. Since for $z \in Z$, $e \geqq 1/|z| \geqq e^{-1}$ it follows that the intersection of all the disks under consideration is the disk with center at the origin and radius e^{-1}.

Before proceeding to show how these value region results can be used to obtain convergence regions for various kinds of infinite processes, we mention briefly that these results can also be applied to other problems. Cowling and Thron [3] used this approach in a study of the location of the zeros of polynomials. Thale [24] and later Perron [17], Scott and Merkes made essential use of value region results in deriving theorems about the univalence of certain functions expressed as continued fractions.

3. Extension of convergence regions by means of the Stieltjes-Vitali theorem. The theorem in question, which in its original form was derived by Stieltjes for the purpose of extending convergence regions for continued fractions, can be stated as follows:

THEOREM (Stieltjes-Vitali). *Let $\{F_n(\zeta)\}$ be a sequence of functions such that for all $n \geqq 1$*

$$F_n(\zeta) \text{ is holomorphic for } \zeta \in D,$$

$$F_n(\zeta) \neq a, \qquad F_n(\zeta) \neq b \quad \text{for all} \quad \zeta \in D,$$

where a and b are two distinct complex numbers. Here D is assumed to be an open region. Further let Δ be an infinite set with at least one limit point in D. Finally assume that the sequence $\{F_n(\zeta)\}$ converges for all $\zeta \in \Delta$. Then $\{F_n(\zeta)\}$ converges uniformly in each compact subset of D.

A proof of this theorem can be found in [28], page 142.

In order to apply this theorem to our problem we have to change the sequence of constants $\{T_n(c)\}$, which we have been considering so far, to a sequence of holomorphic functions. If the generating function f is holomorphic in

both a_n and z in a large enough region, then this can be done by setting $a_n = a_n(\zeta)$, where $a_n(\zeta)$ is a suitably chosen holomorphic function of ζ.

It may be best to illustrate the method by a simple concrete example. Our example will be a theorem of Van Vleck [31], which was originally proved by considering the system of difference equations defining the continued fraction. Even the simplified proof given by Perron ([18], p. 66) takes four pages. The proof presented here was originally published in [27, I].

THEOREM 3.1 (Van Vleck). *The continued fraction $K(1/b_n)$ converges to a value in the right half-plane, provided that for a given $\epsilon > 0$, $-\frac{1}{2}\pi+\epsilon < \arg b_n < \frac{1}{2}\pi+\epsilon$, and $\sum |b_n|$ diverges.*

Proof. We introduce $b_n(\zeta) = |b_n| \exp(i\zeta \arg b_n)$ and observe that the approximants of $K(1/b_n(\zeta))$ are meromorphic functions of ζ. By restricting ζ to lie in a region D, such that for all $n \geqq 1$, $-\frac{1}{2}\pi < \arg b_n < \frac{1}{2}\pi$, we can use Theorem 2.3 to conclude that the approximants of $K(1/b_n(\zeta))$ all lie in the half-plane $\Re(w) > 0$, so that for $\zeta \in D$ the approximants are holomorphic functions and do not assume any value in the left half-plane. A known theorem (see for example [18], p. 46) due to Seidel and Stern, insures the convergence of $K(1/b_n(\zeta))$ if the $b_n(\zeta)$ are positive, that is, if $\Re(\zeta) = 0$, and if

$$\sum b_n(\zeta) = \sum |b_n| \exp\{-\mathscr{I}(\zeta) \arg b_n\}$$

diverges. Since $|\arg b_n|$ is bounded this series diverges if and only if $\sum |b_n|$ diverges. If we let Δ be that part of D for which $\Re(\zeta) = 0$, we have met all the conditions of the Stieltjes-Vitali theorem for the sequence of approximants of the continued fraction $K(1/b_n(\zeta))$ and thus are able to conclude that this sequence converges for $\zeta = 1$, since the set consisting of the point 1 is a compact set in D. The proof of the theorem is completed by noting that $b_n(1) = b_n$.

In general, the Stieltjes-Vitali theorem can be used to obtain convergence-region results for sequences $\{T_n(c)\}$ if the following conditions can be met.

(a) There exists an open region D, such that for all $\zeta \in D$ there exist value regions Z_ζ with corresponding nonempty element regions $E(Z_\zeta)$. Further we require that $\zeta = 1 \in D$, and $E = E(Z_1)$.

(b) A sequence of functions $\{a_n(\zeta)\}$ holomorphic in D can be determined such that $a_n(1) = a_n$ and $a_n(\zeta) \in E(Z_\zeta)$ for all $n \geqq 1$ and all $\zeta \in D$.

(c) A function $c(\zeta)$ holomorphic in D can be found such that $c(1) = c$ and $c(\zeta) \in Z_\zeta$ for all $\zeta \in D$.

(d) Denote by $F_n(\zeta)$ the function obtained if in $T_n(c)$ all a_n are replaced by $a_n(\zeta)$ and c is replaced by $c(\zeta)$. Then we require that $F_n(\zeta)$ is a holomorphic function for all $\zeta \in D$ and for all $n \geqq 1$.

(e) There exists an infinite set Δ with limit points in D so that $\{F_n(\zeta)\}$ converges for $\zeta \in \Delta$.

Then E is a convergence region for the infinite process $\{T_n(c)\}$.

The method has proved extremely useful. We list here a few results which

have been proved in this way, and for which, as yet, no other proofs exist.

THEOREM 3.2. *Let* $-\frac{1}{2}\pi<\alpha<\frac{1}{2}\pi$. *The continued fraction* $K(a_n/1)$ *converges if all its elements lie in the parabola* P_α, *where* $a_n\in P_\alpha$ *if and only if* $|a_n|-\Re(a_n e^{-2i\alpha})\leqq\frac{1}{2}\cos^2\alpha$, *and if in addition at least one of the series*

$$\sum_{\nu=1}^{\infty}\left|\frac{a_2\cdots a_{2\nu}}{a_3\cdots a_{2\nu+1}}\right|,\qquad \sum_{\nu=2}^{\infty}\left|\frac{a_3\cdots a_{2\nu-1}}{a_4\cdots a_{2\nu}}\right|$$

diverges.

The last condition insures that a_n does not tend to ∞ too fast, if it tends to ∞ at all. For $a=0$, Theorem 3.2 was proved by elementary means based on the system of difference equations by Scott and Wall [22] in 1940. Slightly weaker forms of the theorem for all α were obtained by means of the Stieltjes-Vitali theorem by Paydon and Wall [16], and independently by Leighton and Thron [14]. The theorem as stated above, which is a best result, was proved in [26]. Recently [30], elementary proofs (using the methods described in Sec. 6 of this article) of various slightly weaker versions of the theorem were derived.

THEOREM 3.3. *Let* $\alpha(\theta)$ *be a continuous function of period* 2π, *which satisfies the two conditions*

$$|\alpha(\theta)|<\tfrac{1}{2}\pi-\epsilon_1;\quad |\alpha(\theta)-\alpha(\phi)|/|\theta-\phi|<1-\epsilon_2,\ \theta\neq\phi,\ \epsilon_1>0,\ \epsilon_2>0.$$

Let

$$b(\theta)=b_0\exp\int_{\frac{1}{2}\pi}^{\theta}\tan\alpha(x)dx,\qquad b_0>0.$$

Then the continued fraction $K(1/b_n)$ *converges if*

$$|b_{2n}|\geqq b(\arg b_{2n}),\qquad |b_{2n-1}|\geqq 4/b(\pi-\arg b_{2n-1})\quad \textit{for all}\quad n\geqq 1.$$

This extremely general twin-convergence-region result was proved in 1949 ([27], II). The definition of b insures that the regions for the b_{2n} and b_{2n-1} both have convex complements. To apply the Stieltjes-Vitali theorem in this case quite complicated functions were used. For $b_{2n}(\zeta)$ the following function proved to do the job.

$$b_{2n}(\zeta)=b_{2n}\exp\int_{\pi/2}^{\arg b_{2n}}(\tan\zeta\alpha(x)-\tan\alpha(x))dx.$$

THEOREM 3.4. *The infinite radical* $\sqrt{a_1+\sqrt{a_2+\cdots}}$ *converges if*

$$|\arg a_n|<\tfrac{2}{3}\pi-\epsilon,\qquad \textit{for all } n\geqq 1,\ \epsilon>0,$$

and if in addition $\limsup|a_n|^{2^{-n}}<\infty$.

In the proof of this result [21], a criterion due to Herschfeld [9] for positive a_n was used.

In spite of the fact that the convergence extension method by means of the Stieltjes-Vitali theorem has led to some very powerful convergence results, it has some very definite drawbacks. It is only a method of extension of convergence regions. It is undesirable from an aesthetic point of view to use such a deep function-theoretic result in connection with the derivation of convergence criterion for sequences of constants. One cannot prove uniform convergence. This may seem surprising at first glance, since the Stieltjes-Vitali theorem asserts that the convergence is uniform. However this uniformity is only with respect to the—for our purposes—auxiliary variable ζ not with respect to freely varying a_n. Finally one can not obtain any estimates for the speed of convergence of the sequence $\{T_n(c)\}$. For these reasons we now turn to the consideration of other methods for deriving convergence results.

4. Nested sets. Let Z be a value region with $c \in Z$ and let $E(Z)$ be an element region for a certain sequence $\{T_n(c)\}$. Define

$$Z_n = T_n(Z), \qquad n \geqq 1;$$

then the sets Z_n are nested in each other, that is $Z_n \subset Z_m$ if $n > m$. To prove this it suffices to observe that, by Theorem 2.1, $t_n(Z) \subset Z$. It follows that

$$Z_{n+1} = T_n(t_n(Z)) \subset T_n(Z) = Z_n.$$

This device of nested sets was first used in connection with differential equations by Weyl [33]. He dealt with a continuous family of nested circles. Hamburger [5] pointed out its connection with difference equations and continued fractions. Hellinger [8] seems to have been the first to have applied this concept in a convergence proof for continued fractions. All these authors were concerned with sequences of functions rather than with the simpler case of sequences of constants. The first use of the concept in connection with convergence regions of sequences of constants is due to Paydon and Wall [16]. In recent years the author and some of his co-workers have derived a number of convergence region results for various infinite processes employing this device.

The procedure is as follows: Since $c \in Z$, $T_n(c) \in Z_n$ and $T_{n+m}(c) \in Z_{n+m} \subset Z_n$, so that

$$| T_n(c) - T_{n+m}(c) | < \operatorname{diam} Z_n.$$

The sequence $\{T_n(c)\}$ is therefore a Cauchy sequence and hence converges if $\lim \operatorname{diam} Z_n = 0$. Those regions $E(Z)$ for which the diameter of Z_n tends to zero are therefore convergence regions for the sequence $\{T_n(c)\}$.

5. A "hit-and-run" method. The diameters of the sets Z_n are not, in general, easy to compute. Let us denote by C the boundary of Z and by C_n, the boundary of Z_n, and let us assume that Z and the generating function f are such that all C_n are simple, closed and rectifiable curves and that Z_n consists of the interior and part, or all, of the boundary of C_n. Then we have $\operatorname{diam} Z_n \leqq \frac{1}{2} l(C_n)$, where $l(C_n)$

denotes the length of the curve C_n. For it we have, provided that $(d/dz)T_n(z)$ exists,

$$l(C_n) = \int_C |T_n'(z)| \, |dz|.$$

Now let us introduce, for $m \geqq n-1 \geqq 0$,

$$T_{n,m}(z) = T_{n,m-1}(t_m(z)), \qquad T_{n,n-1}(z) = z,$$

that is,

$$T_{n,m}(z) = t_n(t_{n+1}(\cdots t_m(z) \cdots)).$$

From the chain rule of differentiation of a function of a function one obtains

$$T_n'(z) = \prod_{v=1}^{n} t_v'(T_{v+1,n}(z)).$$

Assuming Z has been so chosen, or $E(Z)$ has been further restricted to $E' \subset E$, that for $a_n \in E'$ and $z \in Z$,

$$|t_n'(z)| \leqq 1 - \epsilon, \qquad \epsilon > 0,$$

then, since $T_{v+1,n}(z) \in Z$ for $z \in Z$, $l(C_n) < l(C)(1-\epsilon)^n$, and lim diam $Z_n = 0$ so that E' is a convergence region.

A result that can be obtained by this method is the following.

THEOREM 5.1. *The infinite exponential all of whose elements satisfy the condition* $|a_n| \leqq e^{-1}(1-\epsilon)$, *where* $0 < \epsilon < 1$, *converges. The value* u *to which it converges satisfies the condition* $|\log u| \leqq 1$. *If the* a_n *are functions of any number of variables then the convergence of the continued fraction is uniform, provided that throughout the ranges of the variables the elements* a_n *are subject to the above condition.*

Proof. In Theorem 2.5 we showed that if we take Z to be the region defined by $|\log z| \leqq 1$ then $a_n \in E(Z)$ if and only if $|a_n| \leqq e^{-1}$. We now restrict $|a_n|$ further as indicated in the statement of the theorem and estimate $t_n'(z)$. We have

$$|t_n'(z)| = |a_n| \, |e^{a_n z}| \leqq e^{-1}(1-\epsilon)e^{|a_n z|} \leqq (1-\epsilon),$$

since $|a_n z| \leqq e^{-1} \cdot e$. It follows that diam Z_n tends to zero, independently of the choice of a_n, and the theorem is proved. By a slightly more careful estimate of $T_n'(z)$ one can also prove convergence if one only requires that $|a_n| \leqq e^{-1}$. This proof was carried out in [29].

6. An exact elementary method. If $t_n(z)$ is a linear fractional transformation of the form $a_n/(b_n+z)$, then $\{T_n(0)\}$ is a continued fraction and, as we showed in Section 1,

$$T_n(z) = \frac{A_{n-1}z + A_n}{B_{n-1}z + B_n},$$

where the A_n and B_n satisfy (1.1). We then have, using (1.4) for the last step

$$| T_n'(z) | = \frac{| A_nB_{n-1} - B_nA_{n-1} |}{| B_{n-1}z + B_n |^2} = \frac{\prod_{\nu=1}^{n} | a_\nu |}{| B_{n-1}z + B_n |^2}.$$

If Z is a circular disk or a half-plane one can determine $l(C_n)$ explicitly. This and the further procedure will now be illustrated by proving the following special case of Theorem 3.2.

THEOREM 6.1. *The continued fraction $K(a_n/1)$ converges to a value u which satisfies $\mathcal{R}(u) \geqq -\frac{1}{2}$ provided that*

$$\mathcal{I}^2(a_n) \leqq \mathcal{R}(a_n) + \tfrac{1}{4} \quad \textit{and} \quad | a_n | < M \qquad \textit{for all } n \geqq 1.$$

Here M is an arbitrary large positive quantity. The convergence is uniform in the manner indicated in the statement of Theorem 5.1.

Proof. We begin by considering the quotient $B_n/B_{n-1} = S_n$. From (1.1) with $b_n = 1$ for all $n \geqq 1$ it follows that

$$S_n = 1 + (a_n/S_{n-1}), \qquad n \geqq 2; \qquad S_1 = 1. \tag{6.1}$$

Thus

$$S_n = 1 + \frac{a_n}{1} + \cdots + \frac{a_2}{1}.$$

Now

$$\frac{a_n}{1} + \cdots + \frac{a_2}{1}$$

is in Z, as is seen from repeated application of Theorem 2.1. The quantities S_n therefore satisfy the inequality $\mathcal{R}(S_n) \geqq \frac{1}{2}$. However, a sharper estimate depending on n can be obtained, and will play an important role in this proof. Write $S_n = x_n + iy_n$, $a_n = u_n + iv_n$, where $v_n^2 \leqq u_n + \frac{1}{4}$, $x_n \geqq d_n \geqq \frac{1}{2}$. Then

$$d_{n+1} = \min \mathcal{R}(S_{n+1}) = 1 + \min \mathcal{R}\left(\frac{u_n + iv_n}{x_n + iy_n}\right),$$

where the minimum is taken over all permissible a_n, and all S_n with $\mathcal{R}(S_n) \geqq d_n$. Now set

$$F = \mathcal{R}\left(\frac{u_n + iv_n}{x_n + iy_n}\right) = \frac{u_nx_n + v_ny_n}{x_n^2 + y_n^2}.$$

To determine the minimum of F we first keep u_n, v_n, and x_n fixed and allow y_n to vary. The value of y_n which will minimize F must be, by a simple argument

from calculus, one of the solutions of the equation $v_n y_n^2 + 2u_n x_n y_n - x_n^2 v_n = 0$, that is, $y_n = x_n\{-u_n + \sqrt{(u_n^2+v_n^2)}\}/v_n$. Thus

$$\min_{-\infty < y_n < \infty} F = \frac{-v_n^2\sqrt{(u_n^2+v_n^2)}}{2x_n\{u_n^2+v_n^2+u_n\sqrt{(u_n^2+v_n^2)}\}} = \frac{-v_n^2}{2x_n\{u_n+\sqrt{(u_n^2+v_n^2)}\}}.$$

Continuing to keep u_n and x_n fixed, we now allow v_n^2 to vary. It can only vary between 0 and $u_n + \frac{1}{4}$. There is no critical point in this range so the minimum is attained for $v_n = u_n + \frac{1}{4}$. This leads to

$$\min_{y_n, v_n} F = \frac{-(u_n + \frac{1}{4})}{2x_n(u_n + u_n + \frac{1}{2})} = \frac{-1}{4x_n}\,.$$

Thus $d_{n+1} = 1 - 1/(4d_n)$. Since $d_1 = 1$, one proves easily by induction that

$$d_n = \min \Re(S_n) = \frac{n+1}{2n}\,. \tag{6.2}$$

Having disposed of this we now turn to a study of the regions Z_n and their diameters. In this case the regions Z_n are all circular disks. To establish this we note that the half-plane Z, given by $\Re(z) \geqq -\frac{1}{2}$ is mapped by the linear fractional transformation $T_n(z)$ onto either the inside or the outside of a certain circle or onto a half-plane. Since $Z_n \subset Z$, Z_n must be either a half-plane or a circular disk. In order for Z_n to be a half-plane the point $T_n^{-1}(\infty) = -B_n/B_{n-1}$ would have to be on the boundary of Z. However from our previous result we conclude that

$$\max \Re\left(-\frac{B_n}{B_{n-1}}\right) = -\frac{n+1}{2n} < -\tfrac{1}{2}.$$

Hence, for all $n \geqq 1$, the regions Z_n are circular disks. Let us denote by R_n the radius of Z_n. Then

$$l(C_n) = 2\pi R_n = \int_C \frac{\prod_{\nu=1}^{n} |a_\nu|}{|B_{n-1}|^2 |z + S_n|^2} |dz|\,.$$

We now evaluate the integral. It is

$$\int_C \frac{|dz|}{|z+S_n|^2} = \int_{-\infty}^{\infty} \frac{dy}{|iy - \frac{1}{2} + S_n|^2} = \int_{-\infty}^{\infty} \frac{dy}{(y + \mathscr{I}(S_n))^2 + (\Re(S_n) - \frac{1}{2})^2}$$

$$= \int_{-\infty}^{\infty} \frac{dt}{t^2 + (\Re(S_n) - \frac{1}{2})^2} = \frac{\tan^{-1} t/|\Re(S_n) - \frac{1}{2}|}{|\Re(S_n) - \frac{1}{2}|}\Bigg|_{-\infty}^{\infty}$$

$$= \frac{\pi}{|\Re(S_n) - \frac{1}{2}|}\,.$$

For R_n one thus obtains

$$R_n = \frac{\prod_{\nu=1}^{n} |a_\nu|}{2|B_{n-1}|^2 |\Re(S_n) - \frac{1}{2}|},$$

and hence

$$\frac{R_n}{R_{n-1}} = \frac{|a_n| \, |\Re(S_{n-1}) - \frac{1}{2}|}{|S_{n-1}|^2 |\Re(S_n) - \frac{1}{2}|}.$$

Now

$$\Re(S_n) = 1 + \Re(a_n/S_{n-1}) = 1 + \frac{\Re(a_n\overline{S}_{n-1})}{|S_{n-1}|^2},$$

so that

$$\frac{R_n}{R_{n-1}} = \frac{|a_n| \, |\Re(S_{n-1}) - \frac{1}{2}|}{|\frac{1}{2}|S_{n-1}|^2 + \Re(a_n\overline{S}_{n-1})|}.$$

To estimate the denominator D_n of this quotient, we set as before $a_n = u_n + iv_n$, $S_n = x_n + iy_n$, and note that

$$v_n^2 \leqq u_n + \tfrac{1}{4}, \qquad x_n \geqq \frac{n+1}{2n}.$$

For D_n we can write

$$\begin{aligned} D_n &= |\tfrac{1}{2}(x_{n-1}^2 + y_{n-1}^2) + u_n x_{n-1} + v_n y_{n-1}| \\ &= |\tfrac{1}{2}x_{n-1}^2 + u_n x_{n-1} + \tfrac{1}{2}(y_{n-1} + v_n)^2 - \tfrac{1}{2}v_n^2|. \end{aligned}$$

This expression is not increased if the term $\frac{1}{2}(y_{n-1}+v_n^2)$ is omitted and if $\frac{1}{2}v_n^2$ is replaced by $\frac{1}{2}u_n+\frac{1}{8}$. The remaining expression is clearly positive so that the absolute value signs can be omitted. This leads to the inequality

$$D_n \geqq \tfrac{1}{2}(x_{n-1}^2 + 2u_n x_{n-1} - u_n - \tfrac{1}{4}) = \tfrac{1}{2}(x_{n-1} - \tfrac{1}{2})(x_{n-1} + \tfrac{1}{2} + 2u_n).$$

The inequality is further strengthened if we replace $u_n+\frac{1}{2}$ by $|a_n|$, for we have

$$|a_n|^2 = u_n^2 + v_n^2 \leqq u_n^2 + u_n + \tfrac{1}{4} = (u_n + \tfrac{1}{2})^2.$$

We thus arrive at $D_n \geqq \frac{1}{2}(x_{n-1}-\frac{1}{2})(x_{n-1}-\frac{1}{2}+2|a_n|)$ and hence

$$\frac{R_n}{R_{n-1}} \leqq \frac{2|a_n|(x_{n-1} - \frac{1}{2})}{(x_{n-1} - \frac{1}{2})(x_{n-1} - \frac{1}{2} + 2|a_n|)}$$

$$= \frac{1}{1 + (x_n - \frac{1}{2})/(2|a_n|)} \leqq \frac{1}{1 + 1/(4Mn)}.$$

For R_n, one obtains from this estimate,

$$R_n \leqq 1 \Big/ \prod_{\nu=1}^{M} \{1 + 1/(4M\nu)\} = o(n^{-1/(4M)})$$

and it follows that diam Z_n tends to zero independently of the distribution of the elements a_n, so long as M remains fixed. This completes the proof of the theorem.

This proof is based on the concepts of the nested circles together with an estimate of R_n/R_{n-1}. The latter estimate hinges on a sharp estimate of the location of B_n/B_{n-1}. This same pattern has been used successfully for a number of other convergence criteria, the most recent of which is the following result of Lange and the author [**13**].

THEOREM 6.2. *Let a be a complex number and ρ a positive number and let them satisfy the inequality $|a| < \rho < |1+a|$. Then the continued fraction $K(c_n^2/1)$ converges if its elements c_n satisfy the conditions*

$$|c_{2n-1} \pm ia| \leqq \rho, \qquad |c_{2n} \pm i(1 + a)| \geqq \rho.$$

The proof was based on the methods of this section only for real a. For nonreal a the problem of obtaining a sharp enough estimate for B_n/B_{n-1} has proved as yet unsurmountable so the case for nonreal a was disposed of by using the Stieltjes-Vitali theorem.

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FIBONACCI NUMBER TRIPLES

A. F. HORADAM, The University of New England, Armidale, Australia

1. Introduction. Previously in this MONTHLY [1], I obtained several results relating to a generalized Fibonacci sequence H_{pq}

(1) $$p, \quad p+q, \quad 2p+q, \quad 3p+2q, \quad 5p+3q, \quad 8p+5q, \cdots,$$

for which the nth generalized Fibonacci number is

(2) $$H_n = \frac{1}{2\sqrt{5}}(la^n - mb^n),$$

where $l=2(p-qb)$, $m=2(p-qa)$, $a=\frac{1}{2}(1+\sqrt{5})$, $b=\frac{1}{2}(1-\sqrt{5})$.

The purpose of this article is to find a connection between generalized Fibonacci numbers and Pythagorean number triples. By a Pythagorean (number) triple is meant a set of three mutually prime integers u, v, w for which $u^2+v^2=w^2$. The problem to be solved is this: Given such a triple u, v, w, can we find n, p, q such that the integers whose squares appear in (3) below are these u, v, w? The answer is yes. Viewed in this light, Pythagorean triples may be called Fibonacci (number) triples.

2. A Pythagorean theorem. In [1], I stated without proof the following "Pythagorean theorem" (Theorem 1):

(3) $$(H_nH_{n+3})^2 + (2H_{n+1}H_{n+2})^2 = (2H_{n+1}H_{n+2} + H_n^2)^2.$$

The proof is simple. Starting from the identity $(H_{n+2}-H_{n+1})^2=H_n^2$ *i.e.*, using the recurrence relation for the generalized sequence), add $4H_{n+1}H_{n+2}$ to each side, replace $H_{n+1}+H_{n+2}$ by H_{n+3}, then multiply throughout by H_n^2, and finally add $4H_{n+1}^2H_{n+2}^2$ to each side. Simplification gives the result. Alternatively, Theorem 1 may be proved by using (2).

Equation (3) is expressible in a variety of equivalent forms (some of them simpler) but for our purpose we require it in this form.

3. Fibonacci (number) triples. Firstly, we note the well-known fact that all Pythagorean triples are given by x^2-y^2, $2xy$, x^2+y^2, where $x>y$ and x, y are mutually prime but not simultaneously odd (thus avoiding repetitions) so that $x+y$ is always odd.

THEOREM 2. *All Pythagorean triples are Fibonacci triples.*

Proof. Put $p=x-y$, $q=2y-x$ in (1) to obtain the sequence $H_{x-y,2y-x}$:

(4) $$x-y,\ y,\ x,\ x+y,\ \cdots.$$

For $n=1$, (4) gives $(x^2-y^2)^2+(2xy)^2=(x^2+y^2)^2$.

For example, the triples 3, 4, 5; 5, 12, 13; 15, 8, 17; 7, 24, 25 are associated with the values 2, 1; 3, 2; 4, 1; 4, 3 of x, y, respectively, and therefore with the

sequences H_{10} (the classical Fibonacci sequence), H_{11}, $H_{3,-2}$, H_{12} (the Lucas sequence), respectively, where $n=1$ in each case.

4. Other comments on Fibonacci triples. In the following we use the results

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{5}$$

$$H_n = pF_n + qF_{n-1}, \tag{6}$$

where F_n is the nth Fibonacci number.

(a) Write $x=H_{n+2}$, $y=H_{n+1}$ in (3) whence, after calculation, we have $x^2-y^2=H_nH_{n+3}$ and $x^2+y^2=2H_{n+1}H_{n+2}+H_n^2$. From (6) we have $x=pF_{n+2}+qF_{n+1}$, $y=pF_{n+1}+qF_n$. Solving and using (5), we find

$$p = (-1)^n(yF_{n+1} - xF_n), \qquad q = (-1)^n(xF_{n+1} - yF_{n+2}). \tag{7}$$

When $n=1$ the values of p and q are those in (4). Giving n all positive integral values, we obtain an infinite sequence of sequences H_{pq}, where

$$\begin{aligned} p, q = x - y, 2y - x; \qquad 2y - x, 2x - 3y; \qquad 2x - 3y, 5y - 3x; \\ 5y - 3x, 5x - 8y; \cdots, \end{aligned} \tag{8}$$

corresponding to $n=1, 2, 3, 4, \cdots$, respectively. A given Pythagorean triple may be derived from any of these sequences provided that the correct value of n is associated with it (though it must be remembered that the same set of four numbers $x-y$, y, x, $x+y$ is being operated with in each sequence). For instance, the triple 5, 12, 13 is obtained from the sequences H_{11}, H_{10}, H_{01}, $H_{1,-1}$, $\cdots$ (*i.e.*, $x=3$, $y=2$), when $n=1, 2, 3, 4, \cdots$, respectively.

(b) What happens if, instead of the procedure adopted in (a), we write $x=\frac{1}{2}H_{n+3}$, $y=\frac{1}{2}H_n$, assuming H_n even? (Notice that $H_{n+3}=H_n+2H_{n+1}$ is even or odd along with H_n.) After calculation we find $x^2-y^2=H_{n+1}H_{n+2}$ and, from (6), $x=\frac{1}{2}(pF_{n+3}+qF_{n+2})$, $y=\frac{1}{2}(pF_n+qF_{n-1})$. Solving, using (5), we obtain

$$p = (-1)^n(xF_{n-1} - yF_{n+2}), \qquad q = (-1)^n(yF_{n+3} - xF_n). \tag{9}$$

Defining $F_0=0$ and giving n all positive integral values, we have the infinite sequences H_{pq}, where

$$\begin{aligned} p, q = 2y, x - 3y; \qquad x - 3y, 5y - x; \qquad 5y - x, 2x - 8y; \\ 2x - 8y, 13y - 3x; \cdots \end{aligned} \tag{10}$$

corresponding to $n=1, 2, 3, 4, \cdots$, respectively. Sequences defined by (10) differ from sequences defined by (8) in that they yield a triple which is not a Pythagorean triple, but a Pythagorean triple with each member multiplied by 2. For example, $H_{4,-3}$, $H_{-3,7}$, $H_{7,-10}$, $H_{-10,17}$, $\cdots$, (*i.e.*, $x=3$, $y=2$), all produce the triple 24, 10, 26 = 2(12, 5, 13) (note the changed order) when $n=1, 2, 3, 4, \cdots$, respectively. Associated with the standard sequence $H_{11}(n=1)$ for the triple 5, 12, 13, there is thus the "duplicating" sequence $H_{4,-3}(n=1)$, and this situation

is true for every Pythagorean triple.

(c) Relabeling p, q in (9) p', q' and assuming fixed values x_1, y_1 for x, y, respectively, we deduce from (7) and (9) that

$$(-1)^n p' = (-1)^{n+1} p + x_1 F_{n-1} - (x_1 + y_1) F_n,$$
$$(-1)^n q' = (-1)^{n+1} q - x_1 F_n + (x_1 + y_1) F_{n+1}.$$

(d) Laisant [2] observed the connection between the sides of a right-angled triangle and four consecutive terms of the classical Fibonacci sequence, in effect, the special case $p=1$, $q=0$ of Theorem 1. Apparently, he was the first to do so.

5. Additional remarks on the generalized sequence. (i) H_n/H_{n+1} is expressible as a continued fraction for, if $r_n = H_n/H_{n+1}$, then

$$r_n = \frac{1}{1 + r_{n-1}}, \qquad r_{n-1} = \frac{1}{1 + r_{n-2}}, \cdots.$$

(ii) Writing $\sinh \alpha = \frac{1}{2}$, we find that

$$H_n = [(2p - q) \sinh n\alpha + 2q \cosh \alpha \cosh n\alpha]/[2 \cosh \alpha] \qquad (n \text{ even}),$$
$$H_n = [(2p - q) \cosh n\alpha + 2q \cosh \alpha \cosh n\alpha]/[2 \cosh \alpha] \qquad (n \text{ odd}),$$

reducing to $\sinh n\alpha/\cosh \alpha$ (n even), $\cosh n\alpha/\cosh \alpha$ (n odd) for the classical sequence. Also, $H_{n+1}/H_n \to e^\alpha = \cosh \alpha + \sinh \alpha = a$, which is the same as the limiting value for F_{n+1}/F_n.

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SOME FUNCTIONAL EQUATIONS*

L. CARLITZ, Duke University

Robinson [4] showed that if $f(z)$ is regular for $|z| < r$ and satisfies the functional equation $|f(x+iy)| = |f(x)+f(iy)|$ for real x, y then

$$f(z) = Az, \qquad f(z) = A \sin bz, \quad \text{or} \quad f(z) = A \sinh bz,$$

where A and b are constants and b is real. Earlier, Hille [2] had proved that the same conclusion holds for the equation $|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2$.

Recently, Rosenbaum and Segal [5] have proved that if $f(z)$ is regular for $|z| < r$ and satisfies

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$$f(x+y)f(x-y) = f^2(x) - f^2(y) \tag{1}$$

for complex x, y, then $f(z) = Az$ or $f(z) = A \sinh cz$, where A and c are complex constants. Indeed a somewhat more general result is obtained.

The referee has pointed out that there is a relation between the three functional equations. In the special case when $f(z)$ is a power series containing only odd powers with real coefficients, both Robinson's equation and Hille's equation become $f(x+iy)f(x-iy) = f^2(x) - f^2(iy)$, which has the same form as the equation of Rosenbaum and Segal.

As a possible extension of (1) we may consider the equation

$$f(x+y+z)f(x+\omega y+\omega^2 z)f(x+\omega^2 y+\omega z) = \prod_{r=0}^{2} (f(x) + \omega^r f(y) + \omega^{2r} f(z)), \tag{2}$$

where $\omega^2+\omega+1=0$ and x, y, z are arbitrary complex numbers. We assume that $f(z)$ is regular for $|z| < r$.

If in (2) we replace x, y, z by y, z, x we get

$$f(y+z+x)f(y+\omega z+\omega^2 x)f(y+\omega^2 z+\omega x) = \prod_{r=0}^{2} (f(y) + \omega^r f(z) + \omega^{2r} f(x)).$$

Comparing with (2) we see that

$$f(x+\omega y+\omega^2 z)f(x+\omega^2 y+\omega z) = f(y+\omega z+\omega^2 x)f(y+\omega^2 z+\omega x),$$

so that

$$f(\omega u)f(v) = f(u)f(\omega v). \tag{3}$$

Hence if we put

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{4}$$

it follows from (3) that $c_r c_s \omega^r = c_r c_s \omega^s$, so that $c_r c_s = 0$ unless $r \equiv s \pmod 3$.

Now if we put $y = z$, $x = z+h$ in (2) we get

$$f(3z+h)f^2(h) = \{f(z+h) + 2f(z)\}\{f(z+h) - f(z)\}^2,$$

so that $f(0) = 0$ and

$$f(3z)\{f'(0)\}^2 = 3f(z)\{f'(z)\}^2. \tag{5}$$

If $f'(0) = 0$ it follows that $f(z) \equiv 0$. Thus in (4) we may assume $c_0 = 0$, $c_1 \neq 0$ and $c_r = 0$ unless $r \equiv 1 \pmod 3$. Consequently (4) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n z^{3n+1} \qquad (a_0 \neq 0). \tag{6}$$

Substituting from (6) in (5) we get

$$3^{3n} a_0^2 a_n = \sum_{r+s\leqq n} (3r+1)(3s+1) a_r a_s a_{n-r-s}. \tag{7}$$

For $n=1$, (7) reduces to $3^3 a_0^2 a_1 = a_0^2 a_1 + 4a_0^2 a_1 + 4a_0^2 a_1$, so that $a_1 = 0$. For $n=2$ we get

$$3^6 a_0^2 a_2 = a_0^2 a_2 + 7a_0^2 a_2 + 7a_0^2 a_2,$$

so that $a_2 = 0$. If we assume $a_1 = \cdots = a_{n-1} = 0$, (7) implies $3^{3n} a_0^2 a_n = (6n+3) a_0^2 a_n$ and therefore $a_n = 0$ for all $n \geqq 0$.

We may state

THEOREM 1. *If $f(z)$ is regular for $|z| < r$ and satisfies the functional equation* (2), *then $f(z) = az$, where a is an arbitrary complex constant.*

A different extension of (1) is suggested by the equation ([**3**], p. 84)

$$\operatorname{sn}(u+v)\operatorname{sn}(u-v) = \frac{\operatorname{sn}^2 u - \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

satisfied by the Jacobi elliptic function sn u. We accordingly consider the functional equation

$$f(u+v)f(u-v) = \frac{f^2(u) - f^2(v)}{1 - k^2 f^2(u) f^2(v)}, \tag{8}$$

where k^2 is a constant.

We assume that $f(u)$ is regular for $|u| < r$ and is not constant. It is clear from (8) that $f(u)$ is odd. Put

$$f(u+v) = \sum_{r=0}^{\infty} \frac{v^r}{r!} f^{(r)}(u), \qquad f(u-v) = \sum_{r=0}^{\infty} \frac{(-v)^r}{r!} f^{(r)}(u),$$

so that

$$f(u+v)f(u-v) = f^2(u) + \{f(u)f''(u) - f'^2(u)\} v^2 + \cdots = \frac{f^2(u) - f^2(v)}{1 - k^2 f^2(u) f^2(v)}.$$

If we put $f(v) = av + \cdots$, it follows that

$$f(u)f''(u) - f'^2(u) = \{k^2 f^4(u) - 1\} a^2. \tag{9}$$

Since (8) implies

$$f(2v)f'(0) = \frac{2f(v)f'(v)}{1 - k^2 f^4(v)},$$

it is clear that $a = 0$ implies $f(u) \equiv 0$. Hence if $a \neq 0$ and we put

$$f(u) = ag(u), \tag{10}$$

(9) becomes

$$g(u)g''(u) - g'^2(u) = a^4k^2g^4(u) - 1. \tag{11}$$

Now put

$$g'^2(u) = 1 - (1 + a^4k^2)g^2(u) + a^4k^2g^4(u) + \phi(u), \tag{12}$$

so that

$$g'(u)g''(u) = - (1 + a^4k^2)g(u)g'(u) + 2a^4k^2g^3(u)g'(u) + \tfrac{1}{2}\phi'(u),$$

$$g(u)g''(u) = - (1 + a^4k^2)g^2(u) + 2a^4k^2g^4(u) + \tfrac{1}{2}\{g(u)\phi'(u)\}/g'(u).$$

Combining with (11) we get

$$g'^2(u) - 1 = - (1 + a^4k^2)g^2(u) + a^4k^2g^4(u) + \tfrac{1}{2}\{g(u)\phi'(u)\}/g'(u),$$

and therefore by (12),

$$\phi(u) = \tfrac{1}{2}\{g(u)\phi'(u)\}/g'(u), \qquad \phi'(u)/\phi(u) = 2g'(u)/g(u).$$

Thus $\phi(u) = Cg^2(u)$, where C is constant, and (12) becomes

$$g'^2(u) = 1 - (1 - C + a^4k^2)g^2(u) + a^4k^2g^4(u). \tag{13}$$

If we put $h(u) = \lambda g(\lambda^{-1}u)$, (13) becomes

$$h'^2(u) = 1 - \lambda^{-2}(1 - C + a^4k^3)h^2(u) + \lambda^{-4}a^4k^2h^4(u).$$

This is in Legendre normal form provided $\lambda^{-2}(1-C+a^4k^2) = 1+\lambda^{-4}a^4k^2$. For given C we can choose λ to satisfy this equation. Thus by (10) we get

$$f(u) = \lambda^{-1}a \operatorname{sn}(\lambda u, \lambda^{-4}a^4k^2). \tag{14}$$

It is easily verified that (14) does indeed satisfy (8). This proves

THEOREM 2. *If $f(u)$ is regular for $|z| < r$ and satisfies the functional equation* (8), *then $f(u)$ is determined by* (14), *where a and λ are constants.*

The above proof may be compared with that in [1].

Added in proof. In connection with (8), it may be noted that P. J. Myburg (Arkhimedes, 1959, no. 1, pp. 13–16; MR 22 (1961), 659) proved that all meromorphic functions solutions of the functional equation $\phi(u+v)+\phi(u-v) = R(\phi(u), \phi(v)))$, where R is rational, are of the form $A\phi(ku)+B$, where A, B, k are constants and $\phi(u) = u$, u^2, $1/u^2$, $\cos u$, $1/\cos u$, $\operatorname{cn} u$, $1/\operatorname{cn} u$, or $\wp(u)$ (Weierstrassian elliptic function).

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From (4.7) it is evident that if we assume $\tau(p) \neq 0$ (*i.e.*, c is finite), then $\tau(p^\alpha) \neq 0$ for odd α. Now using the known result that $\tau(p^{2m})$ is odd, if we assume $\tau(p) \neq 0$ (*i.e.*, c is finite), then $\tau(p^\alpha) \neq 0$ for $\alpha \geqq 2$, which is a result due to Lehmer [7].

Finally, if (1.3) is true for all $\theta \geqq 1$, then $c = \theta$ and (1.6) is true. We conclude that the proof of the nonvanishing of the τ-function reduces to the proof of (1.3), which was shown in Section 2 to be true for $\theta = 1, \cdots, 9$.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, University of Washington

Material for this department should be sent to M. H. Protter, Department of Mathematics, University of California, Berkeley 4, California

TOPOLOGY AND TCHEBYCHEFF

B. SCHWEIZER and A. SKLAR, University of Arizona and Illinois Institute of Technology

The purpose of this brief note is to exhibit a curious and rather novel application of Tchebycheff's inequality—namely to show that under certain conditions this inequality implies that the so-called ϵ, λ-topology for a statistical metric space is discrete.

A statistical metric space [3, 4, 5] is a set S in which a cumulative distribution function F_{pq} is associated with every pair of points, p, q. The value of this distribution function for any real number x, *i.e.*, $F_{pq}(x)$, is interpreted as the

probability that the distance between p and q is less than x.* The functions F_{pq} are required to satisfy several simple conditions which, as one readily sees, are generalizations of the usual conditions for an ordinary metric. These are:

I. $F_{pq}(x)=1$ for all $x>0$ if and only if $p=q$. (Identity)
II. $F_{pq}(0)=0$. (Positivity)
III. $F_{pq}(x)=F_{qp}(x)$, for all x. (Symmetry)
IV. If $F_{pq}(x)=1$ and $F_{qr}(y)=1$, then $F_{pr}(x+y)=1$. (Triangle Inequality)

There are also stronger and more interesting possibilities for the triangle inequality [5], but these need not be considered here.

DEFINITION. *Let p be a point in a statistical metric space S. Then the set $N_p(\epsilon,\lambda)$, $\epsilon>0$, $\lambda>0$, of all points q in S for which $F_{pq}(\epsilon)>1-\lambda$ is said to be an ϵ, λ-neighborhood of p.*

Using the interpretation mentioned above, $q\in N_p(\epsilon,\lambda)$ means that the point q lies in an ϵ-neighborhood of the point p (*i.e.*, the distance from p to q is less than ϵ) with probability greater than $1-\lambda$.

From (I) it follows that for every $\epsilon>0$ and every $\lambda>0$, the neighborhood $N_p(\epsilon,\lambda)$ contains p. Thus the smallest topology containing the family of ϵ,λ-neighborhoods in S is a topology for S [2]. We call this topology the ϵ,λ-*topology*.†

LEMMA. *If p is a point of a statistical metric space and if there exist two positive numbers, ϵ_0, λ_0, such that $F_{pq}(\epsilon_0)\leqq 1-\lambda_0$, for all points q different from p, then p is an isolated point in the ϵ, λ-topology.*

Proof. The neighborhood $N_p(\epsilon_0,\lambda_0)$ contains only the point p itself.

THEOREM. *Let p be a point of a statistical metric space such that for all points $q\neq p$ the mean m_{pq} and variance σ_{pq} of the distribution F_{pq} exist. Suppose further that there exist numbers $h>0$ and $t>1$ such that $m_{pq}-t\sigma_{pq}>h$. Then p is an isolated point in the ϵ, λ-topology.*

Proof. Given p and q, we may look upon F_{pq} as the distribution function of a random variable $d(p,q)$, the "distance" between p and q. Then Tchebycheff's inequality [1] states that for every $k>0$,

$$\text{Prob}\left\{\left|d(p,q)-m_{pq}\right|\geqq k\sigma_{pq}\right\}\leqq 1/k^2,$$

i.e.,

$$\text{Prob}\left\{\left[d(p,q)\geqq m_{pq}+k\sigma_{pq}\right]\text{ or }\left[d(p,q)\leqq m_{pq}-k\sigma_{pq}\right]\right\}\leqq 1/k^2,$$

* Thus statistical metric spaces are generalizations of metric spaces in which distances, rather than being certain, are described probabilistically. Possible applications of the theory of statistical metric spaces are briefly discussed in [4].

† Neighborhoods in statistical metric spaces may be defined in several nonequivalent ways. These different definitions and the interesting topological structures to which they lead are the subject of a forthcoming paper by E. Thorp.

so that *a fortiori*,

$$\text{(1)} \qquad \text{Prob}\,\{d(p, q) < m_{pq} - k\sigma_{pq}\} = F_{pq}(m_{pq} - k\sigma_{pq}) \leqq 1/k^2,$$

for every $k>0$. Now by hypothesis, there exist numbers $h>0$ and $t>1$ such that $m_{pq}-t\sigma_{pq}>h$, for every $q\neq p$. Thus from (1) we have that for any $q\neq p$,

$$F_{pq}(m_{pq} - t\sigma_{pq}) \leqq 1/t^2 < 1,$$

from which, on using the preceding lemma (with $\epsilon_0=h$ and $\lambda_0=1-1/t^2$), the result follows.

COROLLARY. *If there exist numbers $h>0$ and $t>1$ such that $m_{pq}-t\sigma_{pq}>h$ for all pairs of points p, q in the statistical metric space S, then the ϵ, λ-topology is discrete.*

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REGULAR POLYGONS OVER $GF[3^2]$

D. W. CROWE, University College, Ibadan, Nigeria

Introduction. Shephard [6] has introduced the notion of regular complex polygon, and has enumerated all such polygons. An example of a regular quaternion polygon has also been given by Crowe [4]. It is, in fact, apparent that any field with a nontrivial involutory automorphism permits an analogous definition of "unitary" and thus of "regular polygon." The purpose of the present note is to illustrate this by finding the regular polygons in the plane over the field with 9 elements, $GF[3^2]$.

Definitions and properties of $GF\ [p^{2n}]$. There is an involutory automorphism, $x\rightarrow x^{p^n}$, in any field $GF[p^{2n}]$ (p prime, $n=1, 2, \cdots$). This defines a *conjugate*, $\bar{x}=x^{p^n}$, in this field. The two-dimensional vector space over this field, with the following unitary structure, will be designated $UG(2, p^{2n})$. Let (x, y) be a point of the space, and let $(x', y')=(x, y)A$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The matrix A is said to be *unitary* if $x\bar{x}+y\bar{y}=x'\bar{x}'+y'\bar{y}'$ for all (x, y). Necessary and sufficient conditions for A to be unitary are $a\bar{a}+b\bar{b}=c\bar{c}+d\bar{d}=1$ and $a\bar{c}+b\bar{d}=0$. In fact:

Every 2×2 unitary matrix of determinant Δ is of the form

$$\begin{pmatrix} a & b \\ -\bar{b}\Delta & \bar{a}\Delta \end{pmatrix},$$

where $a\bar{a}+b\bar{b}=1$.

Proof. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be unitary. Then $a\bar{a}+b\bar{b}=1$, $\bar{a}c+\bar{b}d=0$, and $\Delta=ad-bc$. Thus $a\bar{a}d+b\bar{b}d=d$, and substituting for ad and $\bar{b}d$ from the other two equations yields $\bar{a}(\Delta+bc)-\bar{a}cb=d$, that is, $d=\bar{a}\Delta$. Similarly, $c=-\bar{b}\Delta$.

A (*unitary*) *reflection* in $UG(2, p^{2n})$ is a unitary matrix exactly one of whose eigenvalues is 1. A *regular* (*unitary*) *polygon in* $UG(2, p^{2n})$ is a configuration of points and lines ("vertices" and "edges") which is transformed into itself by two unitary reflections, one, R, which cyclically permutes the vertices on an edge, and another, S, which cyclically permutes the edges at one of these vertices. In practice we usually choose this vertex to be (1, 0), so that S has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}.$$

Regular polygons in $UG(2, 3^2)$. We represent the nonzero elements of $GF[3^2]$ as powers, $\gamma, \gamma^2, \cdots, \gamma^8=1$, of a root, γ, of the irreducible polynomial x^2+x+2 (mod 3). (See, for example, [1], ch. IX.) Thus $\gamma^2+\gamma+2=0$, so that $\gamma^2=2\gamma+1$, $\gamma^3=2\gamma^2+\gamma=(\gamma+2)+\gamma=2\gamma+2$, etc. We note that the only reflections in $UG(2, 3^2)$ are of period two or four. For if

$$P=\begin{pmatrix} a & b \\ -\bar{b}\Delta & \bar{a}\Delta \end{pmatrix}$$

is of period eight then $\Delta=\gamma^r$, r odd. But $(-\bar{b}\Delta)(\overline{-\bar{b}\Delta})+(\bar{a}\Delta)(\overline{\bar{a}\Delta})=\Delta\bar{\Delta}=\gamma^r\overline{\gamma^r}=\gamma^{4r}=-1\neq 1$, so that P is not unitary. The pairs, R, S, of generating reflections for regular polygons in $UG(2, 3^2)$ are thus of three types: (*i*) both reflections of period four, (*ii*) both reflections of period two, and (iii) one reflection each of periods two and four. We treat the three cases separately.

Case (*i*). Let

$$R=\begin{pmatrix} a & b \\ -\bar{b}\Delta & \bar{a}\Delta \end{pmatrix}, \qquad S=\begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}.$$

Since R (or its inverse) has eigenvalues $1, \gamma^2$ we have $\Delta=\gamma^2$, and $a+\bar{a}\Delta=a+a^3\gamma^2=1+\gamma^2=\gamma^3$. The solutions to the latter equation are $a=\gamma^2, 1, \gamma^7$. The first two

solutions yield "degenerate polygons" having only one edge and one vertex respectively. Corresponding to $a=\gamma^7$ there are four matrices, namely

$$R = \begin{pmatrix} \gamma^7 & \gamma \\ \gamma & \gamma^7 \end{pmatrix}, \quad SRS^{-1}, \quad S^2RS^2, \quad \text{and} \quad S^{-1}RS.$$

Thus the only nondegenerate polygon is generated by

$$R = \begin{pmatrix} \gamma^7 & \gamma \\ \gamma & \gamma^7 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}.$$

Its 24 vertices are the images of (1, 0) under R and S. Using Shephard's notation, ${}_nx$, to designate the n numbers obtained by multiplying x by the nth roots of unity, we can write them as $({}_41, 0)$, $(0, {}_41)$, and $({}_4\gamma, {}_4\gamma)$. They are exactly the 24 points of the unit circle, $x\bar{x}+y\bar{y}=1$, in UG $(2, 3^2)$. These vertices lie by fours on the 24 edges, $x={}_4\gamma$, $y={}_4\gamma$, ${}_4x+{}_4y=1$. Any one of the 96 figures consisting of an edge and a vertex on it can be transformed into any other by an element of the group $\{R, S\}$ generated by R and S. Thus $\{R, S\}$ has order at least 96. But R and S satisfy $R^4=I$ and $RSR=SRS$, so that $\{R, S\}$ has order at most 96 ([3], p. 79). It is thus seen that this polygon is an isomorphic copy of the regular complex polygon $4\{3\}4$, or 4(96)4, in Shephard's notation. In fact, its group is the group of all unitary transformations in $UG(2, 3^2)$. As such it is discussed in detail by Edge in [5].

Case (ii). If R and S are both of period two we have $\Delta=-1$, and

$$R = \begin{pmatrix} a & b \\ \bar{b} & -\bar{a} \end{pmatrix}.$$

Thus $a-\bar{a}=1-1=0$, which has solutions $a=\pm 1$, 0. The nonzero solutions yield degenerate polygons, as in Case (i). Corresponding to $a=0$ we get

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad SRS^{-1}, \quad \text{where} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus the only nondegenerate polygon is generated by this R and S, and has the four vertices $(\pm 1, 0)$, $(0, \pm 1)$. It is an isomorphic copy of the square, $2\{4\}2$. The group $\{R, S\}$ is the dihedral group of order eight, defined by $R^2=S^2=I$, $(RS)^2=(SR)^2$.

Case (iii). If R and S have different periods there are two possibilities (aside from degenerate polygons analogous to those in the preceding cases). For one of these we take

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}.$$

Its eight vertices, $({}_41, 0)$ and $(0, {}_41)$, lie in pairs on the 16 edges, ${}_4x+{}_4y=1$, and

there are four of these edges at each vertex. Its group, $\{R, S\}$, is of order 32, and has the abstract definition $R^2=S^4=I$, $(RS)^2=(SR)^2$. This polygon is also an isomorphic copy of a regular complex polygon, $2\{4\}4$, or $2(32)4$ in Shephard's notation. The other polygon is the dual, $4\{4\}2$, of this. It is most efficiently obtained by interchanging the roles of R and S. Thus we put $R'=S$ and $S'=R$, and find the images of a point, say (γ, γ), left invariant by S'. The 16 vertices, $({}_4\gamma, {}_4\gamma)$, thus obtained lie by fours on the eight edges, $x={}_4\gamma$, $y={}_4\gamma$, and there are two of these edges at each vertex.

Another example. We conclude with another "representation" of a complex polygon. Coxeter ([2], pp. 107, 108) has represented the regular complex polygon $3\{3\}3$ in $EG(2, 3)$. But his generators R_1, R_2 cannot qualify as reflections, for their eigenvalues are all 1's. (Furthermore, they are not unitary.) In $EG(2, 7)$ the situation is improved somewhat, for the matrices

$$R=\begin{pmatrix}4 & 1\\ 1 & 6\end{pmatrix} \quad\text{and}\quad S=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

(entries mod 7) each have eigenvalues 1 and 2, that is, 1 and a cube root of 1, and could thus conceivably be called "reflections." Applying them to (1, 0) yields eight vertices, $(\pm 1, 0)$ and $\pm(3, {}_33)$, lying by threes on eight edges, $x+{}_3y=\pm 1$ and $x=\pm 3$. The group $\{R, S\}$ is defined by $R^3=I$, $RSR=SRS$, and has order 24. This is thus another representation of $3\{3\}3$, but this R and S are still not unitary.

The smallest field permitting a representation of the generators of $3\{3\}3$ as unitary reflections is $GF[5^2]$. In fact, if β is a root of the irreducible polynomial, x^2+3x+3 (mod 5), then

$$R=\begin{pmatrix}\beta^{15} & \beta^2\\ \beta^6 & \beta^{11}\end{pmatrix} \quad\text{and}\quad S=\begin{pmatrix}1 & 0\\ 0 & \beta^8\end{pmatrix}$$

yield such a representation. The eight vertices are $\pm(1, 0)$ and $\pm(\beta^3, {}_3\beta^6)$, lying by threes on the eight edges $x=\pm\beta^3$ and $x+{}_3\beta^7y=\pm 1$.

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BOUNDS FOR SOLUTIONS OF THE RICATTI EQUATION

I. I. KOLODNER, University of New Mexico and the Sandia Corporation

Consider the solution ϕ of the Ricatti equation

$$\dot{x} + x^2 = f^2, \qquad f \in C[0, l) \tag{1}$$

reducing to $p>0$ at $t=0$. The maximal interval I on which $\phi \in C^1$ can be defined is not an issue here: one can easily show that under the stated conditions on f and p, $I=[0, l)$ and $\phi>0$ on I. Our concern is upper and lower bounds for ϕ and the main result is the inequality (5) below. Present considerations will go through also, with obvious modifications, if merely $f \in \mathfrak{L}_2$ on every closed subinterval of I and $p \geqq 0$.

1. Derivation of bounds. From (1) it follows that*

$$\phi(t) = p + \int f^2 - \int \phi^2; \tag{1)'}$$

also, that $1/\phi$ is the solution of the differential equation

$$\dot{x} + f^2 x^2 = 1, \tag{2}$$

with initial value $1/\phi(0) = 1/p$. From (2) we conclude that

$$1/\phi(t) = 1/p + t - \int (f/\phi)^2. \tag{2)'}$$

Let now a be a nonnegative number. Multiply (2)′ by a^2 and add to (1)′, obtaining

$$\begin{aligned}\phi(t) + a^2/\phi(t) &= p + a^2/p + \int f^2 + a^2 t - \int (\phi^2 + (af/\phi)^2) \\ &= p + a^2/p + \int (f-a)^2 - \int (\phi - af/\phi)^2 \\ &\leqq p + a^2/p + \int (f-a)^2 = 2z(t, a).\end{aligned}$$

Since $\phi(t)>0$, we can multiply this inequality by $\phi(t)$, getting

$$\phi^2(t) + a^2 \leqq 2z(t, a)\phi(t). \tag{3}$$

The inequality (3) holds for all $t \in I$, and all $a \geqq 0$. Thus the trinomial $y^2 - 2z(t, a)y + a^2$ must have two real zeros, namely $x^*(t, a) = z + \sqrt{(z^2 - a^2)}$ and $x_*(t, a) = z - \sqrt{(z^2 - a^2)}$. (This also follows directly from the fact that $2z \geqq p + a^2/p \geqq 2a$; equality holds only if $p=a$ and $f \equiv a$, in which case $\phi \equiv a$.) From (3) one concludes that

* We write $\int g$ for $\int_0^t g(\tau)d\tau$.

$$x_*(t, a) \leqq \phi(t) \leqq x^*(t, a). \tag{4}$$

Consider now (4) for fixed t. Since x_* is bounded above, and x^* below, $x_+(t) = \inf_{a\geqq 0} x^*(t, a)$ and $x_-(t) = \sup_{a\geqq 0} x_*(t, a)$ both exist. Since (4) holds for all $a \geqq 0$, it follows that

$$x_-(t) \leqq \phi(t) \leqq x_+(t), \tag{5}$$

the main result. Below we shall show that

$$\begin{aligned} x_+(t) &= x^*(t, a_-(t)) = p + \int f^2 - a_-(t) \int f, \\ x_-(t) &= x_*(t, a_+(t)) = p + \int f^2 - a_+(t) \int f, \end{aligned} \tag{6}$$

where

$$2a_\pm(t) \int f = (p^{-1} + t)^{-1} \left[g + 2\left(\int f\right)^2 \pm \sqrt{\Delta} \right], \tag{7}$$

and

$$\begin{aligned} g(t) &= pt + p^{-1} \int f^2 + t \int f^2 - \left(\int f\right)^2, \\ \Delta(t) &= g^2 - 4\left(\int f\right)^2. \end{aligned} \tag{8}$$

Observe that since, by the Cauchy-Schwartz inequality, $t\int f^2 \geqq (\int f)^2$, $g \geqq pt + p^{-1}\int f^2 \geqq 2\sqrt{(t\int f^2)}$. Thus $\Delta \geqq 4(t\int f^2 - (\int f)^2) \geqq 0$, using the Cauchy-Schwartz inequality again.

Consider $x^* \in C^1[0, \infty)$.† We get

$$x^{*\prime} = z' + (zz' - a)/\sqrt{(z^2 - a^2)}, \qquad z' = (p^{-1} + t)a - \int f.$$

Since $z'(0) = -\int f \leqq 0$ and $z > 0$, $x^{*\prime}(0) \leqq 0$. Since $z' \to \infty$ as $a \to \infty$, and for sufficiently large a, $zz' - a > a(z' - 1)$, $x^{*\prime} \to \infty$ as $a \to \infty$. Consequently x^* assumes a minimum at some $\tilde{a} \geqq 0$ and $x^{*\prime}(\tilde{a}) = 0$. Thus $\tilde{a}$ is one of the zeros of the conditional equation

$$(z')^2 = (zz' - a)^2/(z^2 - a^2),$$

which, after some manipulations, reduces to

$$\left(\int f\right)(p^{-1} + t)a^2 - \left[g + 2\left(\int f\right)^2\right]a + \left(\int f\right)\left(p + \int f^2\right) = 0. \tag{9}$$

† We view x^*, x_*, and z as functions of a only.

Equation (9) has two zeros a_- and a_+, both real and nonnegative, given by (7) and (8). One of these is $\bar{a}$; the second need not even be a zero of $x^{*\prime}$. Since we are interested in $x^*(\bar{a})=x_+$, we will rather compute $x^*(a_-)$ and $x^*(a_+)$ and pick the smaller of these. Now,

$$2z(a_\mp) = (p^{-1}+t)a_\mp{}^2 + \left(p + \int f^2\right) - 2a_\mp \int f = ga_\mp\left(\int f\right)^{-1},$$

by using (9). By using the definition of Δ, formula (8), we get $2\sqrt{\{z^2(a_\mp)-a_\mp^2\}} = a_\mp(\int f)^{-1}\sqrt{\Delta}$. Thus $2x^*(a_\mp)=a_\mp(\int f)^{-1}(g+\sqrt{\Delta})$. Since $a_-\leqq a_+$, this already shows that $x^*(a_-)\leqq x^*(a_+)$, so that $x_+=x^*(a_-)$. By using the definition of a_+, formula (7), we now get $x_+=x^*(a_-)=a_-[(p^{-1}+t)a_+-\int f]$, and the first of formulae (6) follows from the formula for the product a_-a_+. The procedure in establishing the second of these is similar.

2. Discussion. Formulae (6) can be put in a form more convenient for discussion, namely

$$(10) \qquad x_\pm(t) = \tfrac{1}{2}(p^{-1}+t)^{-1}\left(g + 2 \pm \sqrt{\left\{g^2 - 4\left(\int f\right)^2\right\}}\right).$$

If $\hat{x}=\frac{1}{2}(x_+-x_-)$, ϕ is within $\frac{1}{2}(x_+-x_-)$ of $\hat{x}$. The relative error δ committed in replacing ϕ by $\hat{x}$ is then

$$(11) \qquad \delta(t) = \frac{x_+ - x_-}{x_+ + x_-} = \frac{\sqrt{\left\{g^2 - 4\left(\int f\right)^2\right\}}}{g+2}.$$

In the limiting case $p=0$ one gets $x_+(t)=\int f^2$, $x_-(t)=0$ and $\delta(t)=1$.

From (10) one obtains $x_+(t)\leqq p+\int f^2$ and $x_-\geqq(p^{-1}+t)^{-1}$, bounds which are also an immediate consequence of (1)′ and (2)′. Our inquiry concerns mainly the conditions on f which insure that $x_+\leqq A$, or $x_-\geqq A$, or $(p^{-1}+t)x_-\geqq A>1$, or that $\delta\leqq A<1$. One finds that

i. as $t\to 0$, $\delta(t)=\frac{1}{2}|p-p^{-1}f^2(0)|t+o(t)$;
ii. $\dot{\delta}\geqq 0$, $\dot{\delta}(t_0)=0$ for some t_0 iff $f(t)=p$ on $[0, t_0]$;
iii. $\dot{x}_+\geqq 0$, $\dot{x}_-\leqq 0$;
iv. if $l<\infty$, $x_+\leqq A$ or $\delta\leqq B<1$ iff $\int^l f^2<\infty$.

The remaining statements concern the case $l=\infty$. They are expressed in terms of the following conditions:

C_1—for some $\alpha\geqq 0$, $\int^\infty(f-\alpha)^2<\infty$,
C_2—for some $\alpha>0$, $\int^\infty(f-\alpha)^2<\infty$,
C_3—for some $t_0>0$, $(t\int f^2)/(\int f)^2\leqq A$ on $[t_0, \infty)$.

When C_1 is satisfied, let $k=\frac{1}{2}(p+p^{-1}\alpha^2+\int^\infty(f-\alpha)^2)$. Thus:

v. $x_+\leqq A$ iff C_1 holds; then $\lim_{t\to\infty} x_+(t)=k+\sqrt{(k^2-\alpha^2)}$;
vi. $(p^{-1}+t)x_-\geqq A>1$ on $[t_0, \infty)$ iff C_3 holds;

vii. $x_- \geqq A > 0$ iff C_2 holds; then $\lim_{t\to\infty} x_-(t) = k - \sqrt{(k^2 - \alpha^2)}$;

viii. $\delta \leqq A < 1$ iff C_2 holds; then $\lim_{t\to\infty} \delta(t) = k^{-1}\sqrt{(k^2-\alpha^2)}$.

The proof of i–iii is achieved by direct computation. (When computing, say, $\dot{x}_+$ one should recall that $x_+(t) = x^*(t, a_-(t))$ and exploit the fact that $(\partial/\partial a)x^*(t, a_-(t)) = 0$.) The proof of iv–viii involves the study of g/t and $g/\int f$ as $t \to l$. In particular, parts v, vii, and viii involve the statement $\int f^2 - t^{-1}(\int f)^2 \leqq A \Leftrightarrow C_1$, the proof of which we leave as an exercise. In order to determine the limits of $x_\pm$ and δ one has to show first that $\int^\infty g^2 < \infty \Rightarrow \lim_{t\to\infty} t^{-1}(\int g)^2 = 0$.

Observe that $C_2 \Rightarrow C_1$ and $C_2 \Rightarrow C_3$, but there are no other implications among the three conditions. The first statement is trivial; the second follows by noting that $\int f^2 \leqq \alpha^2 t + \int (f-\alpha)^2 + 2\alpha\sqrt{\{t\int(f-\alpha)^2\}}$ and $(\int f)^2 \geqq \alpha^2 t^2 - 2\alpha t\sqrt{\{t\int(f-\alpha)^2\}}$; the third from the fact that $f \in \mathfrak{L}_2 \cap \mathfrak{L}_1$ satisfies C_1 but not C_2 or C_3, while a nonconstant periodic f satisfies C_3 but not C_1 or C_2.

While the bounds (10) are relatively simple, they are also not very satisfactory. Particularly disturbing are the results ii, v, and vii. For example, it is well known* that the properties "f is bounded above," "f is bounded away from zero," and "$\lim_{t\to\infty} f(t)$ exists," are shared by ϕ. However, as follows from v, the first of these is not necessarily shared by x_+, and as follows from vii, the second property is not shared by x_-. Still, our considerations show that boundedness of f is not a necessary condition for the boundedness of ϕ. Below we indicate how these bounds may be put to a better use in conjunction with other properties of solutions of (1).

Let $\tilde{\phi} = \phi(t, \tau, q)$ be that solution of (1) which reduces to q at $t = \tau$. Let $x_\pm(t, \tau, q)$ be the corresponding bounds for $\tilde{\phi}$ given by formula (10). It is well known that if $q < \bar{q}$, then $\phi(t, \tau, q) < \phi(t, \tau, \bar{q})$ on $[\tau, l)$ so that $x_-(t, \tau, q) \leqq \phi(t, \tau, \bar{q})$ and $x_+(t, \tau, \bar{q}) \geqq \phi(t, \tau, q)$; and that if $q < f(\tau)$ ($q > f(\tau)$) then there exists a maximal $\bar{\tau} \leqq l$ such that $\tilde{\phi} < f$ ($\tilde{\phi} > f$) on $[\tau, \bar{\tau})$, $\tilde{\phi}(\bar{\tau}) = f(\bar{\tau})$ (if $\bar{\tau} < l$), $\tilde{\phi}'(\bar{\tau}) = 0$, and f is decreasing (increasing) at $\bar{\tau}$. Using this information alone it is obvious how to construct a continuous upper (lower) bound $\bar{x}_+$ ($\bar{x}_-$) for $\phi = \phi(t, 0, p)$ which is piecewise either constant, or equal to f, or equal to $x_+(t, \tau, q)$ ($x_-(t, \tau, q)$) with appropriate τ's and q's. These bounds will share the boundedness property with f; in particular, if $\inf f \leqq p \leqq \sup f$, they will satisfy $\inf f \leqq \bar{x}_- \leqq \bar{x}_+ \leqq \sup f$. However, $\bar{x}_+$ still will be nondecreasing, and $\bar{x}_-$ nonincreasing.

Another kind of improvement may be made on the basis of the observation that if ϕ is a solution of (1) then one can construct functions g, h, τ (defined on $[0, \bar{t})$, $\bar{t} \leqq l$), and ψ such that $\phi(t) = g(t) + h(t)\psi(\tau(t))$ and ψ is again a solution of a Ricatti equation such as equation (1). One would then use bounds (10) for ψ. The choice of g, h, and τ generally depends on f, and their discussion is beyond the scope of this note.

* For an indication of proof see *e.g.* R. E. Bellman, Stability Theory of Differential Equations, New York, 1953, p. 127. This also follows from that in the (t, x) plane the direction field in the half-strip ($t \geqq t_0$, $-\epsilon + \inf_{t \geqq t_0} f \leqq x \leqq \sup_{t \geqq t_0} f + \epsilon$, any $\epsilon > 0$) is strictly confining, and that the graph of ϕ will stay in the interior of a strictly confining strip from some t_1 on.

A SHORT PROOF OF AN EQUIVALENT FORM OF THE SCHROEDER-BERNSTEIN THEOREM

MARSHALL S. HELLMANN, University of Maryland

The usual proof of the Schroeder-Bernstein theorem involves the construction of a mapping by means of an infinite process which many students find hard to grasp. In the following form of the Schroeder-Bernstein Theorem the desired mapping is defined in an explicit fashion which conceals this process and facilitates an easy understanding of the theorem.

THEOREM 1. *If A, B, C are pairwise disjoint sets and f is a one-to-one mapping of A onto $A\cup B\cup C$, there exists a one-to-one mapping g of A onto $A\cup B$.*

Proof. Let f^k be the composite mapping defined as follows:

$$f^1(x) = f(x) \text{ for } x \in A, \qquad f^k(x) = f[f^{k-1}(x)] \text{ if } f^{k-1}(x) \in A.$$

Let $\overline{A} = \{x \mid \text{ for some } k, f^k(x) \in C\}$, $\overline{B} = \{x \mid \text{ for all } k, f^k(x) \notin C\}$. Then $\overline{A}\cap\overline{B} = \phi$ and $\overline{A}\cup\overline{B} = A$. Now define a mapping g on A by $g(x) = x$ for all $x\in\overline{A}$, $g(x) = f(x)$ for all $x\in\overline{B}$. A straightforward verification shows that g is the desired mapping.

The Schroeder-Bernstein theorem in its usual form follows immediately.

THEOREM 2. *If f and h are one-to-one mappings such that f maps B onto $D_1\subset D$ and h maps D onto $B_1\subset B$, there exists a mapping which maps B onto D in a one-to-one fashion.*

Proof. $h(D_1) = A\subset B_1\subset B$. Therefore $hf(B) = A$, and since A, $B_1 - A$, $B - B_1$ are pairwise disjoint and $B = A\cup(B_1 - A)\cup(B - B_1)$ we may now employ the preceding theorem to establish the existence of a one-to-one mapping g such that $g(A) = B_1$. Hence $h^{-1}g(A) = D$ and since $A = hf(B)$, $h^{-1}ghf(B) = D$. Therefore $h^{-1}ghf$ is a one-to-one mapping of B onto D.

A COMMENT ON RYSER'S "NORMAL AND INTEGRAL IMPLIES INCIDENCE" THEOREM

K. GOLDBERG, National Bureau of Standards

1. Introduction. One of H. J. Ryser's well-known results on matrices satisfying the incidence equation states that if such a matrix is normal and integral it is a 0, 1 matrix.

The purpose of this note is to show (in a slightly generalized form) that "integral" can be replaced by "each of its nonzero elements is at least 1 in absolute value." The essence of the condition is thus size and not algebraic type.

As usual, we let I and J denote the identity matrix and the matrix with 1 in every position, respectively. We assume that we are given a real, nonsingular matrix A with at least one positive element, satisfying

$$AA^T = kI + \lambda(J - I) \tag{1}$$

for some nonzero constants k, λ, and that all matrices are of order v.

If k, λ are integers satisfying $k-\lambda+\lambda v=k^2$ then (1) is known as the incidence equation. If, in addition, A is a 0, 1 matrix it is said to represent a v, k, λ design. We shall not assume that $k-\lambda+\lambda v=k^2$.

It is easy to show that the determinant of A is

$$\det A = \pm (k - \lambda + \lambda v)^{1/2}(k - \lambda)^{(v-1)/2}.$$

The condition that A be nonsingular is thus equivalent to $k-\lambda+\lambda v\neq 0$ and $k-\lambda\neq 0$.

It is also easy to show that $k-\lambda+\lambda v$ is nonnegative. Since we are assuming that it is not 0, we can let t denote its positive square root $t=(k-\lambda+\lambda v)^{1/2}>0$. Note that in the incidence equation $t=k$. We shall prove:

THEOREM. *A is a scalar multiple of a* 0, 1 *matrix if and only if it is normal and each of its nonzero elements is at least k/t in absolute value. The scalar is k/t and the* 0, 1 *matrix represents a v, t^2/k, $\lambda t^2/k^2$ design.*

2. **Proof of the theorem.** The necessity is well known, so we proceed with the sufficiency.

Since A is normal we have $AA^T=(k-\lambda)I+\lambda J=A^TA$. Therefore $\lambda AJ=AA^TA-(k-\lambda)A=\lambda JA$.

It follows that all the row and column sums are equal to some constant c. Therefore, all the row and column sums of AA^T are equal to c^2. But all the row and column sums of $(k-\lambda)I+\lambda J$ are equal to $k-\lambda+\lambda v=t^2$. It follows that $c=\pm t$. We can multiply A by -1 if necessary to assure $c=t$, without changing the conditions on A. Note that $c=t>0$ assures that A will have a positive element.

On the other hand, the sum of the squares of the elements in any row of $A=(a_{ij})$ is just the corresponding diagonal element of AA^T. From (1) we see that this is k. That is,

$$\sum_{j=1}^{v} a_{ij} = t, \qquad \sum_{j=1}^{v} a_{ij}^2 = k, \qquad i = 1, \cdots, v. \tag{2}$$

We now use the fact that each nonzero element of A is at least k/t in absolute value, in the following form: $ta_{ij}^2\geqq k|a_{ij}|$, $(i, j=1, \cdots, v)$. Combining this with (2) we have

$$tk = t\sum_{j=1}^{v} a_{ij}^2 \geqq k \sum_{j=1}^{v} |a_{ij}| \geqq k\sum_{j=1}^{v} a_{ij} = kt.$$

Since we have equality throughout it follows that $ta_{ij}^2=ka_{ij}$, $(i,j=1, \cdots, v)$. In other words, every nonzero element of A *equals* k/t: A is k/t times a 0, 1 matrix, and the theorem is proved.

Reference

H. J. Ryser, Matrices with integer elements in combinatorial investigations, Amer. J. Math., vol. 74, 1952, pp. 769–773.

AN EXTENSION OF MIRSKY'S EXISTENCE THEOREM

ALFRED HORN, University of California, Los Angeles

In a previous paper,* Mirsky has determined necessary and sufficient conditions on the sequences ρ_i, σ_i, d_i in order that they be the row sums, column sums, and diagonal elements respectively of a matrix with nonnegative elements. In his paper it was assumed that the row and column sums were finite. It seems desirable to complete Mirsky's result for the case where the ρ_i and σ_i are allowed to be infinite. Without loss of generality we may assume the d_i are all 0.

THEOREM. *Suppose* $0 \leqq \rho_i \leqq \infty$, $0 \leqq \sigma_i \leqq \infty$. *A set of necessary and sufficient conditions that there exist a matrix* (x_{ij}) *with* $x_{ii}=0$, $\sum_j x_{ij}=\rho_i$, $\sum_i x_{ij}=\sigma_j$ *and* $0 \leqq x_{ij} < \infty$ *is*

$$\rho_i \leqq \sum_{j \neq i} \sigma_j \qquad \textit{for all } i; \tag{1}$$

$$\sigma_i \leqq \sum_{j \neq i} \rho_j \qquad \textit{for all } i; \tag{2}$$

$$\sum_i \sigma_i = \sum_i \rho_i; \tag{3}$$

$$\textit{if any } \rho_i = \infty, \quad \textit{then } \sum_{j=k}^{\infty} \sigma_j = \infty \qquad \textit{for all } k; \tag{4}$$

$$\textit{if any } \sigma_i = \infty, \quad \textit{then } \sum_{j=k}^{\infty} \rho_j = \infty \qquad \textit{for all } k. \tag{5}$$

Proof. Note that if any $\rho_i = \infty$, then (4) implies (1) and (3), and similarly (5), when applicable, implies (2) and (3). The necessity of (1), (2), and (3) is clear, as in Mirsky's paper. If some $\rho_i = \infty$, then $\sum_j x_{ij} = \infty$, and therefore $\infty = \sum_{j=k}^{\infty} x_{ij} \leqq \sum_{j=k}^{\infty} \sigma_j$ for all k. To prove the sufficiency, we note that by Mirsky's induction argument we need only show that if ρ_i, σ_i satisfy (1)–(5), then there exist sequences x_i, y_i, $i \geqq 2$, of real numbers such that

$$0 \leqq x_i \leqq \sigma_i \qquad (i \geqq 2); \tag{6}$$

$$0 \leqq y_i \leqq \rho_i \qquad (i \geqq 2); \tag{7}$$

$$\sum_{i=2}^{\infty} x_i = \rho_1; \tag{8}$$

$$\sum_{i=2}^{\infty} y_i = \sigma_1; \tag{9}$$

(10) the sequences $\rho_i - y_i$, $\sigma_i - x_i$ satisfy (1)–(5).

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* This MONTHLY, vol. 68, 1961, pp. 465–469.

If the ρ_i and σ_i are all finite, this has been proved by Mirsky. The remaining case is divided into three cases.

Case 1. Some $\rho_i=\infty$ and some $\sigma_j=\infty$.

We will construct x_i so that (6) and (8) hold, and also

$$\sum_{i=k}^{\infty}(\sigma_i-x_i)=\infty \qquad \text{for all } k\geqq 2. \tag{11}$$

If $\rho_1<\infty$, then by (4) we can obviously find x_i so that (6) and (8) hold, and (4) will imply (11). If $\rho_1=\infty$, set $x_i=\sigma_i/2$ whenever $\sigma_i<\infty$ and $x_i=1$ whenever $\sigma_i=\infty$. If only finitely many σ_i are infinite, then the remaining σ_i have an infinite sum by (4) and so (8) and (11) hold. If infinitely many σ_i are infinite then (8) and (11) are again obvious. In a similar manner we can find y_i so that (7) and (9) hold, as well as

$$\sum_{i=k}^{\infty}(\rho_i-y_i)=\infty \qquad \text{for all } k\geqq 2. \tag{12}$$

Conditions (11) and (12) clearly imply (10).

Case 2. All ρ_i are finite, and $\sigma_n=\infty$ for some $n\geqq 2$.

Let $t=\sum_{i\geqq 2, i\neq n}\sigma_i$. By hypothesis $\rho_n\leqq\sigma_1+t$, and $\sum_{i=1}^{\infty}\rho_i=\infty$. Choose $x_i=0$ for $i\geqq 2$, $i\neq n$, and $x_n=\rho_1$. We now determine y_i so that (7), (9), and (12) hold as well as $\rho_n-t\leqq y_n$, in case $t<\infty$. This is possible (see the method of case 1) since $\rho_n-t\leqq\sigma_1$. Since $\sum_{i=2}^{\infty}(\rho_i-y_i)=\infty$ and $\sigma_n=\infty$, we need only prove $\rho_n-y_n\leqq\sum_{i\geqq 2, i\neq n}(\sigma_i-x_i)$. But this holds because the right side is t.

Case 3. All ρ_i are finite, $\sigma_1=\infty$ and $\sigma_i<\infty$ for $i\geqq 2$.

By hypothesis, $\sum_{i=1}^{\infty}\rho_i=\infty$ and $\rho_1\leqq t$, where $t=\sum_{i=2}^{\infty}\sigma_i$. If $t=\infty$, we choose $y_i=\rho_i/2$, and determine x_i so that (6) and (8) hold. We then have $\sum_{i=2}^{\infty}(\rho_i-y_i)=\sum_{i=2}^{\infty}(\sigma_i-x_i)=\infty$, so that (10) is satisfied. If $t<\infty$, we will determine sequences of real numbers u_i, v_i, $i\geqq 2$, so that $0\leqq u_i\leqq\sigma_i$, $0\leqq v_i\leqq\rho_i$, and $u_i+v_i\leqq\sum_{i=2}^{\infty}u_i=\sum_{i=2}^{\infty}v_i=t-\rho_1$. To do this, first choose u_i so that $0\leqq u_i\leqq\sigma_i$ and $\sum_{i=2}^{\infty}u_i=t-\rho_1$, and determine v_i from the conditions

$$0\leqq v_i\leqq\begin{cases}\rho_i\\ t-\rho_1-u_i,\end{cases}\qquad \sum_{i=2}^{\infty}v_i=t-\rho_1.$$

The conditions on v_i can be satisfied, because if $t=\rho_1$, we take $u_i=v_i=0$, while if $t>\rho_i$, we have $t-\rho_1-u_i$ bounded away from 0 for sufficiently large i. We now set $x_i=\sigma_i-u_i$ and $y_i=\rho_i-v_i$. For these (6)–(9) are clear and (10) holds because the sequences u_i, v_i satisfy (1)–(3).

ON A POLYGONAL INEQUALITY DUE TO L. FEJES TÓTH

F. LEUENBERGER, Zuoz, Switzerland

If O is an interior point of the n-sided convex polygon A_i $(i=1, \cdots, n)$, R_i the distance OA_i, and r_i the distance from O to the side A_iA_{i+1}, then, as is known [1],

$$G(r_i)/G(R_i) \leqq \cos(\pi/n), \tag{1}$$

where $G(x_i)$ denotes the geometric mean of the quantities x_i.

J. Berkes [2] gave an elementary proof for the case $n=3$ and showed that his method is applicable to a simplex of an arbitrary number of dimensions. For such a simplex, another proof by J. Schopp [3] is available. We shall show that (1) can be improved for arbitrary n as follows:

If w_i is the bisector of the angle $A_iOA_{i+1}=2\alpha_i$, then

$$G(w_i)/G(R_i) \leqq \cos(\pi/n). \tag{2}$$

Proof. Let $A(x_i)$ and $H(x_i)$ denote the arithmetic and harmonic means, respectively, of the quantities x_i. Since $H\leqq G\leqq A$, it follows that

$$\cos\alpha_i = \frac{w_i(R_i+R_{i+1})}{2R_iR_{i+1}} \geqq \frac{w_i}{\sqrt{(R_iR_{i+1})}} = a_i,$$

$$A(\cos\alpha_i) \geqq A(a_i) \geqq G(a_i) = G(w_i)/G(R_i).$$

Because of the convexity of A_i and the concavity of the cosine in $(0, \frac{1}{2}\pi)$, it follows from the above relation by Jensen's inequality that $A(\cos\alpha_i)\leqq\cos(\pi/n)$. From this we get (2), the equality sign being obviously valid only when O is the center of a regular polygon.

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APPROXIMATE SOLUTIONS FOR A FIRST-ORDER, NONLINEAR ORDINARY DIFFERENTIAL EQUATION

IWAO SUGAI, ITT Federal Laboratories

Exact solutions of ordinary nonlinear differential equations are not readily obtained for many physical problems. Sometimes it is useful to derive approximate solutions of ordinary nonlinear differential equations as shown by Kalaba, [1]. Although a numerical solution of the equation considered in this note through a high-speed digital computer is routine, it is of some value to have techniques that provide approximate analytical solutions. This is a point which

should be stressed in undergraduate courses in differential equations. This note presents an approximate solution for a first-order differential equation including y^k where k is a constant which is neither one nor zero. The equation under consideration is;

$$y' + A(x)y^k = B(x), \tag{1}$$

where $' = d/dx$, $y = y(x)$, and $A(x)$ and $B(x)$ are arbitrary functions of x. Apply a transform which linearizes a generalized Riccati's nonlinear differential equation [2] such as

$$y = f(x)/g(x), \tag{2}$$

where $f(x)$ and $g(x)$ are arbitrary functions of x.
Equation (2) reduces (1) to

$$f'g - fg' + Af^kg^{2-k} = Bg^2. \tag{3}$$

If the sum of the second and the third terms of the left-hand side of (3) is zero we have

$$g' = Af^{k-1}g^{2-k}, \tag{4}$$

and (3) becomes

$$f' = Bg. \tag{5}$$

When (4) and (5) are solved exactly simultaneously, then (1) is solved exactly. This idea of splitting (1) into two equations and simultaneously solving these two equations exactly is described in detail by another paper [3].

Now from (5) we have $g' = (Bf'' - B'f')/B^2$. From (4) we then obtain $Bf'' - B'f' = Af^{k-1}g^{2-k}B^2$ and hence $f''/f' - B'/B = Af^{k-1}g^{2-k}B/f'$. Since $g = f'/B$ this gives us

$$(f''/f') - (B'/B) = A(Bf/f')^{k-1}. \tag{6}$$

Now let $h(x) = f'/f$, where $h(x)$ is an arbitrary function of x. In terms of $h(x)$, (6) is rewritten as

$$h^k + h'h^{k-2} - (B'/B)h^{k-1} = AB^{k-1}. \tag{7}$$

The processes are exact up to this point. The approximations take place here because (7) does not seem to be readily solvable. Three interesting cases are given below.

Approximation I. If the first term of the left-hand side of (7) is much smaller than the other left-hand terms, (7) can be solved exactly since it becomes a first-order linear differential equation in h^{k-1} if the term h^k is missing. The approximation is valid if

$$h' - (B'/B)h \gg h^2. \tag{8}$$

This gives $y'(x) \gg -B(x)$, and the approximate solution of (1) is obtained from

$$(h^{k-1})' - (k-1)(B'/B)h^{k-1} = (k-1)AB^{k-1}. \tag{9}$$

Approximation II. The converse of (8) is also possible. If the first term of the left-hand side of (7) is much larger than the other left-hand terms, an approximate solution of (1) can be obtained. For $h^2 \gg h' - (B'/B)h$ and this inequality is expressed by $-B(x) \gg y'(x)$, so that the approximate solution of (1) is obtained from

$$h^k = AB^{k-1}. \tag{10}$$

Approximation III. This case results in (9) by Approximation I, although a different approach is taken. Regrouping terms of (7), the following equation is thus obtained:

$$(h^{k-1})' + (k-1)h^{k-1}(h - B'/B) = (k-1)AB^{k-1}. \tag{11}$$

The approximation is to require that $h \ll B'/B$, which is rewritten as $B^2(x) \ll -B'(x)y(x)$. The approximate solution of (1) is obtained from (9), given before.

Approximate methods are selected in the light of the difficulty of finding the exact solution of (1), as shown by [2]. Although $y(x)$ and $y'(x)$ are unknowns, in some problems the knowledge of behaviors of $y(x)$ and $y'(x)$ may be available. In these problems, if conditions of approximations given by the above inequalities are satisfied reasonably well, approximate solutions of (1) are obtained. It is interesting to note that there has been no restriction on constant k except that it is not one and it is not zero. An immediate application of this note is to study a nonlinear differential equation,

$$y' = axy^{-1/3}, \tag{12}$$

where a is constant, and $y = y(x)$. The approximate numerical solution of (12) was discussed from the viewpoint of programming, but the approximate analytical solution was not presented in the previous result [4].

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$$(2p_1 - 1) \cdots (2p_{2m+1} - 1) = 1 - 2m(a + b) - 2a.$$

Hence (10) reduces to

$$c/\{1 - 2m(a + b)\} = 2, \qquad c/\{1 - 2m(a + b) - 2a\} = -2.$$

This shows that $(2-c)/(2a+2b)$ is either $2m$ or $2m+1$, *i.e.*, it is a positive integer. Substituting the corresponding expressions for a, b, c, we have (2), with the minus sign, a positive integer. Now, $dy/dx - Qy + Ry^2 = P$ is solvable if $dy/dx + Qy + Py^2 = R$ is solvable, since one goes into the other by writing $1/y$ for y. In such a case we have (2), with the plus sign, a positive integer. This completes the proof of Case (b).

In Case (a), the final solution is written in the form of a continued fraction

$$y = p_1 \frac{Q}{R} + \cfrac{1}{p_2 \frac{Q}{P} +} \; \cfrac{1}{p_3 \frac{Q}{R} +} \cdots \frac{1}{y_n},$$

where the p's are defined as above and y_n is a solution of the equation in the form (5).

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to J. M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

A CANONICAL BASIS FOR NILPOTENT TRANSFORMATIONS

H. F. TROTTER, Princeton University

A linear transformation of a vector space into itself is *nilpotent* if some power of it is identically 0. In both [1] and [2] the discussion of the Jordan canonical form for matrices is made to depend on certain properties of nilpotent transformations. The most complicated step in the argument is the proof of the theorem stated below, which is equivalent to Theorem 8.4 of [1] or Theorem 2 of Section 57 of [2]. The purpose of this note is to present a slightly simpler proof.

We follow the conventions of [1], using Greek letters for vectors and Latin letters for scalars, and writing ξT for the image of ξ under the linear transformation T. The subspace of all ξ such that $\xi T = 0$ is called the *null-space* of T and denoted by $N(T)$.

T is *nilpotent of index* p if T^p is 0 but T^{p-1} is not. We define the *index* (under a nilpotent transformation T) of a vector ξ as the smallest natural number k for which $\xi T^k=0$. Suppose T a nilpotent transformation defined on a space V, and let $\xi_1, \cdots, \xi_r$ be a set of vectors in V with respective indices $p_1, \cdots, p_r$ under T. We define the set *generated by* $\xi_1, \cdots, \xi_r$ under T to consist of all the vectors $\xi_i, \xi_i T, \cdots \xi_i T^{p_i-1}$ for $i=1, \cdots, r$, and say that the vectors $\xi_1, \cdots, \xi_r$ are a *T-basis* for V if the set they generate is a basis for V in the usual sense. The result we wish to prove is the following.

THEOREM. *If T is a nilpotent transformation on a finite-dimensional space V then V has a T-basis.*

We shall actually prove by induction that the following slightly stronger statement is true for all values of p.

LEMMA (p). *If T is a nilpotent transformation of index p on a finite-dimensional space V, then V has a T-basis. Furthermore if $\xi_1, \cdots, \xi_r$ are any vectors such that $\sum_{i=1}^r a_i\xi_i$ is in $N(T^{p-1})$ only if all $a_i=0$, then there is a T-basis containing $\xi_1, \cdots, \xi_r$.*

To start the induction, note that if T is nilpotent of index 1 then $T=0$. Thus lemma (1) simply asserts that a linearly independent subset of V can be extended to a basis, and is therefore true.

Now assume Lemma (p) as inductive hypothesis and consider a transformation T on V which is nilpotent of index $p+1$. Suppose s vectors $\eta_1, \cdots, \eta_s$ (we include the possibility $s=0$) are given such that $\sum_{i=1}^s a_i\eta_i$ is in $N(T^p)$ only if all $a_i=0$. Let $\alpha_1, \cdots, \alpha_m$ be a basis for $N(T^p)$. Then $\{\alpha_1, \cdots, \alpha_m, \eta_1, \cdots, \eta_s\}$ is a linearly independent set and can be extended (if necessary) to a basis $\{\alpha_1, \cdots, \alpha_m, \eta_1, \cdots, \eta_r\}$ for V. (Note that for the extended set of η's it is still true that $\sum_{i=1}^r a_i\eta_i$ is in $N(T^p)$ only if all $a_i=0$, and that the η's together with *any* basis for $N(T^p)$ will give a basis for V.) Let $\xi_i=\eta_i T$, $i=1, \cdots, r$. The ξ_i are in $N(T^p)$ and $(\sum_{i=1}^r a_i\xi_i)T^{p-1}=(\sum_{i=1}^r a_i\eta_i)T^p$, which is 0 only if all $a_i=0$. Thus the ξ_i satisfy the hypothesis of Lemma (p) relative to $\overline{T}$, the restriction of T to $N(T^p)$. Since $\overline{T}$ is nilpotent of index p, there is a $\overline{T}$-basis for $N(T^p)$ of the form $\xi_1, \cdots, \xi_r, \xi_{r+1}, \cdots, \xi_k$ for some $k\geqq r$. Now consider the vectors $\eta_1, \cdots, \eta_r, \xi_{r+1}, \cdots, \xi_k$. The set they generate under T consists of $\{\eta_1, \cdots, \eta_r\}$ together with the set generated by $\xi_1, \cdots, \xi_k$ under $\overline{T}$. The latter is a basis for $N(T^p)$, and therefore a basis for the entire space is obtained when the η's are added to it. Thus we have obtained a T-basis of the required type and the proof is complete.

This argument is an interesting illustration of the fact that a strong conclusion may be easier to prove by induction than a weak conclusion.

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A NOTE ON CONSECUTIVE COMPOSITE NUMBERS

C. A. GRIMM, South Dakota School of Mines and Technology

The following note is a generalization of problem 4, part II, of the William Lowell Putnam Mathematical Competition, March 5, 1955.

THEOREM. *If* $C_1, C_2, \cdots, C_n$ *are any set of positive integers with* $(C_i, C_j) = 1$, $i \neq j$, *then there are infinitely many sets of* n *consecutive positive integers* $I_1, I_2, \cdots, I_n$ *with* I_j *divisible by* C_j.

Proof by induction. Given C_1, C_2 with $(C_1, C_2) = 1$ and positive, there are an infinite number of positive integer pairs M_2, M_1 such that $M_2C_2 - M_1C_1 = 1$. The proof is immediate by the standard method for solving a Diophantine equation of this type. We obtain an infinite number of solutions of the form $M_1 = C_2\sigma + \rho_1$, $M_2 = C_1\sigma + \rho_2$; ρ_1 and ρ_2 integers, σ the parameter of the solution. The consecutive integers are $I_1 = M_1C_1$, $I_2 = M_2C_2$. We further observe that the coefficients of the parameter, σ, are the two C's, C_2 with M_1, C_1 with M_2. We now assume the theorem for the first K C's. Hence we have the $K-1$ equations, $M_{i+1}C_{i+1} - M_iC_i = 1$, $i = 1, 2, \cdots, K-1$, with solutions of the form $M_j = \alpha_j\lambda + \beta_j$. In the solutions λ is the parameter, the β_j are constants and the $\alpha_j = C_1C_2 \cdots C_{j-1} C_{j+1} \cdots C_k$. The required integers are $I_j = M_jC_j$. Since $(C_{k+1}, C_k) = 1$, there are infinitely many solutions for $M_{k+1}C_{k+1} - M_k' C_k = 1$ of the form $M_k' = C_{k+1}\eta + \delta_1$, $M_{k+1} = C_k\eta + \delta_2$. We require that at least one of the M_k' solutions be identical to one of the M_k solutions. That is, $C_{k+1}\eta + \delta_1 = \alpha_k\lambda + \beta_k$ for at least one η and λ. But since $\alpha_k = C_1C_2 \cdots C_{k-1}$, $(C_{k+1}, \alpha_k) = 1$ and it is easy to show that we have indeed an infinite number of positive integer solutions for η and λ of the form $\eta = \alpha_k\psi + \gamma_1$, $\lambda = C_{k+1}\psi + \gamma_2$. Substitution of the η solution in the solution for M_{k+1} and the λ solution in the solutions for M_j completes the proof.

Example. Let $P_1, \cdots, P_n$ be any n primes, $\alpha_1, \cdots, \alpha_n$ any n positive integers, then there are infinitely many sets of n consecutive integers, $I_1, \cdots, I_n$ such that I_j is divisible by $P_j^{\alpha_j}$.

ON THE EQUATIONS FOR A FLEXIBLE SUSPENSION CABLE

MORRIS MORDUCHOW, Polytechnic Institute of Brooklyn

A standard problem in elementary mechanics, calculus or differential equations is the deflection of a flexible cable suspended between two supports (Fig. 1a). With only rare exceptions (*e.g.*, [1]–[3]), nearly all of the texts the author has seen treat this problem by assuming in advance that the deflection will have a zero (horizontal) slope somewhere between the supports, and the differential equation for the deflection is then obtained by considering equilibrium of the portion of the cable between this point and an arbitrary point of the cable. However, unless supports A and B are on the same level ($h = 0$), in which case Rolle's theorem can be applied, it is not at all true *a priori* that such a point of zero slope along the cable need necessarily exist. It will be seen, in fact, that

such need not necessarily be the actual case. The purpose of this note is to call attention to these facts, and to indicate, for convenient reference, a derivation of the cable equations without assuming in advance the existence of a point of zero slope. The basic difference between the present derivation and that usually given is that here, in developing the equations of equilibrium, an arbitrary "small" element of length Δs, rather than the particular type of element of finite length s indicated above, will be taken.

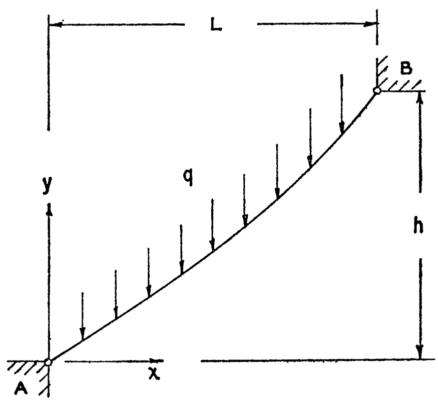

FIG. 1a

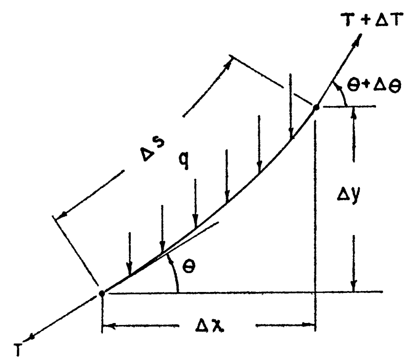

FIG. 1b

Consider now a small arbitrary element Δs of the deflected cable (Fig. 1b). For equilibrium of forces in the x direction,

(1) $$-T\cos\theta + (T+\Delta T)\cos(\theta+\Delta\theta) = 0,$$

where T is the tension in the cable. For equilibrium of forces in the y direction, an external distributed loading $q(x)$ per unit *span* x (parabolic cable for constant q) in the negative y direction will be assumed. Then*

(2) $$-T\sin\theta + (T+\Delta T)\sin(\theta+\Delta\theta) - q\Delta x = 0.$$

By dividing by Δx, and taking the limit as $\Delta x \to 0$, these equations reduce to the form

(3) $$d(T\cos\theta)/dx = 0,$$

(4) $$d(T\sin\theta)/dx = q(x).$$

Equation (3) implies

(5) $$T = H\sec\theta,$$

where H is an arbitrary constant. The constant H denotes physically the hori-

* Although at first independently developed by the author, the author subsequently found (1) and (2) in [2]. Alternative (equivalent) sets of equations are given in [1] and [3].

zontal component of the tension at any point along the cable, and the actual tension at a point of zero slope, *if* such a point exists.

Integrating (4) with respect to x, substituting for T according to (5), noting that $\tan\,\Theta = dy/dx$, and integrating again, one obtains:

$$y = H^{-1}\left[\int\left(\int q(x)dx\right)dx + C_1x\right] + C_2, \tag{6}$$

where C_1 and C_2 are arbitrary constants. Suppose now that the loading is *uniform*, *i.e.*, $q(x) = $ constant. Then, choosing the origin* at support A, and requiring that $y=0$ at $x=0$, while $y=h$ at $x=L$, (6) yields:

$$y = \frac{qx^2}{2H} + \left(\frac{h}{L} - \frac{qL}{2H}\right)x. \tag{7}$$

According to (7), $dy/dx=0$ at $x=x_0$ (say), where

$$x_0 = \tfrac{1}{2}L - (hH)/(qL). \tag{8}$$

From (8) it follows that for the (usual) case of $h \geqq 0$ and $q>0$, $x_0<L$ (in fact, $x_0 \leqq \frac{1}{2}L$). However, it is possible that $x_0<0$, namely when

$$(Hh)/(qL^2) > \tfrac{1}{2}. \tag{9}$$

If condition (9) holds, then the slope of the cable will *not* have any zero slope between its supports.

The case of a uniform loading per unit of *cable* length, leading to a catenary, can be similarly treated.

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* It appears to be almost universal practice to choose as origin the point of zero slope of the cable. However, in case there is no such point on the cable, such a choice appears somewhat artificial, and is (in any case) unnecessary.

SOME SIMPLE CALCULATIONS BASED ON VARIATIONAL PRINCIPLES

DAVID SHELUPSKY, Princeton University

One of the first encounters with the calculus of variations that most students have is the mention made of it in the elementary physics classroom in connection with the principles of least action in mechanics and the principle of least time of Fermat in optics. Since at this first meeting even the simplest deductions of the calculus of variations may be unfamiliar to most students, the following discussion may prove useful.

The method that we shall use is that of Daniel Bernoulli in his solution of

the problem of the brachistochrone [1]. In this treatment the problem of the curve of quickest descent is treated as a problem in ray optics. If the positive y-axis is in the downward direction and a bead is set to sliding on a smooth frictionless wire then the conservation of energy gives us the usual relation $v=(2gy)^{1/2}$ for the speed of the particle when it is at the point (x, y) if the starting position is at the origin. Using the principle of least time in optics the problem may be stated as follows: to find the path of a light-ray in an inhomogeneous medium which has a variable index of refraction which is given by the law

$$n(x, y) = c/v = c(2gy)^{-1/2}$$

(where c is the speed of light in a vacuum). Having formulated the problem in this way Bernoulli solves the equivalent optical problem by dividing the material into a large number of thin slabs parallel to the x-axis, each of which is supposed to have a constant index of refraction. Across the boundaries of these slabs the index is to vary discontinuously so as to approximate the above variation of $n(x, y)$ and here the Snell law of refraction is applied at each boundary. This process gives us in the limit of thin slabs the differential equations of the desired curve, the brachistochrone. This calculation, which is bypassed by the calculus of variations, is suited to the solution of many problems in mechanics and optics as involve indices of refraction and potential energy functions which are functions of one cartesian coordinate.

Here we shall consider the case when these functions are dependent on the radial variable r in the usual (r, θ) plane polar coordinates. Thus, one of the problems which we shall be able to treat using the principle of least action will be the problem of planetary motion.

Let us begin by considering the Fermat principle of least time, according to which (in geometrical optics) a beam of light moves so as to extremize the time of travel between two points, p_1 and p_2, on its path,

$$T = \int_{p_1}^{p_2} ds/v.$$

We shall need Snell's law, the only result of a variational nature in our discussion, and which is frequently given as an exercise in textbooks on elementary calculus: if a ray of light passes through a boundary between two media which we label with the numbers 1 and 2 and in which the speed of light is v_1 and v_2, respectively, then we have $\sin \theta_1/v_1 = \sin \theta_2/v_2$, θ_i being the angle between the interface normal and the ray in the ith medium.

Let us begin with a medium in which v, the velocity of light, is a function of the distance from a given point O. It is clear that every ray path will be a plane curve; if p is the initial point of the path then the plane of the ray will be the plane of the initial direction of the ray and the vector from O to p. In this plane we introduce polar coordinates (r, θ) with O as the origin. Following Bernoulli we divide the plane into thin rings of material in each of which the velocity

function is supposed to be constant. The typical one of these rings is the region with r-coordinate between r and $r+dr$. Considering such a ring, it is clear that the angle at the point (r, θ) which is made by the radius vector to the point and a curve in the plane passing through the point is ϕ, with $\tan \phi = rd\theta/dr$. With this formula we can apply Snell's law. The typical light ray enters the ring at (r, θ) making an angle ϕ with the ring normal (the radius vector to that point) and moving with the speed $v = v(r)$. It leaves the ring at the point $(r+dr, \theta+d\theta)$ making an angle with the ring normal at that point of $\phi+d\phi+d\theta$ and moving with the speed $v+dv$. Here $d\theta$ is the angle change due to the convergence of the radii vectors at the two points of comparison and $d\phi$ is the change in the angle between the ray and local ring normal due to the actual curvature of the path. Snell's law becomes

$$\sin \phi/v = \sin (\phi + d\phi + d\theta)/(v + dv).$$

Neglecting differentials of higher than the first order we have

$$\sin (\phi + d\phi + d\theta) = \sin \phi + (d\phi + d\theta) \cos \phi,$$

so that Snell's law may be written as

$$dv(\sin \phi) = v(\cos \phi)(d\phi + d\theta) \text{ or as } dv/v = \cot \phi(d\phi + d\theta).$$

By the above relation for ϕ, namely $\tan \phi = rd\theta/dr$, we have that $\cot \phi \cdot d\theta = dr/r$, so that the equation becomes

$$dv/v = dr/r + \cot \phi d\phi,$$

which is a differential equation to be integrated. We find, as our fundamental relation, the equation $v = ar\sin \phi$ where a is the constant of integration. On using the relation

$$\sin \phi = r(r^2 + r'^2)^{-1/2} \qquad \text{(where } r' = dr/d\theta\text{)},$$

our fundamental equation becomes the differential equation

$$(r^2 + r'^2)^{1/2}v(r) = ar^2,$$

which we may integrate as

$$\int \frac{v(r)dr}{r(a^2r^2 - v^2(r))^{1/2}} = \theta - \theta_0.$$

Of particular interest is the set of power laws $v(r) = cr^s$, c being a constant. The differential equation becomes

$$\theta - \theta_0 = \int^r cr^{s-2}(a^2 - c^2r^{2s-2})^{-1/2}dr,$$

which is immediately integrable, the resulting equation being

$$r^{s-1} = C \sin ((s - 1)(\theta - \theta_0)) \qquad \text{(where } C = a/c\text{)}.$$

One of the chief attractions of this equation is that for various values of s we get almost all of the curves which are studied in the discussions of polar coordinates in analytic geometry [2]. We remark that the integration performed above is not valid if $s=1$, for then a logarithmic integration enters, the result being

$$r = r_0 e^{b(\theta-\theta_0)} \qquad (\text{where } b = (a^2/c^2 - 1)^{-1/2}).$$

Let us consider the first of these relations. For $s=0$ we have $r \sin(\theta-\theta_0) = -C$ which is the straight line, as we would expect in a medium of constant index of refraction. For $s=1$ we get the exponential spiral as we have seen above, and for $s=2$ we have $r = C \sin(\theta-\theta_0)$ which is the general equation of the circle in polar coordinates which passes through the origin. If we are given any two points in the plane then we can put one unique circle through them which passes through the origin, provided that the three points, the two given points and the origin, are not collinear. If they are collinear then the circle they determine degenerates into the straight line which they determine. Consider the case when they are not collinear. The circle which passes through the two points is divided by them into two arcs on one of which lies the origin. If two rays of light are sent out from one of these points in opposite directions tangent to this circle so as to move along it, then the ray that moves toward the origin will slow down and will spend an infinite time approaching the origin. This is seen if we consider that sufficiently near the origin we can replace the circle by its tangent through the origin. The time it takes for the light to move from a distance $r=\epsilon$ to the origin is then roughly

$$\int_0^\epsilon dr/cr^2 = \int_0^\epsilon dr/v,$$

which is infinite. It is only if the points are on opposite sides of the origin that they cannot be connected by a proper light path, that is, one which has a finite transition time associated with it. The circle in this case, as we have mentioned, tends to the straight line through the origin which connects these two points. Light moving away from the origin moves faster and faster but never gets back to the neighborhood of the origin, while light moving toward the origin slows down and never gets through the central point.

Continuing, now, we set $s=3$ and get the curve $r^2 = C \sin 2(\theta-\theta_0)$ which is the general lemniscate of Bernoulli. In this case we see that a given point p in the medium can be connected with light paths to points whose θ-coordinate differs from that of p by less than $\frac{1}{2}\pi$ radians. For $s=3/2$ we get $r^{1/2} = C \sin \frac{1}{2}(\theta-\theta_0)$ so that

$$r = C^2 \sin^2 \tfrac{1}{2}(\theta - \theta_0) = \tfrac{1}{2}C^2(1 - \cos(\theta - \theta_0)),$$

which is the equation of the cardiod, another famous subject of study in analytic geometry classes. In the case $s=-1$, for which for the first time $v(r)$ is a decreasing function of r, we get the curve

$$K = r^2 \sin 2(\theta - \theta_0) = 2r^2 \sin (\theta - \theta_0) \cos (\theta - \theta_0) = 2xy,$$

where x and y are rectangular cartesian coordinates the axes of which are inclined at an angle θ_0 to the $\theta=0$ radius vector. This is the equation of a rectangular hyperbola with asymptotes passing through the origin. The curve here is convex toward the origin, an indication, if we may be permitted the anthropomorphism, of the attempt of the light to make the greatest part of its journey in the high-speed region without putting in too much extra distance. In the previous examples the curves have been concave toward the origin. These curves illustrate that both maxima and minima occur as solutions of the least time principle and may not exist in some cases (as the illustration of the lemniscate indicates). The case $s=1$ indicates to the student that there may be a large number of relative extrema, for the exponential spiral may wind an arbitrary number of times about the origin before coming to a pre-selected target point for the light signal. It may not be out of place to remark that the advantage of these calculations as a classroom demonstration lies partly in the large number of familiar curves which appear as solutions of the above problem.

Let us consider, now, the principle of least action in mechanics and how our technique may be used to perform calculations here. According to this principle, if we have a system in which the energy is conserved, then of all the paths between two points in space-time (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2), a particle moves so as to extremize the path integral $\int_{P_1}^{P_2} p\,ds$ where p is the momentum of the particle. This principle, like the Fermat principle of least time, can be generalized to describe the motion of systems in general. We see that we can use the formulas developed in our optical considerations if p, the momentum, is identified with the velocity function $1/v(r)$. The differential equation of path which we developed earlier can now be written after this substitution as

$$\int^r \frac{dr}{r(a^2r^2p^2 - 1)^{1/2}} = \theta - \theta_0.$$

The function $p(r)$ can be expressed as a function of r, so that this equation can be integrated, by using the equation of the conservation of energy, according to which the total energy, E, of the particle is the sum of the kinetic energy and the potential energy:

$$\tfrac{1}{2}mv^2 + V(r) = p^2/2m + V(r) = E \qquad \text{(since } p = mv\text{)}$$

in which $V(r)$ is the potential energy function. Thus the path equation becomes

$$\int^r \frac{dr}{r(a^2r^2\cdot 2m(E - V(r)) - 1)^{1/2}} = \theta - \theta_0.$$

A mechanical interpretation of the integration constant, a, can be gotten by considering that our fundamental equation in the mechanical case is to be written as $1/a = pr \sin \phi$ so that it is clear that $1/a$ is the angular momentum of

the particle, the moment of momentum about the central point, and this is a constant in a central force field. We shall replace $1/a$ by the letter L, the usual notation for the angular momentum.

For the case of Newtonian planetary motion we take $V(r)=GMm/r$. Here G is the gravitational constant and M is the mass which is associated with the fixed attracting center at the origin. If we introduce the variable $u=1/r$ then the path integral takes the form

$$-\int(2mE/L^2 - 2GMm^2u/L^2 - u^2)^{-1/2} = \theta - \theta_0,$$

which we may integrate to get

$$u + (GMm^2/L^2) = (2mE/L^2 + G^2M^2m^4/L^4)^{1/2}\cos(\theta - \theta_0),$$

which we may write as

$$r = (L^2/GMm^2)/(e\cos(\theta - \theta_0) - 1),$$

where $e=1+2EL^2/G^2M^2m^3$. This is the well-known conic orbit of Newton with e and the major axis depending in the usual way on E and L. In this way we can derive the equations of motion of a mass in any central force field, the results being the same as when these are arrived at by the direct application of Newton's laws [3]. The results which we have arrived at in the optical case are, of course, not unknown. For derivations of these results which are based on more general optical equations, [4] should be consulted.

As we mentioned earlier, the work of Bernoulli can be used when the potential energy in the mechanical case or the index of refraction in the optical wording of the problem can be written as a function of one cartesian coordinate. This lets us solve the problem of the harmonic oscillator, for example, and of the projectile in a constant gravitational field. The radially symmetrical problems treated here, however, seem to the author to have more appeal.

References

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THE DIAGRAMMATIC EXPANSION OF DETERMINANTS

J. D. BANKIER, McMaster University

Roger Osborn in his note (this MONTHLY, vol. 67, 1960, pp. 682–683) proved that a fourth-order determinant cannot be expanded by making use of a single diagram of the same type used for third-order determinants. A method (ascribed to D. Rebić) for expanding fourth-order determinants, using three diagrams, is given by D. S. Mitrinović in his book *Zbornik Matematickih Problema I* (2nd.

rev. and completed ed., Beograd, 1958, p. 258, problem 78). We shall now show that such a method is possible for determinants of all orders.

Rebić's method of expanding a fourth-order determinant makes use of three diagrams. The first is

$$\begin{array}{cccc|ccc}
(+)\;(-) & (+) & (-) & (+) & (-) & (+) & (-) \\
a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43}
\end{array}$$

where products are taken along the diagonals with the indicated signs. The diagram provides eight terms of the desired expansion. The second diagram is obtained by applying the above method of expansion to the determinant $|C_1, C_3, C_4, C_2|$ obtained from the original determinant $|C_1, C_2, C_3, C_4|$ by two interchanges of the column vectors C_1, C_2, C_3, C_4. The final diagram is obtained from the determinant $|C_1, C_4, C_2, C_3|$. Since the determinants employed in the last two diagrams were derived from the original determinant by two interchanges of pairs of columns, they are identically equal to the original determinant. Different terms are obtained each time so we have the 24 terms which make up the expansion of the fourth-order determinant.

This procedure can be extended to determinants of any order. It leads to terms of the form

$$\pm\, a_{1j_1}a_{2j_2}\cdots a_{nj_n},$$

where $j_1, \cdots, j_n$ is a permutation of the numbers $1, \cdots, n$, obtained by forming all circular permutations of the column subscripts $c_1, \cdots, c_n$ of the elements on the principal diagonal of the basic determinant being used in the diagram. These permutations could be obtained by arranging $c_1, \cdots, c_n$ around a circle, starting with each c_j in turn, and writing down the $2n$ permutations obtained by proceeding in each direction around the circle. The number of diagrams is the same as the number of ways n distinct beads can be arranged to form a necklace, namely, $(n-1)!/2$. The determinants upon which the diagrams are based may be obtained by leaving the first column of the original determinant unchanged, permuting the remaining columns in all possible ways, and discarding any permutation which is a permutation we have already obtained written in reverse order.

It remains to determine the signs to be given to our diagonal products. Consider any one of the diagrams. Let J_0 be the number of interchanges required to restore the column subscripts of the elements on the principal diagonal of the basic determinant to their natural order $1, \cdots, n$. The sign of the corresponding product will be $(-1)^{J_0}$. The sign corresponding to the adjacent parallel diagonal will be $(-1)^{J_0+J}$ where $J=n-1$. The sign corresponding to the kth

parallel diagonal will be $(-1)^{J_0+J}$ where $J=k(n-1)$, $k=1, \cdots, n-1$. Thus, if n is odd, all signs will be the same as that of the principal diagonal while, if n is even, they will alternate in sign. The sign associated with the other diagonal of the basic determinant will be $(-1)^{J_0+J}$ where $J=n(n-1)/2$ and the sign corresponding to the kth parallel diagonal will be $(-1)^{J_0+J}$ where $J=n(n-1)/2+k(n-1)$. Again the sign will be the same as that associated with the other diagonal if n is odd and will alternate if n is even.

Examples. (1) $n=3$. In this case we have two permutations of the second and third columns and one permutation is the reverse of the other. We must discard one of the two corresponding determinants and we decide to keep the original determinant. We have $J_0=0$, n is odd, so the principal diagonal and the two diagonals parallel to it have associated positive signs. The remaining diagonal terms have negative signs.

(2) $n=4$. The subscripts of the permuted columns are (2, 3, 4), (3, 4, 2), (4, 2, 3), (4, 3, 2), (2, 4, 3), (3, 2, 4). We discard the last three permutations since they may be obtained by reversing the first three. For the first three permutations, $J_0=2$, $n(n-1)/2=6$, so both diagonals of the basic determinants have positive signs and the signs alternate for parallel diagonals.

(3) $n=5$. We have 12 diagrams based on the determinants obtained from the determinant $|C_1, C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}|$ by replacing (j_1, j_2, j_3, j_4) with the permutations (2, 3, 4, 5), (2, 4, 5, 3), (2, 5, 3, 4), (3, 2, 5, 4), (3, 4, 2, 5), (4, 2, 3, 5), (2, 3, 5, 4), (2, 4, 3, 5), (2, 5, 4, 3), (3, 2, 4, 5), (3, 5, 2, 4), (4, 3, 2, 5). For the first six permutations, $J_0=0$ and $J_0=1$ for the rest. Since $n(n-1)/2=10$, the other diagonal has the same sign as the principal diagonal in each basic determinant. Since n is odd, all diagonals have plus signs associated with them for the diagrams corresponding to the first six permutations and all diagonals have minus signs for the diagrams corresponding to the second six permutations.

A TRIPLE LONG DIVISION METHOD

LINCOLN TENG, The University of Michigan

Introduction. In the Euclidean domain of integers or polynomials, we are familiar with the long division method in finding the g. c. d. (greatest common divisor) of two numbers, a and m with the Euclidean Algorithm as follows:

$$\begin{aligned} m &= aq_1 + r_1, \\ a &= r_1q_2 + r_2, \\ r_1 &= r_2q_3 + r_3, \\ &\;\vdots \\ r_{n-2} &= r_{n-1}q_n + r_n, \end{aligned}$$

where $r_n=(a, m)$, the g. c. d. of a and m, when $r_{n+1}=0$. In case $(a, m)=1$, *i.e.*, a and m are relative prime to each other, the expression, $sa+tm=(a, m)=1$ is

related to the congruence relation of $ax=b \pmod{m}$, *i.e.*, $ax \pm my = b$ in the linear equation form. In fact, the values of s and t are used in solving the linear equation in x and y with $x = sb \pmod{m}$* and $y = ax - b \pmod{m}$.

Conventionally, the values of s and t in the equation, $sa + tm = (a, m)$ are determined by a method based on step by step observation, for which there is no feasible *a priori* computing procedure. Recently, Albert Newhouse gave a general analytic expression for s and t in a determinantal form which does provide a straightforward computing procedure. The expressions are as follows:†

$$s = (-1)^{k-1} \begin{vmatrix} q_2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & q_3 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & q_4 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & -1 & q_k \end{vmatrix},$$

$$t = (-1)^{k} \begin{vmatrix} q_1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -1 & q_2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & q_3 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & q_k \end{vmatrix}.$$

The purpose of this note is to present a triple long division method to determine the values of s and t as well as g. c. d. simultaneously. It is a synthetic computing scheme as simple as the ordinary long division method.

The method description. Given integers a, b (or polynomials $a(x)$, $b(x)$), by accompanying b with $(1, 0)$ and a with $(0, 1)$, we have triples $(0, a, 1)$ and $(1, b, 0)$ for long division. Assuming that we start dividing $(1, b, 0)$ by $(0, a, 1)$ with the first quotient, q_1 by the Euclidean algorithm, then we obtain: $(1, b, 0) - q_1(0, a, 1) = (1, b - aq_1, -q_1) = (s_1, r_1, t_1)$, a triple remainder. If $r_1 = (a, b)$, $r_2 = 0$ then $s_1 = s$, $t_1 = t$, which verifies the determinantal values for s and t respectively with $k = 1$. If $r_2 \neq 0$, then continue the long division with quotient q_2. We obtain a triple remainder, (s_2, r_2, t_2): $(0, a, 1) - q_2(1, b - aq_1, -q_1) = (q_2, a - bq_2 + aq_1q_2, 1 + q_1q_2) = (s_2, r_2, t_2)$. If $e_2 = (a, b)$, $r_3 = 0$, then $s_2 = s$, $t_2 = t$ which again verifies the determinantal values for s and t with $k = 2$. If $r_3 \neq 0$, we repeat the process until $r_{k+1} = 0$, then $s_k = s$, $r_k = (a, b)$, $t_k = t$, so that $r_k = (a, b) = sa + tb$.

As in the ordinary long division, we can carry out the triple long division in the following form:

* G. Birkhoff and S. MacLane, Survey of Modern Algebra, 1953, New York, page 24.

† This MONTHLY, vol. 62, 1955, p. 657.

	1	b	0	0	a	1	
q_1	0	aq_1	q_1				
	1	$b-aq_1=r_1$	$-q_1$				
				q_2	$bq_2-aq_1q_2$	$-q_1q_2$	q_2
				$-q_2$	r_2	$1+q_1q_2$	
q_3	$-q_2q_3$	r_2q_3	$q_3+q_1q_2q_3$				
	$1+q_2q_3$	r_3	$-q_1-q_3-q_1q_2q_3$				
				$q_4+q_2q_3q_4$	r_3q_4	$-q_1q_4-q_3q_4-q_1q_2q_3q_4$	q_4
$\vdots$				$-q_2-q_4-q_2q_3q_4$	r_4	$1+q_1q_2+q_1q_4+q_3q_4+q_1q_2q_3q_4$	
q_k	$s_k=s$	r_k	$t_k=t$				

where

$$
\begin{aligned}
r_1 &= b - aq_1,\\
r_2 &= a - bq_2 + aq_1q_2 = a - r_1q_2,\\
r_3 &= b - aq_1 - aq_3 + bq_2q_3 + aq_1q_2q_3 = r_1 - r_2q_3,\\
r_4 &= r_2 - r_3q_4,\\
&\vdots\\
r_{i+1} &= r_{i-1} - r_iq_{i+1},\\
&\vdots\\
r_k &= r_{k-2} - r_{k-1}q_k = (a, b), \quad r_{k+1} = 0.
\end{aligned}
$$

Note: In computer coding for the long division, successive subtraction is conceivably faster than straight division.

Similarly, for polynomials $r_k(x) = s_k(x)a(x) + t_k(x)b(x)$ holds for all finite integers k.

Appendix. For the proof of equation, $r_k = s_ka + t_kb$ for all k, a finite integer, we make use of the following recursive relations

$$
\begin{aligned}
s_i &= s_{i-2} - q_is_{i-1},\\
r_i &= r_{i-2} - q_ir_{i-1},\\
t_i &= t_{i-2} - q_it_{i-1}.
\end{aligned}
$$

By finite induction, we have, for $k=1$, $r_1 = s_1a + t_1b$; and we assume that $r_i = s_ia + t_ib$ holds for $i=k$ and $i=k-1$; it remains to show that $r_{k+1} = s_{k+1}a + t_{k+1}b$. But $r_{k+1} = r_{k-1} - r_kq_{k+1}$ and using the equation for r_i we get that $r_{k+1} = (s_{k-1}a + t_{k-1}b) - (s_ka + t_kb)q_{k+1} = (s_{k-1} - q_{k+1}s_k)a + (t_{k-1} - q_{k+1}t_k)b = s_{k+1}a + t_{k+1}b$. Therefore, $r_k = s_ka + t_kb$ holds for all finite integers k.

A MATRIX METHOD FOR LEAST SQUARES

CURT MARCUS, System Development Corporation

This note gives a simple derivation of the well-known result that the expression

$$\|y - Ax\|^2 \tag{1}$$

is minimized for $x=c$ if and only if the normal equations $A'Ac=A'y$ are satisfied. Here y is an n component column vector, x and c are m component column vectors, A is an $n\times m$ matrix, $'$ denotes transpose, and $\|t\|^2=t't$ is the square of the length of the n component column vector t. Other simple proofs of this result exist, but the present one has the virtue of being strictly algebraic, that is, it does not employ differentiation to establish the minimum.

To minimize (1) we seek a vector c such that for any scalar λ and any vector h of the same dimension as c the inequality

$$[y - A(c + \lambda h)]'[y - A(c + \lambda h)] - (y - Ac)'(y - Ac) \geqq 0 \tag{2}$$

is satisfied. By reducing (2) to

$$\lambda^2(Ah)'Ah - 2\lambda[y'A - (Ac)'A]h \geqq 0$$

and substituting e' for $[y'A-(Ac)'A]$ we can write (2) as

$$(Ah)'Ah\left[\lambda - \frac{e'h}{(Ah)'Ah}\right]^2 - \frac{(e'h)^2}{(Ah)'Ah} \geqq 0; \tag{3}$$

which is certainly satisfied for all λ and h when and only when $e'=0$. From the definition of e' we conclude that (3) holds true for all λ and h if and only if the normal equations

$$A'Ac = A'y$$

are satisfied.

We have shown, via the inequality (2), that any c which minimizes (1) must satisfy the normal equations, and that any solution of the normal equations minimizes (1).

ON ISOTROPIC CARTESIAN TENSORS

PHILIP G. HODGE, JR., Illinois Institute of Technology

Although it is well known that the most general isotropic Cartesian tensors of the second, third, and fourth orders are $\lambda\delta_{ij}$, μe_{ijk}, and $\alpha\delta_{ij}\delta_{kl}+\beta\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}$, respectively, proofs of this fact are usually based on a consideration of special cases in which all possible numerical combinations of subscripts are either expressed or implied [1, 2]. Therefore, it may be of some interest to consider a direct proof.

It will be assumed that the Kronecker delta δ_{ij} and alternating tensor e_{ijk}

have already been defined, and that the identity

$$e_{ijk}e_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} \tag{1}$$

has already been established. The general infinitesimal rotation will be written in the form

$$a_{ij} = \delta_{ij} + e_{ijk}\omega_k, \tag{2}$$

where $a_{ij} = \cos(x_i, x_j')$ and ω_k is a vector of infinitesimal magnitude.

The most general isotropic tensor T_{ij} of the second order must satisfy

$$T_{ij}' = T_{ij} = a_{mi}a_{nj}T_{mn}. \tag{3}$$

Substitution of (2) into (3) shows that

$$\omega_r(e_{mir}T_{mj} + e_{mjr}T_{im}) = 0,$$

where higher powers and products of ω_r have been neglected. Since the direction of ω_r is arbitrary,

$$e_{mir}T_{mj} + e_{mjr}T_{im} = 0. \tag{4}$$

Multiplication of (4) by e_{tir} and simplification by means of (1) then leads to

$$2T_{tj} + T_{jt} = T_{mm}\delta_{jt} = 3\lambda\delta_{jt}. \tag{5}$$

Since the right-hand side of (5) is symmetric, the left-hand side must be also, hence

$$T_{tj} = T_{jt} = \lambda\delta_{jt}.$$

For the most general third-order tensor T_{ijk}, a sequence of similar steps leads to

$$e_{mir}T_{mjk} + e_{mjr}T_{imk} + e_{mkr}T_{ijm} \tag{6}$$

in place of (4). Multiplication of (6) by e_{tsr} and setting of s equal to i, j, and k in turn yields the three equations

$$\begin{aligned} 2T_{ijk} + T_{jik} + T_{kji} &= T_{mmk}\delta_{ij} + T_{mjm}\delta_{ik} = 0, \\ 2T_{ijk} + T_{jik} + T_{ikj} &= T_{mmk}\delta_{ij} + T_{imm}\delta_{jk} = 0, \\ 2T_{ijk} + T_{kji} + T_{ikj} &= T_{mjm}\delta_{ik} + T_{imm}\delta_{jk} = 0. \end{aligned} \tag{7}$$

The final step follows from the observation that T_{mmk}, etc. must be isotropic vectors and there are no isotropic vectors; alternatively it may be shown by setting $j = k$, etc. in (7). Obviously any solution of (7) must satisfy

$$T_{jik} = T_{kji} = T_{ikj} = -T_{ijk},$$

which relations precisely describe the alternating tensor μe_{ijk}.

For a fourth-order isotropic tensor T_{ijkl}, the analog of (7) is easily shown to be

$$
\begin{aligned}
(8) \qquad & 2T_{ijkl} + T_{jikl} + T_{kjil} + T_{ljki} \\
&= 2T_{ijkl} + T_{jikl} + T_{ikjl} + T_{ilkj} \\
&= 2T_{ijkl} + T_{kjil} + T_{ikjl} + T_{ijlk} \\
&= 2T_{ijkl} + T_{ljki} + T_{ilkj} + T_{ijlk} \\
&= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}.
\end{aligned}
$$

Relations of the form

$$
(9) \qquad T_{ijkl} = T_{klij} = T_{jilk} = T_{lkji}
$$

may be obtained by subtracting the sum of one pair of (8) from the sum of the other pair. By means of (9), (8) may be written

$$
(10) \qquad 2T_{ijkl} + (T_{ijlk} + T_{ilkj} + T_{ikjl}) = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}.
$$

The addition to (10) of the two equations obtained by cyclic permutation of j, k, l with i remaining unchanged leads to

$$
(11) \qquad \begin{aligned}
&2(T_{ijkl} + T_{iklj} + T_{iljk}) + 3(T_{ijlk} + T_{ilkj} + T_{ikjl}) \\
&\qquad = (\lambda + \mu + \nu)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).
\end{aligned}
$$

Symmetry then demands that each of the group in parentheses on the left-hand side ol (11) be equal to one fifth the right-hand side. Substitution of this value into (10) and definition of new parameters

$$
\alpha = (4\lambda - \mu - \nu)/10, \qquad \beta = (4\mu - \nu - \lambda)/10, \qquad \gamma = (4\nu - \lambda - \mu)/10
$$

leads to the desired result

$$
T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}.
$$

References

1. H. Jeffreys, Cartesian Tensors, Cambridge (England), 1952, pp. 66–70.
2. C. E. Pearson, Theoretical Elasticity, Cambridge (Mass.), 1959, pp. 43–44.

ON THE DEFINITION OF A GROUP

W. E. DESKINS and J. D. HILL, Michigan State University

Recently in a course on concepts in algebra given by J. D. Hill, a discussion arose concerning the "one-sided" definition of a group as a semigroup S having (i) a left identity element e, and (ii) a left inverse element a' for each element a in S, such that $a'a = e$(*cf.*, K. Miller, *Elements of Modern Abstract Algebra*, New York, 1958, p. 4). During this discussion the following question was asked: Is a semigroup S which has (i′) a local left identity element (*i.e.*, for each a in S there is an element ${}_ae$ in S, dependent on a, such that ${}_aea = a$), and (ii′) a left inverse element a'' for each element a in S such that $a''a = {}_ae$, necessarily a group? Since this question, which is a rather natural one to raise, does not seem to be considered in any of the standard textbooks on algebra, we present here a

simple example, which provides a negative answer to the question, and a related theorem and example.

Example 1. Denote by $x_1, x_2, \cdots$, a countable (finite or infinite) set and define a binary composition on the elements by the equation

$$x_i * x_j = x_k, \quad \text{where} \quad k = \min(i, j).$$

The resulting commutative semigroup is not a group, yet it satisfies both (i′) and (ii′) since every element is an idempotent.

THEOREM. *If S is a commutative semigroup with the properties*

(i′) *For each a in S there is an element ${}_ae$ in S such that ${}_aea = a$;*

(ii″) *For each ${}_ae$ in S such that ${}_aea = a$ there is an element f, dependent on ${}_oe$, such that $fa = {}_ae$; then S is a group.*

Proof. First we note that a given element a of S has only one local identity, for suppose x and y are elements such that $xa = ya = a$. Then S contains relative inverse elements f_x and f_y such that $f_xa = x$ and $f_ya = y$. Therefore,

$$xy = (f_xa)y = f_x(ay) = f_xa = x,$$

and also

$$xy = x(af_y) = (xa)fy = af_y = y.$$

From this it follows that each local identity is an idempotent element, for if $xa = a$, then $x^2a = x(xa) = xa = a$. The uniqueness of the local identity implies then that $x^2 = x$.

Finally we see that all the local identities are equal. For consider $b \neq a$, $b \neq {}_ae$. (If no such element exists then S is obviously a group consisting of either one or two elements.) Form ${}_aeb = t$. Then ${}_ae({}_aeb) = {}_ae^2b = {}_aeb = t$, so that ${}_ae = {}_te$. Also ${}_bet = ({}_aeb){}_be = {}_ae(b{}_be) = {}_aeb = t$, so that ${}_be = {}_te$. Therefore ${}_ae = {}_be$.

Thus S is a group since it has a global identity e and each element of S has an inverse relative to e.

That commutativity is essential to the result is demonstrated by our second example.

Example 2. Denote by $x_1, x_2, \cdots$, a countable set and define a binary composition on the elements by the equation $x_i \text{O} x_j = x_i$.

It is evident that while this semigroup possesses both of the properties (i″) and (ii″) it is not a group.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE DEVELOPMENTAL PROJECT IN SECONDARY MATHEMATICS

MORTON R. KENNER, Southern Illinois University

This project was initiated during the school year 1957–1958. The work is being supported by grants from the Marcell Holzer Fund for Education and the Graduate Research Council of Southern Illinois University. The two primary purposes of the project were initially to prepare a set of secondary texts implementing the Report of the Commission on Mathematics and to compare the performance of high school students in this new program with high school stustudents in a good traditional program.

The criteria used in developing text material are the following: (a) Rapid change of high school programs requires a minimum of teacher re-education—in fact, our view is that the job of teacher re-education should be left to the National Science Foundation. (b) There is no point in arguing *if* it is possible to teach certain topics at a certain grade level—the "yes answer" is being given all over the country. The question is where should topics be placed. Our answer followed the Commission and the CUPM. The proper pacing of maturity for high school students is a mathematical development which prepares most college-capable youngsters to begin their college work in a combined analytics and calculus course. (c) Any new secondary program should articulate well in language and general outlook with what is considered good collegiate mathematics. (d) Any new program should—as a responsibility to students who move from school and enter a variety of different colleges—place topics at places in the curriculum which are in general agreement among mathematical educators. (e) In organizing topics, any new program should develop mathematics as a consistent and sequential set of ideas with proper regard for the need of much informality and intuition. This means making a responsible compromise between rigor and rote. (f) Any new program should not develop the important mathematical ideas so that the fields of algebra, geometry, and elementary analysis appear to be completely separated. (g) Any new program should not sacrifice technique learnings to obtain concept learnings.

The evaluation study is being conducted at Central High School, Cape Girardeau, Missouri. A pilot study using three experimental and three control ninth-grade classes was run during 1959–1960.* Answers to the following questions are sought. (1) Does the experimental course produce a detectable increase in proficiency in comparison with a typical good traditional course, or even does

* A detailed report of this pilot study by Dr. Osborn and Dr. Melton is to appear soon in another journal.

the experimental class hold its own with respect to traditional techniques? (2) Can evidence be obtained to show that the experimental class does develop a more conceptual understanding of the abstract structure of algebra? (3) Are different abilities involved in learning abstract mathematical ideas than are involved in learning traditional techniques? (The implication would be that there is no best method or program.)

The pre-tests used were the Iowa Algebraic Aptitude, Orleans Algebraic Prognosis, SRA Primary Mental Abilities, Differential Aptitude Tests. Tests used during the year were proficiency tests constructed by members of the project. The Cooperative Elementary Algebra test was used as a final. The results obtained in the pilot study were as follows: (1) Proficiency of experimental and control students was not significantly different on topics common to both courses. (2) Ability tests were equally valid in predicting proficiency in either course. (3) No significant evidence was obtained showing significant differences in abstract learnings. (4) Ability tests containing a high saturation of spacial visualization were significantly more valid in predicting for experimental students. The proficiency tests have been revised and this study is continuing.

At the present time, text materials developed by the project are being taught in 14 schools. Grades involved are 7–12. Teachers using materials have not been chosen with regard to mathematical backgrounds so that a significant spread of abilities is involved. Although no in-service training is being conducted by the project, our experience agrees with that of others, namely that teachers (of all backgrounds) become motivated to do further study. Further information about the availability of sample materials may be obtained by writing the author.

THE TEACHING OF MATHEMATICS*

The report of the (British) Institute of Physics† to which Professor Mathieu referred, had been circulated to delegates. The following are extracts from it. The report includes a suggested syllabus and specimen examination questions. All the extracts given here refer to the teaching of mathematics to university students of physics.

Some knowledge of mathematics is essential to every experimental physicist. It provides the logic by which he can best develop his ideas, and the language in which his results can often be most conveniently expressed. The acquisition of

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* This second half of Chapter II of the Proceedings of the International Congress on Physics Education, and additional excerpts from this chapter, published in this department last month, are used with the permission of the publishers, The Technology Press, Massachusetts Institute of Technology, and John Wiley and Sons, Inc., New York and London. The Conference was held July 18 to August 4, 1960, in UNESCO House, Paris. The proceedings were edited by Sanborn C. Brown and Norman Clarke.

† The Teaching of Mathematics to Physicists, The Institute of Physics and The Physical Society, London, 1960.

some knowledge of mathematics is, therefore, to physicists a matter of prime importance. In the past there was a feeling amongst physicists that the type of training in mathematics received by students of physics failed in some respects to meet the requirements of the times.

For the average student of experimental physics, a very considerable fraction of his time after entering the university must necessarily be taken up in the study of his main subject, and, in consequence, for him the full discipline demanded of a mathematical specialist must in some way be mitigated. The problem is not a new one, but it is becoming of greater urgency with the increasing extension of the content and technique of both mathematics and physics.

The title of this report does not imply that the kind of mathematics taught to physicists should be different from the kind of mathematics taught to mathematicians. The difference is in the amount of time which a physics student has available for formal mathematical training as compared with a mathematics student.

One way out of this difficulty is to adopt frankly the narrow utilitarian attitude. This would mean concentrating on mathematical techniques and regarding mathematics purely as a tool to be used in solving physical problems, teaching students nothing about differential equations except how to solve some of them, to multiply matrices without saying anything about modern algebra, and to use the results, say, of group theory while only learning the barest minimum about groups. We believe that this method is unduly restrictive and is not an acceptable way out of the difficulty.

It is essential to teach physicists certain mathematical techniques, but it is even more important that they should learn how mathematics is used in physics. Perhaps the most difficult part of the solution of a problem in physics, or in science generally, is the formulation of it in such a manner that known mathematical processes can be applied. By analysis, or sometimes by intuition, the physicist first makes a model of the physical situation, normally by introducing some simplifying assumptions or by accepting the applicability of certain physical laws. It is this model which is expressed in mathematical terms. Too often students do not realize that the mathematics only expresses the physical ideas on which any theory or model is based.

The next step in the process is often comparatively simple; it involves the application of mathematical techniques to get what is often called the result or the solution. This in turn has to be translated back again in terms of the physical characteristics of the problem.

The whole has been picturesquely described by Professor Synge as follows: "the use of applied mathematics in its relation to a physical problem involves three stages: (i) A dive from the world of reality into the world of mathematics; (ii) a swim in the world of mathematics; (iii) a climb from the world of mathematics back into the world of reality, carrying a prediction in our teeth."

The first and third of these stages require as much emphasis in the teaching

of applied mathematics as they do in the teaching of physics; their neglect can be disastrous for the students.

The idea of rigour and the idea of generality are often confused in mathematical expositions intended for physicists. There is often a good reason for sacrificing generality; there is never a good reason for being satisfied with anything less than complete rigour, in the sense that the conclusions should follow from the premises by logical arguments with which no one would disagree. A proof is often presented apologetically by an author who admits it is not rigorous, on the grounds that it is easy and that it is plausible. In any case, it is then said, the result can be shown to be true experimentally. There is no excuse for this. The way to avoid undue difficulty in a proof is usually to state the theorem under more restrictive conditions than the most general under which it is valid. Even with such restrictions the theorem is often sufficiently general for a physical application. Indeed, it may be uneconomical and is sometimes pedantic to insist upon the utmost generality in the statement of a theorem which is needed for a specific physical situation. This, then, is the real difference between the attitude of a physicist and that of a pure mathematician. The former may be satisfied by a proof which is valid in a special case, while the latter is never completely satisfied unless the utmost generality is attained.

A nonrigorous proof, however, is of no use to either. It is just as important to a physicist as to a mathematician that the hypotheses of a theorem should be clearly stated and that the conclusion should follow logically from them. This does not mean, of course, that all logical steps should be explicitly set down. The procedure is well known and well established. Proofs are given in such a way that a logician could insert the missing steps if he wished to do so. Students of physics, even at an elementary level, are confused when assumptions are not clearly labeled as such, and rightly so. Plausible arguments in the guise of proofs merely make the confusion worse confounded. The correct point of view is stated by Sir Harold and Lady Jeffreys in the preface to their *Methods of Mathematical Physics*:

"Some explanation of the standard of rigour and generality aimed at is desirable. We do not accept the common view that any argument is good enough if it is intended to be used by scientists. We hold that it is as necessary to science as to pure mathematics that the fundamental principles should be clearly stated and that the conclusions shall follow from them. But in science it is also necessary that the principles taken as fundamental should be as closely related to observation as possible; it matters little to pure mathematics what is taken as fundamental, but it is of primary importance to science. We maintain therefore that careful analysis is more important in science than in pure mathematics, not less. We have also found repeatedly that the easiest way to make a statement reasonably plausible is to give a rigorous proof. Some of the most important results (e.g. Cauchy's Theorem) are so surprising at first sight that nothing short of a proof can make them credible. On the other hand, a pure mathematician is usually dissatisfied with a theorem until it has been stated in its most general form. The scientific applications are often limited to a few special types. We have therefore often given proofs under what a pure mathematician will consider unnecessarily restrictive conditions, but these are satisfied in most applications. Generality is a good thing but it can be purchased at too high a price."

The broad aims of the mathematical education of a physicist should be to enable him to read, after graduating, the important books in his subject and to use the literature to help him to solve the problems that arise in his chosen field of work.

Since the physicist will be studying mathematics mainly for the purpose of applying it to his own particular problems, the distinction which is often drawn between pure mathematics and applied mathematics becomes meaningless in his case, and may actually be misleading.

In the past, much of the time allotted to applied mathematics has been spent on the solution of problems in which the interest has resided mainly in the ingenuity of the mathematics. Problems have been propounded, not because they were important, or even physically realizable, but because they led to ingenious, and perhaps unexpected solutions. They are out of place and may be omitted, not only without loss, but with actual benefit. It should be kept in mind by the mathematical teacher that much of the time of the physics student will be spent in applying mathematics to problems which arise naturally in his own subject, and that he will obtain a considerable amount of practice in problem-solving in this way.

The success of the course will depend more on the ability and willingness of the teacher to adapt his subject to the technical and cultural interest of his students than to the exact nature of the formal syllabus adopted. It is, for example, important throughout the course to illustrate the uses to which the various mathematical processes and equations can be applied. It might also be helpful to point out that many mathematical topics, of great diversity and interest, have originated in response to the need for quantitative solutions of scientific problems and, conversely, developments in pure mathematics have sometimes pointed the way to advances in physics.

Although we are primarily concerned with the course of training in mathematics for students of physics, we cannot ignore the necessity of examining them also to assess their competence in mathematics. There has undoubtedly been improvement since the war in the type of mathematics paper which physics students are required to take. Nevertheless, remembering the influence which examinations exert on the student, particularly on his private reading, it must be emphasized that the examination should reflect the same outlook and spirit which in this report we have suggested should inspire the whole course. Questions should be straightforward. Problems of which the solution depends merely on some algebraic ingenuity or recondite mathematical device should be avoided. Some questions should, however, test the candidate's ability to use mathematics as a tool, and should require him to draw simple physical conclusions from the mathematical analysis, rather than simply test his memory.

Studies in Teacher Education

In order to encourage experimentation with the professional education part of science and mathematics teacher education, the AAAS Science Teaching Improvement Program has sponsored a project known as Studies in Teacher Education (this MONTHLY, vol. 67, 1960, pp. 373–4) during the past two years. A brochure, available upon request to AAAS, describes investigations at Bucknell University, Emory University, Hunter College, Oklahoma State University, San Francisco State College, University of Arizona, and the University of Tennessee. Two additional cooperating universities, with projects of special interest to mathematics teacher education, have been started during the past year.

At the University of Delaware a study is being made of the nature of problem solving as it pertains to student teacher education in mathematics, social sciences, and English. The purpose of the study is to train student teachers and cooperating teachers in public schools, to which the university sends student teachers, in what is known about how thinking takes place and how this knowledge can be used in teaching secondary school students. Emphasis is also placed on problem solving in methods courses. Participants in 1960–61 included 10 cooperating teachers, five in mathematics and five in the social sciences and English; 14 student teachers, and a group of pupils in grade 11 mathematics, history and English. The student teachers and eleventh grade pupils are helped to see how certain aspects of learning apply in each discipline and by comparison, among the three disciplines. An evaluation is planned as part of the study. Professor John A. Brown of the Departments of Mathematics and Education is Center Coordinator.

The purpose of the study at the University of Hawaii (Hilo Campus), directed by John A. Easley, Jr., is to develop practical operating procedures for a laboratory course in teaching that will provide college freshmen with controlled observation, practice and analysis of selected forms of teaching mathematics and physical science. The course is presently limited to mathematics and physical science in the middle grades (4–8, approximately). Future development of more advanced courses is anticipated in a broader subject matter area and over a wider range of grades. One fifty-minute period each week is devoted to briefing on the lessons assigned for laboratory teaching and analysis of observations and previous laboratory teaching experience. Each member of the class spends one period a week observing a demonstration of techniques by the instructor, and one period a week teaching an assigned lesson to a group of 2 to 4 pupils. Most of these lessons are recorded on tape, but techniques on recording in writing are also being explored. The Department of Public Instruction is cooperating by permitting demonstrations and laboratory teaching in local schools. The first phase of the project utilizes an 8th grade mathematics class. Later phases are being planned for the 5th grade in both physical science and mathematics and a 9th grade class in physical science. Instructional materials in mathematics and physical science are selected for the purpose of demonstrating precise structuring of concepts to be taught. For the present project there include the following:

Unit I of UICSM *High School Mathematics*; selections from the 7th grade course of the SMSG texts in secondary school mathematics: selected "number-line games" developed by the University of Illinois Arithmetic Project; selected physics problems from the University of Illinois Studies in Inquiry Training; and Volume I of the PSSC high school physics course.

Visiting Foreign Scientist Staff Project

Eighteen foreign scientists were brought to this country to lecture at National Science Foundation summer institutes for science and mathematics teachers in 1961. Scientists chosen had been recommended by scientists and educational leaders in this country and in their own countries. The group represents men who are leaders in scientific research

and also others whose principal reputation has been established for their important contributions to science education. One-third of the 1961 lecturers visited mathematics institutes and during the course of the summer 46 such institutes were visited. The mathematicians chosen as lecturers to summer institutes are: Herman Athen, oberstudiendirektor, Bismarckschule, Hamburg, Germany; Gunnar Bergendal, professor of mathematics, University of Lund, Sweden; W. M. Brookes, lecturer in mathematics education, Southampton University, England; Frans Handest, senior master in the secondary school, Hvidore, Denmark; Walter James Langford, Headmaster Battersea Grammar School, London; José Tola Pasquel, Director of Mathematical Sciences, University of San Marcos, Lima, Peru.

The Visiting Foreign Scientist Staff Project is sponsored by NSF and the administration of the program was carried out this year by the American Association for the Advancement of Science.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1481. *Proposed by F. Leuenberger, Zuoz, Switzerland*

If r and R denote the inradius and circumradius of a triangle of perimeter $2s$, show that $2[r(r+4R)]^{1/2}<2s\leqq[4(r+2R)^2+2R^2]^{1/2}$.

E 1482. *Proposed by E. L. Spitznagel, Jr., Xavier University, Cincinnati, Ohio*

Many simple functions are differentiable everywhere except at a single point (e.g., $f(x)=|x|$). Find a function defined on the real line which is differentiable nowhere save at a single point.

E 1483. *Proposed by M. V. Mielke, University of Wisconsin*

For any pair (a, b) of positive integers, show that there are infinitely many pairs (A, B) of positive integers such that $\phi(A)\equiv 0 \bmod a$, $\phi(B)\equiv 0 \bmod b$, $\phi(A+B+AB)\equiv 0 \bmod (a+b)$, where ϕ denotes the Euler ϕ-function.

E 1484. *Proposed by J. H. Heinbockel and R. R. Korfhage, University of North Carolina*

A given point is at distances $\sqrt{2}$, 2, and $\sqrt{3}-1$ from the vertices of a triangle. Find the maximum area of the triangle.

E 1485. *Proposed by C. N. Bhaskaranandha, Chikmaglur, India*

In how many ways can a party of m men, w women, and d dogs be arranged in a row so that neither two women nor two dogs are together?

SOLUTIONS

Error in Partial Fractions

E 1451 [1961, 177]. *Proposed by Anice Seybold, North Central College, Naperville, Illinois*

A student makes an error in breaking a fraction into partial fractions. He writes

$$\frac{x^4 - 3x^3}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}.$$

He then clears of fractions and substitutes in succession -1, 1, and 2 as values of x in order to obtain three equations to solve for A, B, and C. Another student correctly carries out the indicated division and uses the remainder, $-x^2-3x+2$, correctly. He writes

$$\frac{-x^2 - 3x + 2}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}.$$

Both students get the same values for A, B, and C. How does this happen?

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. The result is a consequence of the following generalization of the remainder theorem: "Let r_i, $i=1, \cdots, n$, be distinct numbers, and let $f(x)$ be a polynomial in x of degree $\geqq n$. Then $f(r_i)=R(r_i)$, $i=1, \cdots, n$, where $R(x)$ is the remainder obtained on division of $f(x)$ by $(x-r_1) \cdots (x-r_n)$."

Also solved by A. N. Aheart, Joseph Altinger, J. W. Baldwin, Merrill Barnebey, Brother U. Alfred, J. L. Brown, Jr., Marcus Charles, F. A. Cleveland, D. I. A Cohen, J. L. Cooley, E. I. Deaton, Gus Di Antonio, J. W. Ellis, Jane Evans, D. P. Giesy, Michael Goldberg, L. D. Goldstone, Cornelius Groenewoud, Donald Hall, H. A. Heckart, J. B. Herreshoff, Vern Hoggatt and Charles Phillips (jointly), A. R. Hyde, Lawrence Israel, Erwin Just, J. J. Kim, J. A. Lambert, Betty Levine, D. C. B. Marsh, D. L. Muench, D. E. Myers, C. S. Ogilvy, J. L. Pietenpol, C. F. Pinzka, G. S. Rogers, David Sachs, M. T. Salhab, L. J. Schneider, J. A. Schumaker, R. T. Shannon, Wayne Shepherdson, J. L. Sieber, D. R. Simpson, Sister Mary Denis, E. S. Smith, E. L. Spitznagel, Jr., Guy Torchinelli, W. C. Waterhouse, and the proposer. Late solutions by Marvin Gurber and Frank Harary.

Two Positive Integers with Positive Integral Harmonic Mean

E 1452 [1961, 177]. *Proposed by N. A. Court, University of Oklahoma*

Find two positive integers such that their sum will be a factor of their product.

Solution by D. M. Danvers, Centenary College, Shreveport, La. The smallest

distinct pair is 3, 6. To find all pairs, suppose $(m+n)|mn$ and let $m=dm_1$, $n=dn_1$, where $d=(m, n)$ and $(m_1, n_1)=1$. Then $(dm_1+dn_1)|dm_1dn_1$ implies $(m_1+n_1)|dm_1n_1$, hence $(m_1+n_1)|d$ since $(m_1+n_1, m_1n_1)=1$. Set $d=k(m_1+n_1)$; then $m=km_1(m_1+n_1)$ and $n=kn_1(m_1+n_1)$. On the other hand, any m and n of this form (k, m_1, n_1 being any positive integers) clearly satisfy the requirement.

Also solved by G. D. Adams, Paul Aizley, Louisa R. Alger, Ronald Alter, Joseph Altinger' R. H. Anglin, J. W. Baldwin, C. B. Barfoot, Merrill Barnebey, R. E. Beals, T. L. Beatty, W. R. Becker, W. S. Bishop, Ernest Blaisdell, D. A. Breault, Brother U. Alfred, J. A. Brown, W. E. Buker, Leonard Carlitz, M. N. Channabasappa, A. G. Clark, L. H. Cleveland, I. A. Cohen and S. P. Cohen (jointly), Martin Cohen, J. L. Cooley, F. B. Correia, G. S. Cunningham, T. R. Curry, J. A. Davis, Monte Dernham, G. C. Dodds, J. W. Ellis, Jane Evans, Herbert Fantle and David Zeitlin (jointly), F. E. Fischer, Michael Fischer, J. F. Foley, D. P. Giesey, G. S. Gill, A. F. Gilman, III, Michael Goldberg, L. D. Goldstone, R. B. Grafton, Cornelius Groenewoud, Dunstan Hayden, J. B. Herreshoff, M. Herzberger and E. W. Marchand (jointly), Vern Hoggatt and Dale Ruggles (jointly), J. E. Homer, Jr., J. L. Humphry, A. R. Hyde, Lawrence Israel, V. F. Ivanoff, Gerald Janusz, Diane M. Johnson, J. L. Johnson, E. H. Kanning, III, M. A. Kazim, E. C. Kennedy, R. M. Kennedy, J. J. Kim, A. W. Knapp and Albert Whitcomb (jointly), D. E. Knuth, W. J. Koss, Sidney Kravitz, J. A. Lambert, D. G. Lash, Dean Lawrence, J. F. Leetch, H. R. Leifer, Betty Levine, J. W. Lindsay, Glen Luchau, Jiang Luh, Frank McGee, Barry MacKichan, R. A. MacLeod, D. C. B. Marsh, L. V. Mead, M. V. Mielke, Otto Mond, D. A. Moran, J. B. Muskat, D. E. Myers, J. C. Nichols, H. J. Noble, J. M. O'Connell, C. S. Ogilvy, C. C. Oursler, J. L. Pietenpol, C. F. Pinzka, Anatol Rapoport, W. R. Ransom, Augustin Rien, C. A. Ritcher, J. T. Rosenbaum, G. M. Rosenstein, Jr., David Sachs, M. T. Salhab, J. A. Schumaker, J. W. Sehestedt, M. C. Seibel, R. T. Shannon, D. L. Silverman, D. R. Simpson, Sister Mary Denis, R. I. Snell, O. E. Stanaitis, W. B. Stoval, Jr., R. P. Tapscott, T. N. Tideman, Guy Torchinelli, C. E. Tsai, Patrick Twomey, R. M. Warten, W. C. Waterhouse, R. C. Weger, O. A. Wehmanen, Garry Westly, M. J. Wiedel, W. L. Wong, C. C. Yalavigi, Walter Zayachkowski, Bohdan Zelinka, and the proposer. Late solutions by C. E. Franti, Marvin Gurber, and J. W. Hooper.

Many of the solutions received were only special cases of the general solution. Various solvers established a number of allied results, such as: If $(m+n)|mn$, then (1) m and n cannot both be odd, (2) $(m, n)\neq 1$, (3) $(m+n)|m^2$ and $(m+n)|n^2$, (4) $(m+n)|(m, n)^2$, (5) if n is prime and $n|m$, then $m=n(n-1)$. Kazim showed that the number of all pairs m, n such that $(m+n)|mn$ and $(m, n)=d$ is $d/2$ or $(d-1)/2$ according as d is even or odd. Among suggested generalizations of the given problem were: (1) Find two positive integers such that the ratio of their product to their sum will be a given integral power (Israel); (2) Show that for any positive integer k there are infinitely many k-tuples of distinct positive integers whose sum divides their product (Moran); (3) Find all pairs of positive integers whose arithmetic, geometric, and harmonic means are integers (Ransom). Attention was called to E 1206 [1956, 665] and to Question 1 of the 21st (Dec. 3, 1960) Putnam Mathematical Competition.

Some Inequalities for the Triangle

E 1454 [1961, 177]. *Proposed by Leonard Carlitz, Duke University*

If Δ, R, r denote the area, circumradius, and inradius of a triangle with sides a_1, a_2, a_3, show that (1) $(R+r)^2\geqq\Delta\sqrt{3}$, (2) $(a_1a_2a_3)^2\geqq(4\Delta/\sqrt{3})^3$, (3) $(R\sqrt{3})^3\geqq a_1a_2a_3$, with equality only when the triangle is equilateral.

I. *Solution by F. Leuenberger, Zuoz, Switzerland.* It is known that: (a) the geometric mean of a set of positive quantities $\leqq$ their arithmetic mean, (b) $a_1+a_2+a_3\leqq 3R\sqrt{3}$, (c) $a_1+a_2+a_3\geqq 6r\sqrt{3}$.

(1) By (b) and (a), and setting $a_1+a_2+a_3=2s$,

$$R + r \geqq (s/\sqrt{3} + s/\sqrt{3} + 3r)/3 \geqq (rs^2)^{1/3},$$

whence, from $\Delta = rs/2$ and (c), we find

$$(R + r)^2 \geqq (\Delta^2 s^2)^{1/3} \geqq \Delta(3\sqrt{3})^{1/3} = \Delta\sqrt{3}.$$

(2) Since $4R\Delta = a_1a_2a_3$ it follows, by (b) and (a), that

$$4\Delta/\sqrt{3} = a_1a_2a_3/R\sqrt{3} \leqq 3a_1a_2a_3/2s \leqq (a_1a_2a_3)^{2/3}.$$

(3) Again, by (a) and (b), $(a_1a_2a_3)^{1/3} \leqq 2s/3 \leqq R\sqrt{3}$.

It is easily shown that equality in (1), (2), (3) holds only when the triangle is equilateral.

II. *Solution by the proposer.* Let H be the orthocenter of the triangle $A_1A_2A_3$. It is known (see, e.g., R. A. Johnson, *Modern Geometry*, p. 191) that $A_1H+A_2H+A_3H=2R+2r$. On the other hand, specializing a theorem of Fejes Tóth (*Lagerungen in der Ebene auf der Kugel und im Raum*, p. 11), $A_1H+A_2H+A_3H \geqq 2\sqrt{(\Delta\sqrt{3})}$, with equality only in the case of an equilateral triangle. Consequently

$$(*) \qquad R + r \geqq \sqrt{(\Delta\sqrt{3})}$$

and (1) follows. In the next place, since $R \geqq 2r$, with equality only in the case of an equilateral triangle, (*) implies $3R \geqq 2\sqrt{(\Delta\sqrt{3})}$, or $9R^2 \geqq 4\Delta\sqrt{3}$. Using $R = a_1a_2a_3/4\Delta$, this yields $9(a_1a_2a_3)^2 \geqq 64\Delta^3$ and $9R^3 \geqq a_1a_2a_3\sqrt{3}$, from which (2) and (3) follow.

Also solved by A. N. Aheart, Ragnar Dybvik, Jane Evans, L. D. Goldstone, J. B. Herreshoff, Robert Maier, D. C. B. Marsh, O. E. Stanaitis, and C. C. Yalavigi.

Lattice Points

E 1455 [1961, 177]. *Proposed by M. T. L. Bizley, London, England*

Let O be the origin, X the point $(p, 0)$, and Y the point $(0, p)$, where p is a positive prime number. The triangle OXY is divided into p triangles (of equal area) by the lines joining O to the points $(p-r, r)$ for $r=1$ to $r=p-1$. The interiors of the outermost two of these triangles clearly contain no lattice points (*i.e.*, points whose coordinates are both integers). Prove that the interiors of the remaining $p-2$ triangles all contain equal numbers of lattice points.

I. *Solution by the proposer.* Let R be the point $(p-r, r)$ for any particular value of r from 1 to $p-1$. Since p is prime, $p-r$ and r are coprime and hence OR passes through no lattice points between O and R. Let S be the point $(-r, r)$ and consider the lattice points in the interior of the parallelogram $OXRS$. There are $p-1$ on each of the lines $y=1, \cdots, y=r-1$; hence $(p-1)(r-1)$ in all. But, from symmetry, triangles OXR, RSO contain equal numbers, whence the number of lattice points inside triangle OXR is $f(r) \equiv (p-1)(r-1)/2$. Then,

if R' is the point $(p-r-1, r+1)$, triangle ORR' contains $f(r+1)-f(r)=(p-1)/2$ lattice points. This is independent of r, and the proof is complete.

II. *Solution by C. M. Superko, Michigan College of Mining and Technology.* The area of any simple (nonintersecting) polygon whose vertices are lattice points can be expressed by $A=i+b/2-1$, where i is the number of interior lattice points and b is the number of lattice points on the boundary. Since $(p-r, r)=1$ for $r=1, \cdots, p-1$, the "inner" triangles have boundary lattice points only at the vertices and hence $i=(p-1)/2$.

Also solved by Leonard Carlitz, D. I. A. Cohen and S. P. Cohen (jointly), Irma B. Esrig, Jane Evans, J. B. Herreshoff, D. C. B. Marsh, M. J. Sattinger, George Senge, and Guy Torchinelli.

The proposer pointed out that the problem may be stated in the following nongeometric form: Let all the fractions m/n, where m, n are positive integers and $m<n\leqq p$, p prime, be arranged in order of magnitude, fractions not in their lowest terms being retained as they stand; then the fractions whose denominators are p divide the sequence of other fractions into $p-2$ sets, all having equal numbers of fractions. [E.g., with $p=5$, $1/5<1/4$, $1/3<2/5<2/4$, $1/2<3/5<2/3$, $3/4<4/5$.]

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4983. *Proposed by M. S. Klamkin, AVCO Research, and L. A. Shepp, University of California*

Determine the number of different products, $P_n(r)$, if the factors are to be taken $r+1$ at a time, in $a_1a_2a_3 \cdots a_n$ by inserting parentheses and keeping the order of the elements a_i unchanged. The different products which arise will be due entirely to the nonassociative character of the multiplication. The explicit products for $n=4$, $r=1$ are given by $((a_1a_2)(a_3a_4))$, $(a_1(a_2(a_3a_4)))$, $(((a_1a_2)a_3)a_4)$, $(a_1((a_2a_3)a_4))$, $((a_1(a_2a_3))a_4)$. Whence, $P_4(1)=5$. This problem generalizes the case for $r=1$ (Bateman Project, *Higher Transcendental Functions*, III, 1955, p. 230).

4984. *Proposed by Y. Matsuoka, Kagoshima-shi, Japan*

Prove:

$$\sum_{k=0}^{n}\binom{n}{k}(k+1)^{k-1}(n-k+1)^{n-k}=(n+2)^n.$$

4985. *Proposed by P. T. Bateman, University of Illinois*

Fine and Herstein (*Illinois Journ. Math.*, 2 (1958), 499–504) proved that if K is a finite field with q elements, then the probability that an $n\times n$ matrix over K be nilpotent is q^{-n}. Show that their result implies the following more general fact: If R is a commutative ring with a finite number of elements, the probability that an $n\times n$ matrix with elements in R be nilpotent is r^{-n}, where r is the number of elements in the residue class ring of R modulo its radical.

4986. *Proposed by U. C. Guha, University of Malaya, Singapore*

A familiar result due to Cauchy gives the following necessary and sufficient conditions for the convergence of an infinite series $\sum a_n$: given any positive number ϵ, there exists a positive integer $N\equiv N(\epsilon)$ such that

$$\left|\sum_{k=1}^{p} a_{n+k}\right| < \epsilon, \qquad n \geqq N, \ p \text{ any positive integer.}$$

Prove the following analogous necessary and sufficient condition for the absolute convergence of $\sum a_n$: given any positive number ϵ and any sequence of positive integers $\{\lambda_n\}$ where $\lambda_1<\lambda_2<\cdots$, there exists a positive integer $N\equiv N(\epsilon, \{\lambda_n\})$ such that

$$\left|\sum_{k=1}^{p} a_{n+\lambda_k}\right| < \epsilon, \qquad n \geqq N, \ p \text{ any positive integer.}$$

4987. *Proposed by B.L.T. Dufa Scio, Institute for Defense Analysis*

Let G be a group and H a homomorphism of G into G which commutes with every inner automorphism of G. Let K be the set of elements $x\in G$ such that $H(H(x))=H(x)$. Show that K is a normal subgroup of G and that G/K is Abelian.

4988. *Proposed by W. M. Kantor, Brooklyn College*

Let $I_n=$

$$\int_0^1\int_0^1\cdots\int_0^1\left\{1+\left(\sum_{i=1}^{m} x_i^{2n-2}\right)\left(1-\sum_{i=1}^{m} x_i^{n}\right)^{(2-2n)/n}\right\}^{1/2} dx_1dx_2\cdots dx_m.$$

For given $m\geqq 1$, show that as $n\to\infty$, we have $I_n\to m+1$. What is the geometric interpretation?

SOLUTIONS

Darboux Maps

4903 [1960, 382]. *Proposed by Victor Klee, University of Washington*

A Darboux map of one topological space onto another is one under which the image of a connected set is always connected. With R denoting the real number space, show that there is a map f of R onto R such that f is a Darboux map but the product g of f with the identity map is not. (Here g is a map of the

plane $P=R\times R$ onto itself, given by $g(x, y)=(fx, y)$.)

Solution by E. H. Greene, University of Virginia. Kuratowski (*Topologie* II, page 82) gives an example which may be used to solve the problem.

Define $\omega(x)=\lim\sup(a_1+\cdots+a_n)/n$, $0<x<1$, where $0\cdot a_1a_2a_3\cdots$ is the dyadic expansion of x. The function ω takes on every value between 0 and 1 on every interval, and thus ω is a Darboux map. Likewise the function f, defined on $0<x<1$ by

$$f(x) = 0 \quad \text{when} \quad x = \omega(x), \quad f(x) = \omega(x) \text{ otherwise,}$$

and extended continuously to be a function from R onto R, is a Darboux map.

Define g to be the product of f with the identity map. This is the desired function, for, in particular, the image of the line segment $y=x$, $0<x<1$ is not connected.

Transformation of n-space

4927 [1960, 809]. *Proposed by David Gale, Brown University*

Suppose f is a continuous transformation of n-space into itself, and suppose there is a positive number λ such that $|f(x)-f(y)|\geqq\lambda|x-y|$ for all points x and y, where $|x-y|$ is the Euclidean distance. Prove that f maps onto all of n-space.

Solution by Robert Brown, The University of Chicago. Let A be the image of n-space under f. From the inequality in the hypothesis it follows immediately that f is one-to-one and that f^{-1} is a continuous transformation of A onto n-space. Hence, A is homeomorphic to n-space, and by Brouwer's theorem on the invariance of domain, A is open in n-space. Let a be an accumulation point of A, and let $f(x_1), f(x_2), \cdots, f(x_n), \cdots$ be a sequence in A converging to a. This sequence is a Cauchy sequence, and by the inequality, the sequence $x_1, x_2, \cdots, x_n, \cdots$ is also Cauchy, say with limit x. Then $a=f(x)\in A$. Hence, A is closed, as well as open, and must be the entire n-space.

Also solved by Robert D. Adams, Louis Brickman, E. W. Cheney and C. Farrington, Helen F. Cullen, A. M. Fink, E. R. Gentile, James P. Jans, Harry Sharp, Jr., Robert Spira, D. C. Stevens, Wu Ta Sun, Fred Suvorov, and the proposer. Late solution by W. C. Waterhouse.

Conjugate Roots of a Normal Equation

4928 [1960, 809]. *Proposed by C. C. Faith, Pennsylvania State University*

Show by a concrete example that there exists a normal equation $\phi(x)=0$ of degree n over a field F of characteristic zero having two conjugates v and v' whose difference $v-v'$ has degree less than n over F, when $n=4$, or when $n=6$.

Solution by Daniel A. Moran, University of Illinois. Let $n=4$. Let $\phi(x)=x^4-10x^2+1$, where F is the field of rationals. The roots of this equation are $\pm\sqrt{2}\pm\sqrt{3}$. The difference of two properly selected roots obviously gives an element of degree 2 over F. The equation is easily seen to be normal.

For $n=6$, let $\phi(x)=x^6-3$, where F is the field $Q(i)$, Q being the rationals and

i being a square root of unity. Again, the equation is easily shown to be normal. But the roots of $\phi(x)=0$ are: $\pm\sqrt[6]{3}$, $\pm\omega\sqrt[6]{3}$, $\pm\omega^2\sqrt[6]{3}$, where ω is a primitive cube root of unity. Moreover, $(\omega\sqrt[6]{3})-(\omega^2\sqrt[6]{3})=i\sqrt[3]{9}$, which has degree 3 over $F=Q(i)$.

Also solved by Leonard Carlitz, Harley Flanders, and the proposer.

At various times, contributors to this Department turn their attention to the fund of unsolved past problems and help to cut down the number. (See pages 68, 72, 75 of *The Otto Dunkel Memorial Book*, issued as Part II of vol. 64, 1957, no. 7, this MONTHLY.) The present comments and solutions by C. F. Pinzka will be of interest to a number of readers.

2762 [1919, 170]. *Proposed by N. P. Pandya*

$ABCD$ is a cyclic quadrilateral inscribed in an ellipse. $AB=2BC$ and $CD=2DA$. Find the eccentricity of the ellipse in terms of the sides of the quadrilateral.

Comment by C. F. Pinzka, University of Cincinnati. There is no unique eccentricity—in fact, the circle $ABCD$ may be considered as an ellipse of eccentricity zero. Let an ellipse E passing through A, B, C, D have the equation

$$R \equiv (1-e^2)x^2+y^2-a^2=0,$$

where the axes are appropriately chosen. If the equation of the circle $ABCD$ is $S\equiv x^2+y^2+bx+cy+d=0$, then the equation $R+kS=0$ represents a family of conics through A, B, C, D with eccentricity $e'=e(1+k)^{-1/2}$. Thus there is no unique eccentricity.

2787 [1919, 366]. *Proposed by Warren Weaver*

In the gambling game known as "craps" two dice are thrown. The one throwing the dice makes a bet which is covered by an equal amount by one or more players opposing him. If he throws, on the first throw, a sum of seven or eleven he wins at once; if he throws a sum of two, three or twelve he loses at once; if he throws any other sum he continues throwing until he either duplicates his original sum and wins, or throws a sum of seven and loses. Show that the probability that the one who is to throw the dice will win is 0.49847, so that the game would be very nearly fair if one person were to throw the dice continuously instead of changing about, as is actually done.

Comment by C. F. Pinzka, University of Cincinnati. It is well known that the shooter's probability of winning in a game of craps is $244/495=0.4\overline{92}$. Thus the proposed figure appears to be incorrect.

2816 [1920, 134]. *Proposed by W. H. Echols*

Find two points D and E on the sides AB and CB, respectively, of a triangle ABC such that $AD=DE=EC$. Give a ruler and compass construction.

Comment by C. F. Pinzka, University of Cincinnati. This is solved in Altshiller-Court, *College Geometry*, Section 47, page 44, 1st edition (1925).

2867 [1920, 482]. *Proposed by W. H. Hays*

Show that the probable value of the sum of the squared differences, got by pairing two sets of n numbers each, at random is $n(n^2-1)/6$; and that the coefficient of correlation in such a case is zero.

Solution by C. F. Pinzka, University of Cincinnati. Assume that each set consists of the positive integers from 1 to n. Among the $n!$ ways of pairing the two sets, there will occur $(n-1)!$ cases in which a fixed element x of the first set will be paired with a fixed element y of the second set.

Hence, after direct computations, the expected sum of squared differences is

$$(1/n!)\sum_{x=1}^{n}\sum_{y=1}^{n}(n-1)!(x-y)^2 = n(n^2-1)/6.$$

Similarly, the correlation coefficient is given by

$$s_x s_y r = (1/n)\left[(1/n!)\sum_{x=1}^{n}\sum_{y=1}^{n}(n-1)!xy - (1/n)\sum_{x=1}^{n} x\sum_{y=1}^{n} y\right] = 0.$$

2909 [1921, 326]. *Proposed by J. S. Stroms*

A pinochle pack contains 48 cards, eight each of aces, kings, queens, jacks, tens and nines. When three play they are distributed by giving fifteen to each player and leaving three in the "kitty." When one holds two jacks of diamonds and two queens of spades in the same hand he has what is called double pinochle. When he holds eight aces he wins the game with 1000 points. What are the chances (a) of getting double pinochle; (b) of getting eight aces?

Solution by C. F. Pinzka, University of Cincinnati. Assuming the kitty available for meld and neglecting the effect of the bidding, the probabilities are

$$\text{(a)} \qquad \binom{44}{14}\Big/\binom{48}{18} = \frac{17}{1081}, \qquad\qquad \text{(b)} \qquad \binom{40}{10}\Big/\binom{48}{18} = \frac{221}{1,905,803}.$$

2953 [1922, 81]. *Proposed by C. F. Gummer*

An algebraic equation being known to have exactly r real roots, all simple, is it possible to find an equation of degree $r-1$ whose roots separate those of the given equation and whose coefficients are rational functions of the coefficients of the given equation?

Solution by C. F. Pinzka, University of Cincinnati. It is assumed that the original equation is of degree r (not having n roots of which exactly r are real). If $f(x) \equiv \sum_{k=0}^{r} a_k x^k = 0$ is the given equation, then $f'(x) \equiv \sum_{k=1}^{r} k a_k x^{k-1} = 0$ would have roots which separate the roots of $f(x)=0$, by Rolle's theorem.

3009 [1923, 76]. *Proposed by J. G. Coffin*

Rectangular pieces of cardboard of the same dimensions are piled so that they overhang to the greatest extent possible; what curve do the edges touch? How great a distance between first and last piece can be obtained? And what are the properties of the material volume thus produced?

Comment by C. F. Pinzka, University of Cincinnati. This is essentially Problem E 1122 [1955, 123].

3026 [1923, 275]. *Proposed by B. F. Finkel*

A flat board 12 inches square is suspended in a horizontal position by strings attached to its four corners A, B, C, D, and a weight equal to the weight of the board is laid upon it at a point 3 inches distant from the side AB and 4 inches from AD. Find the relative tensions in the four strings. (From Bowser, *Analytical Mechanics*, p. 92.) Is this a determinate problem? If not, why not?

Solution by C. F. Pinzka, University of Cincinnati. Assuming the strings vertical and denoting by A, B, C, D, the tension in the strings and by W the weight of the board, we have $A+B+C+D=2W$, $12B+12C=10W$, and $12C+12D=9W$ as sufficient conditions for equilibrium. These yield $A=(7W-6D)/6$, $B=(W+12D)/12$, $C=(3W-4D)/4$, where $0\leqq 4D\leqq 3W$, indicating static indeterminancy.

Bowser gives $A:B:C:D=9:6:4:5$, which assumes $D=5W/12$.

3127 [1925, 204]. *Proposed by Harry Langman*

Assuming $x^3+y^3=z^3$ impossible, show that neither of the expressions $4q^3-27p^6$, $12q^3-3p^6$ can be squares.

Comment by C. F. Pinzka, University of Cincinnati. The problem is impossible. We know $x^3+y^3=z^3$ to be impossible (in integers). However, $4q^3-28p^6$ is a square for $(p, q)=(n, 3n^2)$, while $12q^3-3p^6$ is a square for (n, n^2).

3144 [1925, 385). *Proposed by John Biggerstaff*

Find the point from which the sum of the distances to two given straight lines and the distance to a given point is a minimum.

Solution by C. F. Pinzka, University of Cincinnati. Let P be the given point, L_1 and L_2 the given lines (with L_1 the nearer), S their intersection, and ϕ the angle between L_1 and L_2 to which P is interior.

We note the following result from elementary geometry: If T is any point on a rectangle and U, V are the projections of T on the diagonals, then $TU+TV$ is constant. Applying this result, we see that if M is a rectangle whose diagonals lie along L_1 and L_2, then the point on M for which the sum with which we are concerned is a minimum is simply the point nearest P. This will be either a vertex or the nearest projection of P on M. Considering now a family of rectangles whose diagonals lie along L_1 and L_2, we establish a locus of these relative minimum points, along which locus the absolute minimum point or points must lie. This locus is the broken segment PQS, where PQ is parallel to the bisector of ϕ and Q lies on L_1.

A number of cases arise (see figure), which may be summarized as follows:

(1) $\phi<60°$	Minimum point: P
(2) $\phi=60°$	Minimum point: any point on PQ
(3) $60°<\phi<90°$, $\measuredangle PSQ<90°-\phi$	Minimum point: R on QS, where $\measuredangle PRQ=90°-\phi$; PR and the normal from R to L_2 make equal angles with L_1
(4) $60°<\phi<90°$, $\measuredangle PSQ\geqq 90°-\phi$	Minimum point: S
(5) $\phi>90°$, $\measuredangle PSQ\geqq\phi-90°$	Minimum point: S
(6) $\phi>90°$, $\measuredangle PSQ<\phi-90°$	Minimum point: T on QS, where T is the intersection of QS and the normal from P to L_2.

3183 (originally 3179) [1926, 159]. *Proposed by Philip Fitch*

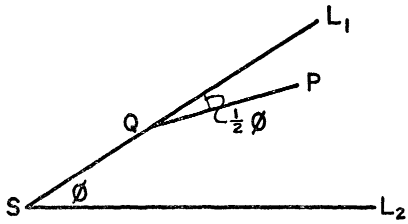
A and B mail letters to $m+n$ people, A mailing m letters and B mailing n letters. If no person can receive more than one letter from either of the senders, and if two letters, no letter, or the wrong letter to a person is counted as an error, how many errors are possible?

Solution by C. F. Pinzka, University of Cincinnati. Let A' and B' denote the groups who would receive letters from A and B, respectively, if there were no errors. Group A' contains m people and group B' contains n people. Assume $m\geqq n$.

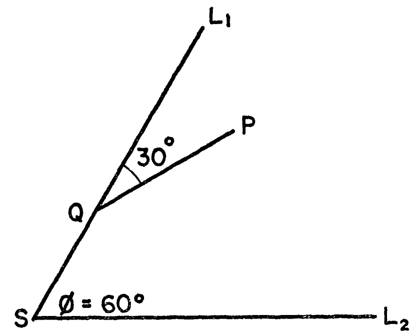
If $n\leqq m\leqq 2n$, $m+n$ errors are possible if the letters are distributed as follows:

Group A': $m-n$ receive no letters,
$m-n$ receive two letters,
$2n-m$ receive letters from B
Group B': n receive letters from A.

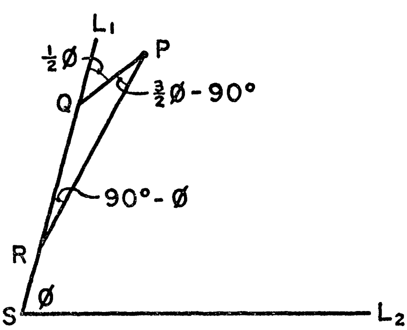
If $m>2n$, let x be the number in A' receiving letters from B only. Then the number in both groups receiving two letters is $\leqq n-x$, the number receiving no letters is also $\leqq n-x$, and the number receiving one wrong letter is $\leqq n+x$; whence the total number of wrong letters is $\leqq 3n-x$. The maximum of $3n$ is achieved as follows (continued on p. 814):



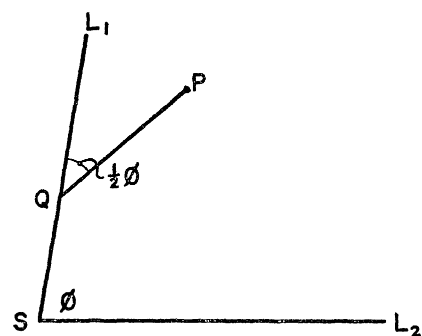
$\phi<60°$, P is minimum point.



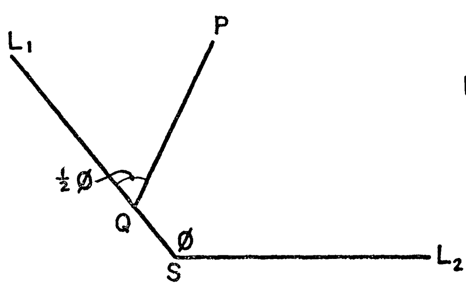
$\phi=60°$, any point on PQ is a minimum point.



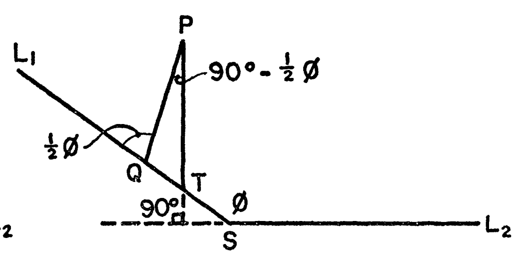
$60°<\phi<90°$, $PSQ<90°-\phi$,
R is minimum point.



$60°<\phi<90°$, $PSQ\geqq 90°-\phi$,
S is minimum point.



$90°<\phi<180°$, $PSQ\geqq\phi-90°$,
S is minimum point.



$90°<\phi<180°$, $PSQ<\phi-90°$,
T is minimum point.

Figures for Problem 3144

Group A': n receive no letters,
n receive two letters,
$m-2n$ receive letters from A.
Group B': n receive letters from A.

3355 [1928, 564]. *Proposed by Elmer Schuyler*

In the *Plane and Solid Geometry* by Wentworth-Smith (revised edition) occurs the following proposition: Of all polygons with sides all given but one (and in a definite order), the maximum can be inscribed in a semicircle which has the undetermined side for its diameter.

Prove that there is one and only one maximum. Given the lengths of the sides and their order, compute the radius of the semicircle.

Solution by C. F. Pinzka, University of Cincinnati. It is well known that the inscriptible polygon has the maximum area of all polygons with given sides, and that there is a unique inscriptible polygon if the order of the sides is specified. When all sides but one are given, the polygon obtained by adjoining to the given sides their reflection about the unknown side must also be a maximum and hence inscriptible. Thus the side to be determined is the diameter of the semicircle in which the given sides are inscribed.

Consider the case of a quadrilateral with given sides a, b, c inscribed in a semicircle of diameter D. The theorems of Ptolemy and Pythagoras yield the equation $D^3-(a^2+b^2+c^2)D-2abc=0$. Even this simplest case gives a somewhat unwieldy formula for D. It is not difficult to find values for a, b, c (*e.g.*, 1, 2, 3) which yield the irreducible case of the cubic, whence D is neither constructible nor expressible in terms of a finite number of real radicals.

3534 [1932, 116]. *Proposed by J. V. Uspensky*

Show that

$$\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t-1}{t^r-1}\right)\right]_{t=1} = [1-r][1-2r]\cdots[1-(n-1)r]r^{-n}.$$

Comment by C. F. Pinzka, University of Cincinnati. For $r=2$, we have

$$\left[\frac{d^{n-1}}{dt^{n-1}}(t+1)^{-1}\right]_{t=1} = [(-1)^{n-1}(n-1)!(t+1)^{-n}]_{t=1}$$

which gives $(-1)^{n-1}(n-1)!2^{-n}$, and does not agree with the proposed formula.

3574 [1932, 549]. *Proposed by Orrin Frink, Jr.*

A company wishes to establish five agencies at five points of a circular region. When this is done there will be a smallest number r such that every point of the circular region will be within a distance r of at least one of the agencies. How should the agencies be located so as to make r a minimum?

Comment by C. F. Pinzka, University of Cincinnati. This is equivalent to the well-known five disc problem, described in Ball, *Mathematical Recreations and Essays*, where reference to a solution is given.

CORRECTION

There are misprints in the solutions to problems (this MONTHLY, vol. 68, 1961, pp. 675–677). In line 2 of the statement of 4919 (p. 675), the upper limit on the integral should be 2π; in line 3 of the solution to 4923 (p. 676), the $\sum$ should be replaced by $\bigcup$; in line 5 (p. 677), the lower summation index should be $n+1$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any of the other editors or officers of the Association.

Cluster Sets. By Kiyoshi Noshiro. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin, 1960. vi+135 pp. DM 36.

Let f be a complex function defined in a plane domain D which has a boundary point z_0. The cluster set of f at z_0 is the set of all complex numbers w (including possibly ∞) such that $f(z_n)$ tends to w for some sequence $\{z_n\}$ in D which tends to z_0. If $\{z_n\}$ is restricted to lie in certain subsets of D, one obtains definitions of other types (radial, angular, etc.) of cluster sets. In the situation which is most thoroughly discussed in this monograph, D is the open unit disc, and f is meromorphic. That is to say, the boundary behavior of meromorphic functions in the unit disc is studied. Proofs of most results are included; they lean heavily on topological (set-theoretic) methods. The scope of the investigation is then enlarged to Riemann surfaces, and an appendix deals with pseudoanalytic functions. The book is a welcome compendium of recent work which has so far been scattered throughout the literature.

WALTER RUDIN
University of Wisconsin

Commutative Algebra, Vol. II. By Oscar Zariski and Pierre Samuel. Van Nostrand, Princeton, N. J., 1960. $9.75.

The material covered in this second volume is of a more advanced and specialized nature than that of the first. The subject matter of volume I is on a level that could be, and generally is, taught in a first year graduate algebra course. Volume II, on the other hand, will appeal mainly to those who are, or want to be, algebraists or algebraic geometers. Although a person who is familiar with basic algebra could read the second volume without having read the first, it would probably be helpful to the reader of this volume to have volume I available.

There are three major topics discussed in this book. They are: i) valuation theory; ii) theory of polynomial and power series rings; iii) local algebra. All three of these topics derive their prime motivation from algebraic geometry, and the authors do an excellent job of discussing the connection between the purely algebraic material and the algebro-geometric situation. For example, when discussing places and the center of a place, they follow this by explaining the notion of the center of a place in algebraic geometry. When discussing polynomial rings (graded and ungraded), they also discuss projective and affine varieties and the relations between them. Their treatment of the algebro-geometric material is more than just illustrative. They go into such topics as

prime divisors in algebraic function fields, the abstract Riemann surface of a field, and derived normal models. A proof of the analytic irreducibility and analytic normality of normal varieties is also included.

A detailed discussion of the three chapters of this book would be too technical, but some general features should be mentioned. In Chapter VI (Valuation Theory), the theory of places is developed first without discussing the associated valuations. The second half of this chapter then goes into valuation theory and treats those properties which depend upon the value group of the valuation. Numerous examples of valuations are given, taken largely from algebraic function fields.

Chapter VII (Polynomial and Power Series Rings) contains the classical material on polynomial and power series rings. This deals mainly with dimension theory in these rings and a thorough formulation of the various normalization theorems (although there seems to be an essential omission in the proof of Theorem 25 in section 7). The chapter also includes an exposition of the characteristic function of graded modules and homogeneous ideals, as well as chains of syzygies (which introduces the basic notion of homological dimension).

Chapter VIII (Local Algebra) treats the general properties of local rings, and includes theorems on the structure of equicharacteristic complete local rings, multiplicity theory, and dimension theory.

The seven appendices treat topics of current interest and are best understood by those actually familiar with work in algebraic geometry (for example, Appendix 4 is related to the concept of a complete linear system on an algebraic variety with assigned base loci).

The book is clearly written, and does an extremely good job of bringing together in one volume the basic material required for working in commutative algebra. Although tastes vary, and one may take exception to some methods of proof or the arrangement of a sequence of ideas, there is no doubt that this volume will be of lasting value to the serious student of algebra.

D. BUCHSBAUM
Institute for Advanced Study

Logic: The Theory of Formal Inference. By Alice Ambrose and Morris Lazerowitz. Holt, Rinehart and Winston, New York, 1961. 78 pp. \$2.00.

The material of this short book corresponds closely with the material of the longer book, *Fundamentals of Symbolic Logic*, published in 1948 by the same authors. The present book has three chapters. The first chapter, on truth-functions, introduces the basic logical connectives, gives some elementary inference rules, and explains truth-table methods. It then illustrates the workings of the propositional calculus by deriving four axioms of Whitehead and Russell which had been used as postulates in the earlier book from three axioms of Lukasiewicz. Questions of consistency and completeness are *not* treated. The second chapter, on quantification, gives formation rules for the first order predicate calculus and enough inference rules so that quantifiers can be brought to the

fore, but the discussion stops short of an axiomatic treatment. Instead, a large portion of this chapter is concerned with the classical A, E, I, and O statement forms and with methods of testing syllogisms for validity. The six syllogistic rules of the earlier book are replaced by four rules, which are shown through Venn diagrams to be independent. The third chapter, on classes, presents an "algebra of classes" based on postulates of Huntington for boolean algebra. Each chapter ends with one or two pages of exercises. Throughout the book the treatments are too brief to permit the development of the important properties of the systems presented. One may wonder whether the space devoted to syllogisms might not nowadays be better spent on the elementary theory of relations.

GEORGE N. RANEY
The Pennsylvania State University

General Theory of Banach Algebras. By Charles E. Rickart. Van Nostrand, Princeton, N. J., 1960. viii+394 pp. $10.50.

As the title indicates, this book is concerned primarily with the general structure of Banach algebras. Indeed, though several special cases are discussed in detail (*e.g.* algebras of operators, group algebras), the emphasis is on the structure of general commutative Banach algebras. The treatment is, intentionally, heavily algebraic with the analytic and topological considerations receiving substantially less emphasis. The principal algebraic ideas used center around the Jacobson-Gelfond definition of the radical, regular (or modular) ideals, semisimplicity, and the concept of an algebra with involution.

The manner of presentation is quite brief and presumes considerable sophistication as well as previous knowledge on the part of the reader. Thus, it seems reasonable to say that this book is not intended for the beginner in Banach algebras, but is rather for the specialist; indeed, this seems to be the author's intention. The specialist will find a wealth of material expertly organized and cross-referenced. An outstanding feature of the book is its extensive bibliography.

E. H. BATHO
University of New Hampshire

Stochastic Population Models in Ecology and Epidemiology. By M. S. Bartlett. Methuen, London, Wiley, New York. 1960. 90 pp. $2.00.

This monograph is one of a series on applied probability and statistics devoted to recent developments in these areas. One must surely applaud this step since short accounts that can give fuller treatment than journal articles provide surely serve a vital function. Unfortunately, this contribution is unlikely to find acceptance either with mathematicians or biologists for somewhat different reasons, but essentially because the book gives every evidence of hasty writing. From the standpoint of the nonprofessional mathematician, this places an unnecessarily heavy burden on the reader.

To document this contention by a few examples, on page 9 the author could easily have added a phrase to indicate that use of L'Hospital's Rule would produce the desired limit. On page 22 the addition of a brief sentence could make it quite clear that the ultimate probability of extinction occurred when the derivative vanished. Again, on page 27 the limit of $\pi(0)$, $\pi[\pi(0)]$, $\cdots$ is given as the minimal root of $\pi(p)=p$. A short graphical argument could not only make this result intuitively obvious but would also reveal the conditions under which the statement is valid. [It is also possible to establish this theorem using generating functions, but this requires more sophisticated, though not difficult, mathematics.]

The mathematician will object to the lack of elegance in the presentation. This even extends to the use of English on occasion, *e.g.*, the use of nonexistent words such as "correlational" page 6 and "infectivity" page 54.

K. A. Bush
Washington State University

Lectures on Linear Algebra (Interscience Tracts in Pure and Applied Mathematics, vol. 9). By I. M. Gel'fand. Interscience, New York, 1961. ix+185 pp. $6.00.

This translation from the second Russian edition (1950) is a welcome addition to an algebraist's English library. By request of the author two appendices are not included in the present publication. The translation of the four main chapters has been done smoothly.

Linear algebra, according to the book's flavor, is the study of linear transformations of vector spaces. The opening chapter establishes the preliminaries concerning finite-dimensional vector spaces in general and Euclidean spaces in particular. Significant examples appear often; some of these illustrate spaces of infinite dimension. Unfortunately it is not always clear just what background has been presupposed for the reader. Bilinear forms and quadratic forms are studied. A final section explains the modifications appropriate for complex vector spaces.

The two middle chapters investigate the properties of linear transformations. Special types are treated in detail: self-adjoint, unitary, normal, orthogonal. Emphasis is accorded the relationship between a linear transformation and the various matrices which represent it relative to assorted bases; much attention is assigned to orthogonal bases and canonical forms.

The material in the last chapter, a brief introduction to the notions of dual space and of multilinear functions, is presented as a natural generalization of earlier ideas.

The style of writing is praiseworthy. Happily the author often pauses to pull threads together. The exercises, sprinkled here and there, may not be numerous enough to serve the purpose of a course text. The lack of an index handicaps the usefulness of the work as a reference.

R. A. Good
University of Maryland

Elements of Statistical Inference. By R. M. Kozelka. Addison-Wesley, Reading, Mass., 1961. 150 pp. $5.00.

This is a small text which manages to cover a surprising amount of elementary statistical theory in a lucid and concise style. The text presupposes one semester of calculus and is designed to serve as a one-semester introductory course in statistical theory at the freshman or sophomore level and is intended primarily for students who intend to follow the social or behavioral sciences.

Chapters 1 and 2 treat probability using the set-function approach. The basic concepts of statistical distribution are developed in chapters 3, 4, and 5. The classical notions of inference, estimation and testing of hypotheses, are introduced in chapters 6 and 7, and chapter 8 very briefly considers the bivariate problems of regression and correlation. The level of exposition is about that of Hoel's *Introduction to Mathematical Statistics*.

To cover so much so briefly has been achieved only at a price. Many of the topics one would expect to find in an introduction to statistics will not be found in this little book. Indeed, the author states in the preface, "It is not designed to teach techniques. Nothing is said about t-tests, F-tests, chi-squared, or other ideas so dear to the heart of the practicing statistician. Rather the material should suggest to the student how a statistician thinks, why he thinks that way and some of the things he is likely to think about." In this reviewer's opinion, it is doubtful that the approach to statistics as developed in this book will be very helpful in suggesting to students, especially those in the social sciences, how and why statisticians think the way they do. It is also unfortunate that the author should raise again the question of defining the sample variance with n or $n-1$ in the denominator, only to dismiss the problem with the comment "the author feels that all writers who define s^2 with $n-1$ in the denominator should be requested by law so to indicate on the front cover of the book."

WILLARD O. ASH
University of Florida

An Introduction to Linear Statistical Models, Volume I. By Franklin A. Graybill. McGraw-Hill, New York, 1961. 463 pp. $12.50.

"This book was written with the intention of fulfilling three needs: (1) for a theory textbook in *experimental* statistics, for undergraduate students; (2) for a reference book in the area of regression, correlation, least squares, experimental design, etc., for consulting statisticians with limited mathematical training; and (3) for a reference book for experimenters with limited mathematical training who use statistics in their research. This is not a book on mathematics, neither is it an advanced book on statistical theory."

The book is Volume I of a two-volume work, the second of which will contain "such topics as sample size, multiple comparisons, multivariate analysis of variance, response surfaces, discriminant functions, partially balanced incomplete block designs, orthogonal latin squares, randomization theory, split-plot

models, and some nonlinear models."

The current volume has four chapters that deal with mathematical and statistical concepts, the multivariate normal distribution, and the distribution of quadratic forms. The remainder of the book, chapters 5–18, is primarily devoted to the mathematical treatment of five linear models that are used in the analysis of variance. All of these models assume infinite populations.

The book is not concerned either with the choice of a correct model or the drawing of conclusions concerning the world of experience—but any author must limit the area he discusses.

WILLIAM G. MADOW
Stanford Research Institute, Menlo Park, California

Linear Algebra. By Kenneth Hoffman and Ray Kunze. Prentice-Hall, Englewood Cliffs, N. J., 1961. ix+332 pp. $7.50.

This is a worthy member of the already long list of recent books on linear algebra and matrices. The authors state that this book is written for mathematics majors at the junior level, and that they have made no particular concession to the fact that the majority of their students are not mathematics majors except to include nearly five hundred examples at various levels. They hope by means of these examples to minimize the number of students who can repeat definition, theorem and proof without grasping the significance of the abstract concepts involved—a situation which they readily acknowledge to exist.

In this last acknowledgement the authors indict many of the recent books on linear algebra, which are too ambitious for the students who are supposed to use them; and in spite of their disclaimer, I am not convinced that the present authors avoid this pitfall. I have no quarrel with the aims of the book, or with the presentation, which is able, careful, and up-to-date. A student who has mastered the contents will have become a modern algebraist, quite prepared to go into a graduate course in algebra. But I believe that for a beginning course to be taken by juniors the expectations are too high and that too much effort is made for great generality. Surely in a first course the concept of greatest common divisor of polynomials can be introduced without ideal theory, and determinants can be introduced without the concept of an n-linear function. Until recently bilinear form and polynomial were simple concepts, but now they seem complicated and, to a beginner, unnatural. These modern ways of looking at familiar concepts seem elegant to persons who are familiar with the older approach, but their great beauty must be lost on those who approach them in this manner before they are in a receptive mood. We feed them cake when they ask for bread.

So much time devoted to definitions and concepts must be purchased at the expense of conventional mathematics, namely techniques (which the authors look down upon) and applications. But somewhere in his career the future mathematician must acquire these techniques, just as did those of an older

generation who afterwards developed the abstract approach. Is it unreasonable, or merely unfashionable, to advocate a middle-of-the-road approach?

C. C. MacDuffee
University of Wisconsin

Calculus and Analytic Geometry. By John F. Randolph, Wadsworth, San Francisco, 1961. xi+618 pp. $8.50.

The forerunners of this book are *Analytic Geometry and Calculus* by Randolph and Kac, Macmillan, 1947, and *Calculus* by Randolph, Macmillan, 1952. Professor Randolph borrows, with permission, from his preceding works, but he revises as he borrows. Thus, the work under review is a sort of third edition based on fourteen years experience, and happily it shows the smoothness and polish one would expect under such circumstances.

There are chapters on vector analysis and differential equations that have not appeared before. In 1952 an appendix was added giving some rigorous "ϵ, δ" arguments. This has been expanded to include an existence proof for the integral of a continuous function.

Randolph and Kac were pioneers in distinguishing between f and $f(x)$. Randolph's discussion of functions has evolved as follows. In 1947 a function was defined as a rule of correspondence and a typical description read, "Let f be the function such that $f(x)=x^2$." In 1952 a function was defined as a set of ordered pairs, and phrases like "the set of all ordered pairs of the form (x, x^2)" are used sparingly. In 1961 the ordered pairs definition stands, and the notation $f=\{(x, y)\,|\,y=x^2\}$ is used systematically. He still has no symbol for the identity function—next time, perhaps.

M. Evans Munroe
University of New Hampshire

Cartesian Tensors: An Introduction. By G. Temple. Methuen, London, 1960. Wiley, New York, 1961. 92 pp. $2.75.

In this small volume the author discusses briefly a number of topics (of particular interest and importance to theoretical physicists) which have expanded the field of cartesian tensors during the thirty years since the appearance in 1931 of the book by Jeffreys. Instead of defining a tensor as an invariant with a set of functions as components which transform cogrediently with the coordinate system with which they are associated, this author defines tensors in the Bourbaki fashion as multilinear functions of direction. Examples are chosen to illustrate the usefulness of the tensor notation in both the algebra and analysis of tensors. There appear concise treatments of the strain tensor, isotropic tensors, and spinors, all of which is accomplished within the confines of three-dimensional Euclidean space. A short discussion of tensors in orthogonal curvilinear coordinates is provided in the closing chapter.

C. E. Springer
The University of Oklahoma

Elementary Logic of Science and Mathematics. By P. H. Nidditch. Free Press of Glencoe, 1960. 339 pp. $4.00.

By "the logic of mathematics" the author understands what is usually called deductive logic, and by the "logic of science" the author understands what is usually called scientific method, which he takes as forming "inductive logic."

The treatment of deductive logic is elementary, and quite a bit of material is covered. The propositional calculus is introduced via truth table, and then the Russell-Whitehead axioms are given and the completeness of these is proved via the methods of Kalmàr. The consistency, but not the completeness, of a version of quantification theory is demonstrated. There is also a chapter on Boolean algebra and elementary set theory. The author begins by using the familiar Peano-Russell symbolism for logic, and later introduces the parenthesis-free Lukasiewcz symbolism. Thereafter, developments are presented in two parallel columns, in which one uses the Peano-Russell symbolism and the other the Lukasiewcz symbolism. There are numerous exercises, many of which a beginning student would find extremely difficult.

The treatment of scientific method discusses the role of observation, measurement (including an introduction to dimensional analysis), elementary probability theory, statistical methods, and the role of deduction in science. There is also a chapter on mathematics and deduction. Here the author states that there is "general agreement that a mathematical theory is, in the now popular phrase, a *hypothetico-deductive system.* It is a system in which there are certain *primitive propositions* (postulates) which are adopted without proof and from which alone all other categorically asserted propositions of the system have to be logically deducible." Although the work of Gödel on undecidability is mentioned in a footnote, there is no discussion of the relevance of this work to the identification of the contents of mathematics with that of axiomatic systems.

MARTIN DAVIS
Yeshiva University

Introductory Formal Logic of Mathematics. By P. H. Nidditch. Free Press of Glencoe, 1960. 186 pp. $3.00.

The preface of this book asserts: "The book initiates a radically new departure in mathematics, since it gives for the first time logically valid proofs of mathematical theorems. *In the whole literature of mathematics there is not a single valid proof in the logical sense.*"

What the book in fact contains is the development by natural deduction methods of the propositional calculus, quantification theory, and certain elementary portions of the theory of sets. None of the deeper results of modern logic (*e.g.* the completeness of quantification theory, the Skolem-Löwenheim theorem, or the Gödel incompleteness theorem) are treated. In fact, the book is original only in its notation which is based on the parenthesis-free notation for the propositional calculus together with the author's own symbols. While there

is little metamathematical exposition, what there is is confused and confusing. For example, speaking of the propositional variable r, the author states that it "can be viewed as the representative of all propositions; it does not denote a specific but unspecified proposition; it is not a name of some particular statement but a name for any statement." Later it is asserted that any set of ordered pairs is a product set.

The author intends this book as a "text book for the second-year logic student and mathematics student." It is to be hoped that the student who wishes to gain an idea of the scope and spirit of modern logic will look elsewhere.

MARTIN DAVIS
Yeshiva University

Norm Ideals of Completely Continuous Operators. By Robert Schatten. Springer-Verlag, Berlin, 1960. 81 pp. $5.66.

This monograph is a concise and careful organization of the currently known theory of ideals (with and without norms) of completely continuous operators on a Hilbert space.

The author presents new results and also draws attention to some existing ones. After presenting preliminary results on operators and some interesting theorems on ideals, the author discusses the important Schmidt-class and trace-class of operators, and presents some nice results on the Banach space (with operator bound as norm) of all completely continuous operators and its conjugate spaces. In the final third of the book he treats norm ideals in general and gives a neat characterization of minimal norm ideals and their conjugates.

The prerequisites for reading the book are few, and the reviewer recommends it to anyone interested in operator theory. A criticism should be made against a certain prevalent carelessness of style, *e.g.*, the use of "either" on page 29, and the corollary on page 12, among numerous instances.

HOWARD H. WICKE
Sandia Corporation

Elements of Linear Spaces. By A. R. Amir-Moéz and A. L. Fass. Lithoprinted by Edward Bros., Ann Arbor, Mich., 1961. vii+149 pp. $5.25.

This book is the outcome of an attempt by the authors to provide an introduction to linear algebra that will bridge the gap between freshman mathematics and modern abstract algebra. In Part 1 (60 pages) the setting is real Euclidean space of three dimensions. The student is introduced, quite rapidly, to vectors, linear dependence, inner products, orthonormal bases, linear transformations, matrices and determinants of order two and three, and their application to systems of linear equations and to the orthogonal reduction of quadratic forms in two and three variables. In Part 2 (55 pages) two generalizations are introduced: the extension of the field of scalars to the complex numbers and the consideration of n-dimensional spaces. A vector is still an n-tuple of complex numbers. The results of Part 1 are extended to n-dimensional unitary spaces

and applied to the classification of quadric surfaces. There is a more thorough discussion of the quadrics than is usual in a book of this sort. In Part 3 (28 pages) we find abstract definitions of group, field, and vector space and of an inner product in a vector space over the complex numbers. The previously described theory is fitted into the abstract setting and carried as far as the decomposition of a Hermitian transformation into a linear combination of projections. A final chapter discusses singular values and their relation to the eigenvalues of a transformation.

The book is mathematically sound and the arrangement of the subject matter makes good pedagogical sense. Nevertheless the beginning student will not read it without some expenditure of blood, toil, tears and sweat. The authors state "We believe that both students and instructors are intelligent and would like to supply details of proof or technique in many places." In accord with this view many gaps are left for the student to fill in, although methods required are usually clear enough. There are extensive problem sets, the working of which should give the student considerable understanding and power. There is an index and an appendix containing a brief review of solid geometry. The diagrams are good. In the reviewer's opinion answers and hints to problems and a short bibliography would have been useful additions. The book should prove a valuable addition to the undergraduate literature in the field.

D. C. Murdoch
University of British Columbia

Fourier Transforms. By R. R. Goldberg. Cambridge Tracts in Mathematics and Mathematical Physics, No. 52. Cambridge University Press, New York, 1961. vii+76 pp. $3.75.

In the preface of this book the author states as his design, " . . . to provide a background in those classical theorems concerning Fourier transforms on the real line which have found fruitful generalization in abstract harmonic analysis."

The book consists of five chapters and an appendix, Chapter 1 being preliminary. In Chapter 2 the L^1-theory is studied, culminating in Wiener's theorem on the closure of the translates of a function whose Fourier transform never vanishes, and in Chapter 3 we find the L^2-theory, the principal result being Plancherel's theorem. Chapter 4 is occupied with some generalizations of Wiener's theorem, and Chapter 5 with a proof of Bochner's theorem on the representation of totally positive functions as Fourier-Stieltjes Transforms. In the appendix the author indicates how Fourier transforms can be generalized to locally compact groups, and reformulates several of the previous theorems in group terminology.

Prerequisites for reading the book are a fair knowledge of Lebesgue and Stieltjes integration, and a certain degree of mathematical sophistication. In this reviewer's opinion the author has accomplished his design admirably.

P. G. Rooney
University of Toronto

Statistical Theory and Methodology in Science and Engineering. By K. A. Brownlee. Wiley, New York, 1960. 570 pp. \$16.75.

As usual with applied statistics works, the book is written with mathematical techniques not far above the freshman college level. As usual, the logic involved in design and analysis of scientific experiments requires a mathematical maturity, apart from techniques, far above this elementary technique level. Thus, as usual, the mathematically unsophisticated reader will delight in the promise of simple mathematics only to find waiting for him a frustrating confusion when he comes on things like Jacobians. This problem is due to a lower bound in the necessary level of concept of the material and the fault lies with the reader rather than with the writer. The only legitimate remedy is to raise the mathematical sophistication of the reader; a reduction in maturity of concept would inevitably lead toward the dangers of Snedecor (which contains a wealth of information which is virtually incomprehensible to the expert due to a complete lack of theory and which is copied willy-nilly by the nonexpert without regard for satisfaction of assumptions, alternative methods, etc.).

Within the framework of this confusing meeting of two worlds, Brownlee presents his well-known mastery of statistical methods in a fashion which could scarcely be improved, from the standpoint of the reader with some mathematical background. His presentation is clear, concise, and explicitly stated, and he derives in detail the theory for each technique so that the reader may decide without uncertainty whether or not this technique is appropriate to his applied problem. This book is very definitely not a "cookbook." For these reasons, and some others, *e.g.* good organization, the book is ideal as a text, will be no less than a gem to the consulting statistician, and will be totally meaningless to the applied research worker not adequately prepared in the mathematics of statistics. Unfortunately this last individual composes a great market for applied books so that it will not be appreciated as it deserves. He is, however, decreasing in proportion so that the book should be better and better received as time goes on.

The book starts with the two chapters: mathematical ideas, statistical ideas. This lays a groundwork invaluable when the book is used as a text and gives easy reference for the research worker. In this case it is required reading for the nonmathematician.

The examples in the text and the exercises are the most unique I have seen. They make very pleasurable reading. The examples are so varied and interesting and deal with problems so comprehensible to everyone that it would take willpower to put down the book without accepting the challenge of at least a few of them.

The notation conforms much more closely with that seen in theoretical texts than it does in most methodology books.

So far as specific criticisms are concerned, the only one so far noticed is that an explicit display of the assumptions and requirements associated with each

technique would be an invaluable addition. These are now rather well buried in the text.

I predict that this book will become my number one reference when consulting on applied problems. If I were teaching an applied statistics course now, Brownlee would unquestionably be my choice as text.

ROBERT H. RIFFENBURGH
Laboratory for Electronics, Inc. and University of Hawaii

Modern Trigonometry. By D. W. Hall and L. O. Kattsoff. Wiley, New York, 1961. 236 pp. $4.95.

As the years pass by trigonometry texts do change, but not very much. After all, the subject has stood in much its present shape for generations. The present volume is fully as good as its competition, well-written, efficiently exercised, and pleasantly printed, but it could not be described as daring or startling. A bit of analytic geometry and a chapter on complex numbers are included. Some readers will find an unfamiliar proof here and there. Like other recent trigonometries the theorem, proof layout is used. This helps, of course, to expose the structure of the subject. However, the definition of angle in terms of amount of rotation should not be described as precise. Just how to start the embryo mathematician down the trigonometric path is quite a question, but we should not destroy his impressions of mathematical rigor and precision (learned at great effort in sophomore geometry) in the process.

FRANCIS SCHEID
Boston University

Finite-Difference Methods for Partial Differential Equations. By G. E. Forsythe and W. R. Wasow. Wiley, New York, 1960. 444 pp. $11.50.

According to the authors, this is a connected account of many of the more important results and methods for difference approximations of solutions of partial differential equations. It is intended primarily for persons interested in the theory of difference methods; that is, the formulation of difference schemes and the analysis of their stability and convergence. Although most of the techniques of stability analysis are described, no serious mention is made of the powerful and elegant technique of von Neumann. Inclusion of this technique would have been appropriate, since most of the equations considered have constant coefficients.

One of the most outstanding features of this book is the treatment of elliptic equations, which includes the theory of positive weights. Over half of the book is devoted to this subject. It includes, in particular, an excellent survey of the numerous iterative methods for solving the linear systems of equations associated with elliptic difference equations.

MILTON LEES
California Institute of Technology

An Introduction to Homological Algebra. By D. G. Northcott. Cambridge University Press, New York, 1960. xii+282 pp. $8.00.

Since this is the first book on a more or less introductory level on the relatively new subject of homological algebra, it is indeed a welcome addition to the mathematical literature.

The ten chapter headings will only give a rough idea of the wealth of material covered in this volume. 1. *Generalities Concerning Modules,* 2. *Tensor Products and Groups of Homomorphisms,* 3. *Categories and Functors,* 4. *Homology Functors,* 5. *Projective and Injective Modules,* 6. *Derived Functors,* 7. *Torsion and Extension Functors,* 8. *Some Useful Identities,* 9. *Commutative Noetherian Rings of Finite Global Dimension,* 10. *Homology and Cohomology Theories of Groups and Monoids.*

An appendix contains twelve pages of notes "meant to give the reader some help in getting his orientation."

The book is written in pleasing and leisurely style. Whereas the notes to each chapter are of great help, the reviewer feels that the complete lack of examples and exercises are extremely detrimental to the understanding of the subject matter for a beginner.

ALBERT NEWHOUSE
University of Houston

Alan M. Turing. By Sara Turing. Heffer, Cambridge, Eng., 1959. xiv+157 pp. 21/–.

This intriguing biography of the British mathematician who invented the Turing machine was written by his mother. It includes many interesting personal details on this eccentric genius whose interests ranged from mathematics to morphogenesis before his untimely and mysterious death. There is some discussion of his ideas and a complete bibliography of his publications.

The harmful results of war-bred secrecy are dramatized by the lack of information on the war years, and one wonders about the influence on Turing's personality and work of this period in which immediate expediency and secrecy rather than innate curiosity and easy communication were the rule.

Of obvious interest to historians and sociologists, the book would be enjoyed by any mathematician. Readers will feel that they have made the acquaintance of a man about whom they would like to know more.

KENNETH O. MAY
Carleton College

BRIEF MENTION

Tables of the Hypergeometric Probability Distribution. By Gerald J. Lieberman and Donald B. Owen. Stanford University Press. 1961. vi+726 pp., $15.00.

An extensive and useful set of tables of the hypergeometric probability distribution printed directly from the output of an IBM 704 computer.

Mathematical Tables of Weber Parabolic Cylinder Functions and Other Functions for Large Arguments. By. L. Fox. The National Physical Laboratory, 1960. iii+40 pp., \$2.40.

Computed tables of large values of the argument of the Airy integrals, the error integral, the gamma, di-gamma, tri-gamma, tetra-gamma, penta-gamma, and hexa-gamma functions, and the Weber functions, as well as the exponential integral $\int_{-\infty}^{x} e^t/t\,dt$ and the integrals $Si(x) = \int_0^x \sin t/t\,dt$ and $Ci(x) = \int_0^x \cos t/t\,dt$.

A series of reprints by Chelsea, New York:

Curve Tracing. By Percival Frost. 1960. 202 pp., \$3.50. (5th ed.)
Higher Plane Curves. By George Salmon. 1960. 389 pp., \$4.95. (3rd ed.)
Calculus of Finite Differences. By George Boole. (4th ed.) Cloth \$3.95, paper \$1.39.
Combinatory Analysis. By Maj. Percy A. MacMahon. 680 pp., \$7.50, Volumes I and II in one binding.

A series of reprints by Dover, New York:

Elements of Projective Geometry. By Luigi Cremona. 293 pp., \$1.75.
Algebraic Equations, An Introduction to the Theories of Lagrange and Galois. By Edgar Dehn. 200 pp., \$1.45.
Transcendental and Algebraic Numbers. By A. O. Gelfond. 177 pp., \$1.75.
Introduction to the Theory of Linear Differential Equations. By E. G. C. Poole. 199 pp., \$1.65.
Theory of Maxima and Minima. By Harris Hancock. 189 pp., \$1.50.
A Treatise on the Calculus of Finite Differences. By George Boole. 336 pp., \$1.85.
Calculus of Variations. By A. R. Forsyth. 646 pp., \$2.95.
A Treatise on the Differential Geometry of Curves and Surfaces. By Luther Pfahler Eisenhart. 564 pp., \$2.75.
Advanced Euclidean Geometry (Modern Geometry). By Roger A. Johnson. 312 pp., \$1.65.
The Theory of Equations, with an introduction to the theory of binary algebraic forms. By William Snow Burnside and Arthur William Panton. Volumes I and II, \$1.85 each.
Great Ideas of Modern Mathematics: Their Nature and Use. By Jagjit Singh. 299 pp., \$1.55.
Mathematical Biophysics, Physico-Mathematical Foundations of Biology. Volumes I and II. By N. Rashevsky. \$2.50 each.

Thinking Machines. By Irving Adler, illustrated by Ruth Adler. The John Day Company, New York, 1961. 184 pp., \$4.00.

A popular introduction to Boolean algebra and computers.

Seminar on Transformation Groups. By Armand Borel. Princeton University Press, 1960. Annals of Mathematics Studies Number 46. 245 pp., \$4.50.

This seminar by A. Borel, G. Bredon, E. E. Floyd, D. Montgomery, and R. Palais, will undoubtedly be reviewed in detail in other publications since it is fairly advanced. We recommend it to our readers interested in advanced transformation theory.

Problem-Solving Methods in Science Teaching. By Lester C. Mills and Peter M. Dean. Bureau of Publications, Teachers College, Columbia University, 1960. 87 pp., \$1.50.

A valuable little book which should be considered by high school teachers, and teachers of high school teachers. Not mathematical, but worth reading.

Soviet Physics-Doklady. An English translation published by the American Institute of Physics.

The Institute has requested that we inform our readers that starting with the July, 1961, vol. 7, No. 1 issue, *Physics-Doklady* will be translated monthly rather than in alternate months, as previously done.

Building Up Mathematics. By Z. P. Dienes. Humanities Press, New York, 1961. 124 pp., $3.00.

A discussion of some of the psychological insights which are generated somewhat spontaneously by young children.

Analytical Quadrics. By Barry Spain. Pergamon, New York, 1960. 114 pp., $5.50.

A three dimensional version of Spain's *Analytical Conics.* Like the earlier work many of the problems involve somewhat lengthy algebraic manipulations.

Recreational Mathematics Magazine, Issue 1, February, 1961. Joseph S. Madachy, Editor and Publisher, Idaho Falls, Idaho. $.65.

This periodical, devoted mainly to the lighter side of mathematics, will prove an interesting addition to the existing literature, if the first issue can be considered a fair sample.

Opportunities in Mathematics. By Harry M. Gehman. Vocational Guidance Manuals, Inc., 1961. 80 pp., $1.65.

This is exactly the book you've been looking for to present to students interested in knowing what mathematics is all about, and whether or not mathematics offers a suitable career.

Elements of Calculus. (2nd Ed.) By Thurman S. Peterson. Harper, New York, 1960. 519 pp., $6.50.

A good book made better.

Calculus With Analytic Geometry. By Thurman S. Peterson. Harper, New York, 1960. 586 pp., $7.50.

A combination of the above book and Peterson's *Analytic Geometry and Calculus,* without very much analytic geometry, as seems to be the current practice.

Calculus. (3rd Ed.) By R. L. Jeffery. University of Toronto Press, 1960. 298 pp., $4.95.

This book seems better known in Canada than in the United States.

Transmission of Information. By Robert M. Fano. The Technology Press of M.I.T. and Wiley, New York, 1961. ix+388 pp., $7.50.

An up-to-date treatment of the statistical theory of communications and coding, including information theory, based on no more esoteric a foundation than probability theory and Fourier analysis. Prepared by photo-offset from typed copy.

Sequential Decoding. By John M. Wozencraft and Barney Reiffen. The Technology Press of M.I.T. and Wiley, New York, 1961. 74 pp., $3.75.

Another book on electrical communications and the problems of coding presented from a probabilistic viewpoint. Published directly from the typed copy.

Adaptive Control Processes. By Richard Bellman. Princeton University Press, 1961. 248 pp., $6.50.

Another welcome Rand Corporation Research study. As Dr. Bellman says, this provides "a guided tour" of a unified approach to the modern field of control theory, involving problems of deterministic, stochastic and adaptive processes of both a linear and nonlinear type. In the end Dr. Bellman succeeds in making his problems amenable to machine solution. Lots of references. Well designed for seminar study.

Information and Decision Processes. Edited by Robert E. Machol. McGraw-Hill, New York, 1960. 181 pp., $5.95.

Papers by G. Brown, H. Chernoff, J. L. Doob, M. H. Flood, Wassily Hoeffding, David Rosenblatt, C. C. Shannon, Milton Sobel, Patrick Suppes, Lionel Weiss, and J. Wolfowitz. Presented at a conference held at Purdue University in 1958–59. The papers are quite varied in approach and difficulty.

Information Retrieval and Machine Translation. Edited by Allen Kent. Interscience, New York, 1960. 667 pp., $23.00.

This extensive collection of papers based on the International Conference for Standards on a Common Language for Machine Searching and Translation sponsored by Western Reserve University and the Rand Corporation in 1959, should certainly be in every computer installation and college library.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Mr. Louis Jaeckel, University of California, Los Angeles, has been awarded the William Lowell Putnam Prize Scholarship for the twenty-first competition.

Arizona State University: Associate Professor W. T. Scott, Northwestern University, has been appointed Professor; Miss Joan Richardson and Mrs. Joan McCarter have been appointed Instructors; Professor E. C. Bryant, University of Wyoming, has been appointed Visiting Professor for the year 1961–62.

Florida State University: Dr. D. R. McMillan, Jr., Louisiana State University, has been appointed Assistant Professor; Messrs. G. W. Johnson and D. J. Kiser have been appointed Instructors; Professor Andrzej Kirkor, Mathematical Institute, Warsaw, Poland, is Visiting Professor for the Fall Semester 1961–62; Research Professor R. L. Wilder, University of Michigan, is Visiting Research Professor during the 1961–62 academic year; Assistant Professor J. J. Andrews, University of Wisconsin, is Visiting Lecturer during the 1961–62 academic year.

Harpur College: Assistant Professor Helen P. Beard, Tulane University, has been appointed Associate Professor; Dr. G. A. Craft, Denison University, has been appointed Assistant Professor.

Lehigh University: Associate Professor Theodore Hailperin has been promoted to Professor; Assistant Professor R. N. Van Arnam has been promoted to Associate Professor; Dr. G. A. Stengle has been promoted to Assistant Professor.

Louisiana State University: Professor J. H. Wahab has resigned as Chairman of the Department of Mathematics and has accepted a Professorship at North Carolina State College. Associate Professor J. W. Ellis has been promoted to Professor and appointed Chairman of the Department of Mathematics.

Michigan State University, Department of Statistics: Professor Leo Katz, Head of the Department, will be on leave at the University of North Carolina, Chapel Hill, for the 1961–62 academic year with a grant from the Ford Foundation for the study of applications of probability and statistics to the area of management theory. He will spend the summer of 1962 at Stanford University in the Department of Statistics. Professor K. J. Arnold will be Acting Head during this period; Associate Professor Gopinath Kallianpur, Indiana University, has been appointed Professor; Dr. Dorian Feldman, University of California, Berkeley, has been appointed Assistant Professor; Dr. B. R. Bhat, University of California and India, and Dr. Klaus Daniel, University of California and West Germany, have been appointed Visiting Assistant Professors.

University of Buffalo: Messrs. R. W. Feldmann, R. J. Marshall, A. D. Polimeni, J. R. Stalder, and D. W. Trasher have been appointed Instructors; Assistant Professor A. G. Fadell has been promoted to Associate Professor; Dr. L. J. Montzingo, Jr., has been promoted to Assistant Professor.

University of California, Riverside: Professor H. D. Brunk, University of Missouri, has been appointed Professor; Assistant Professors C. J. A. Halberg, Jr., and V. A. Kramer have been promoted to Associate Professors.

University of Wisconsin—Mathematics Research Center: Professor R. A. Clark, on leave from Case Institute of Technology, Dr. R. D. Driver, RIAS, Baltimore, Maryland, Professor Roy Gundersen, on leave from Illinois Institute of Technology, Dr. Motoo Kumura, on leave from the National Institute of Genetics, Japan, Professor Alfred Lehman, Case Institute of Technology, Dr. Binyamin Schwarz, Israel Institute of Technology, Haifa, Israel, Dr. B. R. Seth, Indian Institute of Technology, Kharagpur, India, Dr. J. P. Ulrich, and Dr. Norman Zitro, Brown University, are spending this year at the Center; Dr. John Gurland, appointed for the year 1960–61, is now a regular member; Professor Calvin Wilcox, California Institute of Technology, has been appointed a permanent staff member.

Mr. E. D. Anderson, Dana College, has been appoinetd Mathematics Instructor at Blue Earth Public Schools, Blue Earth, Minnesota.

Dr. E. L. Arnoff, Case Institute of Technology, has accepted a position as Director of Operations Research at Ernst & Ernst, Cleveland, Ohio.

Mr. Donald Batman, University of Idaho, has accepted a position in the Mathematics Section of Avco, Wilmington, Massachusetts.

Dr. A. R. Bednarek, University of Buffalo, has accepted a position as Mathematician with Goodyear Aircraft, Akron, Ohio.

Brother Brito, Catholic University of America, has been appointed Teacher at Notre Dame High School, Utica, New York.

Mr. A. J. Carlan, Mellon Institute, Pittsburgh, Pennsylvania, has accepted a position as Supervising Engineer with the Syntron Company, Homer City, Pennsylvania.

Mr. K. E. Carlson, University of Oklahoma, has accepted a position as Statistical Analyst with Autonetics, Downey, California.

Mr. F. W. Dalleska, University of Arizona, has accepted a position as Mathematician at the Pacific Missile Range, Point Mugu, California.

Associate Professor J. N. Eastham, Cooper Union, has been appointed Head of the Department of Mathematics at Queensborough Community College.

Mr. G. R. Ellison, University of Oklahoma, has accepted a position as Staff Member at the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor Joong Fang, St. John's University, has been appointed Assistant Professor at the University of Alaska.

Dr. G. F. Feeman, Massachusetts Institute of Technology, has been appointed Assistant Professor at Williams College.

Assistant Professor John Greever, Florida State University, has been appointed Assistant Professor at Harvey Mudd College.

Dr. J. W. Hamblen, University of Kentucky, has been appointed Director of the Data Processing and Computing Center at the Southern Illinois University.

Mr. R. E. Hughs, Purdue University, has been appointed Instructor at Lehigh University.

Associate Professor Irene Monahan, Keuka College, has been promoted to Professor.

Mr. J. C. Morelock, Naval Weapons Laboratory, Dahlgren, Virginia, has accepted a position as Senior Mathematician at the Huntsville Computer Center of the General Electric Corporation, Huntsville, Alabama.

Professor Amin Muwafi, American University of Beirut, Lebanon, is doing research in number theory this year at the University of Colorado as a Visiting Research Scientist of the National Academy of Sciences.

Mr. Per-Jan Ranhoff, Pomfret School, Pomfret, Connecticut, has been granted a Sabbatical leave for the academic year 1961–62 to teach Mathematics at Ullern High School, Oslo, Norway.

Dr. Mina S. Rees, Dean of Faculty at Hunter College, has been named Dean of Graduate Studies of the newly established City University of New York. This University is made up of City College, Hunter College, Brooklyn College and Queens College and three two-year community colleges.

Associate Professor Seymour Schuster, Carleton College, has been appointed Visiting Associate Professor at the University of North Carolina for the fall term 1961.

Professor James Singer, Brooklyn College, has been appointed Chairman of the Department of Mathematics.

Dr. P. F. Smith, University of Colorado, has been appointed Assistant Professor at Nicholl's State College.

Mr. E. T. Stapleford, Kent State University, has been appointed Assistant Professor at Jamestown Community College.

Dr. S. G. Tellman, Fresno State College, has been appointed Assistant Professor at Pomona College.

Dr. Anthony Trampus, General Electric Company, Cincinnati, Ohio, has accepted a position as a member of the Technical Staff of General Electric TEMPO, Santa Barbara, California.

Mr. R. G. Vinson, University of Alabama, has been appoined Head of the Department of Mathematics at Huntingdon College.

Dr. John Wagner, School Mathematics Study Group, Yale University, has been appointed Associate Professor at Michigan State University.

Mr. J. A. Ward, Jr., University of North Carolina, has accepted a position with the Space Technology Laboratories, Los Angeles, California.

Mr. W. J. Wells, University of Chicago, has accepted a position with the Cornell Aeronautical Laboratory, Buffalo, New York.

Mrs. Betty O. Weneser, Danish Institute of Computing Machinery, Copenhagen Valby, Denmark, has accepted a position at the Applied Mathematics Division of Brookhaven National Laboratory, Upton, New York.

Assistant Professor J. S. Wholey, Newton College of the Sacred Heart, has been appointed Assistant Professor at Rutgers, The State University.

Mrs. Mary L. Yount, Winthrop College, has accepted a position as Research Technician with the Chemstrand Research Center, Research Triangle Institute, Durham, North Carolina.

Mr. K. M. Herstein, President, Herstein Laboratories, New York, New York, died June 1, 1961. He was a member of the Association for six years.

Professor Emeritus F. W. Owens, Pennsylvania State University, died June 22, 1961. He was a charter member of the Association.

Mrs. Georgia C. Smith, Spelman College, died May 6, 1961. She was a member of the Association for eight years.

Associate Professor D. E. Whitford, Polytechnic Institute of Brooklyn, died May 20, 1961. He was a member of the Association for forty years.

MATHEMATICS INSTRUCTORS NEEDED FOR 1962 N S F SUMMER INSTITUTES

Mathematicians available and interested in teaching in a 1962 N S F summer institute for high school mathematics teachers are invited by the Association's Committee on Institutes to send their names—along with brief statements of training, experience, and fields of special interest—to the Committee chairman, E. A. Cameron, University of North Carolina, Chapel Hill, North Carolina. Lists of the names and information submitted will be sent to directors of summer institutes.

With the increase in the number of institutes, the problem of adequate staffing is becoming more acute. Institute teaching offers a real opportunity for service to mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The twentieth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Fordham University on April 15, 1961. The Academic Vice-President of Fordham, Rev. V. T. O'Keefe, S. J., welcomed the gathering. Professor J. P. Russell, Collegiate Vice-Chairman of the Section, presided at the morning session and Dr. George Grossman, High School Vice-Chairman, presided at the afternoon session. One hundred seventy-eight persons, including 97 members of the Association, attended the meeting.

Professor Azelle B. Walcher, Chairman of the Section, presided at the business meeting. Professor C. T. Salkind presented the awards to local winners in the mathematics contest sponsored by the Mathematical Association of America and the Society of Actuaries. He then gave reports as Chairman of the Contest Committee and as Governor of the Section. Reports were also presented by the Treasurer, Mr. Aaron Shapiro, and by Professor J. N. Eastham, Chairman of the Speakers' Bureau. The following officers were elected: Chairman, Professor J. P. Russell, Polytechnic Institute of Brooklyn; Vice-Chairmen, Professor Abraham Schwartz, City College, and Mr. Lester Schlumpf, Andrew Jackson High School; Secretary, Professor Mary P. Dolciani, Hunter College; Treasurer, Mr. Aaron Shapiro, Brooklyn College.

The following papers were presented at the meeting:

1. *A random process arising in air defence*, by Professor W. M. Hirsch, New York University.

A file of attacking aircraft is considered as a queue which is moving toward an objective O.

A defending missile battery is regarded as a server who attempts to perform some operation on each element before that element reaches O. If an element in the queue reaches O without having been served, the server is subject to a risk of disability. The dependence of the process on various parameters (distance between queue elements, probability of disability of server, initial distance of first queue element from objective, etc.) is studied. Various problems concerning the probability distribution of the number of elements served are described. The role of simulation (and its relation to mathematical analysis) in studying such processes is discussed.

2. *Some second thoughts on artificial intelligences*, by Dr. Bradford Dunham, International Business Machines Corporation.

3. *The place of programed instruction in mathematics education*, by Mr. Lewis Eigen, Vice-President, Center for Programed Instruction.

Dr. Bradford F. Hadnot of International Business Machines Corporation announced the formation of the Division of Mathematics of the New York Academy of Sciences and invited all members of the Association to participate in the activities of the Division.

MARY P. DOLCIANI, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The 44th annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, on April 28–29, 1961. The following officers were elected: Chairman, Professor L. C. Barrett, South Dakota School of Mines and Technology; Vice-Chairman, Professor D. W. Robinson, Brigham Young University; Secretary-Treasurer, Professor Leota C. Hayward, Colorado State University.

The following papers were presented:

1. *Approximating the kth derivatives of a function by sums of Sturm-Liouville eigenfunctions*, by Professor F. M. Stein, Colorado State University.

The author uses eigenfunctions of a family of Sturm-Liouville systems as defined by Dunn and Stein, SIAM Review, January, 1961, to prove the existence of a sum, $S_n(x)$, of such eigenfunctions which uniformly approximates an arbitrary differentiable function, $f(x)$, and whose kth derivative at the same time uniformly approximates the corresponding derivative of $f(x)$. That is, it is proved that there exists a sum, $S_n(x)$, such that $|f^{(k)}(x)-S_n^{(k)}(x)|<\epsilon$, $k=0, 1, \cdots, m$, for $\epsilon>0$ and for all x on $[a, b]$, the closed interval over which $f(x)$ and its derivatives are defined.

2. *Separation axioms between T_0 and T_1*, by Mr. C. E. Aull and Professor W. J. Thron, University of Colorado.

3. *A continuation of the zeta series and its implications*, by Professor W. E. Briggs, University of Colorado.

A standard method of continuing the zeta series $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ to the left of $\text{Re}\, s=1$ can be generalized for any integer a greater than 1 by writing $(1-a^{1-s})\zeta(s)=\sum_{n=1}^{\infty} \beta_n n^{-s}$, where $\beta_n=1$ if $a\nmid n$ and $1-a$ if $a|n$. To evaluate the right hand number and its derivatives at $s=1$, first write $\sum_{n\leq x} (\log^k n)/n=(\log^{k+1}x)/(k+1)+\gamma_k+o(1)$. It is now possible to derive the THEOREM. *For integral a and k, $a\geqq 2$, $k\geqq 0$, $\sum_{n=1}^{\infty} (\beta_n \log^k n)/n=(\log^{k+1}a)/(k+1)-\sum_{t=0}^{k-1} \binom{k}{t}\gamma_t \log^{k-t}a$, where the summation on the right is zero for $k=0$.* By solving these equations for γ_t, one immediately obtains the principal result of a paper by Kluyver (Quar. J. Math., vol. 50, 1927, 185–192). In particular this gives $\gamma=\frac{1}{2}\log a-\sum_{n=1}^{\infty} (\beta_n/n) \log_a n$.

4. *Methods of proving mean value theorems*, by Professor L. C. Barrett, South Dakota School of Mines.

The primary purpose of this paper is to emphasize the equivalence of various proofs of the extended law of the mean, including analytic, geometric, vector, and determinant types of proof. A yet more general method of generating mean value theorems is also given.

THE APRIL MEETING OF THE UPPER NEW YORK STATE SECTION

The seventeenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Harpur College, Binghamton, New York, on April 29, 1961. Professor B. H. Gere, the Chairman of the section, presided at the morning session and Professor D. E. Kibbey, Vice-Chairman of the section, presided at the afternoon session. There were 70 persons in attendance, including 55 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor D. E. Kibbey, Syracuse University; Vice-Chairman, Dr. Frank Hawthorne, New York State Department of Education; Secretary-Treasurer, Professor N. G. Gunderson, University of Rochester. The Chairman of the Contest Committee, Professor Nura D. Turner of the State University College of Education at Albany, reported on the 1961 High School Mathematics Contest. The Executive Committee was authorized to set up a Committee on Special Projects to consider projects in which the section might become involved.

The program was as follows:

1. *On Goodman's conjecture*, by Professor M. W. Pownall, Colgate University.

Let A be a set of points and N irreflexive, symmetric, binary relation with domain A, so that (A, N) may be regarded as a graph. Assume that this graph is connected and that for each P in A, the set of Q such that $N(P, Q)$, is finite. A reflexive and symmetric relation M is defined on A by $M=(N\cup I)^k$, where I is the identity relation and k is a positive integer. Now if M is given, the question arises whether N and k are unique, and if so, whether they may be determined from M. It has been shown that for pair-homogeneous cut-point graphs the answer to both questions is affirmative.

2. *A duality in maximum-minimum theory*, by Professor C. S. Ogilvy, Hamilton College.

Many applied elementary extremum problems appear in dual pairs, in the sense that each maximum problem carries with it an associated minimum problem. This duality can be explained in terms of elementary derivatives or of Lagrange's multipliers. Because the two members of a dual pair have equivalent solutions, the duality can be exploited in solving certain problems of first year calculus.

3. *On asymmetry in fields*, by Professor J. D. Reid, Syracuse University.

Let F be a field, G the multiplicative group of nonzero elements of F and A the additive group of F. The question of whether or not G and A can be isomorphic has come up in the problem section of this MONTHLY (Problems 4644 (1955, 447) and E 1410 (1960, 290)). A discussion was given of the structure of the groups Hom (G, A) and Hom (A, G).

4. *The CUPM program for engineers*, by Professor R. J. Walker, Cornell University.

The Physical Science Panel of the CUPM tentatively proposes the following mathematics curriculum for engineering students: 1. In the first two years, twelve semester hours of calculus and differential equations, and three semester hours of linear algebra. Both these subjects should be taught with an awareness of the existence of automatic computers. 2. For the better students, who may be going into research and development work, twelve additional hours selected and arranged to fit the student's specialty. Suitable courses might be vector field theory, advanced ordinary differential equations, complex variables, partial differential equations, probability and statistics, programming and game theory, etc.

5. *On evaluating certain real integrals by Cauchy's residue theorem*, by Professors O. J. Farrell and B. Ross, Union College.

The real integral $\int_0^\infty x^m dx/(x^n+a)$, where $a>0$, m and n are nonnegative integers, $n>m+1$, can be evaluated by Cauchy's residue theorem applied to the integral $\int_C z^m dz/(z^n+a)$, $z=x+iy$, taken around a contour C enclosing just one of the zeros of the denominator of the integrand. Let C be made up of the segment of the real axis from $x=0$ to $x=R$, $R>a^{1/n}$, thence along $|z|=R$ to

the ray $\theta = \exp(2\pi i/n)$, thence along this ray to the origin. The integral along this ray equals $-\exp[(m+1)2\pi i/n]\int_0^R x^m dx/(x^n+a)$. Thus we get $\int_0^\infty x^m dx/(x^n+a) = \pi/\{na^{(n-m-1)/n}\sin[(m+1)\pi/n]\}$.

6. *On iterations with errors*, by Professor Peter Frank, Syracuse University.

The iterates of a contraction mapping T converge to the fixed point of the mapping. While computing T an error can be made. The two cases where the errors are uniformly bounded and "random" were discussed.

7. *Maximality and reflexive-symmetric relations*, by Professor A. R. Bednarek, University of Buffalo.

If R is a reflexive and symmetric relation over the space X, a set $S \subset X$ is called R-scattered if and only if X non Ry for every pair of distinct elements $x, y \in S$. E. J. Mickle and T. Rado (*On covering theorems*, Fund. Math., vol. 45, 1957, pp. 325–331) proved that given R as above there existsan R-scattered subset S of X such that $X = R(s)$; where $R(S) = \bigcup_{x \in S} R(x)$ and $R(x) = \{y \mid y \in X$ and $yRx\}$. In the present paper it is shown that this result is equivalent to the assertion of the existence of a maximal R-scattered set $S \subset X$ and to the proposition that every R-scattered subset of X is contained in a maximal R-scattered subset of X. By a particularization of R, some of the set-theoretic maximality principles were shown to be immediate consequents of the above.

8. *A generalization of the contracting mapping theorem and its numerical application*, by Professor W. C. Rheinboldt, Syracuse University.

The contraction mapping theorem is well known and various generalizations have been proposed. For numerical applications it is very advantageous to consider iterations of the form $x_{n+1} = F_n(x_n)$, where F_n is a convergent sequence of operators in a suitable metric space. A convergence-proof for such a type of iteration has been given by H. Ehrmann. Under rather general conditions another simple proof can be obtained by using the original contraction mapping theorem. Several examples of practical applications underline the usefulness of the method.

9. The M.A.A. films *Mathematical Induction*, with Professor L. A. Henkin, were shown.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the Mathematical Association of America was held on Saturday, May 6, 1961, at Rose Polytechnic Institute, Terre Haute, Indiana. Professor T. P. Palmer of Rose Institute presided at the morning session and Professor John Yarnelle of Hanover College at the afternoon session. The meeting was attended by 62 persons, of whom 42 were members of the Association.

Officers for the year 1961–62, elected at the afternoon session, are Professor John Yarnelle, Hanover College, Chairman; Professor Ernst Snapper, Indiana University, Vice Chairman; and Professor P. T. Mielke, Wabash College, Secretary-Treasurer.

Professor Ernst Snapper delivered the invited hour address entitled "The Foundations of Mathematics" in which he sketched the history of the Russell Paradox and its effect upon the foundations of mathematics. The following short papers were presented:

1. *A preliminary report on the use of teaching machines in teaching mathematics to engineering and science students*, by Professor A. R. Schmidt, Rose Polytechnic Institute.

2. *Dexsinal gauges*, by Mr. Aaron Miller, Indianapolis, Indiana.

3. *A student's eye view of the Rose curriculum*, by Mr. S. D. Burton, Rose Polytechnic Institute.

4. *A comparison of five recent texts in unified calculus*, by Professor P. T. Mielke, Wabash College.

The texts reviewed were those of Johnson and Kiokemeister; Haaser, LaSalle and Sullivan; G. B. Thomas's 3rd Edition; Federer and Jonsson; and J. F. Randolph. The first three have been used at Wabash.

5. *A preliminary report on the Lynn Reeder Astronomical Laboratory*, by Professor I. P. Hooper, Rose Polytechnic Institute.

In addition to the short papers, Professor C. E. Maudlin, Rose Polytechnic Institute, conducted a tour of the Waters Computing Laboratory and supervised a demonstration of the Institute's Bendix G15d computer.

P. T. MIELKE, *Secretary*

PROFESSIONAL OPPORTUNITIES IN MATHEMATICS

A fifth edition of this popular booklet was published by the Association in September 1961. The new edition is a completely revised version of an article which appeared originally in the January 1951 number of this MONTHLY. It was prepared by a committee consisting of A. H. Bowker, C. R. Phelps, Mina S. Rees, S. A. Robertson, C. E. Sealander, and J. S. Frame, Chairman.

Although the new edition has been increased in size from 24 to 32 pages, the price remains at 25 cents for single copies and 20 cents each for five or more copies. Orders with payment should be sent to the Buffalo office of the Association.

CALENDAR OF FUTURE MEETINGS

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, Spring, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962.

KENTUCKY, University of Kentucky, Lexington, Spring, 1962.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 2, 1961.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.

MINNESOTA, Moorhead State College, November 4, 1961.

MISSOURI, Missouri School of Mines, Rolla, Spring, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.

NEW JERSEY, St. Peter's College, Jersey City, November 4, 1961.

NORTHEASTERN, November 24, 1962

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO

OKLAHOMA, Oklahoma City University, October 27, 1961.

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, South Dakota School of Mines, Rapid City, Spring, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

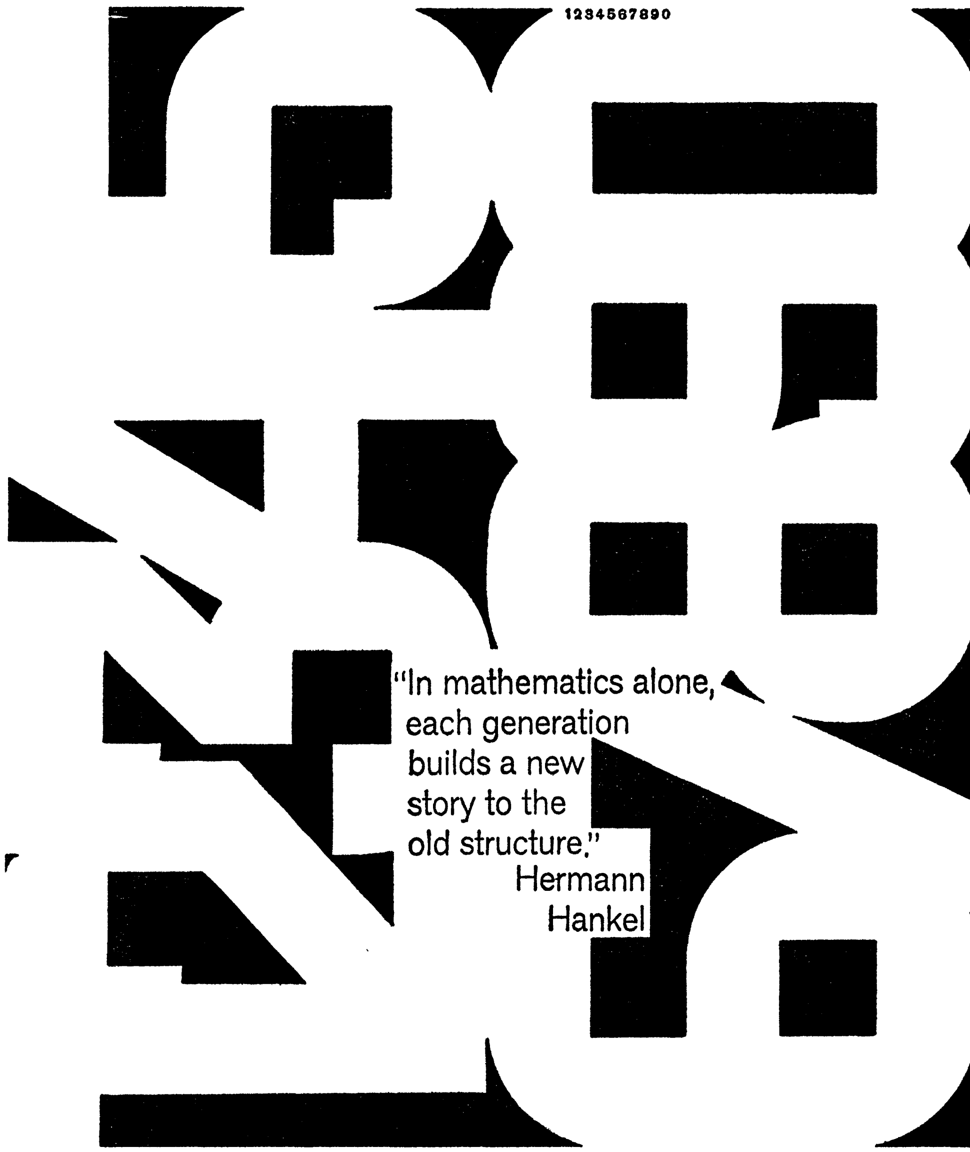
SOUTHERN CALIFORNIA, Long Beach State College, March 9, 1962.

SOUTHWESTERN

TEXAS, Rice University, Houston, April, 1962.

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, Spring, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.



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CONTENTS

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RALPH D. JAMES, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the incoming Editor, F. A. Ficken, Department of Mathematics, New York University, New York 53, N. Y. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

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A UNIQUE APPROACH TO THE APPROXIMATION OF TRIGONOMETRIC FUNCTIONS

LADIS D. KOVACH, Pepperdine College, AND WILLIAM COMLEY, Douglas Aircraft Company, El Segundo, California

Introduction. In present-day science and technology many tedious tasks of the mathematician have been capably assumed by electronic computers, both digital and analog. The increased emphasis and reliance on these modern marvels of complexity imposes upon the mathematician the necessity of carefully considering those operations which the computer does best. This will permit him to arrange his program to take full advantage of the machine's most sophisticated capabilities.

The uninitiated may be surprised to find that many of the classical mathematical methods are not only unsuitable for application to the computer but are entirely beyond its capabilities. In such cases it becomes necessary to retreat to the most rudimentary level and literally lead the machine through a step-by-step approach to the desired result. In other instances, however, unique capabilities of the computer make possible a more sophisticated, yet simpler, approach to a given operation than may be obtained in any other way. Although a certain serendipity may be implied by the last sentence, it is nevertheless true that, with respect to the devising of computer methods, man is slave to the machine.

Function approximation with the digital computer. It is a characteristic of man that, when he designs a machine to make his work easier, the action of the machine is often a copy of the action of the man. Thus it should come as no surprise that, in the early (say, ten years ago) digital computers, trigonometric functions were found by means of a table. The given angle was presented to the computer, which then compared it with a table that was either in the computer or put in at the time. If the given angle did not match any in the table, then the computer performed some sort of interpolation.

It soon became apparent that storing trigonometric tables in a computer's memory would more than exhaust its capacity. The most obvious solution was to use a power-series representation truncated at the proper point to achieve the desired accuracy. Accordingly, Maclaurin's series were used for a time. There was a continued effort, however, to find the "best" approximations to the trigonometric functions. "Best" is defined by the digital computer programmer as having minimum absolute error over the range of interest, using the least amount of storage, and requiring the smallest number of computer operations.

The work of Lánczos and Hastings can be mentioned to illustrate the methods used to find best approximations. By expanding the polynomial approximation to a function in Chebyshev polynomials, Lánczos obtained a polynomial $f(x)$ which, in the range $[-1, 1]$, involved fewer terms for the same accuracy than any other polynomial. He called this method, "economization of power

series" and a good example of it has been given by Hildebrand [1]. Here the expression for e^x is given as

$$e^x \doteq \frac{1}{384}(382 + 383x + 208x^2 + 68x^3) \qquad \text{for } |x| \leqq 1, \tag{1}$$

instead of the truncated Maclaurin's series

$$e^x \doteq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \text{for } |x| \leqq 1. \tag{2}$$

The advantage of Lánczos's method arises from the fact that the error in (2) over the prescribed interval exceeds 0.01 while the error in (1) is less than 0.01 although the expression contains one less term.

In the approximations given by Hastings [2] a different technique is used, which results in the expression

$$e^x \doteq (1 + 0.2507213x + 0.0292732x^2 + 0.0038278x^3)^4. \tag{3}$$

For $0 \leqq x < \infty$ this expression is in error by an amount not exceeding 0.00025. Hastings also gives the approximation

$$\sin \tfrac{1}{2}\pi x \doteq 1.5706268x - 0.6432292x^3 + 0.0727102x^5, \tag{4}$$

which is in error by less than 0.00011 for $|x| \leqq 1$.

More recently, Kogbetliantz [3] has used rational Padé approximations to $\sin N$ in the interval $0 \leqq N \leqq 41\pi/256$ and to $\cos N$ in $0 \leqq N \leqq 87\pi/256$ to compute both functions in $0 \leqq N \leqq \frac{1}{2}\pi$ with the first ten correct significant digits in four multiplications and four divisions only. If the infinite range $0 \leqq N < \infty$ is of interest, one more multiplication can reduce it to the range $0 \leqq N \leqq \frac{1}{2}\pi$ so that the total number of operations is five. The method is flexible and can give any desired accuracy.

Numerical analysts have thus been concentrating on approximations characterized by small absolute error and few terms. This last characteristic usually goes hand-in-hand with few machine operations.

Trigonometric functions with the analog computer. In obtaining the trigonometric functions on an analog computer, a set of completely new problems must be solved. One of these is "frequency response" which has to do with the highest frequency of interest contained in the input signal. For example, the sine of an angle can be obtained by means of a suitably wound potentiometer which produces the proper voltage as a result of a shaft rotation proportional to the angle. Obviously a scheme of this kind is only suitable for slowly changing angles, *i.e.*, frequencies less than 10 cps.

If the angles are changing more rapidly (which is often the case), then it is not possible to use electro-mechanical systems. Hence the most useful type of angle resolver is electronic. In order to obtain reasonable accuracy, electronic

methods of obtaining trigonometric functions are quite complicated. This tends to result in low reliability which, of course, is undesirable.

In discussing error it is necessary to take a different point of view also. The "noise level," *i.e.*, the small random voltages which are present in the equipment, must be taken into consideration. For this reason, it is not possible to talk about *relative* error. If both x and $\sin x$ are represented by voltages, then for $x=0$, the equipment may give $\sin x$ as a few millivolts. Thus a discussion of relative error of $\sin x$ when x is zero (or even when x is small) is meaningless. Considering *absolute* error does not help the situation because of the necessity of using voltages to represent numbers. An absolute error of 0.002 might be fine if the equipment is operating in the 10–100 volt range but unsatisfactory in the 1–10 millivolt range.

Hence the only reasonable way to measure the error of an electronic device that generates trigonometric functions is on the basis of percent of full scale. This means that if the device operates between zero and 100 volts and it is rated as having an error of 1% of full scale, the following can happen:

a) The output may be in error by 1 volt at 100 volts,
b) The output may be in error by 1 volt at 10 volts,
c) The output may be in error by 1 volt at 1 volt, etc.

On the basis of percentage one could say that the equipment gives a result that is in error by 1% at 100 volts, by 10% at 10 volts, and by 100% at 1 volt.

To summarize, a trigonometric approximation is "best" from the viewpoint of the analog computer programmer if it allows the resolution of the highest frequencies of interest with a minimum full scale error and with the most reliability.

The Quadratron. To some extent the mathematical methods used in analog and digital computers must be tailored to suit the equipment. One way in which this has been done is illustrated by the applications of a device known as the Quadratron.

The Quadratron is a small, solid-state, plug-in unit that is used in conjunction with d-c computing amplifiers of the type found in analog computers [**4**]. For the present discussion it is sufficient to think of a Quadratron-amplifier combination as a "black box" that generates the function

$$y = 0.01x^2 (\operatorname{sgn} x). \tag{5}$$

Here sgn x represents the signum function defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{when} \quad x < 0, \\ 0 & \text{when} \quad x = 0, \\ +1 & \text{when} \quad x > 0. \end{cases} \tag{6}$$

A graph of (5) as well as the wiring diagram of the necessary equipment is shown in Figure 1.

The signum function is obtained without the use of complicated electronic equipment. It is a *natural* characteristic of the particular solid-state device being used. Therefore it is necessary to approach the mathematics from a different viewpoint in order to take advantage of this phenomenon. It is possible to use the basic relation in (5) to develop a number of useful functions. The trigonometric functions, in particular, lend themselves to this type of development.

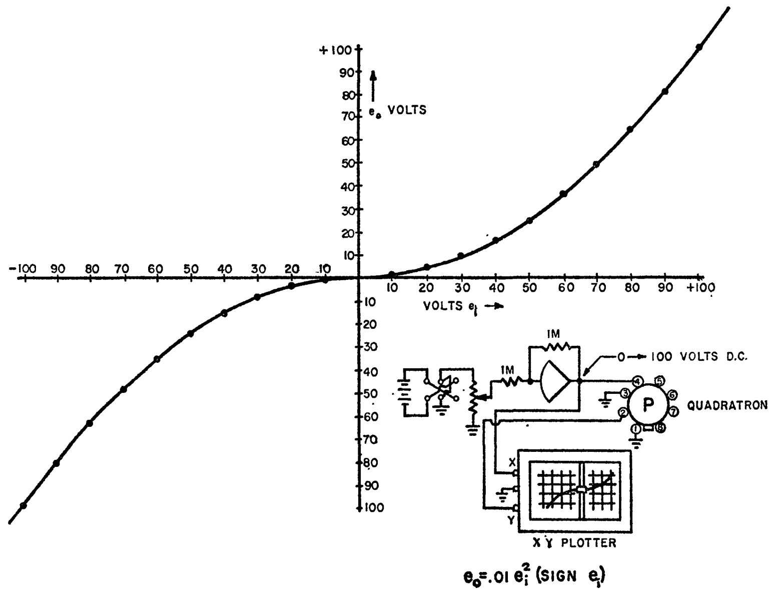

FIG. 1. Basic squaring function given by the Quadratron.

Approximations to sin x. The most simple useful approximation to sin x is given by

$$\sin x \doteq k(\pi x - x^2 \operatorname{sgn} x), \qquad -\pi \leqq x \leqq \pi. \tag{7}$$

The presence of the signum term assures that this function will have the necessary odd symmetry. If the difference between the approximation and sin x is called the error function, then Figure 2 shows the error function resulting from the use of (7). The maximum error is 5.6% for this approximation.

In order to reduce the error, use can be made of the symmetry of sin x about the line $x = \frac{1}{2}\pi$ and also of the capability of the signum function. The result is the approximating polynomial

$$\operatorname{sgn} x \doteq Az + Bz^2 \operatorname{sgn} z, \tag{8}$$

where

$$z = \pi x - x^2 \operatorname{sgn} x, \qquad -\pi \leqq x \leqq \pi, \tag{9}$$

with $A=0.315$ and $B=0.036$. This approximation requires one more operation than (7) but results in a considerable reduction of the error. The maximum error is now 0.073% and has the form shown in Figure 2.

The economy of this method becomes more apparent upon expanding (8) to

$$\sin x \doteq C_1 x + C_2 x^2 \operatorname{sgn} x + C_3 x^3 + C_4 x^4 \operatorname{sgn} x, \tag{10}$$

where the C_i are functions of A and B. In view of the fact that the only nonlinear operation required is the square, the ability to perform this operation accurately is the only requirement for obtaining the full accuracy of the approximation given in (8).

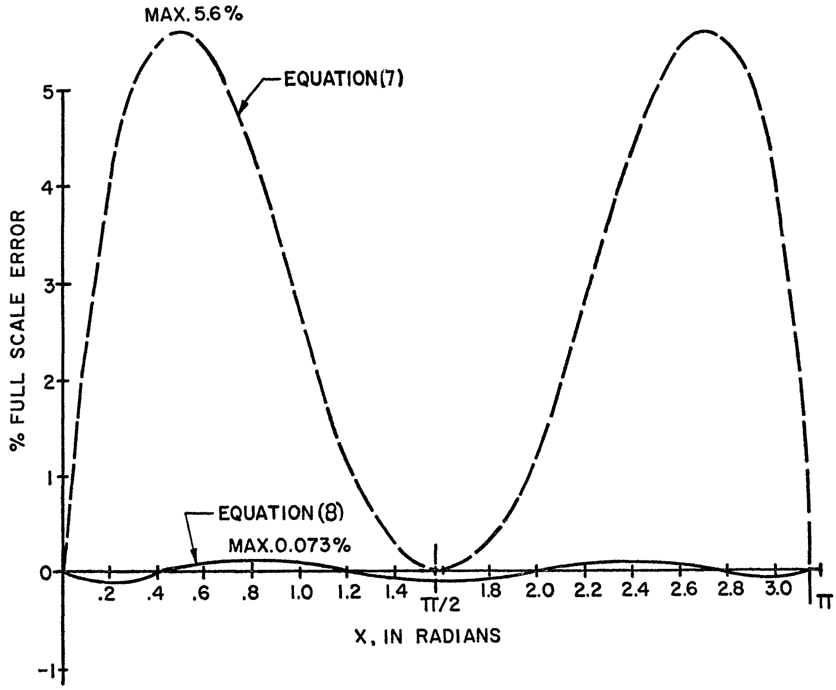

FIG. 2. Error curves for sin x approximations.

Approximation to cos x. The expression for cos x is easily obtained by changing the basic equations (8) and (9) to

$$\cos x \doteq Aw + Bw^2, \tag{11}$$

where

$$w = (\tfrac{1}{4}\pi^2 - x^2), \qquad -\tfrac{1}{2}\pi \leqq x \leqq \tfrac{1}{2}\pi. \tag{12}$$

Note that the range of the independent variable in this approximation is one-half that in the case of sin x. This is due to the fact that cos x is an even func-

tion and it is not possible to take advantage of the properties of the signum function.* The range may be extended, however, by performing an additional squaring operation in making use of the identity

$$\cos 2x = 2\cos^2 x - 1 \tag{13}$$

and substituting for cos x the right side of (11). The error curve for cos x is identical to that given for sin x (Eq. (8), Fig. 2) over the applicable portion of the interval.

Approximations to tan x. Finding an approximating polynomial to tan x presented additional problems. If only squaring operations were to be used, how accurate could the approximation be? What would be a reasonable number of terms to use? What is the range of angles that is of interest in analog computing?

In order to answer these questions a study was undertaken. It was decided to limit the approximating polynomial to three terms and to restrict the variable to the range $-\frac{1}{3}\pi \leqq x \leqq \frac{1}{3}\pi$. The problem then was to study approximating polynomials of the form

$$\tan x \doteq ax + bx^m + cx^n. \tag{14}$$

At $x=0$ this expression gives zero error but it is also convenient in calibrating analog circuits to have zero error at $x=\pm\frac{1}{3}\pi$. With these constraints the coefficients a, b, and c were determined so that the error curve had equal maxima and minima.

The results can best be shown in the form of graphs in which the percent full-scale error is plotted as a function of the exponent n for various exponents m. Figure 3 shows the effect of changing the exponent n in polynomials of the type

$$ax + bx^2(\operatorname{sgn} x) + cx^n(\operatorname{sgn} x). \tag{15}$$

The minimum full-scale error is 0.197% for $n=6$.

A somewhat surprising result is shown in Figure 4 in which the approximating polynomials of the type

$$ax + bx^3 + cx^n(\operatorname{sgn} x) \tag{16}$$

are considered. The minimum error here is 0.044% for $n=7$. In fact for *any* n in the range $6\leqq n\leqq 10$, the error is less than it is for $n=5(0.30\%)$. In other words, the truncated Maclaurin's series is *not* the best approximation to tan x.

Carrying the investigation forward one more step, Figure 5 represents the variation of error plotted against n for polynomials of the form

$$ax + bx^4(\operatorname{sgn} x) + cx^n(\operatorname{sgn} x). \tag{17}$$

Here the minimum error is on the order of 0.16%. Most of the points, however, are below the 0.30% point of the three-term Maclaurin's series.

* A slight modification of the basic circuit yields the function $y=0.01x^2$ rather than the one given in (5), so that even functions can be generated if desired.

On the basis of this investigation, and with a full realization that it is sometimes expedient to trade simplicity for precision, the approximation in (18) was adopted:

$$\tan x \doteq 1.090x - 0.176x^2(\operatorname{sgn} x) + 0.651x^4(\operatorname{sgn} x). \tag{18}$$

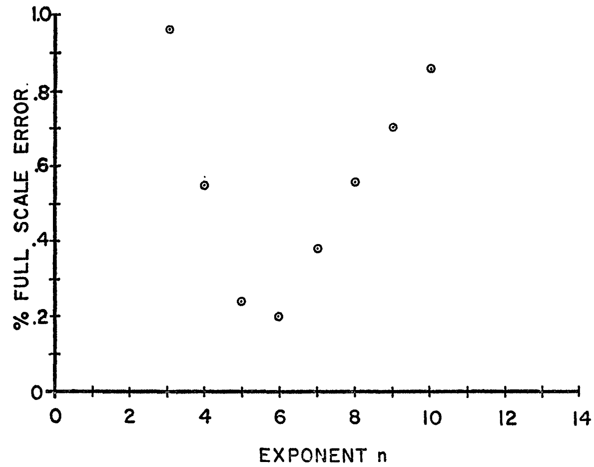

FIG. 3. Variation of error with exponent in $\tan x \doteq ax + bx^2(\operatorname{sgn} x) + cx^n(\operatorname{sgn} x)$.

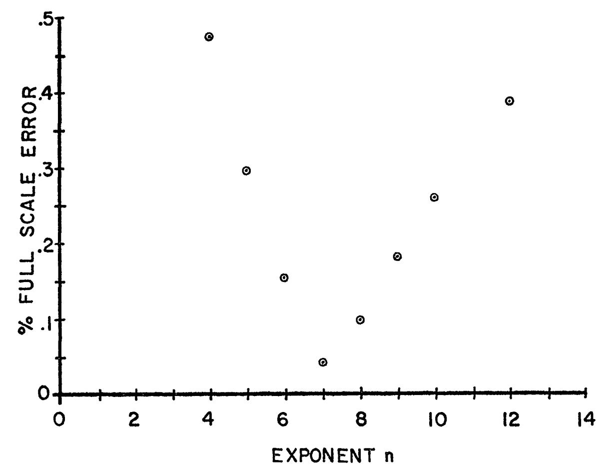

FIG. 4. Variation of error with exponent in $\tan x \doteq ax + bx^3 + cx^n(\operatorname{sgn} x)$.

The error in this approximation is 0.54% of full scale but only two squaring operations are required. Thus the same equipment can be used to obtain $\sin x$, $\cos x$, and $\tan x$ with only a slight variation in circuitry.

Conclusion. In order to take advantage of the capabilities of a computing machine or device, it is advisable sometimes to re-examine the mathematics involved. It may happen that a break with traditional mathematical concepts is required in order to produce results.

Mathematical formulae are sometimes regarded as basic truths that dare not be altered. Examples have been given, however, of instances where some modification is highly desirable in order to utilize the full advantages of computers and the physical properties of certain computing devices.

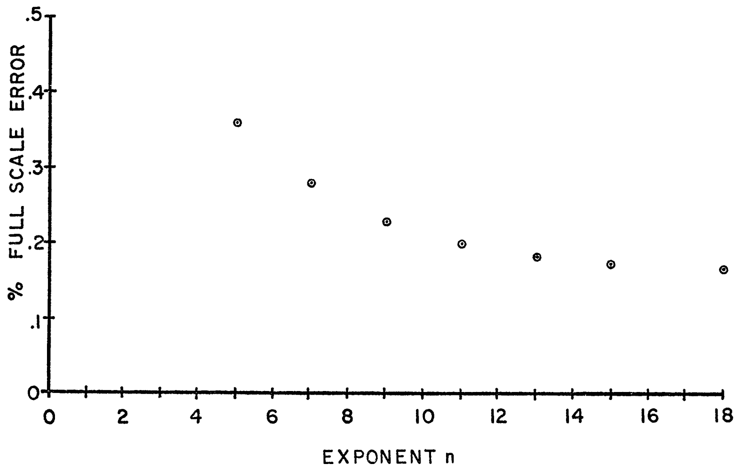

FIG. 5. Variation of error with exponent in tan $x \doteq ax+bx^4(\operatorname{sgn} x)+cx^n(\operatorname{sgn} x)$.

If the mathematician is flexible and is willing to test methods that may at first appear strange to him, he may find that he is rewarded. He may achieve a closer connection between mind and machine. The true reward, however, may lie in the thought that man's ingenuity has overcome the machine's perversity.

Acknowledgments. The authors wish to express their appreciation to Mr. Jack Halliburton for his assistance with the numerical computations and to Miss Irene James for her drawings.

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LOCAL LINEAR DEPENDENCE AND THE VANISHING OF THE WRONSKIAN*

G. H. MEISTERS, University of Nebraska and the Research Institute for Advanced Study

1. Introduction. It is not unusual to find something like the following mistaken statement in elementary text books on differential equations. "A necessary and sufficient condition that n functions $f_1, \cdots, f_n$ be linearly dependent on an interval I is that their Wronskian determinant

$$\begin{vmatrix} f_1 & \cdots f_n \\ f_1^{(1)} & \cdots f_n^{(1)} \\ \cdot & \cdots \cdot \\ f_1^{(n-1)} & \cdots f_n^{(n-1)} \end{vmatrix}$$

vanish identically on I." Now it has long been known (see, for example, [4] and [9]) that the above statement is true if $f_1, \cdots, f_n$ are analytic, or if they are $(n-1)$-times differentiable solutions of a linear homogeneous differential equation with continuous coefficients of the form

$$x^{(m)} + p_1(t)x^{(m-1)} + \cdots + p_m(t)x = 0,$$

on the interval I. However, it has also long been known that for n functions which are only $(n-1)$-times differentiable (so that their Wronskian is defined) the sufficiency part of the above statement no longer holds. Peano [12] seems to have been the first to point this out, and Bôcher [3] has given an example which shows that even if the functions involved are infinitely differentiable on I, the identical vanishing of the Wronskian is still not sufficient to imply their linear dependence on I. It was then recognized that the functions involved must satisfy other conditions, supplementary to the vanishing of their Wronskian, in order to guarantee their linear dependence. Peano [13] and Bôcher [5] have given such conditions (for example, Lemma 3 of Sec. 2 of this paper) and Bôcher has shown that his conditions include those of Peano.

In this paper we look at this problem from a slightly different point of view. Namely, instead of placing linear dependence in the forefront and looking for a condition to supplement the identical vanishing of the Wronskian, rather, we shall put the vanishing of the Wronskian in the forefront and ask for a kind of generalized dependence (necessarily weaker than linear dependence) which is equivalent to it. This point of view has led the author to define a new type of dependence relation for functions of a real variable which he has called "local linear dependence." It is shown that local linear dependence and the identical

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vanishing of the Wronskian are equivalent in the class of infinitely differentiable functions. It is also shown to what extent they are equivalent for n functions which are only $(n-1)$-times differentiable. Next, local linear dependence is discussed from the point of view of abstract dependence relations, and it is seen that although it does not have all the properties of linear dependence in vector spaces it does possess some of the same abstract properties satisfied by functional dependence and by linear dependence in modules.

Throughout this paper we shall be dealing with complex-valued functions of a real variable which are defined on some nonempty open (finite or infinite) interval I.

2. Three classical lemmas. For the proofs of our theorems on local linear dependence in the next section we shall need three elementary classical results. In order to prevent repetition it will be assumed throughout this section that whenever n functions are mentioned in the statement of a theorem involving their Wronskian, they are at least $(n-1)$-times differentiable at every point of I.

LEMMA 1. *If $f_1, \cdots, f_n$ are linearly dependent on I, then their Wronskian vanishes identically on I.*

This is perhaps the oldest and best-known theorem on the Wronskian. Its proof is straightforward and can be found in almost any elementary text on differential equations.

In order to facilitate the statement (and later use) of the next lemma we make the following notational conventions. When functions $f_1, \cdots, f_n$ are under discussion, $W=W(f_1, \cdots, f_n)$ will denote their Wronskian, with $W(f_1)=f_1$. For $n \geqq 2$, W_k will denote the Wronskian of all but f_k and W_{lm} will denote the Wronskian of all but f_l and f_m, while the order of the functions in W_k and W_{lm} is the same as in W. When $n=2$, W_{lm} is defined to be 1. The next lemma is a generalization of the simple formula $(f_2/f_1)'=W/f_1^2$ which corresponds to the case $n=2$ in the lemma.

LEMMA 2. *If $f_1, \cdots, f_n$ are $n \geqq 2$ given functions for which the Wronskian $W_n=W(f_1, \cdots, f_{n-1})$ does not vanish at any point of I, then the following differentiation formulas are valid on I:*

$$(W_i/W_n)' = (W_{in}\cdot W)/W_n^2 \qquad \textit{for } i = 1, \cdots, n-1.$$

The author found need for these formulas in his proof of Theorem 4 of the next section. However they seem to be very old, having been used by Brioschi as early as 1855 (see [11], vol. 2, p. 225). They also appear in an 1873 paper by Frobenius and again in an 1884 paper by Starkoff (see [11], vol. 3, p. 252 and vol. 4, p. 245, respectively). Nevertheless, their use as a tool for proving theorems on the vanishing of the Wronskian seems to have been overlooked. For example, by use of Lemma 2 one can give a very simple proof of the following 1900 theorem of Bôcher, a proof which differs from Bôcher's.

LEMMA 3 (BÔCHER'S FUNDAMENTAL THEOREM [5]). *If the Wronskian of* $f_1, \cdots, f_n$ $(n \geqq 2)$ *vanishes identically on* I, *while the Wronskian of* $f_1, \cdots, f_{n-1}$ *does not vanish at any point of* I, *then* $f_1, \cdots, f_n$ *are linearly dependent on* I, *and in particular there are complex numbers* $c_1, \cdots, c_{n-1}$ *such that* $f_n = \sum_{i=1}^{n-1} c_i f_i$ *on* I.

Proof based on Lemma 2. Given that W is identically zero on I and W_n is zero at no point of I it follows from Lemma 2 that $W_i = k_i W_n$ on I for $i = 1, \cdots, n-1$. Then since the determinant

$$\begin{vmatrix} f_1 & \cdots f_n \\ f_1^{(1)} & \cdots f_n^{(1)} \\ \cdot & \cdots \cdot \\ f_1^{(n-2)} & \cdots f_n^{(n-2)} \\ f_1 & \cdots f_n \end{vmatrix}$$

is identically zero on I, one has upon expansion by the last row the equation

$$W_n \cdot \sum_{i=1}^{n} (-1)^{n+i} k_i f_i = 0$$

on I, where $k_n = 1$. But then, since W_n never vanishes on I, we have

$$f_n = \sum_{i=1}^{n-1} (-1)^{n-1+i} k_i f_i$$

on I, which completes the proof.

3. Local linear dependence. In this section the concept of local linear dependence is defined and its relationship to the vanishing of the Wronskian is established in Theorems 1 through 4.

DEFINITION 1. *Functions* $f_1, \cdots, f_n$, *defined on a nonempty open interval* I, *are said to be* ***locally linearly dependent*** *(l.l.d.) on* I *if and only if for every nonempty open subinterval* J *of* I *there exist* n *complex numbers* $c_1, \cdots, c_n$, *not all zero, and a nonempty open interval* K *contained in* J *such that*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x *in* K. *Functions* $f_1, \cdots, f_n$ *are said to be* ***locally linearly independent*** *on* I *if they are not l.l.d. on* I.

Clearly, if $f_1, \cdots, f_n$ are linearly dependent on I they are *a fortiori* locally linearly dependent on I. However, as the following example shows, even in the class of infinitely differentiable functions, local linear dependence does not imply linear dependence. (In the class of *analytic* functions the two concepts are obviously equivalent.)

Example 1. Let C denote Cantor's ternary set on $[0, 1]$. Let $I_1, I_2, I_3, \cdots,$ be the enumeration of the denumerable number of component open intervals of the complement of C which is obtained by counting them, in decreasing order of length, from left to right. Let $I_k=(a_k, b_k)$, and let $\phi_k(x)$ denote the function

$$\exp\{-(x-a_k)^{-2}-(x-b_k)^{-2}\}.$$

Define $f(x)=(-1)^k\phi_k(x)$ and $g(x)=\phi_k(x)$ on I_k for $k=1, 2, \cdots,$ and define $f(x)=g(x)=0$ on C. Then the following facts are easily verified.

a) f and g are defined and infinitely differentiable at all points of $(0, 1)$.

b) f and g are locally linearly dependent on $I=(0, 1)$.

c) f and g are not linearly dependent on $(0, 1)$; in fact, there exists no partition of $(0, 1)$ into *intervals* on each member of which f and g are linearly dependent.

Definition 1 defines what is meant by a *dependent set* of functions. We give here (although it will not be used until the next section) what is to be meant by one function depending on a set of functions.

DEFINITION 2. *A function g is said to be* ***locally linearly dependent on the functions*** *$f_1, \cdots, f_n$ on I if and only if for every nonempty open subinterval J of I there exists a nonempty open interval K contained in J in which g is linearly dependent on $f_1, \cdots, f_n$.*

The basic relationship between local linear dependence and the vanishing of the Wronskian is expressed in the following theorem.

THEOREM 1. *$(n-1)$-times differentiable functions $f_1, \cdots, f_n$ are locally linearly dependent on a nonempty open interval I if and only if their Wronskian W vanishes on an open dense subset G of I.*

Proof. First suppose that $f_1, \cdots, f_n$ are locally linearly dependent on I. Let G denote the interior of the set of all zeros of the Wronskian W of $f_1, \cdots, f_n$. Let x be an arbitrary point of I and let U be an arbitrary neighborhood of x. By the definition of local linear dependence, there exists a nonempty open interval K, contained in U, on which $f_1, \cdots, f_n$ are linearly dependent. But then by Lemma 1, W is identically zero on K and so U contains points of G. It follows that G is dense in I.

Next suppose that the Wronskian W is identically zero on some open dense subset G of I. We must show that $f_1, \cdots, f_n$ are locally linearly dependent on I. We shall proceed by induction on n. Since the case $n=1$ is trivial we suppose that the statement of our theorem is true for some integer $k\geqq 1$. Let $f_1, \cdots, f_{k+1}$ be $k+1$ complex-valued functions which are defined and k-times differentiable on I, and suppose that their Wronskian W vanishes identically on an open dense subset G of I. Let J be an arbitrary nonempty open subinterval of I. Since G is an open dense subset of I, $J\cap G$ contains a nonempty open interval K. There are two cases.

Case 1. The Wronskian W^* of $f_1, \cdots, f_k$ is identically zero on K. Then by the induction hypothesis $f_1, \cdots, f_k$ are locally linearly dependent on K and therefore they (and *a fortiori* $f_1, \cdots, f_{k+1}$) are linearly dependent on some nonempty open interval $L \subset K \subset J \subset I$.

Case 2. The Wronskian W^* is not identically zero on K. But then, since W^* is continuous on I, there exists a nonempty open interval L contained in K on which W^* does not vanish. Applying Lemma 3 to the functions $f_1, \cdots, f_{k+1}$ on the interval L, it follows that $f_1, \cdots, f_{k+1}$ are linearly dependent on $L \subset K \subset J \subset I$.

Since in either case J is arbitrary, it follows that $f_1, \cdots, f_{k+1}$ are locally linearly dependent on I. This completes the proof of Theorem 1.

The following example shows that this theorem is the best possible in the sense that for every nonempty open interval I and for every positive integer n, there exist n complex-valued functions $f_1, \cdots, f_n$ which are defined and have finite derivatives of the first $n-1$ orders at every point of I and there exists an open dense subset G of I such that the Wronskian of $f_1, \cdots, f_n$ vanishes on G but is not identically zero on I.

Example 2. Denjoy has proved ([6], pp. 237–248) the existence of a real-valued function of a real variable ϕ with the following properties:

a) ϕ is defined and has a finite derivative at every point of I. Hence ϕ is continuous on I.

b) There exists an open dense subset G of I at every point of which $\phi'=0$. $I-G$ is a Cantor discontinuum of positive measure.

c) ϕ' is not identically zero on I.

Now define f_1 to be the $(n-2)$-fold integral of ϕ and $f_i=\sum_{k=0}^{i-2} x^k/k!$ for $i=2, \cdots, n$. Then $W(f_1, \cdots, f_n)$ is equal to

$$\left|\begin{array}{c|ccccc} f_1 & 1 & 1+x & 1+x+\frac{1}{2}x^2 & \cdots & \sum_{k=0}^{n-2} x^k/k! \\ f_1' & & 1 & 1+x & \cdots & \sum_{k=0}^{n-3} x^k/k! \\ \vdots & & 0 & & \ddots & \vdots \\ \phi & & & & & 1 \\ \hline \phi' & 0 & 0 & 0 & \cdots & 0 \end{array}\right| = (-1)^n \phi'.$$

We shall now give some theorems which show under what conditions local linear dependence on an interval I is a sufficient condition for the identical vanishing of the Wronskian on I. A complex-valued function f defined on an open interval I is called *quasicontinuous* at a point x of I if and only if for each open neighborhood $U \subset I$ of x, and for each open neighborhood V of $f(x)$, there

exists a nonempty open interval $J \subset U$ such that $f(J) \subset V$. This concept is due to Kempisty [10] and independently to Bledsoe [2]. The latter called such a function *neighborly*. It is obviously a generalization of continuity.

THEOREM 2. *If $f_1, \cdots, f_n$ are locally linearly dependent on an open interval I and possess finite derivatives of the first $n-1$ orders on I, then their Wronskian W vanishes at every point of I at which it is quasicontinuous.*

Proof. By Theorem 1, W is zero on an open dense subset G of I. Now W is obviously continuous (and *a fortiori* quasicontinuous) at every point of G. Suppose $x_0 \in I-G$ and $W(x_0) \neq 0$. Let ϵ denote $\frac{1}{2}|W(x_0)|$. Now if W is quasicontinuous at x_0, there must exist a nonempty open interval J contained in $\{x: |x-x_0| < \epsilon\}$ such that $|W(x)-W(x_0)| < \epsilon$ for all x in J. But since $|W(x_0)| - |W(x)| \leqq |W(x)-W(x_0)|$, we have

$$|W(x)| > |W(x_0)| - \tfrac{1}{2}|W(x_0)| = \tfrac{1}{2}|W(x_0)| > 0$$

for all x in J, which contradicts the density of G. Thus Theorem 2 is proved.

THEOREM 3. *In the class of infinitely differentiable functions on I the identical vanishing of the Wronskian on I is equivalent to local linear dependence on I.*

Proof. If the Wronskian vanishes identically on I then, by Theorem 1, the functions involved are locally linearly dependent on I (because I is an open dense subset of itself).

On the other hand, if $f_1, \cdots, f_n$ are locally linearly dependent on I, their Wronskian must vanish at all points of I (by Th. 2), since it is continuous on I.

By making use of a theorem of Denjoy on derivative functions we can improve on Theorem 2.

DENJOY'S THEOREM [7]. *If ϕ is a real-valued derivative function on a nonempty open interval I, then for each nonempty open interval J the set $I \cap \phi^{-1}[J]$ is either empty or of positive measure.*

COROLLARY TO DENJOY'S THEOREM. *If g is a complex-valued function of a real variable which is defined and differentiable at each point of a nonempty open interval I, and if $g'=0$ almost everywhere on I, then $g' \equiv 0$ on I.*

Proof. Let $g=u+iv$, where u and v are real-valued functions. Then $g'=0$ a.e. implies u' and v' are zero a.e., so that we need only consider real-valued functions. Suppose g is a real-valued function satisfying the hypotheses of the corollary. If $g'(x_0) \neq 0$ for some point x_0 in I, there exists an open interval J containing $g'(x_0)$ and not containing zero. Then the set $I \cap g'^{-1}[J]$ is not empty, since it contains x_0, and so, by Denjoy's theorem, it must have positive measure. But this contradicts the hypothesis that g' is zero almost everywhere. Thus the corollary is proved.

THEOREM 4. *If $f_1, \cdots, f_n$ are locally linearly dependent on an open interval I, possess finite derivatives of the first $n-1$ orders at every point of I, and if their*

Wronskian W is quasicontinuous almost everywhere on I, then it is identically zero on I.

Proof. In the proof of this theorem we shall use Lemma 2 and the notation introduced there. Let Z_i denote the subset of I on which W_i vanishes, for $i=1, \cdots, n$. Then W is identically zero on $Z=\bigcup_{i=1}^{n} Z_i$, and $I-Z$ is an open subset of I (since $W_1, \cdots, W_n$ are continuous on I). Let x_0 be an arbitrary point of $I-Z$ and let I_0 be the component open interval of $I-Z$ which contains x_0. Then not all of the functions $W_1, \cdots, W_n$ vanish at x_0. We may suppose, without loss of generality, that $W_n(x_0)\neq 0$. Then (again by continuity of W_n) there exists a nonempty open interval J, containing x_0 and contained in I_0, on which W_n is never zero. But then by Lemma 2 we have

$$(W_i/W_n)' = (W_{in}\cdot W)/W_n^2 \tag{1}$$

for $i=1, \cdots, n-1$ and at all points of J. Since $f_1, \cdots, f_n$ are locally linearly dependent on I and W is quasicontinuous almost everywhere on I it follows from Theorem 2 that W is zero almost everywhere on I and therefore also almost everywhere on J. But then, in view of (1) and the corollary to Denjoy's theorem, we have

$$W_{in}\cdot W \equiv 0 \text{ on } J \tag{2}$$

for $i=1, \cdots, n-1$. Now suppose that W is not zero at some point, say x^*, of J. Then by (2) $W_{in}(x^*)=0$ for $i=1, \cdots, n-1$ and so $W_n(x^*)=0$ contrary to the construction of J. Hence W is identically zero on J and so, in particular, $W(x_0)=0$. But since x_0 was an arbitrary point of $I-Z$ and $W\equiv 0$ on Z, it follows that W is identically zero on I. This completes the proof of Theorem 4.

We shall conclude this section by showing that a theorem on the Wronskian given by Bôcher [5] a very simple proof using Theorems 1 and 2.

BÔCHER'S THEOREM ON THE WRONSKIAN [5]. *Let $f_1, \cdots, f_n$ be n complex-valued functions of a real variable which at every point of a nonempty open interval I have continuous derivatives of the first $n-1$ orders; then if the Wronskian of $f_1, \cdots, f_{n-1}$ vanishes identically on I, the Wronskian of $f_1, \cdots, f_n$ also vanishes identically on I.*

Proof. $W(f_1, \cdots, f_{n-1})\equiv 0$ on I implies (by Theorem 1) that $f_1, \cdots, f_{n-}$ are locally linearly dependent on I. But then $f_1, \cdots, f_{n-1}, f_n$ are *a fortiori* locally linearly dependent on I, and therefore (by Theorem 2) $W(f_1, \cdots, f_n)\equiv 0$ on I.

4. *L.l.d.* as a dependence relation. It is perhaps of some interest to see in what ways, if any, *l.l.d.* resembles linear dependence in vector spaces. It has been recognized for some time that the theory of linear dependence in vector spaces can be based on the following four axioms or their equivalent. (See, for example, [1]; [4], p. 81; [8], [15], p. 100; and [16] p. 50.)

Let X be a nonempty set. For $x\in X$ and $A\subset X$ let $x\Delta A$ denote the relation

"x depends on A." Then the basic properties of linear dependence in vector spaces are

(LD1) $x\Delta A$ if and only if $x\Delta F$ for some finite subset F of A.
(LD2) $x\in A$ implies $x\Delta A$.
(LD3) $x\Delta A$ and $y\Delta B$ for each $y\in A$ implies $x\Delta B$.
(LD4) (Exchange axiom). If $x\Delta A$ and if $x\not\Delta A-\{y\}$ for some $y\in A$, then $y\Delta(A-\{y\})\cup\{x\}$.

These are merely an abstract formulation of van der Waerden's "basic theorems" which occur on page 100 and again on page 200 of [**15**].

Now in order to examine the concept of local linear dependence in the light of these axioms we make the following definition.

DEFINITION 3. *Let X denote a family of functions defined on a common nonempty open interval I. Define $g\Delta\{f_1, \cdots, f_n\}$ to mean g is l.l.d. on $\{f_1, \cdots, f_n\}$ on I (cf. Definition 2). If A is an arbitrary subset of X, let $g\Delta A$ mean that g is l.l.d. on some finite subset of A.*

From this definition it is obvious that *l.l.d.* satisfies axioms (LD1) and (LD2). That axiom (LD3) is also satisfied is the content of the following theorem.

THEOREM 5. *If f is l.l.d. on $g_1, \cdots, g_n$ on an interval I and each g_i, $i=1, \cdots, n$, is l.l.d. on $h_1, \cdots, h_m$ on I, then f is l.l.d. on $h_1, \cdots, h_m$ on I.*

Proof. Let J be an arbitrary nonempty subinterval of I. We must show that f is linearly dependent on $h_1, \cdots, h_m$ on some nonempty open subinterval K of J. First of all, since f is *l.l.d.* on $g_1, \cdots, g_n$ on I, there exists a nonempty open subinterval I_1 of J on which f is linearly dependent on $g_1, \cdots, g_n$:

$$f = \sum_{i=1}^{n} \alpha_i g_i \text{ on } I_1 \subset J. \tag{3}$$

Since g_1 is *l.l.d.* on $h_1, \cdots, h_m$ on I, there exists a nonempty open interval $I_2\subset I_1$ such that g_1 is linearly dependent on $h_1, \cdots, h_m$ on I_2. Suppose now that this construction has been carried out $k<n$ times, so that

$$g_i = \sum_{j=1}^{m} \beta_{ij} h_j \text{ on } I_{i+1}$$

for $i=1, \cdots, k$ and $I_{k+1}\subset I_k\subset \cdots \subset I_2\subset I_1\subset J$, where I_i for $i=1, \cdots, k+1$ is a nonempty open interval. Then since g_{k+1} is *l.l.d.* on $h_1, \cdots, h_m$ on I, there exists a nonempty open subinterval I_{k+2} of I_{k+1} such that g_{k+1} is linearly dependent on $h_1, \cdots, h_m$ on I_{k+2}; hence

$$g_{k+1} = \sum_{j=1}^{m} \beta_{k+1} h_j \text{ on } I_{k+2} \subset I_{k+1}.$$

Thus by induction,

$$g_i = \sum_{j=1}^{m} \beta_{ij} h_j \text{ on } I_{i+1}, \tag{4}$$

for $i=1, \cdots, n$, where the β_{ij} are complex numbers, and $I_{n+1} \subset I_n \subset \cdots \subset I_2 \subset I_1 \subset J \subset I$, where I_i for $i=1, \cdots, n+1$ are nonempty open intervals. Thus on $I_{n+1} \subset J \subset I$ we have by combining (3) and (4) that

$$f = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \alpha_i \beta_{ij} \right) h_j.$$

That is, f is linearly dependent on $h_1, \cdots, h_m$ on a nonempty open subinterval (namely I_{n+1}) of J. This completes the proof of Theorem 5.

However, the following simple example shows that axiom (LD4) is not satisfied by local linear dependence.

Example 3. Let $f(t)=1$ for all $t \in (-\infty, +\infty)$, and define g and h on the interval $(-\infty, +\infty)$ as follows.

$$g(t) = \begin{cases} 1 & \text{for } t \leqq 0, \\ t^2+1 & \text{for } t > 0. \end{cases} \qquad h(t) = \begin{cases} t^2+1 & \text{for } t \leqq 0, \\ 1 & \text{for } t > 0. \end{cases}$$

Then the following statements are easily verified.

a) f is *l.l.d.* on $\{g, h\}$ on $I=(-\infty, +\infty)$.
b) f is not *l.l.d.* on $\{g\}$ on $I=(-\infty, +\infty)$.
c) h is not *l.l.d.* on $\{g, f\}$ on $I=(-\infty, +\infty)$.

Thus we see that a general theory of dependence relations based on the van der Waerden axioms would not include the concept of *l.l.d.* The same statement holds with regard to any axiom system, such as the one in [8] or [16], which contains a version of the "exchange axiom." In this respect *l.l.d.* is similar to functional dependence (for example, as defined on pp. 182–186 of [14]) and linear dependence in modules, since also in these theories the exchange axiom does not hold in general.

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ALGEBRAIC NUMBER FIELDS AND THE DIOPHANTINE EQUATION $m^n = n^m$

ALVIN HAUSNER, The City College of New York

1. Introduction. In the 1960 Putnam competition, the problem was posed of determining all integral solutions of the Diophantine equation

$$m^n = n^m \qquad (m \neq n). \tag{1}$$

This question has a long history starting, apparently, with Euler who treated it in *Introductio in Analysin Infinitorum II*, page 294 (see [2], p. 687; [8], pp. 150–151). It is not difficult to show that the pair 2, 4 is the only solution of (1) in positive integers (we do not count 4, 2 as another solution because of the symmetry of the equation). Let us quickly demonstrate this fact (incidentally, not by Euler's procedure). Without limiting generality, suppose $m > n$ and write $m = n + r$, where r is a positive integer. Substituting in (1), we find that $(n+r)^n = n^{n+r}$. This means that $\{(n+r)/n\}^n = n^r$ or $\{1 + (r/n)\}^n = n^r < e^r$. Hence $n = 1$ or 2. If $n = 1$, then $m = 1$ and this case is excluded. The case $n = 2$ yields $m = 4$.

Suppose we seek all integral solutions of (1). Now, $m = 0$ would make $n = 0$ and this is excluded. If (1) holds with m negative, then clearly n must also be negative. A simple discussion, whose details we omit, brings us back to the case already treated, and we get the one additional solution $-2, -4$.

The question of suitably extending (1) to arbitrary algebraic number fields quite naturally presents itself. Let K denote an algebraic number field, *i.e.*, a finite algebraic extension of the field R of rational numbers ([6], p. 35). If $\alpha \in K$, then $N_{K/R}(\alpha) = N(\alpha)$ will denote the norm of α ([6], p. 72). $N(\alpha)$ is a rational number and, if α is an algebraic integer in K, then $N(\alpha)$ is a rational

integer. We note that $N(\alpha)=0$ if and only if $\alpha=0$, and that $N(\alpha\beta)=N(\alpha)N(\beta)$ for any $\alpha, \beta \in K$ ([6], p. 72). Further, if a is a rational integer in K of degree n over R, then $N_{K/R}(a)=a^n$.

A reasonable extension of (1) to the field K is the following: Find the algebraic integers $\alpha, \beta \in K$ which satisfy

$$\alpha^{N(\beta)} = \beta^{N(\alpha)} \qquad (\alpha \neq \beta). \tag{2}$$

This question generalizes the earlier one because $N(m)=m$ for integers in the field of rational numbers. Solutions of (2) certainly exist. For example, in $R(\sqrt{2})$ we have $\alpha=2+\sqrt{2}$, $\beta=6+4\sqrt{2}$; again, in $R(\sqrt{(-2)})$ we have $\alpha=2$, $\beta=\sqrt{(-2)}$. Equation (2) is satisfied by $\alpha=4048+1530\sqrt{7}$, $\beta=45+17\sqrt{7}$, and $\alpha=126+16\sqrt{62}$, $\beta=8+\sqrt{62}$ in $R(\sqrt{7})$ and $R(\sqrt{62})$, respectively. If the degree $[K:R]$ is even, we have infinitely many solutions $\alpha=2k$, $\beta=-2k$, $k=1, 2, \cdots$. This can be proved by substituting in (2) and using the fact that $N(-1)=1$ in fields of even degree. We might term this infinite family of solutions "trivial." Later we will show that there exist infinitely many (real) quadratic fields $R(\sqrt{D})$ such that (2) has infinitely many nontrivial algebraic integer solutions in each of these fields. Several such examples were given above.

2. Solutions in algebraic integers. Suppose K is an arbitrary algebraic number field and that (2) holds for $\alpha, \beta \in K$. Taking the norm of both sides of (2) and using the multiplicativity of the norm, we find that $N(\alpha)^{N(\beta)}=N(\beta)^{N(\alpha)}$. By the Euler problem, three cases can arise:

(a) $N(\alpha)=2$, $N(\beta)=4$; (b) $N(\alpha)=-2$, $N(\beta)=-4$; (c) $N(\alpha)=N(\beta)$, $\alpha\neq\beta$. We shall study each of these possibilities in turn.

In Case (a), we find $N(\alpha)^4=N(\beta)^2$ and this implies $N(\alpha^2)=\pm N(\beta)$ or $N(\alpha^2/\beta)=\pm 1$. Hence $\beta=\eta\alpha^2$, where η is a unit in K. Substituting in (2) we find $\alpha^{N(\eta\alpha^2)}=(\eta\alpha^2)^{N(\alpha)}$ or $\alpha^{4N(\eta)}=\eta^2\alpha^4$; thus $\eta=\pm 1$. Conversely, if α is such that $N(\alpha)=2$ in K and β is defined as α^2, then (2) holds for the pair α, β; this can be seen by direct verification. The possible solution $\alpha, -\alpha^2$ remains if $N(\alpha)=2$. The reader will easily see that $\alpha, -\alpha^2$ is a solution of (2) if and only if $[K:R]$ is even, since the minus sign in $-\alpha^2$ requires that $N(-1)=1$ in K.

Case (b) is similar. If $[K:R]$ is odd so that $N(-1)=-1$, and if $N(\alpha)=-2$, $\beta=-\alpha^2$, then α, β is a solution of (2) as is seen by checking. The other possibility α, α^2 is never a solution-pair quite independently of whether $N(-1)$ is ± 1.

Before discussing Case (c), we complete our study of Cases (a) and (b). We leave to the reader the easy proofs of the following: If α, α^2 is a Case (a) solution of (2) and η is a unit in K with $N(\eta)=1$, then $\alpha\eta, \alpha^2\eta^2$ is also a solution. If $[K:R]$ is even and $\alpha, -\alpha^2$ is a Case (a) solution, then $\alpha\eta, -\alpha^2\eta^2$ is likewise a solution when $N(\eta)=1$. If $N(\alpha)=-2$, then $\alpha\eta, -\alpha^2\eta^2$ is a solution if $[K:R]$ is odd and $N(\eta)=1$ or if $[K:R]$ is even and $N(\eta)=-1$. The preceding statements provide a complete list of all Case (a) and (b) solutions of (2).

Our discussion to this point shows the necessity of dealing with the four Diophantine equations

$$(3) \qquad N(x_1\omega_1 + \cdots + x_n\omega_n) = \pm 1, \pm 2,$$

where $\omega_1, \cdots, \omega_n$ constitute an integral basis of K over R ([6], p. 63) and rational integer solutions of (3) are sought. The first two of equations (3) would be used to determine the units of K. It is therefore of interest to know which of equations (3) are indeed solvable and, if so, to determine all solutions. The equation $N(\alpha)=2$ will be studied below for quadratic fields.

We now deal with Case (c) which is different. If α, β satisfy (2) in Case (c), then $\alpha^{N(\alpha)} = \beta^{N(\alpha)}$ with $\alpha \neq \beta$. This means that $\alpha = \beta\zeta$, where ζ is an $|N(\alpha)|$th root of unity. Every algebraic number field K contains the roots of unity ± 1 and if there are other roots of unity present in K (such roots would be necessarily nonreal), these can occur only in a finite number ([6], p. 133). If $\zeta = 1$, then $\alpha = \beta\zeta = \beta$ and we exclude this case in (2). But if $\zeta = -1$, we have an rth root of unity for all even r. Thus, if both $N_{K/R}(\alpha)$ and $[K:R]$ are even, then $\alpha, -\alpha$ provides a solution-set of (2) as is easily seen. The "trivial" solutions $2k, -2k$, $k = 1, 2, \cdots$, mentioned earlier over fields of even degree belong to this family of solutions. We see that these are included in an even more extensive family of "trivial" solutions when $[K:R]$ is even: $2\lambda, -2\lambda$, where λ is any integer in K. Suppose, however, that K happens to contain a nonreal $|N(\alpha)|$th root of unity ζ. Then the degree of ζ over R is $\phi(q)$, where $q \geqq 3$ and q divides $|N(\alpha)|$. Here ϕ is Euler's totient function and $\phi(x)$ is even if $x \geqq 3$. Thus $[K:R]$ is even because the degree of the field K is divisible by the degrees of the numbers it contains ([6], p. 51). In this case $\alpha, \alpha\zeta$ does actually provide a solution of (2), for $\alpha^{N(\alpha\zeta)} = \alpha^{N(\alpha)N(\zeta)} = \alpha^{N(\alpha)} = (\alpha\zeta)^{N(\alpha)}$ since $\zeta^{N(\alpha)} = \zeta^{|N(\alpha)|} = 1$ and since $N(\zeta) = 1$ for all roots of unity $\zeta \neq -1$. More solutions are present in this case. Suppose γ is divisible by α in the ring of integers of K. Then $\gamma = \alpha\delta$ and $\gamma, \gamma\zeta$ is a solution-set of (2): $(\alpha\delta)^{N(\alpha\delta\zeta)} = \alpha^{N(\alpha\delta)}\delta^{N(\alpha\delta)} = (\alpha\delta\zeta)^{N(\alpha\delta)}$ since $\zeta^{N(\alpha\delta)} = (\zeta^{N(\alpha)})^{N(\delta)} = 1^{N(\delta)} = 1$. The sets γ, $\gamma\zeta$ (α divides γ, ζ an $|N(\alpha)|$th root of unity) exhaust all Case (c) solutions.

In summary, for every $\alpha \in K$ with $N(\alpha) = 2$ we have the solution-set α, α^2 of (2). If $[K:R]$ is even, then $\alpha, -\alpha^2$ is also a solution. If $N(\alpha) = -2$ and $[K:R]$ is odd, then $\alpha, -\alpha^2$ is a solution. If $N(\alpha) = -2$, then α, α^2 is never a solution irrespective of $[K:R]$. Every field K of even degree has the set $\alpha, -\alpha$ as a solution if $N(\alpha)$ is even. If $|N(\alpha)| \geqq 3$ and K contains a nonreal $|N(\alpha)|$th root of unity ζ, then the infinitely many pairs $\gamma, \gamma\zeta$ are solutions for any multiple γ of α. This can happen for only finitely many $\zeta \in K$ and only when $[K:R]$ is even. In fields of odd degree lacking integers of norm ± 2, (2) cannot be solved.

Let us close this section with a more thorough discussion of the fields $R(\sqrt{D})$ in Case (a) as promised. Suppose D is a positive square-free integer. We would like to show that there are infinitely many $R(\sqrt{D})$ each containing infinitely many solutions of Case (a) type and not merely of the $\alpha, -\alpha$ type. In view of our discussion above, we must show that there are infinitely many $\alpha \in R(\sqrt{D})$, for infinitely many D, with $N(\alpha) = 2$. Integers in $R(\sqrt{D})$ are of the form $a + b\sqrt{D}$, where a, b are rational integers (we are excluding those integers $\frac{1}{2}(a + b\sqrt{D})$ with a, b both odd in the case $D \equiv 1 \pmod 4$). The norm of $a + b\sqrt{D}$

over $R(\sqrt{D})$ is $N(a+b\sqrt{D})=(a+b\sqrt{D})(a-b\sqrt{D})=a^2-Db^2$. The question is therefore whether there are infinitely many square-free $D>0$ such that

$$a^2 - Db^2 = 2 \tag{4}$$

has infinitely many solution-sets a, b for each D. That the answer is in the affirmative is shown by taking, for example, $D=2p$, where p is a prime such that $p\equiv 7 \pmod 8$ (there are infinitely many such p). With such D, (4) has a solution set a_0, b_0 ([4], p. 174; [1], p. 450). But from one solution-set we may generate infinitely many more sets by employing the units of $R(\sqrt{D})$. Write

$$a = a_0x + Db_0y, \qquad b = a_0y + b_0x. \tag{5}$$

If we choose x, y in (5) from among the infinitely many integral solutions of the Pell equation $x^2-Dy^2=1$, then a, b from (5) satisfy (4) (for a detailed presentation of the Pell equation, see [5], Ch. 8). If a, b is a solution of (4), then $\alpha=a+b\sqrt{D}$ has norm 2 in $R(\sqrt{D})$ and α, $\beta(=\pm\alpha^2)$ is a solution of (2) as we already saw. Thus, there are infinitely many quadratic fields each containing infinitely many solutions of (2) of the Case (a) type.

3. Solutions in ideals. Our extension of (1) to the form (2) yields results in great contrast to the rational integer case. Let us seek another version of "$m^n=n^m$", for algebraic number fields, which retains the finite-solution character of the rational field case. Suppose (2) is satisfied by a pair α, $\beta\in K$. Then Cases (a), (b) and (c) of Section 2 show that $N(\alpha)$ and $N(\beta)$ have the same sign. If both are positive, then (2) implies $(\alpha)^{N((\beta))}=(\beta)^{N((\alpha))}$ where (α) and (β) denote the principal ideals, in the ring of integers of K, generated by α and β, respectively. $N((\alpha))$ denotes the norm of the ideal (α) and $N((\alpha))=N(\alpha)$ since $N(\alpha)>0$ ([6], p. 107). If, on the other hand, $N(\alpha)$, $N(\beta)<0$, then (2) implies $\alpha^{-N(\beta)}=\beta^{-N(\alpha)}$ and, once again, the ideal equation $(\alpha)^{N((\beta))}=(\beta)^{N((\alpha))}$ holds since $N((\alpha))=-N(\alpha)$ if $N(\alpha)<0$ ([6], p. 107). This suggests that we consider the problem: Find all integral ideals (not necessarily principal) A, B in K such that

$$A^{N(B)} = B^{N(A)} \qquad (A \neq B). \tag{6}$$

Here, $N(A)$ and $N(B)$ denote the norms of A and B ([6], p. 104). We need only consider the case where $N(A)\neq N(B)$ since $A^k=B^k$ implies $A=B$ by the unique factorization theorem for ideals into prime ideals ([6], p. 91, p. 95). We will discover that only *finitely* many pairs of ideals in K can satisfy (6) so that the ideal-theoretic formulation of "$m^n=n^m$" preserves the finite-solution result of the rational case.

Suppose A and B are (integral) ideals in K which solve (6). Since the norm of an ideal is multiplicative ([6], p. 108) we find, by taking norms of both sides of (6): $N(A)^{N(B)}=N(B)^{N(A)}$. Since $N(A)$, $N(B)$ are positive integers and $N(A)\neq N(B)$ by hypothesis, we have $N(A)=2$ and $N(B)=4$. Substituting these values in (6), we get $A^4=B^2$. Now, since the norm of A is 2, A is a prime ideal in K ([6], p. 109) and by the unique factorization theorem for ideals we

find that $B = A^2$. Conversely, if A is an ideal of norm 2 in K (and hence prime) and B is defined as A^2, then (6) is satisfied. Since there are only finitely many ideals in K of fixed norm ([**6**], p. 109) and, in particular, only finitely many of norm 2, we see that (6) has only finitely many solutions (perhaps none).

There is the question of the number of solutions of (6). The preceding discussion shows that the solvability of (6) depends on the existence of (prime) ideals of norm 2 in K and this in turn depends on how the principal ideal (2) factors into prime ideals in K. The reason is that every ideal P of norm 2 divides the ideal (2) since an ideal contains (divides) its norm ([**6**], p. 109). Thus P occurs in the factorization of (2) into prime ideals because of the unique factorization theorem for ideals. In other words, the only place to seek (prime) ideals of norm 2 in K is among the prime ideal factors of (2). Now, the manner in which any rational prime splits into prime ideals in an algebraic number field is known. However, simple and specific results on this question are available in two classic cases, namely, for the quadratic and cyclotomic fields. We must apply these results to the particular prime 2.

In a quadratic field $R(\sqrt{D})$, the decomposition of (2) depends on the Kronecker symbol $(d|2)$, where d is the discriminant of $R(\sqrt{D})$ ([**3**], p. 235; [**4**], pp. 156–158). We simply state the result here. If d is even, then $(d|2) = 0$ and the factorization of (2) is $(2) = P^2$ with $N(P) = 2$. Thus, there is one solution of (6). If d is odd, then $(d|2) = (-1)^{(d^2-1)/8}$ and (2) is itself prime of norm 4 or $(2) = P_1P_2$ with $N(P_1) = N(P_2) = 2$ depending on whether $(d|2)$ is -1 or $+1$, respectively. Thus, (6) has two distinct solution-sets if $d = 8s \pm 1$ and none if $d = 8s \pm 3$. More explicitly, if $D \not\equiv 1 \pmod 4$ (in particular, if D is even), then $d = 4D$ ([**6**], p. 67) and (6) has one solution in $R(\sqrt{D})$. If, however, $D \equiv 1 \pmod 4$, then $d = D$ ([**6**], p. 67) and there is one or there are no solutions of (6) in $R(\sqrt{D})$ depending on the value of $(d|2) = (D|2)$ as we already explained.

If K is a cyclotomic field, *i.e.*, one obtained by adjoining a primitive mth root of unity to the rationals ($m \geqq 2$), then the following is true (see [**3**], pp. 198–199; [**4**], p. 185). The principal ideal (2) remains prime in every cyclotomic field except that $(2) = P^{2^{\nu-1}}$ in the field where $m = 2^\nu$. In this latter case, $N(P) = 2$. As far as (6) is concerned, we see that there are no solutions for any cyclotomic field except those where $m = 2^\nu$ ($\nu \geqq 1$). In this case there is precisely one ideal P of norm 2 and thus P, P^2 is the only solution of (6).

Let us look at further examples. In the fields $R(\sqrt[n]{2})$ and $R(\sqrt[n]{-2})$ there is only one solution of (6). For, in $R(\sqrt[n]{(\pm 2)})$, we see that $(2) = (\sqrt[n]{(\pm 2)})^n$ and the ideals $(\sqrt[n]{2})$ and $(\sqrt[n]{(-2)})$ are such that $N((\sqrt[n]{(\pm 2)})) = 2$. This is true because the norm of an ideal, in a field K of degree n, is a power a^k ($1 \leqq k \leqq n$) of the smallest positive rational integer a contained in the ideal ([**3**], p. 28). Clearly, 2 is the smallest positive integer in $(\sqrt[n]{\pm 2})$ and, since $N((2)) = 2^n$ in $R(\sqrt[n]{(\pm 2)})$, we see that $N((\sqrt[n]{(\pm 2)}))$ is exactly 2^1 and not a greater power of 2. For example, in $R(\sqrt[9]{2})$, $A = (\sqrt[9]{2})$ and $B = A^2$ is the only solution of (6).

If K is any cubic field over R and if (2) is nonprime, then the only possible prime factorizations of (2) are P_1^3, $P_1^2\ P_2$, $P_1P_2P_3$ with $N(P_i) = 2$ ($1 \leqq i \leqq 3$) or

$(2) = P_4P_5$ with $N(P_4) = 2$, $N(P_5) = 4$. This is true because $N((2)) = 2^3 = 8$ over cubic fields. Hence there are either no, one, two or three solutions of (6). We give three examples of the splitting of (2) in cubic fields ([7] pp. 292–293). In $R(\vartheta)$ where ϑ satisfies the irreducible equation $x^3 + x + 1 = 0$, the ideal (2) does not decompose. Equation (6) has no solutions. There are two solutions in $R(\vartheta)$ where $\vartheta^3 + 6\vartheta + 8 = 0$ because $(2) = (2, 1 + \vartheta + \frac{1}{2}\vartheta^2)^2(2, \frac{1}{2}\vartheta^2)$. In $R(\vartheta)$, where $\vartheta^3 + 8\vartheta - 4 = 0$, we find $(2) = P^3$; as a matter of fact, $(2) = (\frac{1}{2}\vartheta^2)^3(132\vartheta^2 + 68\vartheta - 1023)$, where $132\vartheta^2 + 68\vartheta - 1023$ is a unit. Equation (6) has one solution in this latter case.

Having finished with the examples, we may now state the general result. It is known by a theorem of Dedekind that a rational prime p, in an arbitrary field K, is divisible by the square of a prime ideal if and only if p divides the discriminant d of K ([6], p. 101). Clearly, then, the *maximum* number of solution-sets in any K of degree n over R of (6) is n, corresponding to the case where (2) is completely unramified, *i.e.*, (2) can be factored into n distinct prime ideals each of norm 2. A necessary, but not sufficient, condition for this maximum number of solutions is that the discriminant of K be odd.

Summarizing, (6) has solutions if and only if the principal ideal (2) is composite in K. If n is the degree of K, then there are at most n distinct solutions. A necessary condition for the maximum number of solutions is that 2 does not divide the discriminant of K. Hence, if the discriminant of K is even, there are less than n different solution-sets, perhaps none. If K is the quadratic field $R(\sqrt{D})$ and (2) nonprime, then (6) has one or two sets of solutions depending on whether D is even or odd. There are no solutions of (6) in the cyclotomic fields with the exception that one solution is present in each of the fields $R(\zeta)$ where ζ is a primitive 2^νth root of unity. In all cases the number of solution sets, in any field K, is identical with the number of prime ideals P in K with $N(P) = 2$.

In closing, we remark that if we take (2) and make the transition to the principal ideal equation, namely $(\alpha)^{N((\beta))} = (\beta)^{N((\alpha))}$, we do not get infinitely many different ideal solutions even though (2) may have infinitely many solutions in K. This is no contradiction since the resulting principal ideals (α) and (β) give rise to only finitely many different ideals in K because associated numbers α and $\alpha\eta$ (where η is a unit) generate the same principal ideals.

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THE UNDER-OVER-UNDER THEOREM

FRANCIS SCHEID, Boston University

An experimenter has made three measurements of a quantity Q, at time t_1, t_2, and t_3. Conforming to currently accepted ritual of experimentation he plots three points. What they look like is shown in Figure 1.

From discussions with expert theorists in his field our man knows that the points should fall on a straight line. The relationship between Q and t should be linear. The formula $Q=Mt+B$ is indicated. Unfortunately, theory does not say *which* straight line. It does not predict the values of M and B. Indeed the purpose of the present experiment was the obtaining of this missing information. Many experiments are similarly inspired. One of Albert Einstein's accomplishments as a young man (before relativity) was the determination of an unknown coefficient in a theoretically derived formula for Brownian motion, the random motion of suspended particles in a supporting medium (dust in quiet air for example). So, theory often leaves little problems of this sort in its wake.

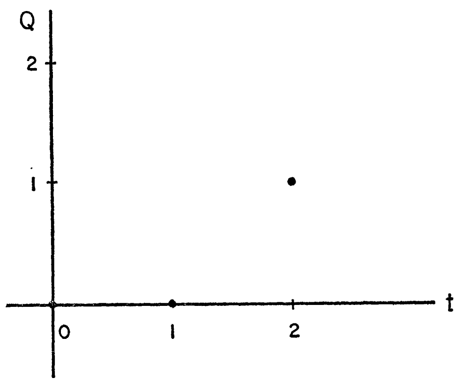

FIG. 1

Apparently, however, experiment also allows room for opinion. The three points in the diagram do not fall on a straight line. Taking his data home with him for the evening our man, in a moment of weakness, wishes he had made only two measurements. Then realizing full well that such thoughts are idle he decides to make the best of what he has, applies a transparent ruler to the diagram, and after a few moments of indecision draws his choice. It passes under, over, and under the three points in turn (Fig. 2). From his diagram he estimates

$$M = \text{rise/run} = .5,\ B = -.2$$

and instantly wonders just how good these values are. Has he made the best choice of straight line?

He is not by any means the first human being to face this question. Indeed, various "best" solutions have been suggested over the passing years. What

many believe to be the best of the best appears first in the work of the nineteenth-century mathematician Čebyšev (alias Tchebysheff, alias Tchebycheff, alias Chebyshev). Chebyshev argues as follows. Choose any line whatever. It will pass over or under the three points by amounts h_1, h_2, h_3 to be called errors (positive for over, negative for under). A direct hit simply makes one error zero. Let H be the largest error size, the worst miss, for the line chosen. For every line there is such an H. Then, of all possible straight lines, the best according to Chebyshev is the one whose H, the biggest miss, is smallest. The maximum error size must be minimized. It is a MINMAX problem.

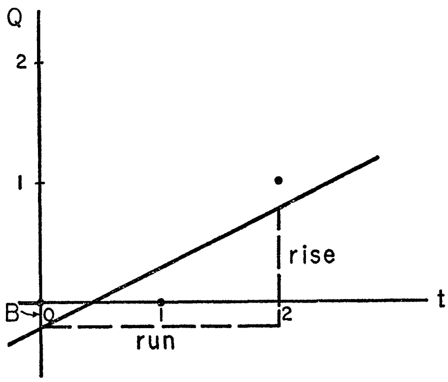

FIG. 2

Now let's take the analytic view. The equation $y=Mt+B$ represents a straight line. With M and B unspecified it could be almost any line. We are to determine M and B by applying the Chebyshev idea of bestness to the experimental data. The values of Y may then be thought of as corrections of the data values Q. Let the data be quite generally

$$\begin{matrix} t_1 & t_2 & t_3 \\ Q_1 & Q_2 & Q_3 \end{matrix}$$

so that the errors become

$$h_1 = y_1 - Q_1, \qquad h_2 = y_2 - Q_2, \qquad h_3 = y_3 - Q_3, \tag{1}$$

with

$$y_1 = Mt_1 + B, \qquad y_2 = Mt_2 + B, \qquad y_3 = Mt_3 + B. \tag{2}$$

A few lines of algebra will now convince the deepest skeptic that

$$(t_3 - t_2)y_1 - (t_3 - t_1)y_2 + (t_2 - t_1)y_3 = 0.$$

(Just substitute y_1, y_2, y_3 from above.) For brevity this is rewritten as

$$\beta_1 y_1 - \beta_2 y_2 + \beta_3 y_3 = 0, \tag{3}$$

where $\beta_1 = t_3 - t_2$, $\beta_2 = t_3 - t_1$, $\beta_3 = t_2 - t_1$. It may be assumed that $t_1 < t_2 < t_3$ so that all three β's are positive.

We now raise the question "Is there a line for which

$$h_1 = h, \qquad h_2 = -h, \qquad h_3 = h,$$

making the three errors of *equal size* and *alternating sign*?" If h is positive this line goes over-under-over the three points, while if h is negative the line goes under-over-under.

This question may seem thoroughly irrelevant and immaterial. But in fashionable courtroom language "it will be connected up." Tradition in mathematical journalism frowns on following the winding paths taken by our mathematical ancestors to their own discoveries. To continue the metaphor, it is considered wasteful of time to re-enact the crime. Our question happens to be a short cut, though this is perhaps not at once obvious.

If a line having the suggested property does exist it is easy to deduce what h must be. For then

$$y_1 = Q_1 + h, \qquad y_2 = Q_2 - h, \qquad y_3 = Q_3 + h,$$

and substituting into (3),

$$\beta_1(h + Q_1) - \beta_2(-h + Q_2) + \beta_3(h + Q_3) = 0,$$

leading quickly to

$$h = -\frac{\beta_1 Q_1 - \beta_2 Q_2 + \beta_3 Q_3}{\beta_1 + \beta_2 + \beta_3}, \tag{4}$$

so that only one h is possible. Our line must therefore pass through the three points

$$\begin{array}{ccc} t_1 & t_2 & t_3 \\ Q_1 + h & Q_2 - h & Q_3 + h \end{array}$$

with h given by (4).

But this is slightly severe. It is common knowledge that passing through just two specified points is all that one should ask of a straight line. It narrows the field of competition to exactly one candidate. That there is a line through the above *three* points is therefore newsworthy, if true. That it is true may be discovered by comparing the slopes of P_1P_2 and P_2P_3 (P_1, P_2, and P_3 being the three points, taken left to right). It requires a little persistence, but using (4) and the fact that $\beta_1 + \beta_3 = \beta_2$ the equality of these slopes emerges and settles the issue beyond further doubt.

The answer to our question is thus, "Yes, there is a line, exactly one, which makes the errors h_1, h_2, h_3 of equal size and alternating sign."

Now comes the connecting up. The line just found will be called the Chebyshev line, for it is the best line in Chebyshev's sense. The proof is easy. As indi-

cated, h, $-h$, h are the errors for the Chebyshev line. Let h_1, h_2, h_3 be the errors for *any other line*. Then using (4), (1) and (3) in succession

$$
\begin{aligned}
h &= -\frac{\beta_1(y_1 - h_1) - \beta_2(y_2 - h_2) + \beta_3(y_3 - h_3)}{\beta_1 + \beta_2 + \beta_3} \\
&= -\frac{(\beta_1 y_1 - \beta_2 y_2 + \beta_3 y_3) - (\beta_1 h_1 - \beta_2 h_2 + \beta_3 h_3)}{\beta_1 + \beta_2 + \beta_3} \qquad (5)\\
&= \frac{\beta_1 h_1 - \beta_2 h_2 + \beta_3 h_3}{\beta_1 + \beta_2 + \beta_3}.
\end{aligned}
$$

But if H is, as suggested before, the maximum of $|h_1|$, $|h_2|$, $|h_3|$, then because all β's are positive the right side of (5) is certainly increased if h_1, h_2, h_3 are replaced by H, $-H$, H respectively. This means

$$
|h| \leqq \frac{\beta_1 H + \beta_2 H + \beta_3 H}{\beta_1 + \beta_2 + \beta_3} = H. \qquad (6)
$$

Thus the maximum error size, $|h|$, of the Chebyshev line is no larger than the maximum, H, for any other line. This proves that $|h|$ is the required minmax.

By example, if the three points of our original diagram are as they seem to be

$$
\begin{array}{lccc}
t: & 0 & 1 & 2 \\
Q: & 0 & 0 & 1
\end{array}
$$

then $\beta_1 = 2-1 = 1$, $\beta_2 = 2-0 = 2$, $\beta_3 = 1-0 = 1$, and by (4),

$$
h = -\frac{(1)(0) - (2)(0) + (1)(1)}{1 + 2 + 1} = -\tfrac{1}{4}.
$$

The Chebyshev line thus passes through

$$
\begin{array}{lccc}
t: & 0 & 1 & 2 \\
y: & -\frac{1}{4} & \frac{1}{4} & \frac{3}{4}
\end{array}
$$

and so has the equation $y = \frac{1}{2}t - \frac{1}{4}$ as may easily be verified. This line passes under, over and under the three points missing each by the same amount, and yields the predictions $M = .5$, $B = -.25$, which are certainly very close to the results our experimenter obtained by eye. As a matter of fact the average person faced with the same problem will draw by eye a line fairly close to the Chebyshev line.

Can there be two best lines? In other words, can some other line have the same maximum error as the Chebyshev line? In this case $H = |h|$, and the *equality* sign holds in (6). At this point a few moments of quiet thought are more useful than a thousand words, and should eventually bring conviction that the substitution which carries (5) into (6) cannot preserve equality unless h_1, h_2, h_3 are of size $|h|$ and of alternating sign. But it is these features which led us to

the three points through which the Chebyshev line passes. Surely there are not *two* straight lines through these three points. The uniqueness of the best line is a fact.

Let's suppose next that our experimenter is not entirely satisfied with his first effort. Returning to his apparatus next morning he discovers that he has forgotten to shut it off. This brings mixed emotions, since it is an expensive affair and he has not been authorized to let it go on so long. However, it would be criminal to waste such a golden opportunity, and accordingly he obtains two additional readings before his conscience forces him to shut down. Adding these two readings to yesterday's plot he has the picture (Fig. 3) before him. (We may assume that his readings are from a counter, so that all have integer values.) It is now plain that his old line is too high. It passes over both new points, missing the first by quite a bit. Having no reason to suppose this par-

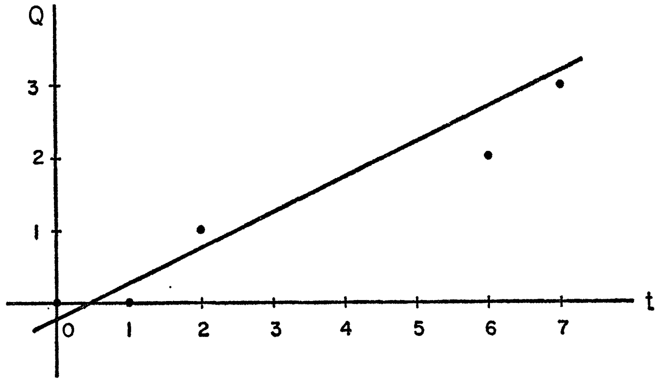

FIG. 3

ticular reading any less worthy than the other four, he once again applies the transparent ruler and draws a new line. Figure 4 shows what the picture now looks like. From the new line he makes the estimates

$$M = \text{rise/run} = 2.75/7 = .39,\ B = -.1,$$

which differ, of course, from his previous estimates but which he suspects are somewhat closer to the fact.

And what does Chebyshev have to say about this more complicated problem? Which line is really the best line? Chebyshev's answer is the same as before. It is the line for which the maximum error size H is the smallest. Becoming analytic once more let the data be rather generally

$$t_1 \cdots t_n$$
$$Q_1 \cdots Q_n.$$

In our example we take the values suggested by the diagram

t:	0	1	2	6	7
Q:	0	0	1	2	3.

For any straight line $y=Mt+B$ the ordinates y_i and errors h_i are again given by equations just like (1) and (2), there simply being more of them. Again H is the maximum of the quantities $|h_i|$ and the best line for the given points, according to Chebyshev, is the line whose H is the smallest. Now, how can this best line be found? The following *exchange method* provides a very effective answer to this question. It proceeds in four basic steps, and does yield the best line as we shall see.

STEP ONE. *Choose any three of the data points. (A set of three data points will be called a triple. This step simply selects an initial triple. It will be changed in Step four.) Proceed to Step two.*

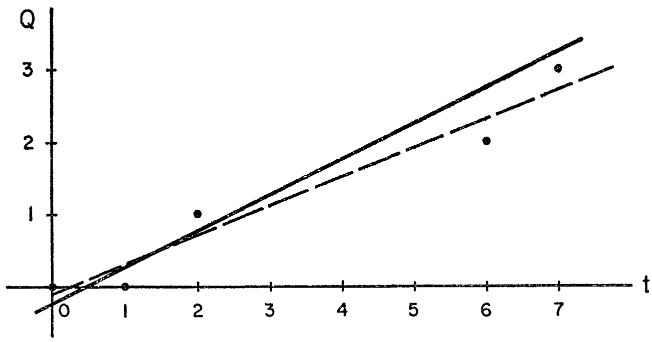

FIG. 4

STEP TWO. *Find the Chebyshev line for this triple as illustrated above. The value h for this line will, of course, be calculated in the process. Proceed to Step three.*

In illustration of these first two steps let us choose the initial triple

t:	0	1	2
Q:	0	0	1

consisting of the first three of the five points. This is, of course, the triple occurring in our original example. The Chebyshev line for this triple was found to be $y=\frac{1}{2}t-\frac{1}{4}$ with $h=-\frac{1}{4}$. This line passes under, over and under the first three of our five points with equal error sizes. But as noted earlier its errors at the remaining two points suggest a new choice of line. The exchange method makes this new choice as follows.

STEP THREE. *Calculate the errors at all remaining points for the Chebyshev line just found. Call the largest of these h_i values (in absolute value) H. If $|h|=H$ the search is over. The Chebyshev line for the chosen triple is also the best line for the entire point set. (We shall prove this in a moment.) If $|h|<H$ proceed to Step four.*

In our example we find

$$h_1 = -\tfrac{1}{4} = h, \qquad h_2 = \tfrac{1}{4} = -h, \qquad h_3 = -\tfrac{1}{4} = h, \qquad h_4 = \tfrac{3}{4}, \qquad h_5 = \tfrac{1}{4}.$$

Thus $H=\frac{3}{4}$ and since this exceeds $|h|$ we are not finished and proceed as instructed to

STEP FOUR. *This is the exchange step. Choose a new triple as follows. Add to the old triple a data point at which the greatest error size H occurs. Then discard one of the former points, in such a way that the remaining three have errors of alternating sign. (A moment's practice will show that this is always possible.) Then return, with the new triple, to Step two.*

In the example we must now include the fourth point and eliminate the first. On the new triple

$$\begin{array}{lccc} t: & 1 & 2 & 6 \\ Q: & 0 & 1 & 2 \end{array}$$

the errors h_2, h_3, h_4 found a moment ago do have the required alternation of sign. With this new triple we return as instructed to Step two:

Again we are to find a Chebyshev line. The computation follows the pattern of our first effort:

$$\beta_1 = 6 - 2 = 4, \qquad \beta_2 = 6 - 1 = 5, \qquad \beta_3 = 2 - 1 = 1,$$

$$h = -\frac{(4)(0) - (5)(1) + (1)(2)}{4 + 5 + 1} = \frac{3}{10}\cdot$$

The Chebyshev line must pass through

$$\begin{array}{lccc} t: & 1 & 2 & 6 \\ y: & \frac{3}{10} & \frac{7}{10} & \frac{23}{10} \end{array}$$

and so has the equation

$$y = \tfrac{2}{5}t - \tfrac{1}{10}. \tag{7}$$

This again completes Step two. Repeating Step three as instructed we find

$$h_1 = -\tfrac{1}{10}, \qquad h_2 = \tfrac{3}{10} = h, \qquad h_3 = -\tfrac{3}{10} = -h, \qquad h_4 = \tfrac{3}{10} = h, \qquad h_5 = -\tfrac{3}{10}.$$

This time $H=\frac{3}{10}$ so that $|h| = H$ and the job is done. The Chebyshev line (7) on the triple at $t=(1, 2, 6)$ is the best line for the set of five points with which we began. Its maximum error, $H=\frac{3}{10}$, is the smallest possible for any line. This will now be proved in the general case. Let us stop, however, to record the latest predictions of M and B, namely,

$$M = .4, \qquad B = -.1,$$

which should certainly tend to boost sales of transparent rulers.

There are two open questions. The exchange method can come to a halt only in Step three, and then only if no error exceeds $|h|$. How do we know this condition is ever satisfied? (Conceivably we could be computing forever.) That is the first question. The second is, of course, assuming the condition *is* satisfied at some stage, why is the last Chebyshev line also the best line in the Chebyshev sense for the whole point set?

To answer the first question, recall that after any particular exchange the old Chebyshev line has errors of size $|h|$, $|h|$, H on the new triple. Also recall that $|h| < H$ (or we would have stopped) and that these errors alternate in sign. The Chebyshev line for this new triple is then found. Call its errors on the triple h^*, $-h^*$, h^*. Returning to (5) now, with the old Chebyshev line playing the role of "any other line," we have

$$h^* = \frac{\beta_1 h_1 - \beta_2 h_2 + \beta_3 h_3}{\beta_1 + \beta_2 + \beta_3},$$

where h_1, h_2, h_3 are the numbers h, h, H with alternating sign. Because of this alternation of sign all three terms in the numerator of this fraction *have the same sign*, so that

$$|h^*| = \frac{\beta_1 |h| + \beta_2 |h| + \beta_3 H}{\beta_1 + \beta_2 + \beta_3}$$

if we assume that the error H is at the third point, just to be specific. (It really makes no difference in which position it goes.) In any event, $|h^*| > |h|$ because $|h| < H$. The new Chebyshev line has a *greater* error size on its triple than the old one had (on its own triple). This result now gives excellent service. If it comes as a surprise, look at it this way. The old line in our example gave excellent service ($h = \frac{1}{4}$) on its own triple, but poor service ($H = \frac{3}{4}$) elsewhere. The new line gave good service ($h = \frac{3}{10}$) on its own triple, and just as good service ($H = h$) on the other points also.

We can now prove that the exchange method must come to a stop sometime. For, there are only so many triples. And *no triple is ever chosen twice*, since as just proved the h values increase steadily. At some stage the condition $|h| = H$ will be satisfied.

The second question is almost as easily answered as the first. For the last Chebyshev line, and the h value of its triple, the maximum error size on the whole point set is $H = |h|$, (or we would have proceeded by another exchange to still another triple). If $h_1, \cdots, h_n$ are the errors for any other line, then by (6) $|h| \leqq \max |h_i|$, where h_i is restricted to the three points of the last triple. But then certainly $|h| \leqq \max |h_i|$, where h_i is unrestricted, for including the remaining points can only make the right side even bigger. Thus the maximum error, $H = |h|$, of this final Chebyshev line is the smallest maximum error of all. It is the minmax. The best line in the Chebyshev sense is one which has errors of equal size and alternating sign on a properly chosen triple. This is what we have

light-heartedly designated as the under-over-under theorem. Taking one moment more, let us again show that the best line is unique. If a second line had maximum error (on the entire point set) equal to that of the final Chebyshev line, which is $|h|$, then *on the final triple* its maximum error would certainly be no greater. But it could also be no less, for, as we saw earlier, no line can outdo the Chebyshev line on its own triple. So on the final triple both lines would have the same maximum error. Our earlier proof of uniqueness now applies. The two lines must be the same.

As a final example we may suppose that our laboratory scientist has received a new research grant, which restores his fiscal respectability and permits him to repeat his experiment on a grander scale. The results are the following experimenter's delight.

t:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
Q:	0	1	1	2	1	3	2	2	3	5	3	4	5	4	5	
t:	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Q:	6	6	5	7	6	8	7	7	8	7	9	11	10	12	11	13

With this wealth of expensive information he abandons his transparent ruler and consults higher authority on best-fitting straight lines. The search for the correct triple and corresponding Chebyshev line is on. There are 31 points to choose among. First graders may soon be learning that the number of possible triples in this accumulation is $C(31, 3)$ or $31!/(28!\ 3!) = 4495$. Finding the correct triple in this haystack appears at first glance to be a task well left to the arithmetical wizardry of automatic computing machines. Fortunately the exchange method does have the repetitious nature so easy to explain to (program for) such machines. And fortunately too the method wastes very little time on inconsequential triples. It hastens to its target like a hungry lion. The IBM 650 at Boston University took less than five minutes and only three exchanges, starting from the horrible initial triple at $t = (0, 1, 2)$, to produce the Chebyshev line (shown in Fig. 5).

$$y = .38095t - .28571$$

on the triple at $t = (9, 24, 30)$. The value of h for this line is 1.85714 and the maximum error outside the triple is only 1.61905 so that truly this is the best line in Chebyshev's sense. It yields the predictions

$$M = .38095, \qquad B = -.28571$$

for the long-sought coefficients. These may be compared with the cruder values found earlier, and presumably will serve the purpose for the present. If the forward march of science leads to the suspicion that these predictions are grossly in error, then perhaps a better experimental method can be devised, or hopefully more funds will become available. But for the given data the consensus of current mathematical literates is that the Chebyshev line is the best.

The exchange method has rooted out the correct triple so brilliantly in this example that the use of high-speed machines seems more a luxury than a necessity. It is interesting to watch the progress of the computation through the points included in the successive triples and the corresponding h and H values.

Triple at $t=$	h	H
(0, 1, 2)	.25000	5.25000
(0, 1, 24)	.35417	-3.89583
(1, 24, 30)	-1.75862	-2.44828
(9, 24, 30)	-1.85714	-1.85714

Note that in this case no unwanted point is ever brought into the triple. Three points are needed, three exchanges suffice. Note also the steady increase of $|h|$, as forecast by theory.

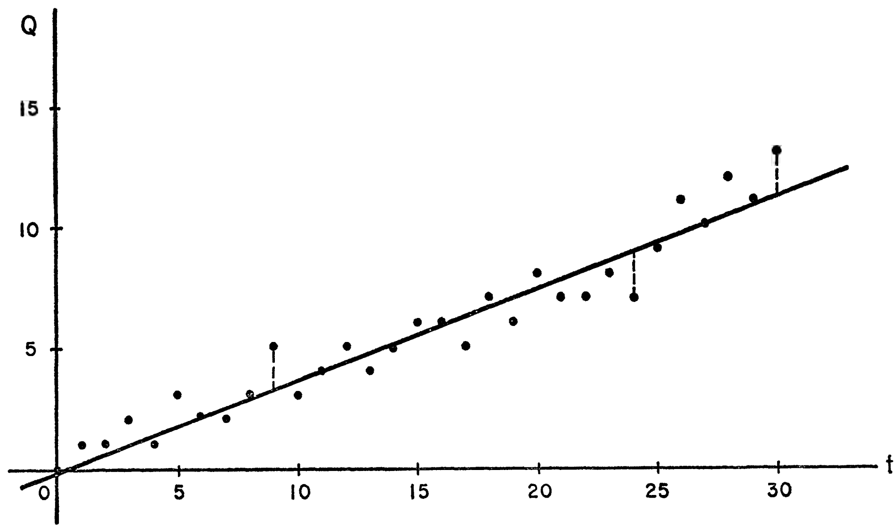

FIG. 5

Though the machine is really unnecessary for the above calculations, the existence of modern computers has led during the past decade to a rapid spread of the Chebyshev method to more complicated problems of approximation. Not only may straight lines be fitted by the minmax criterion, but a variety of more sophisticated curves as well. For these the search may be much more laborious (perhaps for a correct octuple). For various reasons such problems of approximation are fundamental in modern computation, and it is easy to believe that the machine becomes man's indispensable colleague.

PROBLEM. *Show that the best line of our last example (shown in Fig.* 5) *makes direct hits at* $t=6$ *and at* $t=27$. *There are no other direct hits.*

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INDEXED SYSTEMS OF NEIGHBORHOODS FOR GENERAL TOPOLOGICAL SPACES*

A. S. DAVIS, University of Oklahoma

0. Introduction. From the very beginning, topologies have been defined in terms of neighborhood systems. Yet, even after the publication of Weil's paper [3], no one seems to have noticed that "indexed" neighborhood systems may be used in the most general space as well as in the relatively special uniform space. This is strange for at least two reasons. First, as will be seen, indexed systems of neighborhoods provide a natural bridge from topological spaces in general to uniform spaces and metric spaces in particular. Indeed, starting in a metric space, point-set topology is reached by traversing this bridge in the reverse direction; when the metric is last seen, it is seen in the form of an indexed system of neighborhoods, *viz.*, the system of open spheres $N_\epsilon(x)$ of radius ϵ centered at x which provide the definition of open set. It was this observation, no doubt, that led Weil to formulate the notion of uniformity. Secondly, much of the work in point-set topology has been concerned with generalizing results from analysis—results which more often than not require concepts like uniform continuity, Cauchy sequences, totally bounded sets, contraction maps, etc., and these concepts in turn require that one be able to compare the "sizes" of neighborhoods at different points in the space. But one does not need uniform spaces to meet this requirement. All of the concepts named can be defined in the most general topological space in terms of indexed systems of neighborhoods.

It is the purpose of this paper to call attention to these possibilities. In the context of what has gone before, the results obtained are not profound; but it is hoped that they will be of use to someone.

1. Spaces in general. A topological space $(X, \mathfrak{J})$ is defined in the usual way: X is a set and $\mathfrak{J}$ is a family of subsets which includes unrestricted unions and finite intersections of its members.

THEOREM 1. *The pair $(X, \mathfrak{J})$, where X is a set and $\mathfrak{J}$ is a family of subsets, is a topological space if and only if there exists a family $\{N_\alpha\}_{\alpha\in I}$ of functions which assign to each $x\in X$ a subset $N_\alpha(x)\subseteq X$, such that*

(i) *for each $x\in X$, $x\in\bigcap_{\alpha\in I} N_\alpha(x)$;*

* This research is supported by a National Science Foundation Cooperative Fellowship.

(ii) *to each pair* $\alpha\in I$, $\beta\in I$, *there corresponds at least one* $\gamma\in I$, *such that for all* $x\in X$, $N_\gamma(x)\subseteq N_\alpha(x)\cap N_\beta(x)$;

(iii) $G\in\mathfrak{I}$ *if and only if for each* $x\in G$, *there is an* $\alpha\in I$, *such that* $N_\alpha(x)\subseteq G$. *Such a family* $\{N_\alpha\}_{\alpha\in I}$ *may always be chosen so that* $\{N_\alpha(x):\alpha\in I,\ x\in X\}$ *is a base for* $\mathfrak{I}$; *that is, so that the following also holds*:

(iv) *given* $\alpha\in I$, $x\in X$, *and* $y\in N_\alpha(x)$, *there is a* $\beta\in I$, *such that* $N_\beta(y)\subseteq N_\alpha(x)$.

Proof. The proof of sufficiency is standard and is omitted. Assume, then, that a topological space $(X,\mathfrak{I})$ is given and defined for each $G\in\mathfrak{I}$, the subset $N_G=[X\times G]\cup[(X-G)\times X]$ in the product space. Let I be the class of all finite subfamilies of $\mathfrak{I}$ and define, for each $\alpha\in I$, $N_\alpha=\bigcap_{G\in\alpha}N_G$. Then, with $N_\alpha(x)$ defined to be $\{y\in X:\ (x,y)\in N_\alpha\}$, the family $\{N_\alpha\}_{\alpha\in I}$ is easily seen to meet all requirements of the theorem, and $\{N_\alpha(x):\alpha\in I,\ x\in X\}$ is a base for $\mathfrak{I}$.

A family $\{N_\alpha\}_{\alpha\in I}$ with Properties (i) through (iv) will be called an *indexed system of open neighborhoods for* X which *defines* the topology $\mathfrak{I}$, the adjective "open" being omitted if Property (iv) is replaced by

(iv)′ *given* $\alpha\in I$, *there is a* $\gamma\in I$, *such that for* $x\in X$, *and* $y\in N_\gamma(x)$, *there is a* $\beta\in I$, *such that* $N_\beta(y)\subseteq N_\alpha(x)$.

If $\{N_\alpha\}_{\alpha\in I}$ and $\{N'_{\alpha'}\}_{\alpha'\in J}$ are two indexed systems of neighborhoods for X, the second will be said to be *as fine as* the first if for each $\alpha\in I$, there is some $\alpha'\in J$, such that $N'_{\alpha'}\subseteq N_\alpha$. Two such systems will be called *equivalent* if each is as fine as the other. Clearly, equivalent systems define the same topology, although a given topology may be defined equally well by nonequivalent systems. (Throughout the discussion, as was the case here, the N_α will be interpreted both as functions and as subsets $\{(x,y)\in X\times X:\ y\in N_\alpha(x)\}$. All the usual conventions concerning the latter interpretation will be assumed. *E.g.*, $(x,y)\in N_\alpha^{-1}$ iff $(y,x)\in N_\alpha$.)

The usefulness of Theorem 1 is severely limited by the fact that the family $\{N_\alpha\}_{\alpha\in I}$ cannot in general be made to satisfy either of the following "symmetry" conditions:

(v) *for each* $\alpha\in I$, $x\in N_\alpha(y)$ *implies* $y\in N_\alpha(x)$;

(v)′ *for each* $\alpha\in I$, *there is a* $\beta\in I$, *such that* $x\in N_\beta(y)$ *implies* $y\in N_\alpha(x)$.

The reason for this limitation can be found in the fact that the N_α of Theorem 1, while containing the diagonal $\Delta=\{(x,y)\in X\times X:\ y=x\}$, are not necessarily neighborhoods of it in the product topology. (See Theorem 2.)

2. R_0-spaces. The next question then is "For which spaces can the N_α be chosen as neighborhoods of the diagonal?" It turns out that only a very mild "regularity" condition is needed. Since it seems that the spaces which satisfy this condition have not appeared explicitly in the literature before, several characterizations will be given in the next theorem.

In a topological space $(X,\mathfrak{I})$ let A and B be subsets of X. A is said to be *separated from* B by an open set G if $A\subseteq G$ and $G\cap B=\varnothing$. This is the case if and only if $A\cap\overline{B}=\varnothing$. A and B are *entirely separated* if they are separated by disjoint open sets. In the present context singletons will be referred to as

"points." Thus a T_1-space is one in which distinct points are separated from one another.

THEOREM 2. *The following statements about a topological space* $(X, \mathfrak{I})$ *are equivalent*:

(a) *closed sets are separated from the points that they exclude*;

(b) *every open set contains the closure of each of its points*;

(c) $\mathfrak{I}$ *is defined by an indexed system* $\{N_\alpha\}$ *of neighborhoods for* X, *such that each* N_α *is an open neighborhood of* Δ;

(d) $\mathfrak{I}$ *is defined by an indexed system of neighborhoods for* X *which satisfies* (v);

(e) *for all* $x \in X$, $y \in X$, *either** $\{x\}^- \cap \{y\}^- = \varnothing$ *or* $\{y\}^- = \{x\}^-$;

(f) $\mathfrak{I}$ *is isomorphic* (*lattice-theoretically*) *to the topology of a* T_1-*space*.

An R_0-*space, by definition, is one which satisfies any, hence all, of these conditions.*

Proof. (a) implies (b): If G is an open set containing x, then $X-G$ is a closed set excluding it. Hence $(X-G)\cap\{x\}^-=\varnothing$, *i.e.*, $\{x\}^-\subseteq G$.

(b) implies (c): Let $\{N_\alpha\}_{\alpha\in I}$ be the family of all open neighborhoods (in the product topology) of the diagonal. Property (i) is equivalent to $\Delta\subseteq\bigcap_{\alpha\in I} N_\alpha$. (ii) is satisfied with $N_\gamma = N_\alpha \cap N_\beta$. Now let $G\in\mathfrak{I}$ and $x\in G$ be given. Define $U=[X\times G]\cup[(X-\{x\}^-)\times X]$. Since $\{x\}^-\subseteq G$, U is an open neighborhood of Δ. Hence $U=N_\alpha$, for some $\alpha\in I$, and $N_\alpha(x)=G$. Thus (iii) holds. Then so does (iv), since $N_\alpha(x)$ is open in X whenever N_α is open in $X\times X$.

(c) implies (d): Property (v) is equivalent to the statement that $N_\alpha^{-1}=N_\alpha$, for all $\alpha\in I$. Now if N_α is an open neighborhood of Δ, then so are $N_\alpha\cap N_\alpha^{-1}$ and $N_\alpha\cup N_\alpha^{-1}$. Since $N_\alpha\cap N_\alpha^{-1}\subseteq N_\alpha\subseteq N_\alpha\cup N_\alpha^{-1}$, the family of all open neighborhoods of Δ is equivalent to the subfamily consisting of all symmetric open neighborhoods.

(d) implies (e): If $z\in\{x\}^-\cap\{y\}^-$, then $\{z\}^-\subseteq\{x\}^-\cap\{y\}^-$. Also, for all $\alpha\in I$, $x\in N_\alpha(z)$, whence $z\in N_\alpha(x)$. Then $x\in\{z\}^-$, and so $\{x\}^-\subseteq\{z\}^-\subseteq\{y\}^-$. Similarly, $\{y\}^-\subseteq\{x\}^-$. In summary, if $\{x\}^-\cap\{y\}^-\neq\varnothing$, then $\{y\}^-=\{x\}^-$.

(e) implies (f): For each $G\in\mathfrak{I}$, define $G^*=\{x^*\subseteq X: x^*=\{x\}^-$, for some $x\in G\}$. Let $\mathfrak{I}^*$ be the family of all such sets. Then $(X^*, \mathfrak{I}^*)$ is a T_1-space and $G\to G^*$ is a one-one mapping of $\mathfrak{I}$ onto $\mathfrak{I}^*$ which preserves unions and intersections. The proof of the latter depends only on the properties of open sets and the fact that if $x\in X-G$, with $G\in\mathfrak{I}$, then $\{x\}^-\subseteq X-G$, which is true in any space. For the proof that $\mathfrak{I}^*$ is a T_1-topology, note that if $\{x\}^-\neq\{y\}^-$, then $\{x\}^-\subseteq X-\{y\}^-$, whence $\{x\}^-\in(X-\{y\}^-)^*$. Similarly $\{y\}^-\in(X-\{x\}^-)^*$, so that $\{x\}^-$ and $\{y\}^-$ are separated from each other in $(X^*, \mathfrak{I}^*)$.

(f) implies (a): Suppose that $\mathfrak{I}^*$ is the T_1-topology to which $\mathfrak{I}$ is isomorphic. Clearly, the corresponding lattices $\mathfrak{Z}$ and $\mathfrak{Z}^*$ of closed sets are also isomorphic. Now the lattice of closed sets for a T_1-space is characterized by the fact that each of its members is a union of minimal nonvoid closed sets, namely, the

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* We denote the closure of a set A by A^-.

wherever they do not coincide, and R_2-spaces are those which are properly called *regular*—those in which points and closed sets are entirely separated wherever the former are not contained in the latter. Each solid arrow represents an implication (absence represents absence), and each fractured arrow represents the existence of isomorphisms between topologies. Finally, each separation axiom is defined as the conjunction of two weaker axioms: $T_k = R_{k-1} \wedge T_{k-1} = R_{k-1} \wedge T_0$. (But the usual definition of "normality" must be modified slightly if R_3 is to be the axiom for normal spaces.)

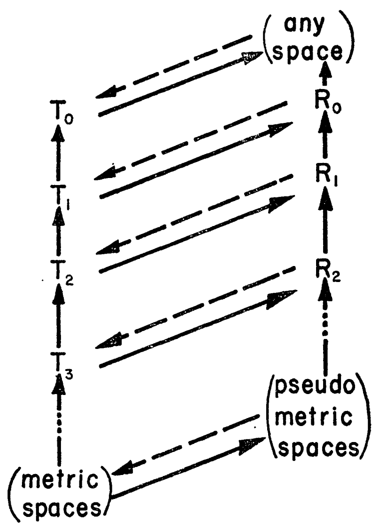

FIG. 1

4. Regular spaces. Topological spaces were created by banishing the metric from metric spaces. The problem of metrization of topological spaces is essentially the problem of describing in nonmetrical terms—in fact, in terms of topological invariants—the amount of structure a space must have in order to re-admit a metric. But the question may be turned around: What metric-like properties does a space with a given "amount" of topological structure have?

One virtue of indexed systems of neighborhoods is that they often show very clearly which metric or pseudo-metric properties are restored with each step toward metrizability. Regular spaces provide a good example of this. Since they are but a half-step away from uniform (completely regular) spaces, the following result should not be too unexpected:

THEOREM 4. *A space* $(X, \mathfrak{J})$ *is regular if and only if there exists an indexed system of neighborhoods satisfying Properties* (i) *through* (v) *and the following "local triangle inequality"*:

(vi) *given* $x \in X$ *and* $\alpha \in I$, *there is a* $\beta \in I$, *such that if* $z \in N_\beta(x)$ *and* $y \in N_\beta(z)$, *then* $y \in N_\alpha(x)$.

That is, a system $\{N_\alpha\}_{\alpha\in I}$ *exists such that, given* $N_\alpha(x)$, *there is a* $\beta\in I$ *for which* $N_\beta(N_\beta(x))\subseteq N_\alpha(x)$.

Proof. We use the fact that a space is regular if and only if each neighborhood of a point contains a closed neighborhood of that point. Now if $(X, \mathfrak{J})$ is regular, then *a fortiori* it is an R_0-space and so we may choose $\{N_\alpha\}_{\alpha\in I}$ to be the family of all open neighborhoods of Δ in the product topology. Given $N_\alpha(x)$ from this system, choose a closed neighborhood U of x contained in $N_\alpha(x)$. Choose a second closed neighborhood V satisfying $x\in V\subseteq U^0$ (the interior). Define $W=[(X-U)\times X]\cup[(X-V)\times N_\alpha(x)]\cup[X\times U^0]$. (See Fig. 2.) Then W contains Δ and is open, whence $W=N_\beta$, for some $\beta\in I$. Moreover, $N_\beta(N_\beta(x))=N_\beta(U^0)=N_\alpha(x)$, as desired.

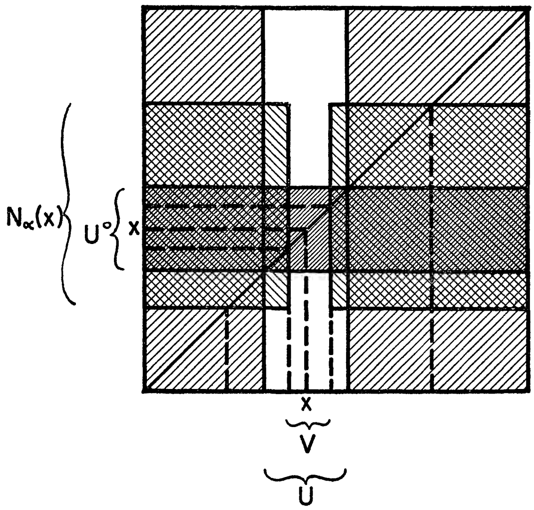

FIG. 2

Conversely, suppose $\{N_\alpha\}_{\alpha\in I}$ satisfies Properties (i) through (vi). Then given any neighborhood U of x, there exists an $\alpha\in I$ and a $\beta\in I$, such that $x\in N_\beta(x)\subseteq\overline{N_\beta(x)}=\bigcap_{\gamma\in I}N_\gamma(N_\beta(x))\subseteq N_\beta(N_\beta(x))\subseteq N_\alpha(x)\subseteq U$, and so the space is regular.

The fact that the β in Property (vi) depends in general not only on α, but on x, makes (vi) a local property. If, for each α, the β could be chosen "uniformly," *i.e.*, independently of x, then of course we would be in a uniform space.

5. Concepts definable in terms of indexed systems of neighborhoods. It should be evident that most of the concepts encountered in a metric space and definable in a uniform space can just as well be defined in any topological space whatsoever. But what can be done with these concepts after they have been defined is quite another question! Some results will require so many metric-like properties for their proof that one might as well stay in a metric space to

begin with. For other results this need not be the case. Many questions remain to be answered. In this last section we can do no more than give a very brief discussion of a few fundamental ideas.

Uniform continuity is defined just as it is in analysis. Let $(X, \mathfrak{J}_1)$ and $(Y, \mathfrak{J}_2)$ be topological spaces with indexed systems $\{U_\delta\}_{\delta\in I}$ and $\{V_\epsilon\}_{\epsilon\in J}$ of neighborhoods defining, respectively, $\mathfrak{J}_1$ and $\mathfrak{J}_2$. A function $f\colon X\to Y$ is *uniformly continuous re* $\{U_\delta\}$ *and* $\{V_\epsilon\}$ *if*

(u.c.) *for each* $\epsilon\in J$, *there is a* $\delta\in I$, *such that for all* $x\in X$, $f(U_\delta(x))\subseteq V_\epsilon(f(x))$.

If $\{U'_{\delta'}\}$ is another system as fine as $\{U_\delta\}$, and if $\{V_\epsilon\}$ is as fine as $\{V'_{\epsilon'}\}$, then the function f is also uniformly continuous *re* $\{U'_{\delta'}\}$ and $\{V'_{\epsilon'}\}$. Thus uniform continuity is well defined with respect to equivalence of indexed systems of neighborhoods. (In this regard (u.c.) is typical of all definitions quoted in this section.) The basic properties of uniform continuity hold in any topological space. For instance, the set of all uniformly bicontinuous one-one mappings of a space onto itself forms a group under composition. (A u.c. function of a u.c. function is a u.c. function.)

Moreover, the Cauchy property of a sequence, net, or filter is preserved under uniformly continuous maps. A *Cauchy filter* in a space $(X, \mathfrak{J})$ with topology defined by $\{N_\alpha\}_{\alpha\in I}$ is a filter $\mathfrak{F}$ which contains "small sets":

(c.f.) *for each* $\alpha\in I$, *there is an* $x(\alpha)\in X$, *such that* $N_\alpha(x(\alpha))\in\mathfrak{F}$.

This is the usual definition. To get the same results with nets as are obtainable with filters, one must work with nets $\{x_n\}_{n\in D}$ which satisfy

(s.n.) *for each* $\alpha\in I$, *there is an* $x(\alpha)\in X$, *and an* $n(\alpha)\in D$, *such that for all* $n\geqq n(\alpha)$, $x_n\in N_\alpha(x(\alpha))$.

These might be called *semiregular* in deference to the fact that *Cauchy nets* are sometimes called "regular." The latter are defined by

(c.n.) *for each* $\alpha\in I$, *there is an* $n(\alpha)\in D$, *such that for all* $m\geqq n(\alpha)$ *and* $n\geqq n(\alpha)$, $x_m\in N_\alpha(x_n)$.

Semiregular nets are not necessarily Cauchy. In fact, even convergent nets (which are always semiregular) may not be Cauchy. On the other hand, semiregular nets and Cauchy filters are intimately related through the points $\{x(\alpha)\}_{\alpha\in I}$ which, if I is directed by defining $\alpha\geqq\beta$ to mean $N_\alpha\subseteq N_\beta$, also form a net. (*Dragnet* might be an appropriate name, since it "drags" the given net or filter along with it.) By means of this relationship one may prove that spaces in which Cauchy filters always converge are precisely those spaces in which semiregular nets always converge. Such spaces are *complete.*

Thus completeness may be studied in any topological space. One may ask: Which topological spaces have "completions?" Cohen [1], for example, succeeded in embedding a space somewhat more general than a uniform space in a space in which sequences satisfying (s.n.) converge. But a general answer has not yet been developed.

In spite of the generality, at least two basic facts about completeness can be recited: Any closed subspace of a complete space is complete (with respect to

the relativization of the given neighborhood system). In a Hausdorff (T_2) space, complete subspaces are always closed (provided that a neighborhood system satisfying (v)′ is used).

It was mentioned that contraction maps may be defined in a general topological space. A likely definition might be: Given a system $\{N_\alpha\}_{\alpha \in I}$ defining the topology of the space $(X, \mathfrak{I})$, a *contraction map with coefficient* c $(0<c<1)$ is a map $A: X \rightarrow X: x \rightarrow xA$ which satisfies

(c.m.) *for each* $N_\alpha(x)$, *there is a positive rational number* $r/s \leqq c$ *and a* $\beta \in I$, *such that* $N_\beta^s(xA) \subseteq N_\alpha(xA)$ *and* $N_\alpha(x)A \subseteq N_\beta^r(xA)$;

where $N_\beta^s(y)$ is $N_\beta(\cdots N_\beta(N_\beta(y)) \cdots)$ with s repetitions. According to Theorem 4, the image XA of X must be regular for A to be a contraction. It seems to this author doubtful, however, that useful or interesting results—a fixed-point theorem, for example—can be obtained without imposing some kind of countability condition on the space.

Axioms which concern the existence of countable or finite coverings and bases, local or global, are as numerous as those which are stated in terms of separation. For this reason, questions of countability and compactness have been deliberately, if unjustly, avoided. A study of such questions in terms of indexed systems of neighborhoods might be rewarding.

We conclude with a corollary to Theorem 4 on uniform convergence. Let $\{f_n\}_{n \in D}$ be a net of functions from a topological space $(X, \mathfrak{I}_1)$ to a space $(Y, \mathfrak{I}_2)$ whose topology is defined by an indexed system of neighborhoods $\{N_\alpha\}_{\alpha \in I}$. Then $\{f_n\}_{n \in D}$ is said to *converge uniformly* (*re* $\{N_\alpha\}$) to a function $g: X \rightarrow Y$ if

(c.u.) *for each* $\alpha \in I$, *there is an* $n(\alpha) \in D$, *such that for all* $n \geqq n(\alpha)$ *and all* $x \in X$, $f_n(x) \in N_\alpha(g(x))$.

THEOREM 5. *If the range space is regular, its topology can always be defined by an indexed system of neighborhoods relative to which any uniformly convergent net of continuous functions converges to a continuous function.*

Proof. If $(Y, \mathfrak{I}_2)$ is regular, then Theorem 4 asserts the existence of a system $\{N_\alpha\}_{\alpha \in I}$ defining $\mathfrak{I}_2$ and satisfying Properties (i) through (vi). Let $\{f_n\}_{n \in D}$ be a net of continuous functions $f_n: X \rightarrow Y$, converging uniformly *re* $\{N_\alpha\}$ to g. We shall show that for all $x \in X$ and all $\alpha \in I$, $g^{-1}(N_\alpha(g(x)))$ is a neighborhood of x, by showing that there exists a $\gamma \in I$ and an $n \in D$, such that $f_n^{-1}(N_\gamma(f_n(x))) \subseteq g^{-1}(N_\alpha(g(x)))$. By Property (vi) we may choose $\beta \in I, \gamma \in I$, so that $N_\gamma(N_\gamma(g(x))) \subseteq N_\beta(g(x))$ and $N_\beta(N_\beta(g(x))) \subseteq N_\alpha(g(x))$. Then we choose $n \geqq n(\gamma)$ so that for all $z \in X, f_n(z) \in N_\gamma(g(z))$. It is then not difficult to see (making use of Property (v)) that if $f_n(z) \in N_\gamma(f_n(x))$, then $g(z) \in N_\alpha(g(x))$, as desired.

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ADDENDUM

Both the editor and the author, A. Oppenheim, of the paper *The Erdös inequality and other inequalities for a triangle* (this MONTHLY, vol. 68, 1961, pp. 226–230) overlooked earlier papers *Cyclic properties of Miquel polygons* by B. M. Stewart (this MONTHLY, vol. 47, 1940, pp. 462–466) and *Bibliography on cyclic properties of Miquel polygons* by V. Thébault (this MONTHLY, vol. 48, 1941, p. 541). Thébault calls attention to prior work by Boutin, Dieudonné, Bernes, Goormaghtigh, and himself.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, University of Washington

Material for this department should now be sent to M. H. Protter, Department of Mathematics, University of California, Berkeley 4, California.

A THEOREM ON NORMAL FAMILIES

S. P. FRANKLIN, University of California, Los Angeles

An application of the Riemann mapping theorem provides us with the following rather general theorem on normal families.

THEOREM. *Let Ω be a region, Ω^* be a simply connected region not the whole plane. If $F=\{f\}$ is any family of analytic functions such that each f maps Ω into Ω^*, then F is normal.*

Proof. By the Riemann mapping theorem there exists a univalent analytic function which maps Ω^* onto the unit disk Δ. Let Γ be such a function and let $C=\{g=\Gamma\circ f \mid f\in F\}$. Then each $g\in C$ maps Ω into Δ and hence is uniformly bounded on Ω, and C is, therefore, normal.

Now if $\{f_n'\}$ is any sequence of functions from F, and $\{g_n'=\Gamma\circ f_n'\}$ the corresponding sequence from C, there exists a subsequence of $\{g_n'\}$, say $\{g_n\}$, which converges uniformly on compact subsets of Ω. Clearly $\{g_n\}$ does not approach ∞ and the limit function g is analytic. Then if $E\subseteq\Omega$ is compact, $g_n(z)\rightarrow g(z)$ uniformly on E. Now, given any $\epsilon>0$, we wish to show that for all $z\in E$ and sufficiently large n, $|f(z)-f_n(z)|<\epsilon$, where $f=\Gamma^{-1}\cdot g$.

Since Γ^{-1} is continuous, it is uniformly continuous on $g(E)$. That is, there exists a δ such that if $w_1, w_2\in g(E)$ and $|w_1-w_2|<\delta$, then $|\Gamma^{-1}(w_1)-\Gamma^{-1}(w_2)|<\epsilon$. Now choose n so large that $|g(z)-g_n(z)|<\delta$ for all $z\in E$. Then $|f(z)-f_n(z)|<\epsilon$ for all $z\in E$ and hence F is normal.

The hypothesis can be weakened to include Ω^* as any finitely connected region, not the whole plane, by using a more powerful theorem [1] to produce the function Γ.

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THE MATRIX EQUATION $AA^*=sA$

J. L. BRENNER, Stanford Research Institute

In [1], K. Goldberg proved that if $AA^*=sA$ and if A has only zeros and ones for elements, then for some permutation matrix P, $PAP^*=A_1\oplus A_2\oplus\cdots$, where each A_i is either all zeros or all ones. This theorem is susceptible of some generalization.

THEOREM 1. *A necessary and sufficient condition that the real matrix A, with nonnegative elements, satisfy*

$$AA^* = sA \tag{1}$$

is that for some permutation matrix P, $PAP^*=A_1\oplus A_2\oplus\cdots$, where the elements of each A_i are either all 0 or all positive; $A_iA_i^*=sA_i$.

Proof. Assume $s>0$ and (1) holds. Note first that $A=(a_{ij})$ is square and symmetric. Let t be the minimum number of nonzero elements in a row of A. If $t=0$, it is clear that a permutation matrix P exists with $PAP^*=0\oplus B$, and induction applies to the $n-1\times n-1$ matrix B. Assume $t>0$. Take P so that some row (the kth) of PAP^* consists of t nonzero elements followed by zeros. It follows from (1) that $a_{kk}>0$ (thus $t\geqq k$), and the first t columns of A contain only zeros in all rows after the tth. Thus $PAP^*=A_1\oplus B$; all t^2 elements of A_1 must be positive. But from (1), $PAA^*P^*=sPAP^*$; thus $A_1A_1^*\oplus BB^*=s(A_1\oplus B)$, $A_1A_1^*=sA_1$. The necessity is proved; the sufficiency is clear.

For applications, a more attractive formulation of Theorem 1 is

THEOREM 2. *Let S be a real symmetric unitary matrix. If the elements of $A=I-S$ are nonnegative, there is a permutation matrix P such that $PSP^{-1}=S_1\oplus S_2\oplus\cdots$, where either S_i is the identity matrix I_i, or else $A_i=I_i-S_i$ contains only positive elements. In either case S_i is real, symmetric, and unitary.*

Matrices S of this kind occur as Hamiltonian matrices in quantum theory.

The referee pointed out that the matrix A_i must have rank 1 and hence be representable in the form $A_i=Z_iZ_i^*$, where Z_i is a vector every element of which is positive. This is clear from the following considerations. The equation $A_i^2=sA_i$ states that every column of the symmetric matrix A_i is a characteristic vector† of A_i with root s. Moreover, any two columns x, y must be proportional to a single vector Z_i. Otherwise a constant c would exist such that $w=x-cy$ is a nonnegative vector with some zero entries. The equations $A_ix=sx$, $A_iy=sy$ carry with them $A_iw=sw$, which would be impossible because every entry of the vector A_iw must be positive, since A_i is positive and w is nonnegative. Further facts concerning matrices of positive elements appear in [2] (Ch. III).

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1. K. Goldberg, The incidence equation $AA^T=aA$, this MONTHLY, vol. 67, 1960, p. 367.
2. F. R. Gantmacher, Applications of the Theory of Matrices, New York, 1959.

† Remark of Dr. C. M. Ablow.

A CONJECTURE OF ERDÖS CONCERNING CONSECUTIVE INTEGERS

W. R. UTZ, University of Missouri

According to a theorem of Sylvester [5] (also [4]), if k is a positive integer, then the product of k consecutive integers all greater than k is divisible by a prime greater than k. That is, for every positive integer k and $n>k$, the product $n(n+1)\cdots(n+k-1)$ is divisible by a prime $p>k$. If $f(k)$ denotes the least integer such that each product of $f(k)$ integers all greater than k has a prime factor greater than k, then the cited theorem of Sylvester states that $f(k)\leqq k$.

Erdös [2] has shown that for some real constant $c_1>1$, $f(k)\leqq c_1k/\log k$ and Rankin [3] has shown that for some real constant $c_2>0$,

$$f(k) > c_2 \log k(\log\log k)(\log\log\log\log k)/(\log\log\log k)^2.$$

It is clear that if k is composite, the value of $f(k)$ is that of $f(p)$, where p is the largest prime smaller than k. Obviously, $f(2)=2$. By the theorem of Sylvester, $f(3)\leqq 3$ and in view of the sequence 8, 9, we have $f(3)>2$; hence $f(3)=f(4)=3$.

To see that $f(5)=f(6)=4$, it is convenient to use the following theorem of Erdös and Turán [1]: *The sums* (a_i+b_j) *formed of the two sets*

$$0 < a_1 < \cdots < a_{k+1} \quad \textit{and} \quad 0 < b_1 < \cdots < b_s$$

cannot be composed of only k primes if $b_s>a_{k+1}^k$.

In this theorem, take $k=3$, $a_i=i$, $i=1, 2, 3, 4$; $s=1$ and we see that for all $b_1>4^3=64$,

$$b_1+1, \qquad b_1+2, \qquad b_1+3, \qquad b_1+4$$

cannot be made up of 3 primes, in particular not of the primes 2, 3, 5; hence, since the cases $b_1\leqq 64$ are easily disposed of by inspection, $f(5)\leqq 4$. However, because of the sequence 8, 9, 10, we have $f(5)>3$ to show that $f(5)=f(6)=4$.

Erdös [2] has conjectured that $f(7)=f(8)=f(9)=f(10)=4$ and $f(13)\geqq 6$. That $f(13)\geqq 6$ is easily verified by exhibiting the sequence 24, 25, 26, 27, 28 which shows that $f(13)>5$, or $f(13)\geqq 6$.

We now establish the remainder of the conjecture.

THEOREM. $f(7)=f(8)=f(9)=f(10)=4$.

Proof. We need only show that $f(7)=4$ and it is clear that $f(7)>3$ because of the sequence 14, 15, 16. Let $n>10$ and consider the quadruple n, $n+1$, $n+2$, $n+3$. Suppose that for some $n>10$ these four numbers are made up only of the primes 2, 3, 5, 7. At least three of these numbers must be free of 7 and of these three, at least two must be free of 5 and so at least two of the numbers must take the form 2^u3^v, 2^x3^y.

Clearly, $u\leqq 1$ or $x\leqq 1$. Without loss of generality we assume that $u\leqq 1$. Also, $v\leqq 1$ or $y\leqq 1$. In case $v\leqq 1$, the resulting numbers are 1, 2, 3, 6 since $u\leqq 1$ and the quadruple in which they appear cannot start with $n>10$. Thus, we take $y\leqq 1$ and $u\leqq 1$.

It is convenient to take $x \geqq 1$, $v \geqq 1$. The cases $x=0$ and $v=0$ are easily disposed of by inspection.

The cases that arise are

$$3^v \pm i = 2^x, \tag{1}$$

$$2(3^v) \pm i = 2^x, \tag{2}$$

$$3^v \pm i = (2^x)3, \tag{3}$$

$$2(3^v) \pm i = (2^x)3, \tag{4}$$

for $i=1, 2, 3$. If $i=1$, (2), (3), (4) are impossible and we secure, from (1), the possibility of

$$3^v \pm 1 = 2^x. \tag{5}$$

If $i=2$, (1), (3), (4) are impossible and (2) becomes (5). If $i=3$, (1), (2), (4) are impossible and (3) becomes (5). Thus, we need only solve (5).

One of the equations in (5) is

$$2^x = 3^v + 1 = (2^2 - 1)^v + 1 = 2^{2v} - \cdots \pm v2^2 \mp 1 + 1. \tag{5a}$$

If v is odd, $2^x = 2^2Q$, Q odd, and $x=2$, $v=1$, which yields a number too small for our quadruple. If v is even, $2^x=2Q$, Q odd; hence $x=1$, $v=0$, which also gives a number too small for our quadruple.

The other equation from (5) is

$$2^x = 3^v - 1 = (2^2 - 1)^v - 1 = 2^{2v} - \cdots \pm v2^2 \mp 1 - 1. \tag{5b}$$

If v is odd, $2^x=2Q$, Q odd, and $x=v=1$, which gives a number too small. If v is even,

$$2^x = 3^v - 1 = 2^{2v} - \cdots - v2^2.$$

Suppose $v=2^sR$, R odd, $2^x=2^{s+2}Q$, Q odd; hence $x=s+2$ and we have $v=2^{x-2}R$. Thus $v \geqq 2^{x-2} = \frac{1}{4}(3^v-1)$, which has solutions $v=0, 1, 2$, of which only $v=2$ is meaningful in this case and for which we have $x=3$. These values of x and v give numbers standing in sequences having prime factors greater than 7. This completes the proof of the theorem.

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THE ANTICENTERS OF ABELIAN GROUPS

PAUL D. HILL, Auburn University

In a recent note [2], N. Levine introduced the anticenter of a group and gave a few results, most of them for abelian groups, concerning this subgroup. The definition of anticenter given in [2] is equivalent to the following. The rim of a group G consists of those elements g in G with the property that $\{g, x\}$, the group generated by g and x, is cyclic for all x in the centralizer of g. The subgroup of G generated by the rim of G is defined as the anticenter and is denoted by $AC(G)$. Actually, if G is abelian, the anticenter coincides with the rim [2]. The present note gives complete information regarding anticenters of abelian groups.

THEOREM. *An abelian group G has zero anticenter unless G is (isomorphic to) a subgroup of the additive rationals, R, or is torsion. Moreover, a subgroup of R coincides with its anticenter; a torsion abelian group G decomposed into its p-primary components $G=\sum\oplus G_p$ has anticenter $AC(G)=\sum\oplus AC(G_p)$, where $AC(G_p)=G_p$ if G_p is cyclic or of type p^∞, and in all other cases $AC(G_p)=0$.*

Proof. Suppose that $AC(G)\neq 0$. Then G is either torsion or torsion free, for a nonzero element of G can not be contained in both a finite and infinite cyclic group. However, if G is torsion free, it is immediate that the rank of G does not exceed 1. Hence G is isomorphic to a subgroup of R [1] and $AC(G)=G$.

If G is torsion, then $AC(G)=AC(\sum\oplus G_p)=\sum\oplus AC(G_p)$, where G_p denotes the p-primary component of G. It is a simple exercise to show that $AC(G_p)=0$ if G_p is decomposable. But G_p is indecomposable if and only if G_p is cyclic or of type p^∞, and in this case $AC(G_p)=G_p$.

Added in proof: The above theorem also contains some of the main results of H. H. Johnson, *On the anticenters of a group*, this MONTHLY, vol. 68, 1961, pp. 469–472.

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A CONDITION FOR A GROUP TO BE COMMUTATIVE

TREVOR EVANS, Emory University

Let G be an additive groupoid, that is, a set and a binary operation with respect to which the set is closed. In [3], Frink "almost" said that if the sum of any two endomorphisms of G is an endomorphism, then G satisfies the identity (called the entropic law by Etherington):

$$(x+y)+(z+w)=(x+z)+(y+w) \qquad \text{for all } x, y, z, w.$$

The converse of this statement, that is, that this identity implies endomorphism closure in G, is easy to prove. In [1], Etherington gave examples to

ON THE PARITY OF SOME QUANTITIES RELATED TO THE EUCLIDEAN ALGORITHM

FRANK PAULSEN, Space General Corporation, AND BASIL GORDON, University of California, Los Angeles

Suppose x and y are positive integers with $x<y$ and $(x, y)=1$. As is well known, the fraction x/y has a unique continued fraction expansion

$$x/y = \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots \frac{1}{b_n} \qquad (b_n > 1),$$

which can be obtained from the Euclidean algorithm

$$\begin{aligned} y &= b_1 x + r_1, && 0 \leqq r_1 < b_1; \\ x &= b_2 r_1 + r_2, && 0 \leqq r_2 < r_1; \\ &\cdots\cdots && \cdots\cdots\cdots \\ r_{n-3} &= b_{n-1} r_{n-2} + r_{n-1}, && 0 \leqq r_{n-1} < r_{n-2}; \\ r_{n-2} &= b_n r_{n-1}. \end{aligned} \tag{1}$$

In this note we obtain simple criteria for the parity of n and of $B=\sum_{i=1}^{n} b_i$. Of course, parity questions concerning infinite continued fractions are of great importance in connection with the solvability of $u^2-Dv^2=-1$, but the much less difficult problem treated here seems to have been overlooked. Our main results are the following.

THEOREM 1. *Let $\bar{x}$ be the unique integer satisfying $0<\bar{x}<y$, $x\bar{x}\equiv 1 \pmod y$. Then n is odd or even according as $0<\bar{x}/y<\frac{1}{2}$ or $\frac{1}{2}\leqq \bar{x}/y<1$.*

THEOREM 2. *Suppose y is odd. Then $B\equiv 1+x+\bar{x} \pmod 2$. If y is even, define $\bar{y}$ to be the unique integer such that $0<\bar{y}\leqq x$, $y\bar{y}\equiv 1 \pmod x$. Then $B\equiv 1+\bar{y} \pmod 2$.*

Proof of Theorem 1. By induction on n. If $n=1$ then $x|y$, which implies $x=1$ since $(x, y)=1$. Hence $\bar{x}=1$ so that $0<\bar{x}/y\leqq\frac{1}{2}$. Assume the theorem true for $n-1$ and let x, y be such that the length of (1) is n. Then the induction hypothesis can be applied to the pair x, r_1, where r_1 is defined by (1). If n is odd, then $n-1$ is even, and so $\frac{1}{2}x<\bar{r}_1<x$, where $r_1\bar{r}_1\equiv 1 \pmod x$. Writing $r_1\bar{r}_1=1+kx$, we have $\frac{1}{2}r_1\leqq k<r_1$. From the identity $(y-\bar{r}_1b_1-k)x=1+(x-\bar{r}_1)y$, we see that $\bar{x}=y-\bar{r}_1b_1-k$. But $y-\bar{r}_1b_1-k>y-xb_1-r_1=0$ and $y-\bar{r}_1b_1-k<y-b_1(\frac{1}{2}x)-\frac{1}{2}r_1=\frac{1}{2}y$. Thus $0<\bar{x}<\frac{1}{2}y$. If n is even, then $n-1$ is odd and hence $0<\bar{r}_1<\frac{1}{2}x$. Again putting $r_1\bar{r}_1=1+kx$, we have $0\leqq k<\frac{1}{2}r_1$. Thus $y-\bar{r}_1b_1-k>y-b_1(\frac{1}{2}x)-\frac{1}{2}r_1=\frac{1}{2}y$, and $y-\bar{r}_1b_1-k<y$ so that $\frac{1}{2}y<\bar{x}<y$.

Proof of Theorem 2. Again by induction on n. If $n=1$, then $x=\bar{x}=1$, $\bar{y}=1$, and $B=b_1=y$. So if y is odd, then $B\equiv 1\equiv 1+x+\bar{x} \pmod 2$, while if y is even, then $B\equiv 0\equiv 1+\bar{y} \pmod 2$. Assume the theorem for $n-1$, and let x, y be a pair of integers with Euclidean algorithm (1). Induction can be applied to the pair x, r_1, and there are three cases to consider.

Case 1. y odd, x odd. Then $b_2+\cdots+b_n\equiv 1+r_1+\bar{r}_1 \pmod 2$, where $0<\bar{r}_1<x$, $r_1\bar{r}_1\equiv 1 \pmod x$. As in the proof of Theorem 1, $\bar{x}=y-\bar{r}_1b_1-k$ (where $r_1\bar{r}_1=1+kx$). Hence

$$\begin{aligned}1+x+\bar{x} &\equiv 1+1+b_1x+r_1-\bar{r}_1b_1-k\\ &\equiv b_1+r_1+\bar{r}_1b_1+1+r_1\bar{r}_1\equiv b_1+1+r_1+\bar{r}_1\equiv B \pmod 2.\end{aligned}$$

Case 2. y odd, x even. Then $b_2+\cdots+b_n\equiv 1+z \pmod 2$, where $0<z\leqq r_1$, $xz\equiv 1 \pmod{r_1}$. Putting $xz=1+mr_1$, we have $x(z+b_1)=my+1$, and $0<z+b_1<y$, so that $\bar{x}=z+b_1$. Hence $1+x+\bar{x}\equiv 1+\bar{x}\equiv b_1+\cdots+b_n \pmod 2$.

Case 3. y even. Then $b_2+\cdots+b_n\equiv 1+r_1+\bar{r}_1$. Defining k as before, we have $\bar{r}_1y=1+(b_1\bar{r}_1+k)x$, from which it follows that $\bar{y}=\bar{r}_1$. Also $0\equiv y\equiv b_1x+r_1\equiv b_1+r_1 \pmod 2$. Hence $1+y\equiv 1+\bar{r}_1\equiv 1+r_1+\bar{r}_1+b_1\equiv B \pmod 2$, completing the proof.

We note that two cases of Theorem 2 can be combined by writing $B\equiv 1+x+y+\bar{x}+\bar{y}+xy \pmod 2$.

In conclusion it should be mentioned that similar but more complicated criteria can be developed for the parity of other quantities, such as b_n and the higher elementary symmetric functions of $b_1,\cdots,b_n$.

THE SUM OF THE ANGLES IN AN n-DIMENSIONAL SIMPLEX*

L. CARLITZ, Duke University

Let s_k denote the sum of the angles at its $(n-k)$-cells of a simplex in spherical or Euclidean space and put

$$a_k=\binom{n+1}{k}^{-1}s_k.$$

Then the a_k satisfy the following relations:

$$a^r(1-a)^{n-r+1}=a^{n-r+1}(1-a^r) \qquad (0\leqq r\leqq \tfrac{1}{2}n), \tag{1}$$

$$(1-a)^r=a^r \qquad (1\leqq r\leqq 2[\tfrac{1}{2}n]+2), \tag{2}$$

$$(B+a)^r=(2B+a)^r \qquad (r \text{ even}, 2\leqq r\leqq n+2), \tag{3}$$

due to Sommerville, Hohn, and Peschl, respectively. (For references see [1].) In these formulas it is understood that after expansion by the binomial theorem a^k is replaced by a_k; also in (3), B^k is replaced by B_k, the Bernoulli number in the even suffix notation.

Guinand [1] has given a simple proof of the equivalence of the sets of equations (1), (2), (3).

The occurrence of the Bernoulli numbers in (3) seems somewhat surprising. We should like to point out that (3) can be replaced by a more general statement that does not involve Bernoulli numbers (see (8) below).

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We recall that ([2], p. 24)

$$B_k(x) - 2^k B_k(\tfrac{1}{2}x) = \tfrac{1}{2}kE_{k-1}(x), \tag{4}$$

where $B_k(x) = (x+B)^k$ and $E_{k-1}(x)$ is the Euler polynomial of degree $k-1$. Since (3) can be written as $B_r(a) = 2^r B_r(\frac{1}{2}a)$, it follows from (4) that (3) can be replaced by

$$E_r(a) = 0 \qquad (r \text{ odd, } 1 \leqq r \leqq n+1). \tag{5}$$

We have also ([2], p. 25)

$$E_r(x) = \sum_{s=0}^{r} \binom{r}{s} 2^{-s} E_s (x - \tfrac{1}{2})^{r-s},$$

where the E_s are the Euler numbers in the even suffix notation, so that $E_s = 0$ $(s = 1, 3, 5, \cdots)$. Consequently (5) is equivalent to

$$(a - \tfrac{1}{2})^r = 0 \qquad (r \text{ odd, } 1 \leqq r \leqq n+1). \tag{6}$$

But, as Guinand proved, (6) is equivalent to (2) and thus (5) is equivalent to (2).

It is clear from the above that if $f_r(x)$ denotes an arbitrary sequence of polynomials of odd degree r such that

$$f_r(1 - x) = -f_r(x), \tag{7}$$

then the set of equations

$$f_r(a) = 0 \qquad (r \text{ odd, } 1 \leqq r \leqq n+1) \tag{8}$$

is equivalent to (3). For if we put

$$f_r(x) = \sum_{s=0}^{r} c_{r,s}(x - \tfrac{1}{2})^{r-s}$$

then (7) implies

$$\sum_{s=0}^{r} c_{r,s}(x - \tfrac{1}{2})^{r-s} = -\sum_{s=0}^{r} c_{r,s}(\tfrac{1}{2} - x)^{r-s} \qquad (r \text{ odd}),$$

so that $c_{r,s} = 0$ when s is odd.

We remark that $E_k(x)$ satisfies $E_k(1-x) = (-1)^k E_k(x)$, so that (5) is included in (8). Since $B_k(1-x) = (-1)^k B_k(x)$, (8) also implies

$$B_r(a) = 0 \qquad (r \text{ odd, } 1 \leqq r \leqq n+1). \tag{9}$$

Thus (3) may be replaced by (9).

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CHARACTERISTIC ROOTS OF A MIXED DIFFERENCE-DIFFERENTIAL EQUATION

ROBERT D. LARSSON, Clarkson College of Technology

Pinney [1] has analyzed the characteristic equation of

$$y'(t) + ay'(t-1) + by(t-1) = w(t) \tag{1}$$

in some detail, and in particular has pointed out that a triple root can occur only when $a=e^{-2}$ and $b=4e^{-2}$. Furthermore, he has shown that this root is $z=-2$, and, in fact that all characteristic roots then have their real parts equal to -2,

$$z_{\pm p} = -2 \pm iy_p,$$

where $y_0=y_1=0$ and $y_2, y_3, \cdots$, are the successive nonzero roots of the equation $y=2\tan\frac{1}{2}y$.

The purpose of this paper is to give a method whereby it is possible to determine all cases where the real parts of all characteristic roots are equal; further, to give the value of such a real part, and the equation which will determine the imaginary parts.

The characteristic equation of (1) is, of course, $z+e^{-z}(az+b)=0$, which on writing $z=x+iy$ can be separated into

$$-xe^x = (ax+b)\cos y + ay\sin y, \tag{2}$$

$$-ye^x = -(ax+b)\sin y + ay\cos y. \tag{3}$$

These in turn can be written in the form

$$-xe^x = \sqrt{[(ax+b)^2 + a^2y^2]}\sin(y+v), \tag{4}$$

$$-ye^x = \sqrt{[(ax+b)^2 + a^2y^2]}\cos(y+v), \tag{5}$$

where $v=\arctan(ax+b)/ay$. Squaring (4) and (5) and adding we obtain

$$e^{2x}(x^2+y^2) = (ax+b)^2 + a^2y^2. \tag{6}$$

We wish to find values of a and b such that for $x=c$, (6) is satisfied by an infinite number of values of y, or geometrically that the curve given by (6) is intersected by the line $x=c$ in an infinite number of different points. There is no need to sketch the curve.

By inspection, if $x=c$ then $a^2=e^{2c}$ is the necessary condition, since the terms in y drop out. Therefore $a=\pm e^c$ and from (6) we obtain $b=\mp 2ce^c$.

To show that the conditions on a and b are sufficient we substitute their values and $x=c$ into the equations (2) and (3), obtaining

$$-c = \mp c\cos y \pm y\sin y, \tag{7}$$

$$-y = \pm c\sin y \pm y\cos y. \tag{8}$$

Now (7) and (8) must have identical solutions and therefore solving for y in (7) and (8) we find the condition

$$y = \frac{-c(\pm 1 - \cos y)}{\sin y} = \frac{-c \sin y}{\pm 1 + \cos y}.$$

But

$$\frac{\pm 1 - \cos y}{\sin y} \equiv \frac{\sin y}{\pm 1 + \cos y}$$

for all y. Therefore we have established the sufficiency for our conditions on a and b. But $\sin y/(1+\cos y) = \tan \frac{1}{2}y$ and thus

$$(9) \qquad y = \mp c(\tan \tfrac{1}{2}y)^{\pm 1}.$$

It is easily seen that Pinney's case of $c = -2$ is a special case; and as he has shown it is the only case of a triple root.

A double root will occur when

$$(10) \qquad -(1 + c) = \pm 1 \quad \text{and} \quad c^2 = \mp 2c,$$

since in [1] it is shown that the necessary and sufficient conditions for a double root are $a = -(1+c)e^c$ and $b = c^2e^c$.

Solutions of (10) are $c = -2$, which is the triple root, and $c = 0$. For $c = 0$ we have $a = -1$ and $b = 0$. Notice that this is the case for $a = -e^c$.

From (7) and (8), being careful to use the lower sign, we obtain $0 = -y \sin y$ and $-y = -y \cos y$. The possible solutions are $y = 0$ and $y = \pm 2n\pi$ and we have all the roots exactly.

The foregoing procedure may be used in many different mixed differential-difference equations.

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TWO REMARKS ON THE KANTOROVICH INEQUALITY*

PETER HENRICI, University of California, Los Angeles

L. V. Kantorovich [4] stated a result which is trivially equivalent to the following. Let $0 < m \leqq \mu_i \leqq M$, $\xi_i \geqq 0$ $(i = 1, \cdots, n)$, and suppose that $\xi_1 + \cdots + \xi_n = 1$. Then

$$(1) \qquad (\xi_1\mu_1 + \cdots + \xi_n\mu_n)(\xi_1\mu_1^{-1} + \cdots + \xi_n\mu_n^{-1}) \leqq (M + m)^2/(4Mm).$$

This inequality is of importance in discussions of the rate of convergence of methods of steepest descent for solving systems of equations; see [4] and [3]. A number of proofs, some of considerable complexity, of the inequality or of generalizations of it have been given [1, 2, 4, 6, 8, 10]. The present note contains: (i) an easy derivation of (1) from a result known in 1914; (ii) a short direct

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proof of (1), starting from first principles.

1. It is stated in [4] that (1) is a special case of an inequality given by Pólya and Szegö ([7], p. 57). While this is not strictly true (see the editorial comment in [5] and the introduction of [2]), we shall show that (1) is a simple consequence of the following special case of the inequality stated by Pólya and Szegö. If the numbers μ_i satisfy the same hypotheses as above, then

$$\frac{\mu_1 + \cdots + \mu_n}{n} \frac{\mu_1^{-1} + \cdots + \mu_n^{-1}}{n} \leqq \frac{(M+m)^2}{4Mm}. \tag{2}$$

This inequality (2) is due to P. Schweitzer [9].

It evidently suffices to establish (1) for all sets of *rational* ξ_i such that $\xi_i + \cdots + \xi_n = 1$. In order to deduce (1) from (2) for rational ξ_i, denote by $Q > 0$ a common denominator of the ξ_1 and let $\xi_i = P_i/Q$, where the positive integers P_i satisfy $P_1 + \cdots + P_n = Q$. By letting each μ_i appear P_i times in the sequence of the μ's, we can write the expression on the left of (1) in a form where all P_i are 1. But then the expression reduces to that on the left of (2) (with n replaced by Q), and the desired result follows immediately.

2. The following direct proof of (1) is modelled after the proof of the Pólya and Szegö inequality given in [7], pp. 213–214. Our proof permits an easy discussion of the case of equality in (1). There is essentially no simplification in the special case (2).

We may assume that $m < M$. Determine p_i and q_i from the equations

$$\mu_i = p_iM + q_im, \qquad \mu_i^{-1} = p_iM^{-1} + q_im^{-1}, \qquad i = 1, \cdots, n.$$

An easy computation shows that $p_i \geqq 0$, $q_i \geqq 0$, $i = 1, \cdots, n$. Furthermore from

$$1 = (p_iM + q_im)(p_iM^{-1} + q_im^{-1}) = (p_i + q_i)^2 + p_iq_i(M - m)^2/(Mm)$$

it follows that $p_i + q_i \leqq 1$. Setting $p = \sum \xi_ip_i$, $q = \sum \xi_iq_i$, we thus have $p + q = \sum \xi_i(p_i + q_i) \leqq \sum \xi_i = 1$. Hence, using the inequality of the arithmetic and geometric mean,

$$\begin{aligned}
&(\xi_1\mu_1 + \cdots + \xi_n\mu_n)(\xi_1\mu_1^{-1} + \cdots + \xi_n\mu_n^{-1}) \\
&\qquad = (pM + qm)(pM^{-1} + qm^{-1}) = (p+q)^2 + pq\frac{(M-m)^2}{Mm} \\
&\qquad \leqq (p+q)^2\left[1 + \frac{(M-m)^2}{4Mm}\right] = (p+q)^2\frac{(M+m)^2}{4Mm} \leqq \frac{(M+m)^2}{4Mm}.
\end{aligned}$$

Equality is attained in (1) if and only if the two following conditions are simultaneously fulfilled (we assume here $\xi_i > 0$, $i = 1, \cdots, n$ without loss of generality):

a) $p + q = 1$. This implies that $p_i + q_i = 1$ or $p_iq_i = 0$ for $i = 1, \cdots, n$. Thus, for equality *every* μ_i *must equal either* M *or* m.

b) $p + q = 4pq$. This implies that $p = q$ or, by a),

$$\sum_{\mu_i=m} \xi_i = \sum_{\mu_i=M} \xi_i.$$

Thus, *the weights attached to m and M must be the same.*

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* The author is indebted to A. S. Householder for calling his attention to the references [1], [6], and [8].

A NOTE ON THE SECULAR EQUATION OF THE PRODUCT OF TWO MATRICES

HARRY LASS AND CARLETON B. SOLLOWAY, Jet Propulsion Laboratory, California Institute of Technology

It is known that if A and B are square matrices of order n, then the secular equations for AB and BA are identical. In this note we show that if A is an $m\times n$ matrix and B is an $n\times m$ matrix, $m\geqq n$, with elements in the complex number field, then

$$|\lambda I_m - AB| = \lambda^{m-n}|\lambda I_n - BA| \tag{1}$$

with I_m and I_n unit matrices of orders m and n, respectively. Setting $\lambda=1$ it follows that $|I_m-AB|=|I_n-BA|$, and for $m>n$ it follows that $|AB|=0$, by setting $\lambda=0$.

First we give a simple proof of (1) for the case $m=n$. Let the zeros of $|\lambda I_n-BA|$ be distinct, say $\lambda_1, \cdots, \lambda_n$. If $\lambda=0$ is an eigenvalue of BA it is an eigenvalue of AB, since $|BA|=|AB|$. For $\lambda_1\neq 0$, $X_1\neq 0$, it follows from $BAX_1=\lambda_1X_1$ that $AX_1\neq 0$. Hence $AB(AX_1)=\lambda_1(AX_1)$, so that λ_1 is an eigenvalue of AB. Thus every eigenvalue of BA is an eigenvalue of AB, and (1) holds for $m=n$. If multiple zeros of $|\lambda I_n-BA|$ exist, one need only add small quantities to the elements of A and B such that the zeros of $|\lambda I_n-BA|$ separate and become distinct. Thus $|\lambda I_n-A(\epsilon)B(\epsilon)|=|\lambda I_n-B(\epsilon)A(\epsilon)|$ with $A(0)=A$, $B(0)=B$, and ϵ

represents a set of small elements. From continuity considerations it follows that (1) holds for $m=n$.

We consider next the case $m>n$, or $m=n+p$, $p>0$. Let α be the augmented matrix $\alpha=[A, \phi_1]$, with ϕ_1 the $m\times p$ null matrix, and let

$$\beta = \begin{bmatrix} B \\ \phi_2 \end{bmatrix},$$

with ϕ_2 the $p\times m$ null matrix, $\phi_1=\phi_2^T$. Thus, α and β are square matrices of order m. It follows that

$$(2) \qquad | \lambda I_m - \alpha\beta | = | \lambda I_m - \beta\alpha | .$$

One notes that

$$\alpha\beta = [A, \phi_1]\begin{bmatrix} B \\ \phi_2 \end{bmatrix} = AB, \qquad \beta\alpha = \begin{bmatrix} B \\ \phi_2 \end{bmatrix}[A, \phi_1] = \begin{bmatrix} BA & \phi_3 \\ \phi_4 & \phi_5 \end{bmatrix}.$$

Hence (2) becomes

$$| \lambda I_m - AB | = \begin{vmatrix} \lambda I_n - BA & \phi_3 \\ \phi_4 & \lambda I_p \end{vmatrix} = \lambda^{m-n} | \lambda I_n - BA |,$$

which concludes the proof of (1).

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

Material for this department should now be sent to J. M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

UNIQUE FACTORIZATION OF GAUSSIAN INTEGERS

WALTER RUDIN, University of Wisconsin

The usual proof of the unique factorization theorem in $R[i]$, the ring of all complex numbers of the form $m+ni$, where m and n are integers, depends on the existence of a Euclid algorithm in $R[i]$.* In the present note an elementary fact from plane geometry is exploited to yield a very simple and short proof of the theorem.

LEMMA. *Suppose C is a circle of radius r and Q is a square whose center lies on C and whose diagonal is not longer than $2r$. Then at least one vertex of Q lies in the interior of C.*

To see this, let t be the radius of Γ, the circle which passes through the vertices of Q. Let Γ' be the intersection of Γ with the interior of C. Since $r\geqq t$, the length of Γ' is at least one third of the circumference of Γ, and hence one or two vertices of Q lie on Γ'.

* See, for instance, W. J. LeVeque, Topics in Number Theory, vol. I, Reading, Mass., 1956.

The *units* of $R[i]$ are the numbers i^n ($n=0, 1, 2, 3$). For θ in $R[i]$, the four numbers $i^n\theta$ are the *associates* of θ, and θ is a *prime* if θ is not a unit and if θ is not the product of any two members of $R[i]$ neither of which is a unit. If α is in $R[i]$, $|\alpha|>1$, and α is not a prime, these definitions imply that $\alpha=\alpha'\alpha''$, where $|\alpha'|<|\alpha|$ and $|\alpha''|<|\alpha|$; finitely many repetitions of this process lead to a factorization of α into primes. This factorization is unique in the following sense:

THEOREM. *If $\theta_1, \cdots, \theta_r$ and $\phi_1, \cdots, \phi_s$ are primes in $R[i]$ and if $\theta_1 \cdots \theta_r = \phi_1 \cdots \phi_s$, then $r=s$ and the numbers ϕ_j can be so ordered that ϕ_j is an associate of θ_j ($1 \leqq j \leqq r$).*

Proof. Suppose the theorem is false. Since there are only finitely many elements of $R[i]$ in every bounded region of the complex plane, there exists α in $R[i]$ such that

(A) α has two *distinct* factorizations into primes

$$\alpha = \theta_1 \cdots \theta_r = \phi_1 \cdots \phi_s, \tag{1}$$

(B) no β in $R[i]$ with $|\beta|<|\alpha|$ has property (A).

Note that no θ_j in (1) is an associate of any ϕ_k, for otherwise α/θ_j would satisfy (A), in contradiction to (B). We may assume that $|\theta_1| \geqq |\phi_1|$. Applying the lemma to the square whose vertices are $\theta_1+i^n\phi_1$ ($n=0, 1, 2, 3$), we see that ϕ_1 has an associate, say ϕ_1^*, such that

$$|\theta_1 - \phi_1^*| < |\theta_1|. \tag{2}$$

Put

$$\beta = (\theta_1 - \phi_1^*)\theta_2 \cdots \theta_r. \tag{3}$$

By (2), $|\beta|<|\alpha|$. Since $\beta=\alpha-\phi_1^*\cdot\theta_2 \cdots \theta_r$, we see that ϕ_1 divides β. Since ϕ_1 is not an associate of any of the primes $\theta_2, \cdots, \theta_r$, (3) and (B) imply that ϕ_1 divides $\theta_1-\phi_1^*$. Hence ϕ_1 divides θ_1, and we have a contradiction.

It may be of interest to students to pinpoint just where the above proof breaks down in the ring of all numbers of the form $m+ni\sqrt{3}$ (to give just one example); in this ring, 4 has two distinct factorizations into primes:

$$4 = 2\cdot 2 = (1+i\sqrt{3})(1-i\sqrt{3}).$$

Note, however, that the unique factorization theorem does hold in the ring of all numbers of the form $m+n\theta$ if $2\theta=1+i\sqrt{3}$, and that it can be proved in the above manner (with regular hexagons in place of squares).

TAKING CONSECUTIVE HULLS

ALBERT WILANSKY, Lehigh University

1. (The setting for the first part of the article may be taken to be the Euclidean plane. It is actually valid for any linear topological space.)

The smallest closed convex set which includes a set A is the closure of the convex hull of A but not necessarily the convex hull of the closure. We tell our

PARTICULAR SOLUTIONS FOR SYSTEMS OF NONHOMOGENEOUS, LINEAR ORDINARY DIFFERENCE EQUATIONS

LOUIS C. BARRETT AND RICHARD A. JACOBSON, The South Dakota School of Mines and Technology

An earlier Classroom Note* explains how particular solutions of nonhomogeneous, linear, ordinary difference equations may be determined by utilizing a suitable form of Lagrange's identity. The present note extends the method and concepts there encountered to systems of such equations.

Define $\mathbf{y}_k=[y_1(k),\cdots, y_n(k)]$; $\mathbf{v}_k=[v_1(k),\cdots, v_n(k)]$; $A=[a_{ij}(k)]$; $i, j=1,\cdots, n$; and set

$$(1)\qquad \text{(a)}\quad L = D + A, \qquad\qquad (b)\quad \bar{L} = \bar{D} + A^T,$$

where D and $\bar{D}$ are the $n\times n$ diagonal matrices $[E]$ and $[E^{-1}]$, respectively. It follows that

$$(2)\qquad \mathbf{v}_kL\mathbf{y}_k^T = \Delta(\mathbf{v}_{k-1}\mathbf{y}_k^T) + \mathbf{y}_k\bar{L}\mathbf{v}_k^T.$$

The similarity of this identity and of the bilinear form $P=\mathbf{v}_{k-1}\mathbf{y}_k^T$ to the Lagrange identity and the bilinear concomitant of ordinary differential equation theory is at once evident. The operators (1) will be called adjoint difference matrix operators and the systems of equations

$$(3)\qquad \text{(a)}\quad L\mathbf{y}_k^T = \mathbf{0}, \qquad\qquad \text{(b)}\quad \bar{L}\mathbf{v}_k^T = \mathbf{0}$$

will be said to be adjoint systems of difference equations.

From (2) it is evident that any solution of (3b) makes $\mathbf{v}_kL\mathbf{y}_k^T$ an exact difference, whereas any solution of (3a) makes $\mathbf{y}_k\bar{L}\mathbf{v}_k^T$ exact.

Now, suppose that we are required to find a particular solution of the nonhomogeneous system

$$(4)\qquad L\mathbf{y}_k^T = \mathbf{f}_k^T, \quad \text{where}\quad \mathbf{f}_k = [f_1(k),\cdots, f_n(k)].$$

This may always be accomplished if a set of solutions

$$(5)\qquad {}_i\mathbf{v}_k = [v_{i1}(k),\cdots, v_{in}(k)], \qquad\qquad i = 1,\cdots, n,$$

of (3b) is known such that $V=[v_{ij}(k)]$; $i, j=1,\cdots, n$; is nonsingular and such a set may always be found by use of (2) if the general solution of (3a) is known.

Assuming, then, that solutions (5) have been found, we substitute $\bar{L}_i\mathbf{v}_k^T=\mathbf{0}$ and $Ly_k^T=f_k^T$ into (2) and subsequently take indefinite sums over k to obtain,

$$(6)\qquad {}_i\mathbf{v}_{k-1}\mathbf{y}_k^T = \sum_k {}_i\mathbf{v}_k\mathbf{f}_k^T, \qquad\qquad i = 1,\cdots, n.$$

Since V is nonsingular, a particular solution $\mathbf{Y}_k$ of (4) may be determined by

* L. C. Barrett and F. Dristy, Particular solutions for nonhomogeneous, linear, ordinary difference equations, this MONTHLY, vol. 67, 1960, pp. 71–73.

solving the linear algebraic system (6).

Example. The adjoint of the homogeneous system corresponding to

$$\begin{bmatrix} y_1 & (k+1) \\ y_2 & (k+1) \end{bmatrix} + \begin{bmatrix} -2^{k-1}-1 & -1 \\ 2^{2k-1} & 2^k-2 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 \\ 2^k \end{bmatrix} \tag{7}$$

is

$$\begin{bmatrix} v_1 & (k-1) \\ v_2 & (k-1) \end{bmatrix} + \begin{bmatrix} -1-2^{k-1} & 2^{2k-1} \\ -1 & 2^k-2 \end{bmatrix} \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for which two linearly independent solutions are

$$\mathbf{v}_1 = [2^k - 1,\ 1], \qquad \mathbf{v}_2 = [1,\ 2^{-k}].$$

Utilizing (6) we obtain

$$(2^{k-1}-1)y_1 + y_2 = \sum_{n=1}^{k-1} (2^n - 1 + 2^n) = 2^{k+1} - k - 3,$$

$$y_1 + 2^{1-k}y_2 = \sum_{n=1}^{k-1} (1+1) = 2(k-1),$$

where the constants of summation have been omitted since we require merely any particular solution of (7). Solution of these equations for y_1 and y_2 leads to the particular solution

$$\mathbf{Y}_k = [(k-3)2^k + k + 3,\ (k-5)2^{k-1} - (k-3)2^{2k-1}].$$

ACUTE ISOSCELES DISSECTION OF AN OBTUSE TRIANGLE

VERNER E. HOGGATT, JR., San Jose State College, AND RUSS DENMAN,
Lockheed Missiles Space Division, Sunnyvale, California

Introduction. The following problem appeared as problem E 1406:* Cut an obtuse triangle into the least number of acute triangles. Here we give the construction for the dissection of an obtuse triangle into at most eight acute isosceles triangles.

A fundamental construction. From Figure 1, let $B > 90°$ and draw a circle with center at I, the incenter of ΔABC, and passing through vertex B. The following observations are easy to establish:

1. $\Delta C'IB$, $\Delta A'IB$, and $\Delta A''IC''$ are isosceles and congruent with central angles at I of $180° - B < 90°$.

* Problem E 1406 was proposed by Michael Goldberg and the solution by Wallace Manheimer appeared in this MONTHLY, vol. 67, 1960, p. 923. There were more related results in an editorial comment. The given problem was proposed by Martin Gardner in "Mathematical Games," Scientific American, vol. 202, Feb., 1960, p. 150, and a dissection into seven parts appeared in the March 1960 issue, pp. 177–178, but without proof.

2. $\Delta AA'A''$ and $\Delta CC'C''$ are isosceles with nonequal angles A and C, respectively, each $<90°$.

3. $\Delta A'IA''$ and $\Delta C'IC''$ are isosceles with central angles at I of $B-A$ and $B-C$, respectively.

Thus, if $B-A<90°$ and $B-C<90°$, the above construction yields a seven-piece acute-isosceles-triangle dissection of an obtuse triangle.

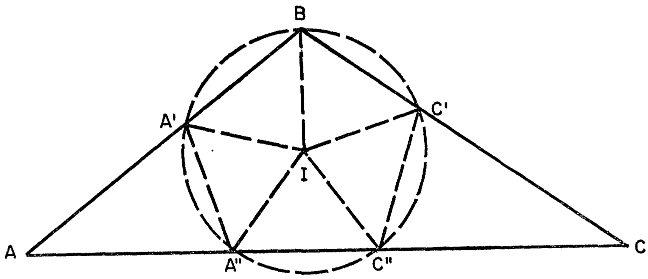

FIG. 1

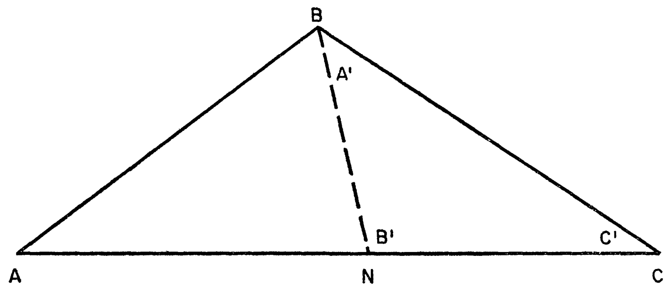

FIG. 2

An important condition. If $0<A\leqq C<90°<B<180°$, then, in Figure 2, ΔBAN is isosceles and acute. In ΔBNC the following are easy to establish: 1. $\measuredangle B'=\measuredangle N=90°+\frac{1}{2}A$; $\measuredangle C'=\measuredangle C$; 3. $\measuredangle A'=\measuredangle CBN=90°-\frac{1}{2}A-C$. Thus in $\Delta BNC=\Delta A'B'C'$, $B'-C'=90°+\frac{1}{2}A-C<90°$ since $C\geqq A$; and $B'-A'=90°+\frac{1}{2}A-(90°-\frac{1}{2}A-C)=A+C<90°$ since $A+B+C=180°$ and $B>90°$.

Therefore, either ΔABC has $B>90°$, $B-A<90°$, $B-C<90°$, and the fundamental construction yields a seven-piece dissection; or ΔABC is ΔBAN and $\Delta A'B'C'$, where ΔBAN is isosceles and acute and $B'>90°$, $B'-C'<90°$, $B'-A'<90°$ so that the fundamental construction yields a seven-piece dissection of $\Delta A'B'C'$. We have thus established the

THEOREM. *An obtuse triangle can be dissected into eight acute isosceles triangles. If $B>90°$, $B-A<90°$, and $B-C<90°$, only seven are needed.*

Question. Are eight isosceles acute triangles necessary for the general obtuse triangle?

THE i-CENTROID OF AN n-SIMPLEX

BEN CARTER, Stanford University

This paper generalizes the following results: *The centroid of a triangle divides each median in the ratio* 2:1, *and the centroid of the edges of a triangle is the incenter of the medial triangle.**

Let $R=\bigcup_{i=1}^{m} S_i$ be the union of m k-simplices in E^n. For $k=0$, each S_i is a point $x_i=(x_i^1, \cdots, x_i^n)$ and the centroid of R is the point $\bar{x}(R)=(\sum_{i=1}^{m} x_i)/m$. For $k>0$, the centroid is $\bar{x}(R)=(\int_R x dv)/V(R)$, where dv is the k-dimensional volume element and $V(R)=\int_R dv$ is the volume of R. If R' is another union of k-simplices such that $R\cap R'$ is of dimension $<k$, then

$$\bar{x}(R\cup R') = \frac{V(R)\bar{x}(R) + V(R')\bar{x}(R')}{V(R) + V(R')} \tag{1}$$

lies on the line segment joining $\bar{x}(R)$ and $\bar{x}(R')$.

DEFINITION. *The i-centroid of a k-simplex is the centroid of the union of its* $\binom{k+1}{i+1}$ *i-faces.*

THEOREM 1. *The* 0*-centroid is the k-centroid of a k-simplex; it lies on each median* (*line segment joining a vertex to the centroid of the opposite* $(k-1)$*-face*) *and divides each median in the ratio* $1:k$.

Proof. Let S be a k-simplex with vertices $p_0, p_1, \cdots, p_k$, and let B be the face opposite p_0. Choose a coordinate system $x^0, \cdots, x^{k-1}$ spanning S with the x^0-axis perpendicular to B and the origin at p_0. For any x in B, $x^0=a$, where a is a constant assumed positive. The vertices of B are $p_i=(a, p_i^1, \cdots, p_i^{k-1})$. The 0-centroid of S is $(p_0+\cdots+p_k)/(k+1)$ which is $k/(k+1)$ times the 0-centroid of B. We shall see that the k-centroid of S bears the same relationship to the $(k-1)$-centroid of B. Hence, if the $(k-1)$-centroid of B is the 0-centroid of B, then the k-centroid of S is the 0-centroid of S. The case $k=1$ is trivial so that the theorem will follow by induction.

If $0<t\leqq a$, the $(k-1)$-space $x^0=t$ intersects S in a $(k-1)$-simplex B' whose vertices are $(t/a)p_1, \cdots, (t/a)p_k$. Thus x is in B iff $y=(t/a)x$ is in B', and

$$\begin{aligned}\int_S x dv &= \int_S x dx^1 \cdots dx^{k-1}dx^0 = \int_0^a dt \int_{B'} y dy^1 \cdots dy^{k-1}\\ &= \int_0^a dt \int_B \frac{t}{a} x d\left(\frac{t}{a}x^1\right) \cdots d\left(\frac{t}{a}x^{k-1}\right) = \int_0^a \left(\frac{t}{a}\right)^k dt \int_B x dx^1 \cdots dx^{k-1}\\ &= \frac{a}{k+1}\int_B x dx^1 \cdots dx^{k-1}.\end{aligned} \tag{2}$$

* For definitions see, *e.g.*, N. A. Court, College Geometry, 1925.

Similarly,

$$\int_S dv = \frac{a}{k}\int_B dx^1 \cdots dx^{k-1}. \tag{3}$$

Dividing (2) by (3) we find that $\bar{x}(S) = \{k/(k+1)\}\bar{x}(B)$, *i.e.*, $\bar{x}(S)$ lies on the median between $p_0 = (0, \cdots, 0)$ and $\bar{x}(B)$, k times as far from p_0 as from $\bar{x}(B)$.

THEOREM 2. *The i-centroid of a k-simplex can be constructed by ruler and transferrer-of-segments.*

Proof. Let $S_1, \cdots, S_m$ be the i-faces of a k-simplex. They intersect by pairs in simplices of dimension $<i$ (when they intersect at all); hence by repeated use of (1),

$$\bar{x}(S_1 \cup \cdots \cup S_m) = \frac{V(S_1)\bar{x}(S_1) + \cdots + V(S_m)\bar{x}(S_m)}{V(S_1) + \cdots + V(S_m)}. \tag{4}$$

Each S_j is an i-simplex $(1 \leqq j \leqq m = \binom{k+1}{i+1})$. If an i-simplex S has vertices $p_0, \cdots, p_i$, then by Theorem 1, $\bar{x}(S)$ has coordinates

$$\bar{x}^r(S) = (p_0^r \cdots p_i^r)/(i+1), \qquad r = 1, \cdots, k. \tag{5}$$

The distance from p_u to p_v is

$$a_{uv} = \sqrt{\{(p_u^1 - p_v^1)^2 + \cdots + (p_u^k - p_v^k)^2\}}, \tag{6}$$

in terms of which*

$$V^2(S) = (-1)^i 2^{i-1}(i-1)!^2 \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & a_{01}^2 & \cdots & a_{0i}^2 \\ 1 & a_{10}^2 & 0 & \cdots & a_{1i}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{i0}^2 & a_{i1}^2 & \cdots & 0 \end{vmatrix}. \tag{7}$$

Using (4), (5), (6), (7), the coordinates of the i-centroid can be written in terms of those of the vertices of the k-simplex using only rational operations and square roots of sums of squares.†

The construction implied by the above equations would be difficult to describe in terms of ruler and transferrer-of-segments operations; a better method is described by the following two constructions.

Construction 1. To construct the centroid of the union of two k-simplices

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* D. M. Sommerville, An Introduction to the Geometry of n Dimensions, New York, 1958, pp. 124–125.

† The equivalence of ruler and transferrer-of-segments operations to those producing points whose coordinates are functions of given points involving rational functions and square roots of sums of squares is discussed in Hilbert's Foundations of Geometry for the two-dimensional case.

which have a common $(k-1)$-face and span E^{k+1}: Consider the perpendicular projection $f\colon E^{k+1}\to E^2$ obtained by dropping perpendiculars to a plane E^2 which is perpendicular to the common face B of the two k-simplices S, S'. Since S and S' are not contained in any E^k, their separate vertices (those not in B) v and v' are sent into points not collinear with $f(B)$. The centroids $m=\bar{x}(v)$, $m'=\bar{x}(v')$ are sent into points on $f(v)f(B)$, $f(v')f(B)$, respectively. Construct the parallelogram with $f(B)f(m)$ and $f(B)f(m')$ as adjacent sides (Fig. 1). The angle bisector of the newly formed vertex of the parallelogram intersects $f(m)f(m')$ at some point p. By an elementary theorem on angle bisectors,

$$\frac{pf(m')}{pf(m)}=\frac{f(B)f(m)}{f(B)f(m')}=\frac{f(B)f(v)}{f(B)f(v')}=\frac{V(S)}{V(S')}\,.$$

$f^{-1}(p)$ is a $(k-1)$-space which intersects m' in a point which divides $m'm$ in the ratio $V(S):V(S')$. By (1), this point is the centroid of $S\cup S'$.

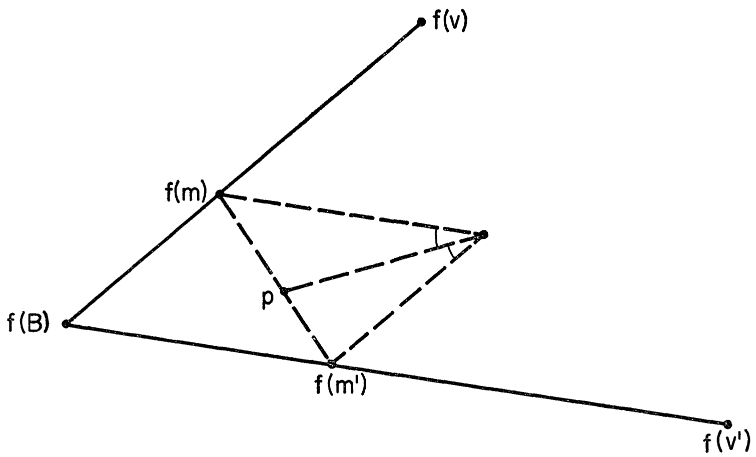


FIG. 1

Construction 2. The i-centroid of an n-simplex. Denote the vertices of the n-simplex by $1,\cdots,n+1$. Let $(1\cdots k;\,k+1\cdots n+1)$ denote the i-centroid of the union of all i-simplices which contain the vertices $1,\cdots,k$. Each i-simplex which contains $1,\cdots,k$ either contains $k+1$ or is a subsimplex of the $(n-1)$-simplex $1,\cdots,k,k+2,\cdots,n+1$. Hence for $k\leqq i$, $(1\cdots k;\,k+1\cdots n+1)$ is on the line joining $(1\cdots k+1;\,k+2\cdots n+1)$ and $(1\cdots k;\,k+2\cdots n+1)$. One such line may be drawn for each of the vertices $k+1,\cdots,n+1$ unless $n=k+1$, in which case there is only one line. If $n\geqq k+2$, the intersection of the lines is the desired point. Thus if $P(k,n)$ is the proposition that $(1\cdots k;\,k+1\cdots n+1)$ is constructible, then for $k\leqq i$ and $n-2\geqq k$,

$$P(k+1,n)\quad\text{and}\quad P(k,n-1)\quad\text{imply}\quad P(k,n). \tag{8}$$

$P(k, n)$ can be proved by induction on n and k: (i) If $n=i$, $P(k, n)$ from Theorem 1. (ii) If $n=i+1$, $P(i+1, n)$ from Theorem 1, $P(i, n)$ from Construction 1, and $P(k, n)$ from $P(k+1 \cdot n)$, (i), and (8) for $0 \leqq k \leqq i-1$. (iii) If $n \geqq i+2$, assume $P(k, n-1)$ for all $k=0, 1, \cdots, i+1$. $P(i+1, n)$ from Theorem 1, $P(k, n)$ from $P(k+1, n)$, $P(k, n-1)$, and (8) for $0 \leqq k \leqq i$.

The desired construction is $P(0, n)$.

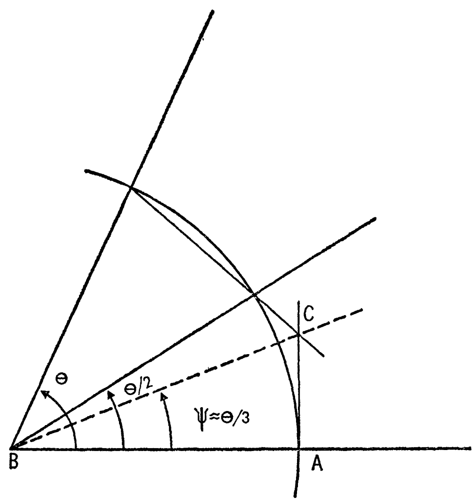
AN APPROXIMATE TRISECTION

WILLIAM R. RAIFORD, IBM, Washington, D.C.

The simple ruler-and-compass constructions indicated in the figure give an approximate trisection of an angle $0° \leqq \theta \leqq 240°$. We have

$$\frac{AC}{AB} = \tan \psi = \cot \tfrac{3}{4}\theta(\cos \theta - 1) + \sin \theta.$$

The error, $\psi - \frac{1}{3}\theta$, computed on the IBM 709 for each degree, indicates that it is monotonic increasing, is $0°21'40''$ at $\theta = 90°$, is $3°26'6''$ at $\theta = 180°$, and is $10°0'0''$ at $\theta = 240°$.



DERIVATION OF A FACTORIAL FUNCTION BY METHOD OF ANALOGY

T. A. NEWTON, Washington State University

In a recent article by Davis [2], we find a nice example (the derivation of formula (15) of [2]) of Euler's use of "naive questioning" and "uninhibited play with symbols" as a means of creative thinking. An excellent example of a related type of reasoning, the inductive approach, is given by Long in [5]. We will in this paper give an example of the method of analogy, a method which has long been used to discover new and interesting results. Furthermore, our example will

result in an infinite series expansion of a factorial function which satisfies the same difference relation as Hadamard's factorial function mentioned by Davis ([2], p. 865).

We shall exploit those well-known analogies which exist between the differential-integral calculus and the calculus of finite differences. Corresponding to the symbols D_x, $\int$, and $\int_a^b$ of the differential-integral calculus, we have the analogous symbols Δ (where $\Delta a_n = a_{n+1} - a_n$), Δ^{-1}, and $\sum_a^b$ respectively. Also, the functions of n, $n^{(j)} = n(n-1) \cdots (n-j+1)$, $j = 1, 2, \cdots$, and 2^n have properties relative to the operator Δ analogous to properties that the functions x^j and e^x have relative to the operator D_x. That is

$$\Delta n^{(j)} = jn^{(j-1)} \tag{1}$$

for $j = 1, 2, \cdots$, and $\Delta 2^n = 2^n$.

We now recall the integral definition of the gamma function,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{2}$$

Using the method of analogy to construct the new function for investigation, we would replace in (2): $\int_0^\infty$ by $\sum_0^\infty$; t by n; e by 2; t^{z-1} by $n^{(z-1)}$. However, we obtain a richer result if we replace e^{-t} by w^{n+1}. As a result of our reasoning, we define

$$H(z, w) = \sum_{n=0}^{\infty} n^{(z-1)} w^{n+1}. \tag{3}$$

Thus far, $H(z, w)$ is defined by (3) only for positive integral values of z. We further extend our definition by defining

$$n^{(z)} = \Gamma(n+1)/\Gamma(n-z+1) \tag{4}$$

for $n = 0, 1, \cdots$, and complex z. In particular, $n^{(0)} = 1$ for $n = 0, 1, \cdots$, and $0^{(z)} = 1/\Gamma(1-z)$. A discussion of this definition for all real z is found in [1]. For proof that $1/\Gamma(z)$ is an entire analytic function of the complex variable z and has zeros at $z = 0, -1, -2, \cdots$, see page 440 of [4]. This definition of $n^{(z)}$ still satisfies difference equation (1) with j replaced by z. The terms of the series in (3) are now defined for all complex z.

By repeated application of the recurrence relation for the gamma function, $\Gamma(z+1) = z\Gamma(z)$, we can rewrite (3) as

$$H(z, w) = \frac{1}{\Gamma(1-z)} \sum_{v=0}^{\infty} \frac{v! w^{v+1}}{(1-z)(1-z+1) \cdots (1-z+v)}. \tag{5}$$

Referring to pages 177–180 of [3], one recognizes this series to be a *factorial* series. The convergence region for such series has been extensively dealt with in [3], however it is quite easy for us to determine for ourselves the convergence region of (3).

Let $a_n(z) = n^{(z)}w^{n+1}$. Then

$$\frac{|a_{n+1}(z)|}{|a_n(z)|} \leqq \frac{|w|}{|1 - |z|/(n+1)|}.$$

Let $\delta > 0$ be given. If $|z| \leqq \delta$ and $n \geqq \delta$, then $|1 - |z|/(n+1)| \geqq |1 - \delta/(n+1)|$ and hence

$$\frac{|a_{n+1}(z)|}{|a_n(z)|} \leqq \frac{|w|}{|1 - \delta/(n+1)|}.$$

Finally, it follows that for any w such that $|w| < 1$, if we choose any r such that $|w| < r < 1$, then an $N > 0$ can be determined such that $n \geqq N$ implies that $|a_{n+1}(z)|/|a_n(z)| \leqq r < 1$ for all z in the circle $|z| \leqq \delta$. Consequently for each w such that $|w| < 1$, series (3) converges uniformly over any circle $|z| \leqq \delta$. It now follows that for each w with $|w| < 1$, $H(z, w)$ is analytic everywhere in the finite complex z-plane.

In the usual development of the gamma function which begins with the integral definition (2), integration by parts is applied to give the recurrence relation $\Gamma(z+1) = z\Gamma(z)$. By analogy, we shall apply the summation by parts formula

$$\sum_{n=k}^{m-1} u_n \Delta v_n = (u_m v_m - u_k v_k) - \sum_{n=k}^{m-1} v_{n+1} \Delta u_n. \tag{6}$$

We first note that

$$w^{n+1} = \frac{w}{w-1} \Delta w^n,$$

whence

$$\sum_{n=0}^{m-1} n^{(z)} w^{n+1} = \frac{w}{w-1} \sum_{n=0}^{m-1} n^{(z)} \Delta w^n.$$

Applying (6) to the sum on the right, it follows that

$$\sum_{n=0}^{m-1} n^{(z)} w^{n+1} = \frac{w}{w-1} \left[m^{(z)} w^m - 1/\Gamma(1-z) - z \sum_{n=0}^{m-1} n^{(z-1)} w^{n+1} \right]. \tag{7}$$

Since $\sum n^{(z)} w^n$ converges for $|w| < 1$, $\lim_{m \to \infty} m^{(z)} w^m = 0$, and hence if we pass to the limit in (7), it follows that

$$H(z+1, w) = \frac{w}{1-w} [1/\Gamma(1-z) + zH(z, w)]. \tag{8}$$

Now consider positive integral values for z, say $z = m = 1, 2, \cdots$. In this case, $1/\Gamma(1-m) = 0$, and

$$H(m+1, w) = \frac{w}{1-w} mH(m, w)$$

from which it follows that

$$H(m+1, w) = \frac{w^k}{(1-w)^k} m^{(k)} H(m-k+1, w)$$

for $k=1, \cdots, m$. If $k=m$, then $m^{(m)}=m!$, $H(1, w)=w/(1-w)$ and

$$H(m+1, w) = \frac{w^{m+1}}{(1-w)^{m+1}} m!$$

Consequently, $H(m+1, \frac{1}{2})=m!=\Gamma(m+1)$ for $m=1, 2, \cdots$.

Returning to (8), we see that for complex z,

$$H(z+1, \tfrac{1}{2}) = 1/\Gamma(1-z) + zH(z, \tfrac{1}{2}),$$

this being precisely the functional equation satisfied by Hadamard's factorial function ([2], p. 865). Our solution of this equation, $H(z, \frac{1}{2})$, is also an entire function which coincides with the gamma function at the positive integers. However, in conclusion we shall show that $H(\frac{1}{2}, \frac{1}{2}) \neq \Gamma(\frac{1}{2})$.

Letting $z=\frac{1}{2}$ in (5), using the fact that $\Gamma(\frac{1}{2})=\sqrt{\pi}$ (set $z=\frac{1}{2}$ in formula (24) of [2]) and simplifying, it follows that

$$H(\tfrac{1}{2}, w) = \frac{1}{\sqrt{\pi}} \sum_{v=0}^{\infty} \frac{v!}{1\cdot 3 \cdots (2v+1)} (2w)^{v+1}. \tag{9}$$

It can be shown (exercise 123, p. 271, [4]) that

$$\frac{\sin^{-1} w}{w\sqrt{(1-w^2)}} = \sum_{v=0}^{\infty} \frac{v!}{1\cdot 3 \cdots (2v+1)} (2w^2)^v,$$

this series converging for $|w|<1$. Substituting into (9), we get

$$H(\tfrac{1}{2}, w^2) = \frac{2w\cdot \sin^{-1} w}{\sqrt{\pi}\sqrt{(1-w^2)}}$$

and, letting $w=1/\sqrt{2}$, $H(\frac{1}{2}, \frac{1}{2})=\frac{1}{2}\sqrt{\pi}=\frac{1}{2}\Gamma(\frac{1}{2})$.

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ON A GENERALIZATION OF THE FACTOR THEOREM

DAVID ZEITLIN, Remington Rand Univac, St. Paul, Minnesota

Hillman [1], using a simple property of finite differences, obtained a necessary and sufficient condition for a polynomial $p(x)$ to be divisible by $(x-r)^m$. Since $p(x)$ is divisible by $(x-r)^m$ if and only if $p(rx)$ is divisible by $(x-1)^m$, only the case $r=1$ was considered. The theorem may be stated as follows:

THEOREM 1. *A polynomial* $p(x)=\sum_{k=0}^{n} a_{n-k}x^k$ *is divisible by* $(x-1)^m$ *if and only if*

$$\sum_{k=0}^{n} a_{n-k}(k+1)^s = 0, \qquad s=0,1,\cdots,m-1. \tag{1}$$

In this note, we will establish the result as follows:

THEOREM 2. *A polynomial* $p(x)=\sum_{k=0}^{n} a_{n-k}x^k$ *is divisible by* $(x-1)^m$ *if and only if*

$$D^s p(1) \equiv \sum_{k=0}^{n} k^{(s)} a_{n-k} = 0, \qquad s=0,1,\cdots,m-1, \tag{2}$$

where $k^{(s)}=k(k-1)(k-2)\cdots(k-s+1)=\sum_{j=1}^{s} S_s^j k^j$, $k^{(0)}=1$, S_s^j [2] *are Stirling numbers of the first kind, and* $D^s p(1)=[D^s p(x)]_{x=1}$.

Necessity. Let $p(x)=q(x)(x-1)^m$, where $q(x)$ is a polynomial in x of degree $n-m$. Using Leibnitz's rule, we have

$$D^s p(x) = \sum_{j=0}^{s}\binom{s}{j} D^j(x-1)^m D^{s-j}q(x) = \sum_{j=0}^{s}\binom{s}{j} m^{(j)}(x-1)^{m-j}D^{s-j}q(x). \tag{3}$$

For $x=1$ and $s=0,1,\cdots,m-1$, (3) yields (2).

Sufficiency. Since $p(x)=\sum_{s=0}^{n}[D^s p(1)](x-1)^s/s!$, we find, using (2), that

$$p(x) = \sum_{s=m}^{n}[D^s p(1)](x-1)^s/s! = (x-1)^m q(x),$$

where $q(x)=\sum_{j=0}^{n-m}[D^{j+m}p(1)](x-1)^j/(j+m)!$

We now state the main result:

THEOREM 3. *Let* $p(x)=\sum_{k=0}^{n} a_{n-k}x^k$. *Then* (1) *is true if and only if* (2) *is true.*

Sufficiency. Since $k^r=\sum_{i=0}^{r}\mathfrak{S}_r^i k^{(i)}$, where $\mathfrak{S}_r^i$ [2] is a Stirling number of the second kind, $\mathfrak{S}_r^0=0$, $r>0$, and $\mathfrak{S}_0^0=1$, we have

$$\sum_{k=0}^{n} a_{n-k}(k+1)^s = \sum_{k=0}^{n} a_{n-k}\sum_{r=0}^{s}\binom{s}{r}k^r$$

$$= \sum_{r=0}^{s} \binom{s}{r} \sum_{k=0}^{n} a_{n-k} k^r = \sum_{r=0}^{s} \binom{s}{r} \sum_{k=0}^{n} a_{n-k} \sum_{i=0}^{r} \mathfrak{S}_r^i k^{(i)},$$

$$= \sum_{r=0}^{s} \binom{s}{r} \sum_{i=0}^{r} \mathfrak{S}_r^i \left\{ \sum_{k=0}^{n} k^{(i)} a_{n-k} \right\} = 0$$

for $s=0, 1, \cdots, m-1$.

Necessity. Since $S_0^0=1$ and $S_s^0=0$ for $s>0$, we have

$$\sum_{k=0}^{n} k^{(s)} a_{n-k} = \sum_{k=0}^{n} a_{n-k} \sum_{j=0}^{s} S_s^j (k+1-1)^j$$

$$= \sum_{k=0}^{n} a_{n-k} \sum_{j=0}^{s} S_s^j \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r} (k+1)^r$$

$$= \sum_{j=0}^{s} S_s^j \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r} \left\{ \sum_{k=0}^{n} a_{n-k} (k+1)^r \right\} = 0$$

for $s=0, 1, \cdots, m-1$. This completes the proof of Theorem 3, which has the following generalizations:

THEOREM 4. *Let $r \neq 0$, c, and b_k, $k=0, 1, \cdots, n$, be constants. Then*

$$\sum_{k=0}^{n} b_k r^k (k+c+1)^s = 0, \qquad s = 0, 1, \cdots, m-1, \tag{4}$$

if and only if

$$\sum_{k=0}^{n} b_k r^k (k+c)^{(s)} = 0, \qquad s = 0, 1, \cdots, m-1. \tag{5}$$

THEOREM 5. *For $f(x) \not\equiv 0$ and c, a constant,*

$$\int_a^b (x+c+1)^s f(x)dx = 0, \qquad s = 0, 1, \cdots, m-1, \tag{6}$$

if and only if

$$\int_a^b (x+c)^{(s)} f(x)dx = 0, \qquad s = 0, 1, \cdots, m-1. \tag{7}$$

The proofs of Theorems 4 and 5 are similar to the proof of Theorem 3, and, therefore, are omitted. For $r=1$, $c=0$, and $b_k=a_{n-k}$, Theorem 4 yields Theorem 3.

References

1. A. P. Hillman, A generalization of the factor theorem, this MONTHLY, vol. 67, 1960, pp. 777–778.

2. C. Jordan, Calculus of Finite Differences, New York, 1950, p. 142, p. 168.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D.C.

UNDERGRADUATE TRAINING FOR GRADUATE STUDY

ANDREW M. GLEASON, Harvard University

The Ph.D. degree in any subject represents two kinds of attainment, a professional knowledge of the discipline as a whole and the ability to advance the subject itself through research. Correspondingly, the candidate for the degree must usually negotiate two hurdles, the general examination and the thesis. The thesis requirement is basically the same everywhere, but the general examination varies tremendously from subject to subject and from university to university. In mathematics at Harvard, to be specific, the general examination is normally taken at the end of two years of graduate study and covers real and complex variable theory, advanced calculus, differential equations, algebra, geometry and topology.

The requirements for a major in mathematics also vary widely from one institution to the next, but probably all impose some restriction on about one-third of a student's undergraduate work. The minimum requirement for the major at Harvard is one-fourth of one's work in mathematics itself and an additional one-eighth in mathematics or a related discipline; most of the approved related courses are in physics.

For purposes of discussion let us imagine a program for a student majoring in mathematics. In each of his first two years he will take one course in mathematics covering one of the standard texts in elementary calculus. As a Junior he will take advanced calculus and a modern algebra course. The Senior year he chooses four half-year courses from such fields as differential equations, mechanics, probability and statistics, number theory, geometry, and real or complex variables. (I am excluding the student at a university college who is able to elect a graduate course. He, after all, is merely starting his graduate training early.) At some time during his four years, he will probably take at least one course in physics.

If we appraise this hypothetical program as direct preparation for a Ph.D. in pure mathematics we find that its contribution is very small. Toward the thesis requirement, nothing; it is extremely rare that a student makes any progress towards writing his thesis as an undergraduate. And if we attempt to measure the factual content in terms of concepts introduced and theorems proved, we will surely be disappointed. Less than twenty percent of the topics required of the Ph.D. candidate, are likely to be covered. Although the times involved are comparable, the graduate student is expected to learn more than

four times as much mathematics in the two years preceding his general examination as he did in four years of part-time study!

The source of this imbalance is not hard to find. To understand pure mathematics at all, requires a substantial preparation. There is no logical bar to teaching mathematics on a completely abstract plane from the freshman year, but it is well-known that it simply doesn't work. Each student must be shown enough of the model-building process which connects the real world with the abstract to justify the need for and exhibit the place of rigorous mathematical thinking. The primary goal of undergraduate training is not knowledge but savoir faire.

One aspect of mathematical savoir faire is the ability to read and write mathematics. I imagine that most students proposing to study French literature in graduate school come prepared with a working knowledge of the French language. It seems reasonable to expect the equivalent of prospective mathematics students; yet, in my experience, the inability to write a correct mathematical argument has been the most common cause of failure in graduate school. I do not refer to any subtleties of mathematical logic; I ask only that students should be able to write correct proofs involving mathematical induction and elementary ϵ-δ technique.

Perhaps the most important subgoal of the undergraduate program should be to describe what mathematics is. Certainly mathematics is widely misunderstood. The oldest and widest-spread misconception, that a mathematician is someone very handy at addition and subtraction, is now giving way to the more sophisticated, but ultimately more dangerous error that mathematics is primarily concerned with curves drawn on pretzels. It is obviously desirable, for our own sakes as well as our students', to propagate a more accurate impression of our profession.

Occasionally a student gets to graduate school very much in love with mathematics, only to discover that there has been a mistake of identity. He doesn't have any idea of what the subject is really about, only that he enjoyed solving the problems in his calculus book. Here the failure of his college to teach what mathematics is, has caused a real tragedy because it is probably now impossible for him to start back and take up a subject better suited to his temperament. The same story frequently occurs during the college years, most often when the student first encounters abstraction in a modern algebra course. The later it happens, the harder it is for him to switch his major to a more congenial subject, and the more serious the effect on his career.

What should be done to change undergraduate mathematics instruction? I think that the worst feature of the present standard curriculum, as I have tried to describe it above, lies in the sequence of the courses. The sequence matters comparatively little to those who indeed go on to graduate school, but we have an equal responsibility to warn off those for whom mathematics is not a suitable career. In the first two years of study we must offer the student the broadest possible insight into the subject in order that he may make a rational choice: for or against mathematics. On this ground alone a strong case can be made for

introducing the modern algebra course not later than the second year. This suggestion can also be justified on the purely mathematical ground that one can teach a better course in calculus to those who know vector algebra.

Among the so-called modern approaches to college mathematics are those that put altogether too much emphasis on formal logic and set theory. In an effort to present abstract mathematics early, these courses paint too cheap a picture. It does not take very much formal manipulation of nonsense statements or cup and cap signs to convince a bumptious freshman that mathematics is trivial and impractical. He is likely to drop the subject, read a semipopular book, and become a firm believer in the curves-on-pretzels canard. Let us not start any subject unless we can take it far enough to justify the effort. Set theory, for example, hardly qualifies on its own merits, so let us teach it only as a tool and shy away from rhapsodies on the uncountability of the real numbers.

In all our considerations of the undergraduate curriculum, we must not lose sight of our fundamental goal: To teach what mathematics is and how it works. The graduate schools want not so much students with formal knowledge as those who think like mathematicians.

ANALYTIC GEOMETRY AND THE CALCULUS

HARRY LEVY, University of Illinois

During the past thirty years, we have seen the standard freshman course in analytic geometry steadily going downhill, and what remains now is of so little importance in the development of a student's mathematical talents that the current vogue is to begin freshman mathematics with the calculus while introducing the indispensable minimum of analytics in small scattered doses through the year. In part, at least, this program reflects a national weakness; we must get things done in a hurry, and everyone knows that the scientists of tomorrow must study calculus today. But the advantages and disadvantages of relegating analytic geometry to the background cannot be dismissed so lightly. We cannot avoid the fundamental question: Is the program of a combined calculus and analytic geometry course a program that can raise the level of mathematical achievement of our youth?

One of the dangers in the combined course stems from the increased opportunities for self-delusion that it affords. No teacher of analytic geometry can fail to be aware of how much or how little of the basic ideas have been mastered by his students because there are so many problems in analytic geometry, of every level of difficulty, which require independent thought and independent analysis. There is, perhaps I should say there should be, very little formal computation to obscure the facts. But formalism in the calculus is considerable; its computations are varied and often quite complicated and even ingenious. Its mathematical theory, to a much greater extent than in analytic geometry, is divorced from everyday practice, and even the better-than-average student finds that many of its theorems are (except in the study of series) only distantly

related to what he is doing. In this setting, I contend that courses in combined calculus and analytic geometry will inevitably tend more and more to emphasize computation; the present high-quality texts will be replaced by more "teachable" ones where ideas are treated quite superficially.

We would like the student to study calculus as early as possible; to help him achieve that goal, we, as teachers, must understand the character of the troubles that plague him. We all know that it is more difficult for him to "Find the volume of a cone, by single integration" than to "Find the volume generated by revolving about the x-axis the area bounded by the lines $y=0$, $x=h$, $y=mx$." We are all aware of the difficulties that arise even with simple verbal problems in maxima, minima, and related rates. Teachers of physics as well as of mathematics are often able to pinpoint a student's troubles with problems in elementary dynamics by asking "How have you chosen coordinates?"

Can we find a common denominator of the troubles that confront an average student in the calculus? I would suggest two sources of his difficulties:

1. The concept of a variable—even a real-valued variable—is intrinsically difficult; the student needs every opportunity to work with this concept in concrete settings.

2. Formulating a scientific question mathematically requires, at every stage of mathematical and scientific development, a high degree of sophistication.

The student of the calculus is called upon to create this interplay between the world of abstract mathematical ideas and the world of reality at a level considerably in advance of anything he encountered in high school. Consider, for example, the difference between knowing the numerical properties of a numerical quadratic polynomial and applying that kind of knowledge to determine the extremals of $ax^2+2bxy+cy^2$.

I suspect that one of the major difficulties in our teaching of analytic geometry may be due to our reluctance to throw overboard the seventeenth century concept of an absolute space. Yet the importance of geometry to the scientist as well as to the mathematician is due to its abstract character as a mathematical system. Thus, when a physicist speaks of the ellipsoid of inertia of a rigid body, he is thinking like a mathematician, and when he wishes to consider the motion of a billiard ball on a flat table, he may again think like a mathematician and associate with that motion a curve in five-dimensional space.

Consider the wealth of ideas that could present themselves in the first weeks of our present courses in analytic geometry. In his first few lessons, the student could be confronted with a variety of applications of Descartes' idea: different coordinatizations of the plane; metric and affine coordinates on a line; the slope as a coordinate in a pencil of lines; other coordinatizations of pencils of lines and of the set of all lines. He could begin to understand that Descartes' arithmetization of geometric concepts opened the door to the arithmetization (and subsequent mathematization) of nongeometric ones, and laid a basis for the science of today. He could meet the several distance functions and slope functions (with domains of definition not subsets of the real line). He could learn

that he can command variables to hold fast for his own purposes, and to subject them to other conditions of his choosing. These are profound ideas, and the student who has not experienced them will hardly learn more than the techniques of the calculus.

My answer then to the question raised at the end of my first paragraph is a flat "No." A generation of students who have been trained in the calculus without having been subjected to a systematic program of algebraic-geometric analysis of the kind that analytic geometry should provide will forever be "forgetting" which way the parabola $y^2=4x$ turns.

A CONFERENCE ON UNDERGRADUATE RESEARCH IN MATHEMATICS

(Preliminary Report)

A Conference on Undergraduate Research in Mathematics, supported by a grant from the National Science Foundation, was held at Carleton College June 19 through June 23, 1961. Seventy-one colleges and universities were represented. Invited addresses were given by R. L. Wilder, Paul C. Rosenbloom, Lloyd B. Williams, Frank L. Griffin, Robert Z. Norman, and Kenneth O. May. Lewis Pino and William Rosen spoke on behalf of the N.S.F., Donald Western reported on the work of the Committee on Undergraduate Research Participation of the M.A.A., and R. J. Wisner represented the C.U.P.M. Sixty-six institutions reported a wide variety of activities designed to help students begin to think and work like mathematicians. Some speakers and discussants suggested using the word "research" in a broader sense than is now customary among mathematicians and in a manner consistent with its use in other fields. Resolutions were passed urging continued attention to independent study and research by undergraduates, improved communication between teachers supervising such activity, the provision of additional opportunities for publication of student work by utilizing existing publications and the establishment of a journal directed only to undergraduates and written largely by them, and the increased use of sectional meetings of the M.A.A. for presentation of undergraduate work of high quality. A complete report of the conference will be sent to department chairmen listed in the Administrative Directory of the A.M.S. Others may request copies by writing to the Department of Mathematics, Carleton College, Northfield, Minnesota.

SHELL COMPANIES FOUNDATION, INCORPORATED, RESIDENCIES IN SCIENCE AND MATHEMATICS

A new aid-to-education program designed to help improve the quality of science and mathematics teaching in high schools and elementary schools, announced in the spring by Shell Companies Foundation, Incorporated, provides postgraduate training for six key teaching-improvement leaders a year. These teachers, returning to schools or school systems, will insure a flow of fundamental knowledge and new teaching techniques into science and mathematics courses. The Foundation has established a program of Shell Merit Residencies for High School Sciences and Mathematics teachers at Stanford and Cornell. Three teachers will devote a minimum of 12 months to special graduate-level study and leadership experiences at each institution. The residents will do advanced work in science and mathematics, become acquainted with new curriculum and teaching materials produced by national committees on science and mathematics teaching, gain experience in research techniques, curriculum development and supervision. The first residents entered Cornell and Stanford in the fall of 1961 and will continue through the summer of 1962.

The Residencies are divided into two categories: two senior and four junior. The Foundation will award each senior $6,000 and each junior $4,600 to cover living expenses and will also pay tuition and fees. In addition, the Foundation will make a grant to Stanford and a grant to Cornell. Selection of six teachers will be made by the two institutions. First consideration for the awards will be given to the 550 high school teachers chosen to attend the Foundation-sponsored Shell Merit Fellowship summer seminars held annually at Cornell and Stanford for the past six years. The Shell Foundation hopes that teachers finishing the advanced work in the science and mathematics residencies will return to their schools as supervisors or curriculum coordinators or in other ways provide leadership for developing greater excellence in science and mathematics teaching.

The new program is an extension of long-term support of education. The other aids include an annual donations program of 61 graduate fellowships and 25 research grants at 56 colleges and universities; 100 Shell Merit Fellowships, full-term summer programs at Stanford and Cornell for in-service high school science and mathematics teachers; 100 Shell Merit Scholarships for undergraduates planning careers as teachers of high school science or mathematics; and a program of Shell Assists, designed to provide helpful "extras" to teachers, departments and administrations in approximately 100 additional colleges throughout the United States not receiving other Shell Foundation support.

INTER-AMERICAN CONFERENCE ON MATHEMATICAL EDUCATION

Under the direction of the International Commission on Mathematical Education and the Organization of American States, an Inter-American Conference on Mathematical Education will be held in Bogota, Colombia, December 4 to 9, 1961. Attendance will be limited to invited participants and persons sent as delegates or observers from their governments, scientific organizations, and sponsoring bodies. Participants have been invited from each of twenty-four nations in the western hemisphere.

The conference will concern itself with the present status of mathematical education at the secondary and university level in each of the countries, with the needs for improved mathematical instruction, and with procedures each country can initiate to move forward in mathematics, especially in preparing teachers and research workers.

Speakers include: Laurent Schwartz, Gustave Choquet, of France; Sven Bundgaard of Norway; Rafael LaGuardia of Uruguay; Leopold Nachbin, Thomas Carvalho of Brazil; Gonzalez Dominquez, Andres Valeiras of Argentina; Enrique Cansada of Chile; Guillermo Torres of Mexico; Saunders MacLane, E. G. Begle and Howard F. Fehr of the United States of America.

The conference was made possible by grants from the Ford Foundation, Rockefeller Foundation, National Science Foundation, UNESCO, Organization of American States, and the Colombia Government. The proceedings of the conference will be published. The committee organizing the conference is: Marshall H. Stone, Chairman, Howard F. Fehr, Secretary (U.S.A.), Guillermo Torres (Mexico), Jose Babini (Argentina), and Leopold Nachbin (Brazil).

TEACHER PREPARATION-CERTIFICATION TO BE CONTINUED

The Carnegie Corporation of New York has announced a second grant of $100,000 to the American Association for the Advancement of Science in support of a project concerned with the development of guidelines for state departments of education, to use in approving college programs for the preparation of mathematics and science teachers. The program is under the sponsorship of the National Association of State Directors of Teacher Education and Certification. It is carried on with the cooperation of AAAS with funds being granted to the latter organization since NASDTEC is not incorporated. The project started in December 1959 with Dr. William P. Viall, formerly of the New York State Department of Education, as director. During the 18 months of the study guide-

lines have been prepared for the preparation of science and mathematics teachers in the secondary schools. A final conference, at which the guidelines were offered for approval by the state directors of teacher education, was held at Pennsylvania State University July 18–20, 1961. Among participants were some 25 mathematicians.

The new grant is for extension of study and use of the guidelines for secondary school programs and to embark on a study of programs in mathematics and science for elementary school teachers.

Dr. E. G. Begle is a member of the Advisory Board of the Teacher Preparation-Certification Study. The offices are in the AAAS building in Washington.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1486. *Proposed by Aboulghassem Zirakzadeh, University of Colorado*

Given a conic E and a line l in the plane of the conic, choose a point K on l and not inside E, and draw the tangents KA and KB to the conic. Consider a point P on the conic and draw the lines PA and PB and find the points R and S, the respective intersections of PA and PB with l. Consider another point P' on the conic and draw the lines $P'R$ and $P'S$ and find the points A' and B', the respective intersections of $P'R$ and $P'S$ with the conic. Prove that $A'B'$ passes through the pole of line l with respect to the conic E.

E 1487. *Proposed by N. V. Glick and Gregory Sheridan, North American Aviation, Inc.*

Prove that any rational number can be expressed as a finite sum of distinct terms of the harmonic series.

E 1488. *Proposed by R. G. Winter, Pennsylvania State University*

For square matrices of order n, prove that any matrix M: (1) has zero trace if $MQ=-QM$, where Q is any nonsingular matrix, (2) can be written as a matrix of zero trace plus a multiple of the identity.

E 1489. *Proposed by C. G. Fain, Technical Operations, Inc., Burlington, Mass.*

From a square array of n^2 objects, n objects are chosen randomly without replacement. What is the probability that no two of the objects chosen came from adjacent positions, whether in a row or column, of the array?

E 1490. *Proposed by Marlow Sholander, Western Reserve University*

Three towns A, B, C are joined by roads AP, BP, CP which cost respectively r, s, t dollars per mile. Locate P that minimizes cost.

SOLUTIONS

An Inequality Among the Sides of a Triangle

E 1456 [1961, 294]. *Proposed by J. F. Darling, Woodstown, N. J.*

Prove that in a triangle with sides a, b, c and semiperimeter s,

$$a^2 + b^2 + c^2 \geqq (36/35)(s^2 + abc/s),$$

with equality only if the triangle is equilateral.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* By expanding both members one may verify that

$$\sum a^2 - (36/35)(s^2 + abc/s) \equiv (1/35s)[13s \sum (a - b)^2 + 4 \sum a(b - c)^2],$$

where the summations are over cyclic permutations of a, b, c. The right hand side is nonnegative, being zero if and only if $a = b = c$, so that

$$\sum a^2 \geqq (36/35)(s^2 + abc/s),$$

equality holding only for equilateral triangles.

II. *Solution by William Moser, University of Manitoba.* Using the well-known inequalities

$$a^2 + b^2 + c^2 \geqq (a + b + c)^2/3 \quad \text{and} \quad abc \leqq [(a + b + c)/3]^3,$$

where $a, b, c > 0$ and there is equality only if $a = b = c$, we have (with $2s = a + b + c$)

$$\begin{aligned} a^2 + b^2 + c^2 &\geqq (a + b + c)^2/3 = 4s^2/3 = (36/35)\{s^2 + (2s/3)^3/s\} \\ &= (36/35)\{s^2 + [(a + b + c)/3]^3/s\} \geqq (36/35)(s^2 + abc/s), \end{aligned}$$

with equality only if $a = b = c$.

Also solved by A. N. Aheart, Samuel Beatty, Brother Alfred, Leonard Carlitz, T. R. Curry and Jeff Raskin (jointly), D. M. Danvers, K. M. Das, G. C. Dodds, F. J. Duarte, H. M. Feldman, Curt Gilchrist, L. D. Goldstone, C. A. Green, J. B. Herreshoff, Erwin Just and Norman Schaumberger (jointly), Leonard Klosinski, A. W. Knapp and Albert Whitcomb (jointly), M. LeLeiko, F. Leuenberger, R. J. Lewyckyj, M. J. Pascual and C. D. Sutherland (jointly), J. L. Pietenpol, David Sachs, Mitchell Secondo, Philip Smedley, E. L. Spitznagel, Jr., G. B. Torchinelli, Patrick Twomey, W. C. Waterhouse, Charles Wexler, and the proposer. Late solution by A. K. Bagchi.

A Set of Numbers Containing Internally Composite Members

E 1457 [1961, 294]. *Proposed by Aaron Herschfeld, Pennsylvania State University, Hazleton, Pennsylvania*

Show that the set of numbers $J_m = m^2 + 1$, $m = 1, 2, \cdots$, contains an infinity

of composite $J_N = J_m J_n$. In fact, for arbitrary m, find two pairs of corresponding integers n, N.

Solution by J. H. Avila, Jr., University of Santa Clara. Since, for arbitrary a,

$$(a^2 + 1)[(a + 1)^2 + 1] = (a^2 + a + 1)^2 + 1,$$

we have, for every $m \geqq 2$, $J_{m-1}J_m = J_{m^2-m+1}$ and $J_m J_{m+1} = J_{m^2+m+1}$. For $m = 1$, $J_1 J_2 = J_3$ and $J_1 J_{12} = J_{17}$.

Additional composite J_N are obtained from the formula $J_m J_{2m^2} = J_{2m^3+m}$.

Also solved by Winifred Asprey, K. A. Baker, R. C. Beach, William Becker, D. R. Breach, D. A. Breault, J. L. Brenner, A. M. Broshi, Brother Alfred, Brother Joseph Heisler, J. A. Brown, J. L. Brown, Jr., R. V. Budny, Leonard Carlitz, H. S. Cash, D. I. A. Cohen, J. L. Cooley, Gus DiAntonio, Underwood Dudley, J. W. Ellis, Loszlo Engleman, J. A. Faucher, N. J. Fine, Curt Gilchrist, Michael Goldberg, R. P. Goldberg, L. D. Goldstone, A. S. Gregory, Cornelius Groenewoud, N. G. Gunderson, W. J. Halm, J. B. Herreshoff, Vern Hoggatt, J. E. Homer, Jr., Aughtum Howard, J. A. H. Hunter, Lawrence Israel, Erwin Just, M. S. Klamkin, Kenneth Kloss, A. W. Knapp and Albert Whitcomb (jointly), A. G. Konheim, W. J. Koss, Sidney Kravitz, J. A. Lambert, L. J. Lardy, Dean Lawrence, Joseph Lehner, Jiang Luh, C. R. MacCluer, Barry MacKichan, R. A. McGuigan, Jr., D. C. B. Marsh, M. V. Mielke, J. W. Mean, Otto Mond, D. A. Moran, D. L. Muench, J. B. Muskat, C. S. Ogilvy, Walter Penney, D. J. Persico, J. L. Pietenpol, C. F. Pinzka, E. H. Primoff, W. H. Richardson, David Sachs, R. T. Shannon, Nancy Shera, D. L. Silverman, A. Sinkov, Sister Mary Denis, Denis Sjerve, Bob Snell, Leon Steinberg, Eric Sturley, Paul Stygar, Suvorov, E. H. Theil, G. B. Torchinelli, Patrick Twomey, E. W. Wallace, R. M. Warter, W. C. Waterhouse, Charles Wexler, Rodney Wilton, Xavier University Mathematics Study Group, F. H. Young, David Zeitlin, and the proposer. Late solution by C. C. Oursler.

Gunderson pointed out that putting $m = N$ and calculating an N_1 and n_1 gives $J_{N_1} = J_N J_{n_1} = J_m J_N J_{n_1}$. Then iteration shows that for any integer $r \geqq 2$, the set J_m contains an infinity of composite $J_M = J_{m_1} J_{m_2} \cdots J_{m_r}$. There is also an infinity of $J_N = J_1 J_m$, since this is equivalent to the Pell equation $N^2 - 2m^2 = 1$, which is known to have an infinity of positive integral solutions.

Beach gave the additional pairs of corresponding integers: $(n, N) = (8m^4 + 4m^2,\ 8m^5 + 8m^3 + m)$, $(32m^6 + 32m^4 + 6m^2,\ 32m^7 + 48m^5 + 18m^3 + m)$, $(128m^8 + 192m^6 + 80m^4 + 8m^2,\ 128^9 + 256m^7 + 160m^5 + 32m^3 + m)$, $(512m^{10} + 1024m^8 + 672m^6 + 160m^4 + 10m^2,\ 512m^{11} + 1280m^9 + 1120m^7 + 400m^5 + 50m^3 + m)$, a list which can be continued indefinitely.

An Application of Partial Fractions

E 1458 [1961, 295]. *Proposed by Hans Schwerdtfeger, McGill University*

Let $x_1 < \cdots < x_n$ be n points on the x-axis. Let P be any point in the (x, y)-plane with ordinate different from zero. If d_i is the distance of P from x_i show that

$$\sum_{j=1}^{n} a_j d_j^2 = \begin{Bmatrix} 1 \text{ if } n = 3 \\ 0 \text{ if } n > 3 \end{Bmatrix},$$

where $a_i = 1/f'(x_i)$, $f(x) = (x - x_1) \cdots (x - x_n)$.

Solution by A. G. Konheim, IBM, Yorktown Heights, N. Y. Let the coordinates of P be (a, b). If we expand $(n \geqq 3,\ b > 0) F(x) \equiv [(a - x)^2 + b^2]/f(x)$ in partial fractions we obtain

$$F(x) = \sum_{j=1}^{n} a_j d_j^2/(x - x_j), \tag{1}$$

by virtue of the simplicity of the zeros of $f(x)$. Multiplying both sides of (1) by x and letting $x \to +\infty$ we obtain the desired result.

Also solved by Samuel Beatty, P. R. Bender, R. L. Bohuslov, J. L. Brown, Jr., H. S. Cash, Michael Goldberg, L. D. Goldstone, J. B. Herreshoff, L. J. Lardy, R. J. Lewyckyj, D. C. B. Marsh, David Sachs, L. J. Schneider, O. E. Stanaitis, G. B. Torchinelli, W. C. Waterhouse, R. J. Whitley, David Zeitlin, and the proposer.

Goldstone found a more general result in Burnside-Panton, *The Theory of Equations*, vol. 1, prob. 4, p. 172. He also showed that the case $n=3$ is equivalent to Stewart's theorem of college geometry.

Arithmetic Progressions of Primes

E 1459 [1961, 295]. *Proposed by Azriel Rosenfeld, Yeshiva University*

Let $S(m, N)$ be the statement: "There exist m primes not exceeding N which are consecutive terms of an arithmetic progression." For example, $S(3, N)$ is true for $N \geqq 7$ (use 3, 5, 7); $S(5, N)$ is true for $N \geqq 29$ (use 5, 11, 17, 23, 29). Prove that $S(7, N)$ is false for $N < 900$, and that $S(11, N)$ is false for $N < 10{,}000$.

Solution by J. B. Muskat, University of Pittsburgh. Let p be a prime. Let $a, a+d, a+2d, \cdots, a+(p-1)d$ be an arithmetic progression of p terms. If $p \nmid d$, the arithmetic progression runs through a complete set of residues, modulo p. Thus if all the terms of the arithmetic progression are to be prime, either $p \mid d$, or else p itself is the first term of the arithmetic progression. A similar argument shows that all the primes less than p must divide d.

First let $p=7$. Now 2, 3, 5 all divide d, so 30 divides d. If 7 divides d, d is a multiple of 210, and the seventh term $>(6)(210)=1260>900$. Thus it suffices to consider $a=7$. If $d=30$, 60, or 90, the term $187=(11)(17)$ is encountered. If $d=120$, the term $247=(13)(19)$ is encountered. If $d=150$, then 7, 157, 307, 457, 607, 757, 907 are all prime, and $907>900$.

Now let $p=11$. The numbers 2, 3, 5, 7 divide d, so 210 divides d. If 11 divides d, d is a multiple of 2310, and the eleventh term $>(10)(2310)=23100>10{,}000$. Thus it suffices to consider $a=11$. If $d=210$, 420, or 840, then $851=(23)(37)$ is encountered. If $d=630$, then $3791=(17)(223)$ is encountered. Thus $d \geqq 1050$, so the eleventh term is at least $10511>10{,}000$.

Also solved by Merrill Barnebey, William Becker, Brother Alfred, J. B. Herreshoff, D. C. B. Marsh, David Sachs, Paul Stygar, G. B. Torchinelli, W. C. Waterhouse, David Zeitlin, and the proposer. Late solution by A. S. Gregory.

The basic facts used in the above solution were located by Becker and Zeitlin in a theorem of A. Guibert; see L. E. Dickson, *History of the Theory of Numbers*, vol. 1, pp. 425–6.

A Number-Theoretic Function

E 1460 [1961, 295]. *Proposed by Masao Arai, Jiyu Gakuen, Tokyo, Japan*

Let x, y, n be positive integers and let $\chi(n)$ denote the number of pairs of

integers x, y satisfying $(x, n)=(y, n)=(x+y, n)=1$, $x<n$, $y<n$. Find a formula for $\chi(n)$.

Solution by Leonard Carlitz, Duke University. Let $(m, n)=1$. If $(r', m)=(r'', m)=(r'+r'', m)=1$, $(s', n)=(s'', n)=(s'+s'', n)=1$, then $(nr'+ms', mn)=(nr''+ms'', mn)=(n(r'+r'')+m(s'+s''), mn)=1$, so that

$$\chi(mn) = \chi(m)\chi(n), \qquad (m, n) = 1. \tag{1}$$

Next, if p is a prime we can select r in $p^{e-1}(p-1)$ ways such that $1\leqq r\leqq p^e$, $(r, p^e)=1$, and for each r we can select s in $p^{e-1}(p-2)$ ways such that $1\leqq s\leqq p^e$ and $(s, p^e)=(r+s, p^e)=1$. Hence

$$\chi(p^e) = p^{2e-2}(p-1)(p-2). \tag{2}$$

Combining (1) and (2) we get

$$\chi(n) = n^2 \prod_{p|n} (1 - 1/p)(1 - 2/p).$$

Note that $\chi(2n)=0$.

Also solved by Brother Alfred, N. J. Fine, J. B. Herreshoff, Betty Levine, D. C. B. Marsh, J. W. Moon, J. B. Muskat, Walter Penney, David Sachs, G. B. Torchinelli, W. C. Waterhouse, and the proposer. Late solution by Gerald Janusz.

Editorial Note. The relation (1) above requires a slight modification of the proposer's definition to make $\chi(1)=1$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4989. *Proposed by Yoshio Matsuoka, Kagoshima-shi, Japan*

Let $f(x)$ be a polynomial of degree n, such that $\int_0^1 x^k f(x)dx=0$ $(k=1, 2, \cdots, n)$. Prove that

$$\int_0^1 \{f(x)\}^2 dx = (n+1)^2 \left\{ \int_0^1 f(x)dx \right\}^2.$$

4990. *Proposed by R. G. Medhurst, Wembley, Middlesex, England*

Prove the identity

$$\int_{(p+p_0)/2}^{p_m} dy \int_{p-y+p_0}^{y} f(y)f(x)f(y+x-p)dx$$

$$= \int_{p_0}^{2p_m-p} dy \int_{(y+p)/2}^{p_m} f(y)f(x)f(y-x+p)dx,$$

and establish under what conditions it holds. (As originally formulated, $f(x)$ was any positive real function of x over the range p_0 to p_m, $p_m > p_0$ and $2p_m - p_0 \geqq p \geqq p_m$.)

4991. *Proposed by Oscar Varsavsky, Caracas, Venezuela*

Find the infimum of all Boolean topologies defined on a given set T. (A topology is called Boolean if it is Hausdorff, compact and totally disconnected.)

4992. *Proposed by Seth Warner, Duke University*

Let $(E, +, \leqq)$ be an infinite totally ordered semigroup, that is, an infinite commutative semigroup with a total ordering $\leqq$ such that $x \leqq y$ implies $x+z \leqq y+z$ for all elements x, y, z of E. Prove that $(E, +, \leqq)$ is isomorphic (as an ordered semigroup) to the ordered semigroup of nonnegative integers if the following condition is satisfied: There exist elements 0 and 1 satisfying $0<1$ such that if S is any subset of E containing 0 and containing $x+1$ whenever it contains x, then $S=E$.

4993. *Proposed by A. G. Konheim and R. A. Willoughby, IBM Research, Yorktown Heights, N. Y.*

Given a positive integer N, find all real sequences $\{a_i\}_{i=-\infty}^{\infty}$ which satisfy

$$a_{i+N} = a_i \qquad (i = 0, \pm 1, \pm 2, \cdots), \tag{1}$$

$$\sum_{i=0}^{N-1} a_i a_{i+\mu} = \begin{cases} 0 & \text{if } \mu \not\equiv 0 \pmod{N}, \\ 1 & \text{if } \mu \equiv 0 \pmod{N}. \end{cases} \tag{2}$$

4994. *Proposed by Peter Ungar, New York University*

For $a<x<b$, let

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \lim_{n\to\infty} f_n'(x) = \phi(x).$$

Prove: if all the functions named above and also $f'(x)$ are continuous, then $f'(x) = \phi(x)$.

SOLUTIONS

Area of A Plane Polygon

3087 [1924, 306]. *Proposed by H. W. Bailey, Champaign, Illinois*

Given a polygon of n sides with vertices $(x_1, y_1), \cdots, (x_n, y_n)$. Set up a determinant of the nth order which shall represent its area.

Note by D. A. Moran, University of Illinois. Let $V_1(x_1, y_1), \cdots, V_n(x_n, y_n)$ be the vertices in cyclic order. From elementary analytic geometry, we know that the area A is given by the following equation:

$$(1) \qquad 2A = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}.$$

We wish to find a single determinant to represent the above sum. For $n=3$ and $n=4$, the following determinants satisfy:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \qquad \begin{vmatrix} x_1 & y_1 & 1 & 0 \\ x_2 & y_2 & 1 & 1 \\ x_2 & y_3 & 1 & 0 \\ x_4 & y_4 & 1 & 1 \end{vmatrix}.$$

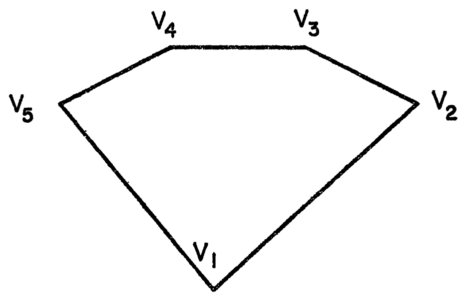


FIG. 1

For $n \geqq 5$, no analogous determinant exists. We shall show that this is the case for $n=5$; similar methods will easily be seen applicable in the general case.

Consider the pentagon $V_1V_2V_3V_4V_5V_1$, (Fig. 1) and write:

$$(2) \qquad \Delta = 2A = \begin{vmatrix} x_1 & y_1 & a_1 & b_1 & c_1 \\ x_2 & y_2 & a_2 & b_2 & c_2 \\ x_3 & y_3 & a_3 & b_3 & c_3 \\ x_4 & y_4 & a_4 & b_4 & c_4 \\ x_5 & y_5 & a_5 & b_5 & c_5 \end{vmatrix}.$$

Now consider the special case: $V_1 = V_2 = V_3 = (0, 0)$. From (2) we find that

$$\Delta = \begin{vmatrix} 0 & 0 & a_1 & b_1 & c_1 \\ 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & a_3 & b_3 & c_3 \\ x_4 & y_4 & a_4 & b_4 & c_4 \\ x_5 & y_5 & a_5 & b_5 & c_5 \end{vmatrix} = \left| a_1, b_2, c_3 \right| \cdot \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix}.$$

It follows, by comparison with (1), that $|a_1, b_2, c_3| = 1$.

On the other hand, suppose that $V_1 = (0, 0)$, and let $V_4 \to V_1$ along segment V_4V_1. In the end, the area becomes simply that of triangle $V_1V_2V_3$, or $|x_2, y_3|$. (See Fig. 2.) From (2) we obtain

$$\left| a_1, b_4, c_5 \right| \cdot \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \left| a_1, b_3, c_4 \right| \cdot \begin{vmatrix} x_2 & y_2 \\ x_5 & y_5 \end{vmatrix} + \left| a_1, b_2, c_4 \right| \cdot \begin{vmatrix} x_5 & y_5 \\ x_3 & y_3 \end{vmatrix}.$$

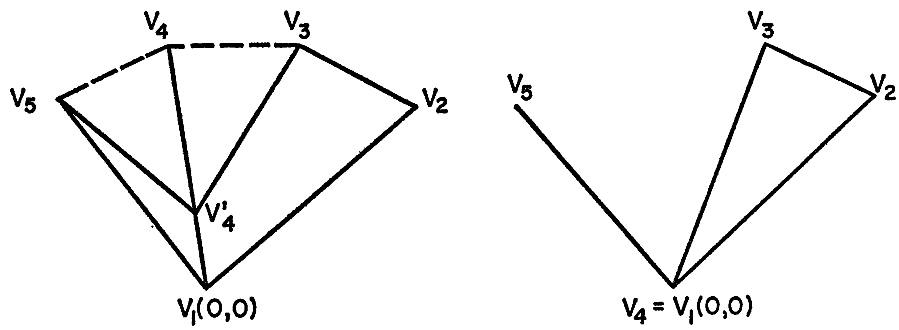


FIG. 2

Therefore $|a_1, b_4, c_5| = 1$; $|a_1, b_3, c_4| = |a_1, b_2, c_4| = 0$. From this last pair of equations we can write

$$\lambda_1(a_1, b_1, c_1) + \lambda_3(a_3, b_3, c_3) + \lambda_4(a_4, b_4, c_4) = 0,$$

$$\mu_1(a_1, b_1, c_1) + \mu_2(a_2, b_2, c_2) + \mu_4(a_4, b_4, c_4) = 0.$$

Eliminating (a_4, b_4, c_4) between these last two relations, we obtain a linear relation

$$\nu_1(a_1, b_1, c_1) + \nu_2(a_2, b_2, c_2) + \nu_3(a_3, b_3, c_3) = 0.$$

This relation contradicts $|a_1, b_2, c_3| = 1$, found above. Thus no determinant such as (2) can represent the area.

Editorial Note. The sum (1) can, however, easily be expressed as a determinant of order n of different form, viz:

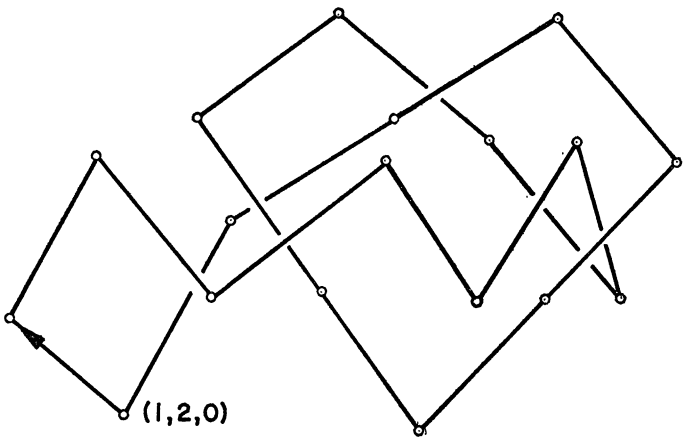
$$2A = \begin{vmatrix} x_1y_2 - x_2y_1 & -1 & -1 & \cdots & -1 & -1 \\ x_2y_3 - y_3x_2 & 1 & 0 & \cdots & 0 & 0 \\ x_3y_4 - y_4x_3 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1}y_n - x_ny_{n-1} & 0 & 0 & \cdots & 1 & 0 \\ x_ny_1 - x_1y_n & 0 & 0 & & 0 & 1 \end{vmatrix}.$$

Minimum Number of Segments for a Knotted Path

3147 [1925, 385]. *Proposed by Norman Miller, Queen's University, Canada*

If space of three dimensions be divided into cubes by means of three sets of equidistant parallel planes and a closed circuit be formed of their diagonals, what is the least number of diagonals necessary in order that this broken-line curve be knotted?

Solution by D. A. Moran, University of Illinois. A segment of such a broken-line curve must be a lattice diagonal $\overline{MN}$, where M has coordinates (m_1, m_2, m_3) and N has coordinates (n_1, n_2, n_3), and $|m_i - n_i| = 1$. Any closed circuit must consist of an even number of lattice diagonals. Note that many lattice points



are inaccessible from a given point by means of lattice diagonals. For instance, from the point (1, 2, 0), only points of the forms (odd, even, even) or (even, odd, odd) are accessible.

Consider the knotted broken-line curve in the figure. Starting from the point (1, 2, 0) in the direction of the arrow, the lattice points defining this loop are the following:

(1, 2, 0) – (0, 3, 1) – (1, 4, 2) – (2, 5, 1) – (3, 4, 2) – (4, 3, 1) – (5, 2, 2)
(4, 1, 1) – (3, 0, 2) – (2, 1, 3) – (1, 2, 2) – (2, 3, 1) – (3, 4, 0) – (4, 5, 1)
(5, 4, 2) – (4, 3, 3) – (3, 2, 2) – (2, 1, 1) – (1, 2, 0).

This broken-line curve forms a closed circuit knotted in a simple overhand knot, as can be seen from the figure, or by building a three-dimensional model. It is easy to convince oneself of the minimality of the length of this knot by noting that the convex hull of the eighteen lattice points defining it contains only these eighteen points, together with points inaccessible from the point (1, 2, 0). Hence the answer is eighteen.

Linear Transformation in a Hilbert Space

4926 [1960, 809]. *Proposed by S. Berberian, State University of Iowa*

If $\mathfrak{H}$ is an infinite-dimensional (say separable) Hilbert space, construct a densely defined transformation T for which the domain of the adjoint transformation T^* is the subspace $\{0\}$.

Solution by the proposer. Let $e_1, e_2, \cdots$ be an orthonormal basis of $\mathfrak{H}$. Let $\{e_{ij}: i, j=1, 2, \cdots\}$ be any double indexing of these basis vectors. For the domain of T, take the linear subspace generated by the e's, and define $Te_{ij}=e_i$, and extend by linearity to finite linear combinations. Suppose y is a vector in the domain of T^*. Then $(Tx|y)=(x|T^*y)$ for all x in the domain of T, and, in particular, $(e_{ij}|T^*y)=(Te_{ij}|y)=(e_i|y)$. By Bessel's inequality,

$$\sum_{j=1}^{\infty} |(e_{ij}|T^*y)|^2 \leqq \|T^*y\|^2.$$

Since $(e_{ij}|T^*y)$ is independent of j, we conclude that $(e_{ij}|T^*y)=0$, that is, $(e_i|y)=0$. Since i is arbitrary, $y=0$.

Also solved by E. W. Cheney and C. Farrington, James T. Joichi, John V. Ryff, and Fred Suvorov.

Laplace Transform of Solution of an Integral Equation

4929 [1960, 926]. *Proposed by James W. Brown, University of Michigan*

Given that $y(t)$ satisfies the integral equation

$$y(t) = t^k + \int_0^t y(t-\tau)J_0(\tau)d\tau, \tag{1}$$

where $J_0(t)$ is the Bessel function of first kind and order zero, and $k>-1$. Show that

$$\int_0^\infty e^{-t}y(t)dt = \frac{\sqrt{2}\Gamma(k+1)}{\sqrt{2}-1}.$$

Solution by James E. Potter, Massachusetts Institute of Technology. If $y(t)$ is restricted to the class of locally integrable functions on $[0, \infty)$ it is known from the theory of Volterra integral equations with bounded kernels that (1) has a unique solution. Formally, Laplace-transforming (1), letting $Ly=Y$, using the convolution theorem, and noting $LJ_0(t)=(s^2+1)^{-1/2}$, $Lt^k=\Gamma(k+1)/s^{k+1}$, we have

$$Y(s) = \Gamma(k+1)/s^{k+1} + Y(s)(s^2+1)^{-1/2},$$

$$Y(s) = \frac{\Gamma(k+1)}{s^{k+1}} + \frac{\Gamma(k+1)}{s^{k+1}} \cdot \frac{1}{\sqrt{(s^2+1)} - 1}. \tag{2}$$

Let

$$y_1(s) = \frac{\Gamma(k+1)}{s^{k+1}} \cdot \frac{1}{\sqrt{(s^2+1)} - 1}.$$

Then $y_1(s) = O(1/s^{k+2}) = O(1/s^{1+\epsilon})$ for small enough $\epsilon > 0$ as $|s| \to \infty$, and is regular in the half plane re $s > 0$. By a known theorem (see R. V. Churchill, *Modern Operational Mathematics in Engineering*, p. 159, Th. 5) there exists a $u_1(t) = L^{-1}y_1(s)$ and $u_1(t)e^{-at} \in L[0, \infty)$ for all $a > 0$. Therefore, if $u(t) = t^k + u_1(t)$, then $Lu(t) = Y(s)$. Since $Lu(t)$ satisfies (2), u satisfies (1) and $u(t)$ is the unique solution of (1). Since $e^{-at}t^k$ and $e^{-at}u_1(t)$ are in $L[0, \infty)$ for all $a > 0$, the integral $\int_0^\infty e^{-at}u(t)dt$ exists for all $a > 0$, and by definition of Y,

$$Y(a) = \int_0^\infty e^{-at}u(t)dt = \frac{\Gamma(k+1)}{a^{k+1}} + \frac{\Gamma(k+1)}{a^{k+1}} \cdot \frac{1}{\sqrt{(s^2+1)} - 1}.$$

In particular, taking $a = 1$, we have the proposed result.

Also solved by R. D. Adams, H. D. Arnett, D. A. Breault, Robert Breusch, J. L. Brown, Jr., E. A. Burfine, R. G. Buschman, F. V. Cavoto, P. R. Chernoff, A. E. Danese, P. J. de Doelder, J. A. Faucher, Newman Fisher, M. L. Glasser, Samuel Goldberg, S. H. Greene, Emil Grosswald, Jan Grzesik, J. P. Jones, D. G. Kabe, P. G. Kirmser, A. G. Konheim, W. E. Lawrence, G. Leibowitz, M. E. Levenson, E. W. Marchand, D. C. B. Marsh, Immanuel Marx, J. R. Modeer, D. H. Moore, S. J. Pagano, Thomas Porsching, D. A. Russo, G. A. Sabin, V. M. Sakhara, Jeff Scargle, E. J. Scott, M. R. Spiegel, Theodore Teichmann, Donato Teodoro, C. J. Tranter, F. R. Urbanus, J. S. White, J. Ernest Wilkins, Jr., K. L. Yocum, David Zeitlin, and the proposer.

Distinct Prime Divisors

4930 [1960, 926]. *Proposed by D. F. Rearick, University of Colorado*

For a fixed positive integer n, consider the numbers $a_m = (m, n)$ as m ranges over a complete residue system (mod n). Prove that the number of a_m having an even number (including none) of distinct prime factors exceeds the number having an odd number of distinct prime factors if n is odd, and that the two are equal if n is even.

Solution by Robert Breusch, Amherst College. Let $e(m, n) = (-1)^s$ if (m, n) contains s distinct prime factors. Let $g(n) = \sum_m e(m, n)$ where m ranges over a complete residue system (mod n).

Assume first that $(n_1, n_2) = 1$, so that $e(m, n_1n_2) = e(m, n_1) \cdot e(m, n_2)$. If now m_1 ranges over a complete residue system (mod n_1) and m_2 over a complete residue system (mod n_2), then $x = m_1n_2 + m_2n_1$ ranges over a complete residue system (mod n_1n_2), and $e(x, n_1) = e(m_1, n_1)$, $e(x, n_2) = e(m_2, n_2)$. Therefore $e(x, n_1n_2) = e(m_1, n_1) \cdot e(m_2, n_2)$, $g(n_1n_2) = \sum_x e(x, n_1n_2) = \sum_{m_1}\sum_{m_2} e(m_1, n_1) \cdot e(m_2, n_2)$, or

$$g(n_1n_2) = g(n_1)\cdot g(n_2), \qquad (n_1, n_2) = 1. \tag{1}$$

Also, if p is a prime, $g(p^a)$ equals the number of m relatively prime to p minus the number of m divisible by p; thus

$$g(p^a) = \phi(p^a) - p^{a-1} = (p-1)p^{a-1} - p^{a-1} = (p-2)p^{a-1}. \tag{2}$$

(1) and (2) imply that if $n = \prod_{k=1}^{r} p_k^{a_k} (2 \leqq p_1 < p_2 < \cdots < p_r)$ then

$$g(n) = \prod_{k=1}^{r} (p_k - 2)p_k^{a_k-1}.$$

Thus $g(n) = 0$ if $p_1 = 2$, $g(n) > 0$ if $p_1 > 2$.

Also solved by G. S. Cunningham, D. C. B. Marsh, D. G. Mead, J. B. Muskat, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should now be sent directly to R. A. Rosenbaum, Department o, Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Boundary and Eigenvalue Problems in Mathematical Physics. By Hans Sagan. Wiley, New York, 1961. xviii+381 pp. Prof. Ed. $9.50, College Ed. $8.00.

This book is sheer delight—the sort of book that one teacher at least has been looking for. Part of the ground covered is familiar; but topics such as Fourier series, Bessel, Legendre, and Laguerre functions, which are frequently taken up—more or less incidentally—in books on advanced calculus, appear here in a unified treatment. The theme of the title is firmly developed, chapter by chapter, from the first which deals with the classical use of variational methods to the last which deals with Green's function for ordinary and partial differential equations. The treatment of eigenvalues and eigenfunctions in conjunction with the Sturm-Liouville theorems (Chapter V) and later by means of a variational principle (Chapter VII) is exceptionally thorough; indeed, the reviewer knows of no other English text on differential equations in which the variational concepts which stem from Rayleigh and Ritz, Courant and Weinstein are so fully discussed.

The author has plainly made it his aim to interest the reader. When a theorem or result requires careful proof or examination, the author explains to the reader what he is about and then proceeds to a lucid proof or discussion.

The typography is excellent.

JOHN MCNAMEE
University of Alberta

Arithmetic: An Introduction to Mathematics. By L. Clark Lay. Macmillan, New York, 1961. xiii+323 pp. $4.50.

Written for the multitude of "those who have had instruction in arithmetic in the elementary grades but find themselves not prepared to begin the study of algebra," this book accordingly ranges from addition to the extraction of roots, with excursions into inequalities, decimal fractions and measurement. Extensive use is made of graphical methods to illustrate concepts. The more than 5,000 remarkably diverse exercises include many relatively challenging verbal problems.

Distinctive features include the introduction of signed numbers by means of operators (*e.g.*, subtracting 5 and adding 3 is equivalent to subtracting 2; thus, $-5+(+3)=-2$) and the treatment of simplification in terms of inverse operations. The presentation is generally meticulous—sometimes to excess, it seems, as with an eight-case analysis of the relative size of F and P in $MF=P$ for M nonnegative rational. Also, although true in the context of natural numbers, repeated declarations of the type "3–5 has no meaning" will undoubtedly confuse many students.

The reviewer feels that those who employ this text as an exercise book should attain considerable proficiency in working with numbers, but he doubts that those in greatest need of the arithmetical review will appreciate the sophisticated approach.

D. C. B. Marsh
Colorado School of Mines

Stochastic Processes. By Lajos Takács. Methuen, London, and Wiley, New York, 1960. xi+137 pp. $2.75.

This book is primarily a collection of problems involving Markov chains, Markov processes, and non-Markovian processes. Among the topics considered in these problems are the following: random walks, waiting time, chain reactions, disintegration, spatial distribution of stars, counter tube problems, birth and death process, and renewal theory. A solution is given for each problem.

The theory sections are very concise and employ about one-fourth of this small book; the necessary definitions and theorems are stated, and occasionally a proof is outlined. There are a few instances of broken or missing type, but the reader can usually make the corrections.

In order to follow the solutions of the problems with some ease a familiarity of the following subjects is required: probability, matrices, difference equations, real and complex variables, partial and ordinary differential equations, integral equations, and transform calculus. Needless to say, this is an advanced book; and the author's fine collection of problems, and their solutions, makes it a valuable addition to the literature of stochastic processes.

Robert V. Hogg
University of Iowa

Statistical Inference for Markov Processes. By Patrick Billingsley. University of Chicago Press, 1961. 75 pp. $4.00.

This slim volume is emphatically a monograph for the expert. It is as close-packed as an article in the Annals of Mathematical Statistics, and indeed the author suggests that a recent paper of his in the Annals (vol. 32, 1961, pp. 12–40) should be read as an introduction to the present work. I would hesitate to recommend the book to the average graduate student unless he had first read Halmos's *Measure Theory* and Doob's *Stochastic Processes*, not to mention Cramér's *Mathematical Methods of Statistics*. Such concepts as absolute continuity of a set function, Radon-Nikodym derivatives, and martingales, are taken as too familiar to need explanation. But the well-read statistician will find a great deal of interest in these pages.

A theory of statistical inference is developed for a stationary Markov process in which the transition probabilities depend on an unknown parameter θ. Given a sample from the process, the author obtains estimates of θ (by the method of maximum likelihood) and tests of various hypotheses regarding θ using the Neyman-Pearson criterion. A new central limit theorem for martingales is proved, and applied in order to prove asymptotic normality for various statistics. The word martingale, by the way, is not used in the dictionary sense of a gambling system, but means a stochastic process in which the conditional expected value of x at time t_{n+1}, given the values at $t_1, \cdots, t_n$, is equal to $x(t_n)$, with probability 1. The general theorems are specialized to obtain many known results about finite Markov chains.

In Part II, a discontinuous Markov process $x(t)$ with continuous time parameter t is treated, and is reduced to the problem of Part I. About one-third of the book is devoted to a mathematical appendix in which proofs are given of various theorems which are merely quoted in the main body of the text.

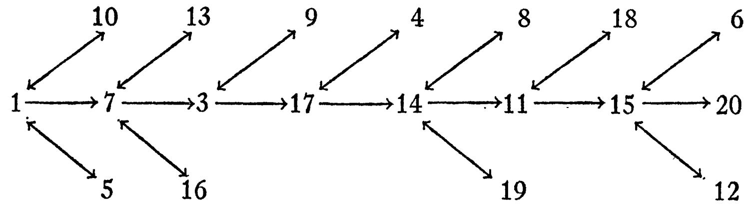
The book is quite attractively printed and produced, as befits a work sponsored jointly by the Institute of Mathematical Statistics and the University of Chicago.

E. S. Keeping
University of Alberta

Adventures in Algebra. By Norman A. Crowder and Grace C. Martin. Doubleday, Garden City, N. Y., 1960. 350 pp. $4.95.

Any adequate review of this book must approach the task from three different directions; the format, the contents, and the intended purpose. For, this is what has come to be called a "scrambled book." The pages are not meant to be perused in their serial order; indeed, an intelligent reader might encounter only half of its pages. In general, each page ends in a question addressed to the reader, for which two or three possible answers are listed. Depending upon the reader's selection, he will find himself directed to as many possible pages where the main stream of discussion is continued, or where his error is explained and corrected.

The following diagram shows the structure of Chapter 1:



Some of the later chapters show a more elaborate structure.

The mathematical content is familiar and can be described briefly. In order, the authors take up something about the use of sentences and variables in mathematics, carry the reader through Euler's proof of the infinitude of primes, discuss briefly how to solve a linear equation, how to work with negative integers and rational numbers, and how to multiply simple binomials; use mathematical induction to prove several familiar formulae, and conclude by discussing square roots and proving that $\sqrt{2}$ is irrational.

These topics are excellent and well suited to illustrate the use of the special scrambled book format. Most of the incorrect responses are natural, and the additional explanation which is found on the pages off the main stem of the tree should be successful in putting the reader back onto the main line.

As a device for inducing learning, I think this has considerable promise; my daughter will gladly supply a testimonial to its fascination. However, the worth of this instrument must be measured against the extravagant use of space. I sincerely regret that this particular example suffers so greatly from defects in the treatment of content, arising no doubt from the lack of mathematical maturity on the authors. The situation could have been improved had the publisher processed the book in the usual manner, having it reviewed by competent mathematicians before publication. Proofreading such a book as this must indeed be hard; there are only two misprints that I could find; no answer on page 85 is correct, and there is an obvious error on page 218.

Most disturbing to me, and I think more damaging to the book, are the errors of mathematical fact and judgment that occur throughout.

Starting with Chapter 1, and indeed permeating much of the book, one finds a confused attitude toward statements, with and without free variables, and quantified in various ways. For example, $n-n=0$ is called a true universal statement, while (p. 39) $n-a=0$ is said to be meaningless. In answer to the question: "How would you express the fact that if any number (except 0) is divided by itself, the result is 1," the "correct" response is

$$\frac{a}{a} = 1,$$

with no quantification and no restrictions.

The statement (p. 46) $n+1=n$ is called false, and on page 47, the statement

$p+2=9$ is expected to be selected as true. (Presumably, the reader is asked to supply quantifiers such as "for all n," "for some p" etc. from context.)

The book also abounds in awkward phrases. In the question cited above, the answer "$a \div a = 1$" is labeled as "arithmetically correct" but "in algebra" we use a/a. Again, in response to the question: "What is the product of the algebraic factors a and b?" we find (p. 71) "We don't know what numbers a and b are so we cannot write down the arithmetic product of a and b. We can write the algebraic product ab."

More fundamental, however, are the errors that surround the proof of the infinitude of primes in Chapters 3 and 4. On page 85, we find: "The decomposition of 24 into factors would be $24 = 6 \times 4$." One wonders why not 8×3 or 12×2. The key to this confusion seems to be the lack of understanding of the role of *uniqueness* of prime factorization. (This does not even appear in the summary given on p. 87.) Moving to the proof of the main result, there are in general two avenues; on one, the assumption that there are only finitely many primes, leads to the construction of an impossible number $R = (2 \times 3 \times \cdots \times P) + 1$ which cannot exist since it is neither prime nor composite. On the other road, one assumes that all primes less than N have been found, and a process is found for constructing a prime larger than N. In the present approach, the authors do not know which road they are on; in particular, page 110 would seem to have two correct responses.

In Chapter 5, and later work, the ignorance of the authors in matters of the number system and its axiomatics is a handicap. On page 123, one finds

$2+(3+5)$ means: "find the sum of 3 and 5 and add that to 2."

This expression can in fact only mean the number 10; it is neither a sentence nor a command.

More serious is the constant confusion of "$-$" as a sign (when placed before either a number or a letter, it makes them negative—see pp. 154, 158, 162), as a unary operation (as in $-(a-b+c)$ on page 187) and as a binary operator in $a-b$. Thus, one finds perpetuated such nonsense as the following *definitions*: [*sic*]

$-a$	is $0-a$	page 146
$a-b$	is $-(b-a)$	page 162
$a+(-b)$	is $a-b$	page 170
$a-(-b)$	is $a+b$	page 177

"when you move a to the other side, you change its sign"
"a fraction is an implied division" (page 203)

Mathematical induction is presented as a way to prove things, not as a property of the number system. I was, however, pleased to see $\sqrt{a}$ defined as the positive root of $x^2 = a$ (although existence problems here, and elsewhere, were avoided).

Summarizing, I find much of value in the device; I think that it is a worth-

while experiment. But, I strongly urge the authors and the publishers to make sure that what they are wrapping in such a fancy container is not so outmoded or incorrect as to be worthless. Perhaps they can seek ready refuge in the sales figures; however, the present crisis in mathematics education does not allow any reputable firm to shirk its responsibility in such matters.

R. C. BUCK
University of Wisconsin

Linear Algebra. By G. Hadley. Addison-Wesley, Reading, Mass., 1961. ix+290 pp. $6.75.

This book is written for people who are *not* mathematicians. The emphasis is upon an intuitive rather than an axiomatic development. For example, several linear models are described in the first chapter. Computational procedures are stressed; numerical exercises are included. The assertion that the mathematical prerequisites include only college algebra and elements of analytic geometry seems exaggerated unless the student has acquired considerable mathematical maturity while studying other fields.

Vectors, matrices, determinants, linear transformations, systems of linear equations, characteristic value problems, and quadratic forms are included, as would be expected. The inclusion of a chapter on convex sets and n-dimensional geometry is noteworthy and provides a basis for considering properties of sets of equations and inequalities. This introduction of linear programming reflects a major interest of the author. One of his aims is to prepare persons with a limited mathematical background for the study of linear programming. Persons sharing the author's aim of introducing linear algebra to students from other fields will find this book worthy of serious consideration.

BRUCE E. MESERVE
Montclair State College

Real Variable. By James M. Hyslop. Interscience, New York, 1960. viii+136 pp. $1.95.

This is another readable little book in the University Mathematical Texts Series. It is, as the author points out, intended for students at the beginning of their study of analysis and ranges over such elementary topics as bounds of sets and of functions, the theory of limits, continuity and differentiation, and the properties of the simple functions of analysis.

Although much of the text follows the classical approach, the last chapter is of special interest in its derivations of the circular functions and their properties through the use of limits.

Few errors are to be found. However, on page 94, exercise (2), there is an obvious error in asking the reader to show that for $x \geqq 0$, $1+x+\frac{1}{2}x^2 < e^x < (1+x+\frac{1}{2}x^2)/(1-\frac{1}{6}x^3)$.

FRANK R. OLSON
University of Buffalo

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., *University of Buffalo*

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Idaho: Mr. S. S. Mitra, University of Washington, has been appointed Assistant Professor; Associate Professor Hans Sagan has been promoted to Professor and Head of the Department of Mathematics.

Miss Margaret Bullock, Agnes Scott College, has accepted a position as Mathematician with the Army Map Service, Arlington, Virginia.

Dr. M. L. Coffman, United Aircraft Corp., Windsor Locks, Connecticut, has been appointed Head of the Department of Physics at Oklahoma City University.

Assistant Professor R. E. Ekstrom, University of Florida, has been appointed Associate Professor at the University of Nebraska.

Professor Ky Fan, Wayne State University, has been appointed Professor at Northwestern University.

Professor G. E. Forsythe of Stanford University has been named Director of a newly formed Computer Science Division within Stanford's Department of Mathematics. This Division is planned as a center of activity in the use of automatic digital computers in all areas of research and education. Professor Forsythe also serves as Director of the Stanford Computation Center.

Mr. R. C. Frascatore, University of Maine, has been appointed Instructor at St. John's University.

Dr. T. C. Fry has retired as Vice President for Research and Engineering of the Sperry-Rand Corporation and is now a consultant to the director of the National Center for Atmospheric Research (NCAR), Boulder, Colorado.

Dr. Seymour Geisser, National Institute of Mental Health, has been appointed Chief of Biometry of the National Institute of Arthritis and Metabolic Diseases.

Assistant Professor John Greever, Florida State University, has been appointed Assistant Professor at Harvey Mudd College.

Professor L. Aileen Hostinsky, Pennsylvania State University, will be on leave for the academic year 1961–62 as Visiting Professor at Mt. Holyoke College.

Mr. J. F. Keoski, Fresno State College, has accepted a position as Research Engineer with North American Aviation, Space Information and Systems Division, Downey, California.

Mr. R. E. Kirsammer, University of Michigan, has accepted a position on the Technical Staff of the Mitre Corporation, Bedford, Massachusetts.

Professor L. D. Kovach, Pepperdine College, has received a part-time appointment as Visiting Professor at the University of California, Los Angeles. He will work with the Curriculum Study Committee in the Department of Engineering.

Dr. L. A. MacColl has retired as a member of the Technical Staff of the Bell Telephone Laboratories, New York, New York and has been appointed Professor at the Polytechnic Institute of Brooklyn.

Professor Harriet F. Montague, University of Buffalo, has been appointed Acting Chairman of the Department of Mathematics to succeed Professor H. M. Gehman who has resigned in order to serve as Executive Director of the Mathematical Association of America. Professor Gehman will continue to teach on a limited basis.

Mr. C. Q. Moo, Convair, San Diego, California, has accepted a position as Senior Reliability Engineer with International Electric, Paramus, New Jersey.

Assistant Professor A. L. Rabenstein, Allegheny College, has been appointed Assistant Professor at Pennsylvania State University.

Associate Professor R. G. Selfridge, Miami University, has been appointed Associate Professor at the University of Florida.

Assistant Professor Konrad Suprunowicz, University of Idaho, has been appointed Associate Professor at Utah State University.

Dr. Patricia A. Tucker, University of Wisconsin, has been appointed Instructor at the University of Illinois.

Mr. M. N. Vesely, Senior Supervisor of Mathematics in the Pittsburgh Public Schools, has been appointed Assistant Director of Curriculum Development and Research.

Asistant Professor J. H. Walter, University of Washington, has been appointed Associate Professor at the University of Illinois.

Dr. F. J. Weyl has been appointed Deputy Chief and Chief Scientist of the Office of Naval Research.

Dr. P. M. Whitman, Johns Hopkins University, has been appointed Professor and Chairman of the Mathematics Department of Rhode Island College.

Mr. C. R. Woodrow, Austin College, has been promoted to Assistant Professor.

Associate Professor W. L. Zlot, Paterson State College, has been appointed Assistant Professor of Mathematics Education in the Department of Mathematics and Science Education of Yeshiva University.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of the General Mathematics Examination of the 1961 Preliminary Actuarial Examinations are as follows:

FIRST PRIZE OF $200

Wells, John C.	Massachusetts Institute of Technology

ADDITIONAL PRIZES OF $100 EACH

Bender, Edward A.	California Institute of Technology
Butler, William A.	Queen's University
Corwin, Lawrence J.	Harvard University
Hochster, Melvin	Harvard University
Mather, John N.	Harvard University
Segal, David M.	Harvard University
Waterhouse, William C.	Harvard University
Weiss, Norman J.	Harvard University

The Society of Actuaries has authorized a similar set of nine prizes for 1962. The Preliminary Actuarial Examinations consist of two examinations: The General Mathematics Examination (based on the first two years of college mathematics), and The Probability and Statistics Examination. The 1962 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians, and will be administered by the Society of Actuaries at centers throughout the United States and Canada on November 16, 1961 and on May 9, 1962. The closing date for applications is April 1, 1962. Further information concerning these Examinations can be obtained from the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

GRADUATE LABORATORY DEVELOPMENT PROGRAM

The National Science Foundation announces that March 1, 1962 is the next closing date for receipt of proposals in the Graduate Laboratory Development Program. This program requires at least 50 percent participation by the institution with funds derived from non-Federal sources.

Purpose of the grants is to aid institutions of higher education in modernizing, renovating, or expanding graduate-level basic research laboratories used by staff members and graduate students. Only those departments having an on-going graduate training program leading to the doctoral degree in science at the time of proposal submission are eligible at present.

Proposals, as well as requests for additional information, should be addressed to: Office of Institutional Programs, National Science Foundation, Washington 25, D.C.

FELLOWSHIP AND RESEARCH OPPORTUNITIES

The Division of Mathematics, National Academy of Sciences-National Research Council calls attention to a variety of fellowships and other support for basic research in mathematics to be awarded by agencies of the Federal Government during the year 1961-62. The bulletin "A Selected List of Major Fellowship Opportunities and Publications for Educational Support" is available from the Fellowship Office, National Academy of Sciences-National Research Council, 2101 Constitution Avenue, Washington 25, D.C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-SECOND SUMMER MEETING OF THE ASSOCIATION

The Forty-second Summer Meeting of the Mathematical Association of America was held at Oklahoma State University, Stillwater, Oklahoma, from Monday, August 28 through Wednesday, August 30, 1961, in conjunction with summer meetings of the American Mathematical Society, the Society for Industrial and Applied Mathematics, the Econometric Society, the Pi Mu Epsilon Fraternity, and Mu Alpha Theta. There were registered 580 persons (not including members of their families), of whom 394 were members of the Association.

Sessions of the MAA were held on Monday morning and afternoon, on Tuesday morning and on Wednesday afternoon. All sessions were held in the Upper Ballroom of the Union Building at Oklahoma State University. Presiding officers at the three Earle Raymond Hedrick Lectures were President A. W. Tucker, First Vice-President A. S. Householder, and Second Vice-President R. A. Rosenbaum, at the remainder of the first session Professor H. M. MacNeille, at the remainder of the second session Professor D. E. Richmond, at the session on applied mathematics Professor M. E. Shanks, and at the session on computers Dr. C. F. Kossack. The tenth series of Earle Raymond Hedrick Lectures were delivered by Professor R. H. Bing of the University of Wisconsin. The Program Committee for the meeting consisted of H. M. MacNeille, Chairman, Jacob Korevaar, C. F. Kossack, G. E. Latta, M. E. Shanks.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *Topology of 3-Space*, Lecture I, by Professor R. H. Bing, University of Wisconsin.

These lectures will be published in one of the publications of the Association.

Progress Report by the Committee on the Undergraduate Program in Mathematics, by Professor R. J. Wisner, Executive Director, Committee on the Undergraduate Program in Mathematics.

Reporting on the first full year of operation of the Committee on the Undergraduate Program in Mathematics under its 1960 grant from the National Science Foundation, the speaker discussed briefly the past accomplishments, the present activities, and the future plans of the Committee. While most Committee effort has been concentrated on the Panel on Teacher Training, it was also shown that considerable progress has been made by the Panel on Mathematics for the Physical Sciences and Engineering; the Panel on Mathematics for the Biological, Management, and Social Sciences; and the Panel on Pre-Graduate Training. The program for the future activities of the Committee includes a Consultants Bureau, the publication of action manuals, the establishment of experimental centers, and the institution of cooperative seminars for college teachers.

Undergraduate Research in Mathematics, by Professor Seymour Schuster, Carleton College.

The speaker presented a substantive report of a conference on undergraduate mathematical research that was sponsored by the National Science Foundation and held at Carleton College June 19–23, 1961. General remarks were made in answer to questions posed to the conference relative to the aims, criteria and role of mathematical research at the undergraduate level. The speaker's own views were expressed on the possibilities of—and the need for—introducing *research type* of activity at all levels of mathematical education to emphasize the creative and dynamic aspects of the subject in contrast to the pattern of dwelling merely on the achievements of predecessors.

Implications of the New School Mathematics for Colleges, by Professor Paul C. Rosenbloom, University of Minnesota.

Calculus, supplemented by analytic geometry, will be the standard course for college freshmen. Students will be no more gifted than before, but will be more sophisticated mathematically. They will have worked with sets, inequalities, perhaps with matrices and vectors, and will have had much more experience than our present students with proofs and nonroutine problems. They will be bored with the usual calculus course devoted primarily to formal manipulations. They will expect mathematics to deal with ideas. It is not clear yet how soon they will be ready for a rigorous discussion of the real numbers. Colleges must prepare now.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Bing.

Mathematical Training in Soviet Institutes and Universities, by Professor N. D. Kazarinoff, University of Michigan.

An account of mathematics curricula at Soviet teachers colleges, engineering institutes, and universities was given. The quality of professors and students was discussed along with what is expected of them. Mention of Soviet efforts to improve qualifications of teachers on the job and of Soviet "SMSG" activities was made.

The Mathematical Requirements for the Non-Teaching Mathematician, by Dr. Morris Ostrofsky, Westinghouse Research Laboratories.

This paper is based on results collected in the 1960 survey by the Bureau of Labor Statistics. The survey was prepared for the National Science Foundation in cooperation with the Mathematical Association of America. The mathematical requirements were indicated by the individuals working in the field of mathematics and by their supervisors. Greatest unfilled needs appear to be in probability, statistics, numerical analysis, and calculus of finite differences.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Bing.

Session on Applied Mathematics

Certain Questions in the Theory of Nonlinear Oscillations, by Professor W. S. Loud, University of Minnesota.

A problem of interest from many points of view is that of finding periodic solutions of a nonlinear ordinary differential equation in which time, the independent variable, enters in a periodic manner if it enters at all. Existence theory for such solutions can involve topological fixed-point methods. One constructive approach is the perturbation of known periodic solutions. There are many approximate methods which predict periodic solutions in a nonrigorous manner, and which have ample experimental verification. It is likely that important theoretical progress on such problems can be made with the use of machine studies.

The Propagation of Error in the Numerical Solution of Ordinary Differential Equations, by Professor Peter Henrici, University of California, Los Angeles.

The paper presents some recent results on convergence and error propagation in the numerical solution of initial value problems for systems of ordinary differential equations by linear multistep methods in the sense of G. Dahlquist (Math. Scand., *4*, 33–53), including asymptotic formulas (for small stepsizes) for the discretization error and for both mean and covariance matrix of the rounding error. As an example, the paper discusses the integration of the equations of the two-body problem. Some unsolved problems are mentioned, and some suggestions concerning undergraduate research participation in numerical analysis are made at the end of the paper.

Recent Advances in the Numerical Solution of Elliptic and Parabolic Partial Differential Equations, by Professor D. M. Young, Jr., University of Texas.

Recent work on the numerical solution of elliptic and parabolic partial differential equations has been largely concerned with effective procedures for solving finite difference equations obtained in replacing the domain of the independent variables by a finite point set and the differential equation by a difference equation. For many problems involving elliptic equations, the successive (point) overrelaxation iterative method for solving large systems of linear equations is being replaced by faster methods involving simultaneous modifications on one or more lines, and/or the use of Chebyshev polynomials, and by alternating direction methods, the latter being useful for parabolic equations as well.

FOURTH SESSION OF THE ASSOCIATION

Business Meeting of the Association.

Session on Computers

The Logic and Mathematics of Automatic Programming, by Dr. M. de V. Roberts, International Business Machines Corporation.

Compilers are designed to make man-machine communication easier. There are three aspects of the compiler that can be examined. Firstly the meta-language is chosen. Secondly the translation from the meta-language to machine-language. Thirdly the optimization of the machine language program. There are two philosophics extant. One calls for the fastest possible compiling at the expense of a good object program. The other calls for very good object programs. It seems possible to achieve both ends in one system. In the future, optimization will probably become so important that the idea of a fast inefficient compiler will become obsolete.

What Goes on in a University Computer Center? by Professor G. E. Forsythe, Stanford University.

The Stanford Computation Center, with its Burroughs 220 computer, is described as an example of a university computing center. It houses divers individuals linked mainly by the mis-

sionary's faith that automatic digital computers are the leading technological development of the century. Our activities include processing the files for university fund-raising, advice and computing for engineering and science projects, data reduction for research in medicine, teaching at least 500 students per year how to write BALGOL (our ALGOL-like input language), writing in BALGOL a program to translate BALGOL into an intermediate language (and thence later into another machine language), developing "range arithmetic" as a step in the automation of error analysis, programming football card stunts on the 220, simulating business enterprises, testing Mordell's conjecture (this MONTHLY, vol. 68 (1961), pp. 472–473), using the 220 to grade student homework and examinations, etc., etc. Total computer usage will double approximately each year.

Can Computers Think? by Professor John W. Carr III, University of North Carolina.

To a mathematician, any reasonable operational or physiological definition of "to think" must be a very elaborate one, perhaps even unattainable. At any rate, under any reasonable definition now proposable, the answer must be unqualified: General-purpose digital computers, of the type with which university mathematicians are familiar, cannot "think." Operational development of "thinking machines" under the criteria that Turing first proposed is dependent on the elaboration of very complex theories not yet more than begun. Reproduction of physiological thought activity awaits combination of much mathematical analysis and very detailed experiment. Nevertheless, the present ability of computer programs to perform certain commonly taught nonnumerical mathematical algorithms: differentiation, anti-differentiation, simple theorem proving, etc., should be a challenge to teachers of mathematics to re-evaluate much of the rote manipulation that is required of mathematics students in such areas.

SPECIAL SESSIONS OF THE ASSOCIATION

On Tuesday afternoon at 3:15 P.M. an open meeting on High School Contests was held in the Senate Chambers with 27 persons present. Second Vice-President R. A. Rosenbaum presided. Since the results of this year's examination indicated that it was more difficult than last year's, the Committee was urged to lower the level of difficulty, in particular to make a strong effort to make the first 10 questions easier. The introduction of three questions on "modern mathematics" into this year's examination was favorably received; it was felt, however, that the introduction of additional such questions should be at a moderate pace.

The three films produced by the Committee on Production of Films were shown in the Classroom Auditorium as follows: On Monday evening at 7:30 P.M. a sound film in color on "Mathematical Induction" with Professor Leon Henkin as lecturer, made by Palmer Films, San Francisco, and consisting of two parts, each 30 minutes long; on Tuesday at 3:15 P.M. a black and white sound film "Theory of Limits" with Professor E. J McShane as lecturer, made by Mr. Herbert Kerkow, New York, and consisting of four parts of length 34, 14, 11, and 13 minutes, respectively; on Tuesday at 4:30 P.M. and also at 7:30 P.M. a kinescope film "Integration" with Professor Edwin Hewitt as lecturer, made at the Educational TV studio of the University of Washington.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday at 4:30 P.M. in the Senate Chambers of the Union Building at Oklahoma State University with twenty-seven members present.

The Board elected the following associate editors of the MONTHLY for five year terms effective January 1, 1962 with special responsibilities indicated in parentheses: A. A. Blank (general cognizance of editorial office, with responsibility for action, when feasible and appropriate, in the absence or incapacity of the Editor), J. A. Brown (Mathematical Education Notes), Herbert Busemann, J. H. Curtiss, Howard Eves (Elementary Problems), Marshall Hall, Jr., L. M. Kelly, J. R. Mayor (Mathematical Education Notes), L. J. Montzingo, Jr., (official in the publication office at Buffalo responsible for the

MONTHLY), J. M. H. Olmsted (Classroom Notes), M. H. Protter (Mathematical Notes), R. A. Rosenbaum (Recent Publications), E. P. Starke (Advanced Problems), H. S. Zuckerman.

The Board also elected the following Editorial Board of the MATHEMATICS MAGAZINE for a three year term effective January 1, 1961, with special responsibilities indicated in parentheses: Roy Dubisch (Editor, Miscellaneous Notes), Homer V. Craig (Editor, Current Papers and Books), Ali R. Amir-Moez, Paul W. Berg, Charles W. Trigg, the latter three as Associate Editors. The Board at its previous meeting had elected Professor R. E. Horton as Editor of the MATHEMATICS MAGAZINE.

The Board voted to invite Professor Andrew M. Gleason of Harvard University to deliver the eleventh series of Earle Raymond Hedrick Lectures at the 1962 Summer Meeting.

Upon the recommendation of the Committee on the Structure of the Government of the Association under the chairmanship of Professor H. M. Gehman, the Board voted to instruct the Secretary to prepare the necessary amendments to the By-Laws to provide for the following:

1. That members of the Finance Committee be ex-officio members of the Board of Governors;

2. That the President shall be an ex-officio member of the Finance Committee;

3. That the President shall be Chairman of the Executive Committee and of the Finance Committee;

4. That the office of President-elect be established as soon as convenient; the President-elect to be elected by the same procedures as those now specified for the election of President; the President-elect to serve for one year and then automatically to become President; the President-elect to be a member of the Board of Governors and of the Executive Committee; the President to continue as a member of the Executive Committee for one year after the end of his term as President.

The Board approved the following schedule of future meetings: Sheraton-Gibson Hotel, Cincinnati, Ohio, January 1962; University of British Columbia, August 1962; University of California, Berkeley, January 1963; University of Colorado, August 1963; Miami, Florida, January 1964; University of Michigan, August 1964; Cornell University, August 1965; Rutgers-The State University, New Brunswick, August 1966.

The Board gave approval to the establishment of two types of institutional memberships, namely academic and corporate (for dues and privileges see page 956 of this issue of the MONTHLY).

The Board also authorized the establishment of reciprocity agreements with foreign mathematical organizations and voted that for foreign members of organizations having established such agreements with the Association payment of the initiation fee shall be waived.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Wednesday afternoon with President Tucker presiding. The Secretary reported that the membership of the Association was 11,110, an increase of 12% since the corresponding date last year.

The Secretary then reported on some of the actions taken by the Board of Governors on Monday.

The amendments to the By-laws which were printed on page 527 of the May 1961 issue of the MONTHLY were unanimously adopted.

MEETING OF SECTION OFFICERS

A meeting of representatives of the Sections of the Association was held on Tuesday evening in the Senate Chambers. Second Vice-President R. A. Rosenbaum presided.

Approximately fifty persons were present, representing 25 of the 27 Sections of the Association. Professor L. J. Montzingo, Jr., gave a report as Chairman of the Committee on Sections, Professor C. O. Oakley as Chairman of the Committee on Secondary School Lecturers, and Professor C. T. Salkind as Chairman of the Committee on High School Contests.

A "Discussion of the Role of the Sections in Implementing the Recommendations of CUPM" was led by Professor R. J. Wisner, Executive Director of CUPM. A "Program for Meeting CUPM recommendations in Oklahoma Institutes of Higher Learning" was presented by Professor R. R. Murphy of Panhandle Agricultural and Mechanical College; the "Iowa Colleges' Reactions to CUPM Recommendations" were presented by Dean E. L. Canfield of Drake University, and the "CUPM Recommendations in West Viriginia" were discussed by Professor Evan Johnson, Jr., of Pennsylvania State University. At his talk, Professor Murphy presented a report on a meeting of Oklahoma College Teachers on the CUPM recommendations held May 12 and 13, 1961, at Oklahoma State University at which motions were approved setting a time table for Oklahoma Colleges in meeting the CUPM suggestions no later than 1965. Copies showing the details of this plan were distributed at the meeting.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. The colloquium speaker was Professor G. W. Mackey of Harvard University, who spoke on "Infinite Dimensional Group Representations," and invited addresses were given by Professor Leon Ehrenpreis of Yeshiva University on "Some Applications of the Theory of Distributions to Lacunary Series" and Professor Stephen Smale of Columbia University on "A Survey of Some Recent Developments in Differential Topology."

The Society for Industrial and Applied Mathematics held three sessions on Thursday. At the second session at 2:00 P.M., the Second John von Neumann Lecture was presented by Professor Marc Kac of the Rockefeller Institute on "Probability Theory: its Role and its Impact"; at the third session at 8:00 P.M., an invited address was given by Professor G. E. Forsythe of Stanford University on "Highlights from the Gatlinburg Matrix Symposium."

The Econometric Society met from Wednesday through Friday. Professor Jack C. Kiefer of Cornell University delivered a lecture on "Current Directions in Statistical Theory" at a joint session with the American Mathematical Society on Thursday at 3:30 P.M.

The Pi Mu Epsilon Fraternity held a Council meeting at breakfast and a luncheon meeting at noon on Tuesday. A session for five contributed papers was held starting at 1:30 P.M. on Tuesday.

Mu Alpha Theta, the National High School and Junior College Mathematics Club, held a breakfast meeting at 7:00 A.M. on Wednesday.

The Conference Board of the Mathematical Sciences held a public session on Friday at 10:30 A.M., at which reports of its activities were presented.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of L. W. Johnson, Chairman; H. L. Alder, R. A. Hultquist, C. E. Marshall, R. D. McDole, H. S. Mendenhall, G. L. Walker, J. W. T. Youngs.

Registration headquarters were located in the second floor corridor of the Union Building. Dormitory and cafeteria accommodations were provided by Oklahoma State University. The Mathematical Sciences Employment Register was maintained in the Terrace Room and textbook exhibits in Parlors D, E, and F.

The Department of Mathematics was host at a tea on Tuesday from 4:00 P.M. to 6:00 P.M. in the Chinese Lounge of the Student Union. A Western-style barbecue was held Wednesday at 6:00 P.M. in the Theta Pond area of the Campus. The usual SIAM social evening was held on Thursday evening at 9:00 P.M. at Lake Carl Blackwell.

The following resolution was adopted at the Business Meeting of the American Mathematical Society on Wednesday morning: "Be it resolved that the American Mathematical Society, of itself, and on behalf of the Mathematical Association of America, the Society for Industrial and Applied Mathematics, the Econometric Society, the Pi Mu Epsilon Fraternity, and Mu Alpha Theta express its thanks to Oklahoma State University, to the members of the Department of Mathematics, and, in particular, to Professor L. Wayne Johnson, for the facilities and the arrangements which have made the meetings a success."

HENRY L. ALDER, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The Annual Spring Meeting of the Minnesota Section of the Mathematical Association of America was held on May 13, 1961, at St. Cloud State College, St. Cloud, Minnesota. Professor Rowland Anderson, St. Cloud State College, presided at the morning session, and the Section Chairman, Professor Fulton Koehler, University of Minnesota, presided at the afternoon session. There were 77 persons registered for the meeting, of whom 66 were members of the Association.

At the business meeting Professor Koehler very briefly described the 1961 Minnesota High School Mathematics Contest, for which the Minnesota Section was a sponsoring agency. The following officers were elected to serve during the school year of 1961–1962: Chairman, Professor Charles Hatfield, University of North Dakota; Secretary-Treasurer, Professor Murray Braden, Macalester College; Members of the Executive Committee, Professor H. M. Anderson, Gustavus Adolphus College, and Professors Elizabeth Carlson and Fulton Koehler, University of Minnesota.

Professor Warren Thomsen, Mankato State College, distributed and briefly discussed a summary of the results of a survey of the thinking of Minnesota College Mathematics teachers regarding the high school preparation of college mathematics students.

By invitation of the Section, Professor S. E. Warschawski, of the University of Minnesota, gave the main address of the morning, which was entitled: *Existence proofs and constructive methods in conformal mapping.*

One feature of the afternoon program was a discussion of the proposed Doctor of Arts degree for mathematicians. Professor Warren Loud, University of Minnesota, presented arguments in favor of the new degree, and Professor Gerhard Kalisch, University of Minnesota, spoke against it. A general discussion followed. Though no vote was taken, the majority of those present seemed opposed to the new degree.

The following short papers were presented:

1. *Cross-ratio and the notion of order*, by Professor Seymour Schuster, Carleton College.

A description is given of the role of cross-ratio in defining order in geometry. Further, it is shown how these notions can be carried over into certain nonorderable geometries to introduce a quasi-ordering. In doing so, it is interesting to observe which properties of the field, that defines the geometry, contribute to obtain the various order properties.

2. *The Hamline University class plan for St. Paul "D" Program students*, by Professor Walter Fleming, Hamline University.

Twenty outstanding high school seniors who are participating in the St. Paul Developmental Program enrolled in one regular college course in science or mathematics at Hamline University while continuing their pre-college training at their high schools. Seventeen of these enrolled in an accelerated freshman mathematics course. In the eleventh grade these students had used the School

Mathematics Study Group textbook (11th grade). The performance of these students was as good as that of the selected college students (average high school percentile rank 95.1), in the same course. Both groups of students maintained a high level of enthusiasm.

3. *On a certain type of perfect number*, by Mr. Barry Mackichan, Central High School, Grand Forks, North Dakota, introduced by Professor R. C. Staley, Macalester College.

The author showed by inequalities that there are no odd perfect numbers with only two distinct prime factors. Restrictions were placed on odd perfect numbers with three or more distinct prime factors.

4. *On Hermann Weyl's interpretation of "normalizers,"* by Professor Joon Fang, St. John's University, Collegeville, Minnesota.

This is an examination of Hermann Weyl's effort to bluff Kant on the strength of a mathematical concept, viz. "normalizer" (*cf.* H. Weyl, *Classical Groups*, p. 22f.), which in itself has as little epistemological weight as the apocryphal case of Euler vs. Diderot (*cf.* this MONTHLY, vol. 61, 1954, p. 77ff.). In general, the familiar disclaimer: "without making any recourse whatever to our intuitive knowledge" (*cf. e.g.* R. H. Bing, *Elementary Point Set Topology*, p. 1), is almost as fictitious as movie disclaimers. No abstract mathematics is, or will ever be, absolutely abstract.

5. *On the Riccati and Bessel differential equations*, by Professor F. J. Arena, North Dakota State University.

The speaker discussed a differential equation of the Riccati type that appeared in a paper on curves published by John Bernoulli in 1694, the infinite series solution of this equation found by James Bernoulli in 1703, Daniel Bernoulli's solution of Riccati's equation, and finally the relation between the Riccati and the Bessel differential equations.

MURRAY BRADEN, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The 14th annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Washington, Seattle, Washington on June 17, 1961 jointly with the Pacific Northwest Section of the Society for Industrial and Applied Mathematics. There were 83 members of the two organizations registered.

At the business meeting the following officers were elected: Chairman, Professor R. E. Gaskell of Oregon State College; Vice Chairman, Professor H. M. Gelder of Western Washington College; Secretary-Treasurer, Professor L. H. McFarlan of the University of Washington. It was decided to skip the usual June meeting of the section for the year 1962 on account of the Summer Meeting of the Association to be held later in the year at Vancouver, B. C.

The morning portion of the program consisted of six 20 minute papers followed by an invited one-hour address entitled, "Singular Perturbations of Differential Equations", by Professor Wolfgang Wasow of the University of Wisconsin. Professor O. W. Rechard of Washington State University presided. The afternoon portion of the program, presided over by Professor J. M. Kingston of the University of Washington, consisted of a Progress Report on the Recommendations of the Panel on Teacher Training by Professor J. H. McKay of Michigan State University Oakland. This was followed by a discussion period.

Abstracts of the papers follow:

1. *A method for solving algebraic equations with coefficient parameters*, by Professor R. D. Stalley, Oregon State University.

A reducible algebraic equation with coefficient parameters is defined and a storage-saving procedure is described for obtaining the roots of a reducible equation corresponding to the sets of values for the coefficient parameters within severe time limits. Two very general examples are given. A nonreducible equation may be solved similarly by finding and using an approximating reducible equation.

2. *Remarks on analytical methods for fuel-payload programming*, by Dr. J. D. Esary and Mr. D. L. Johnson, Boeing Scientific Research Laboratories, Seattle, Washington.

A problem which occurs in commercial aircraft design and operation is that of deciding the maximum payload that can be carried over a given route, and also the minimum fuel load required to do this, when conditions governing the performance of the aircraft, such as weather parameters and operating procedures, are fixed. The objective is the establishment of a mathematical model for the relationships between gross initial weight, gross terminal weight and fuel consumption for any fuel burning vehicle. The model treats fuel consumption as a function of either of these two weights. The application of the model to the load programming problems of the type represented by the fuel-payload problem for commercial aircraft is illustrated.

3. *On higher-order information about the output of nonlinear devices*, by Professor W. M. Stone, Oregon State University and Boeing Airplane Company.

4. *A stochastic treatment of a control system with breakdown and repair*, by Dr. R. W. Rishel, Boeing Airplane Company, Seattle, Washington, introduced by the Secretary.

5. *On an improved solution to the problem of the crossing of an arbitrary level by noise plus a fixed sinusoid*, by Dr. J. D. Wheelock, Boeing Airplane Company, Seattle, Washington, introduced by the Secretary.

The problem described in the title was solved by S. O. Rice (Bell System Tech. J. vol. 27, 1948). This paper resolves the problem, reducing the exposition to elementary terms. The approach used suggests an asymptotic approximation for the expected number of crossings. The series developed has computational advantages over the exact solution which requires a numerical integration of tabulated functions. In addition, with suitably restrictive assumptions, the well-known anomaly associated with Markovian processes is removed in a natural way.

6. *The algebra of semimagic matrices*, by Professor F. D. Parker, University of Alaska.

Semimagic matrices (squares matrices whose row and column sums are all equal) are closed under multiplication and addition. Their algebra has interesting properties. For example, if the matrix elements are chosen from a ring R, the matrices themselves form a ring homomorphic to R. Results of previous investigators are discussed. Some of the author's current results are described as well as questions now under investigation.

7. *Singular perturbations of differential equations*, by Professor Wolfgang Wasow, University of Wisconsin. (By invitation)

Singular perturbations are perturbations that increase the order of a differential equation. A summary of the history, principal results and methods of this theory for ordinary differential equations was given. This included a discussion of the phenomena of nonuniform convergence occurring in these problems and their connection with the boundary layer theory of fluid dynamics, and an explanation of the role played by the asymptotic series in the actual solution of such problems.

L. H. McFarlan, *Secretary*

ESTABLISHMENT OF INSTITUTIONAL MEMBERSHIPS IN THE MAA

The Board of Governors at its meeting on August 28, 1961, at Oklahoma State University gave approval to two types of institutional memberships, namely academic and corporate. Academic memberships are limited to teaching institutions. Annual dues are $25. Each academic member is entitled to receive two copies of the Monthly or one copy with the privilege of nominating one person to ordinary membership in the Association. Such a person if not already a member of the Association shall be elected to ordinary membership in the usual fashion and shall be exempt from the initiation fee and from the payment of annual dues as long as he is a nominee of the academic member. He shall have all the rights and privileges of other ordinary members.

Corporate members pay annual dues of $200. Each corporate member is entitled to receive two copies of the MONTHLY or one copy with the privilege of nominating one person to ordinary membership in the Association. The statements made about a nominee of an academic member shall be equally applicable also for a nominee of a corporate member.

Any institution interested in an institutional membership should write for an application blank to Professor H. M. Gehman, Executive Director, Mathematical Association of America, University of Buffalo, Buffalo 14, N.Y.

HENRY L. ALDER, *Secretary*

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Annual Meeting at the Sheraton-Gibson Hotel in Cincinnati, Ohio, on January 23, 24, and 25, 1962. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

There is no charge for registering either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $1 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position-description forms, which must be completed and returned to Providence not later than January 3, 1962, in order to be included free of charge in the listings at the Annual Meeting in Cincinnati. Forms which arrive after this closing date, but before January 15, will be included in the register at the meeting for a late registration fee of $3.00, and will also be included in the printed listings, but not until ten days after the meeting. The printed listings will be available for distribution both during and after the meeting.

It is essential that applicants and employers register at the employment register desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

CUPM ESTABLISHES CONSULTANTS BUREAU

The Committee on the Undergraduate Program in Mathematics announces the establishment of a Consultants Bureau for the purpose of aiding colleges and universities in upgrading and revising their present undergraduate mathematics offerings or with the planning of new curricula. Upon request, consultants will visit departments of mathematics for a period of two days, the expenses and honoraria being shared by the host institution and by CUPM. Only a limited number of such visits are possible during the academic year 1961–62, it being expected that the thirty consultants can serve a total of approximately 150 schools. The consultants are:

Richard D. Anderson, Louisiana State University
John D. Baum, Oberlin College
Roy Dubisch, University of Washington
Lincoln K. Durst, William Marsh Rice University
Philip Dwinger, Purdue University
John V. Finch, Beloit College
Marion K. Fort, Jr. University of Georgia
Leon A. Henkin, Institute for Advanced Study
James A. Hummel, University of Maryland
Bernard Jacobson, Franklin and Marshall College
Paul B. Johnson, University of California, Los Angeles
Burton W. Jones, University of Colorado
Paul J. Kelly, University of California, Santa Barbara
Eugene E. Kohlbecker, University of Utah
Arthur E. Livingston, Montana State University
Anil Nerode, Cornell University

Robert Z. Norman, Dartmouth College
William R. Orton, University of Arkansas
Billy J. Pettis, University of North Carolina
Mina Rees, City University of New York
Charles E. Rickart, Yale University
Gerald S. Rogers, University of Arizona
Hartley Rogers, Jr., Massachusetts Institute of Technology
Seymour Schuster, University of North Carolina (until December 1961—then Carleton College)
Leland L. Scott, Southwestern at Memphis
William R. Scott, University of Kansas
E. Baylis Shanks, Vanderbilt University
John Wagner, Michigan State University
Elbert A. Walker, New Mexico State University
James H. Zant, Oklahoma State University

Further information and applications for a visit by a consultant may be obtained by writing: Professor Robert J. Wisner, Executive Director, Committee on the Undergraduate Program in Mathematics, Michigan State University Oakland, Rochester, Michigan.

CALENDAR OF FUTURE MEETINGS

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, Spring, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962.

KENTUCKY, University of Kentucky, Lexington, Spring, 1962.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D.C., December 2, 1961.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.

MINNESOTA

MISSOURI, Missouri School of Mines, Rolla, Spring, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.

NEW JERSEY

NORTHEASTERN, November 24, 1962

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, South Dakota School of Mines, Rapid City, Spring, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

SOUTHERN CALIFORNIA, Long Beach State College, March 10, 1962.

SOUTHWESTERN

TEXAS, Rice University, Houston, April, 1962.

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, Spring, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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Edited by JOHN A. BROWN, University of Delaware, and JOHN R. MAYOR, AAAS and University of Maryland

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Numbers refer to pages, **boldface** type indicating a problem solved and solution published; *italics*, a problem solved but the complete solution not published; ordinary type, a problem proposed.

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Edited by LLOYD J. MONTZINGO, JR., University of Buffalo

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EDITORIAL CORRESPONDENCE should be addressed to the incoming Editor, F. A. Ficken, Department of Mathematics, New York University, New York 53, N. Y. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

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INVERTIBLE SPACES

P. H. DOYLE AND J. G. HOCKING, Michigan State University

Our characterization of the n-sphere in terms of inversion [2] led us to define and study the concept of topological invertibility. While modest in scope, invertibility is typical of many new developments and provides an easily accessible example of mathematical research. In the hope that both the results and the means of discovery of those results will be of interest, we present this rambling account of our investigation.

To show how our new subject started, we must describe the characterization of the n-sphere at least in part. Suppose that U is a nonempty open subset of the n-sphere S^n (and we mean the geometric set $\{x: |x| = 1\}$ in E^{n+1}). Regardless of the size and shape of U, we can find an $(n-1)$-sphere S^{n-1} centered at any point of U and lying inside of U. An inversion of S^n in S^{n-1} (essentially a reciprocal radii transformation) interchanges the two components of the complement $S^n - S^{n-1}$ of S^{n-1}, leaving S^{n-1} fixed. Since such an inversion is a one-to-one, continuous mapping of S^n onto itself, the following statement is true:

For each nonempty open set U in S^n there is a homeomorphism h of S^n onto itself with the property that the image $h(S^n - U)$ of the complement of U lies in U.

When we found that the n-sphere is the only n-manifold with this property (see Th. 1 of [2]), an obvious question arose. Are there any other topological spaces (not manifolds, of course) with this same kind of topological invertibility? We immediately noted that the rational points on the n-sphere are inverted (essentially) by the inversion of the n-sphere. It follows immediately that the rational points in n-space constitute a space which may be inverted topologically. This example is not compact, however, and we wanted a compact example.

Suppose one wishes to study compact subspaces of E^1 which contain no 1-cells. The first example which comes to mind is, perhaps, the Cantor set. And we leave it as a simple exercise to show that the Cantor set has the desired topological invertibility. This immediate success in uncovering examples convinced us that a property which characterizes the n-sphere among n-manifolds would be worthy of investigation. A name was chosen and the formal definition set up.

A topological space S is *invertible* if for each nonempty open subset U of S there is a homeomorphism h of S onto itself such that $h(S-U)$ lies in U. The homeomorphism h is called an *inverting homeomorphism* for the open set U.

In this terminology the characterization of the n-sphere given in [2] may be stated as "The n-sphere is the only invertible n-manifold." Indeed, in [2] we show that if an invertible space is locally Euclidean at any point, then it is a sphere.

The first property of the general invertible space we located was part of the proof of our characterization theorem.

THEOREM 1. *If the invertible space S contains a nonempty open set U whose closure $\overline{U}$ is compact, then S is compact.*

Proof. An inverting homeomorphism h for the open set U will carry the closed complement $S-U$ in U. Then, as a closed subset of the compact set $\overline{U}$, $h(S-U)$, and hence $S-U$, is compact. Therefore S is the union of the two compact sets $\overline{U}$ and $S-U$.

COROLLARY. *A locally compact invertible space is compact.*

This very easy result immediately led us to ask for a comparable result involving connectedness in place of compactness. Attempting to prove such a conjecture led quickly to the following result.

THEOREM 2. *If S is an invertible space and contains an open connected set U, then S consists of at most two components. If S is not connected, then U and $S-U$ are the components of S and they are homeomorphic.*

Proof. An inverting homeomorphism h for U carries $S-U$ into U and, since h is onto, $S-U$ must lie in $h(U)$. If S were not connected, then the connected open set would be separated unless $h(U)=S-U$. Thus if S is not connected, the components of S are U and $S-U$ and they are homeomorphic.

It was necessary to introduce a separation property in order to complete the proof of the desired analogue of Theorem 1.

THEOREM 3. *If the invertible space S is a T_1-space and contains a nondegenerate, open, connected set, then S is connected.*

Proof. If S is not connected, then by Theorem 2, U is a component of S and $S-U$ is homeomorphic to U. Since S is a T_1-space, the removal of a point p from U leaves an open set $U-p$. Then there is an inverting homeomorphism g for $U-p$ and $g(S-U)$ lies in $U-p$. But then $g(S-U)$ is a component of S properly contained in the component U. This is contradictory.

COROLLARY. *A locally connected invertible T_1-space either is connected or is the zero-sphere.*

(We are still amused by the complicated characterization of the zero-sphere implicit in this corollary.)

It was at this stage of our investigation that we formulated the following heuristic principle: *If an invertible space has a given local property, then it also has the corresponding global property.* Note that this is not a theorem.

The next two results are typical of those suggested by this principle and there are many we omit because they grow wearisome.

THEOREM 4. *If the invertible space S has a nonempty open set U which, as a subspace, is a T_i-space, $i=0, 1, 2$, then S is a T_i-space.*

Proof. Suppose that U is a T_0-space and let p and q be points of S. If p and q lie in U, the separation of p and q can be made since U is a T_0-space. If p lies in U and q in $S-U$, then the separation of p and q is already accomplished.

Finally, if both points lie in $S-U$, then an inverting homeomorphism for U carries the points into U where the desired separation can be made. If either of the stronger separation properties is assumed, the proof is just as easy.

THEOREM 5. *If the invertible T_1-space S is locally arcwise connected, then S is arcwise connected.*

Proof. By definition, given any point p of an open set U, there is an open set V contained in U and containing p such that every pair of points in V can be joined by an arc lying in U. Let x and y be any two points in S. If x and y lie in V, the desired arc is already assumed. If both x and y lie in $S-V$, an inverting homeomorphism h for V carries x and y into V, an arc joining them is chosen and then the entire arc carried back with the inverse h^{-1} of the homeomorphism h. If x is in V and y in $S-V$, we choose an inverting homeomorphism for the open set $V-x$ and play the same game.

The addition of the T_1 separation property to the hypotheses of our theorems is apparently unavoidable. That is, the property of invertibility seems to have no relation to the separation properties themselves. One of the first examples of a T_1-space that is not a T_2 (Hausdorff)-space which a student might see consists of a countably infinite set S in which a subset is defined to be open if and only if its complement is finite. It is easy to show that a one-to-one mapping of S onto itself which moves only a finite number of points is a homeomorphism. It follows that S is invertible; indeed, this is an example of a connected, compact invertible T_1-space which is not T_2.

In discussing our work here we noted how often we said "map the closed set C into the open set U." This observation led to the following result.

THEOREM 6. *A space S is invertible if and only if, for each proper closed subset C and each nonempty open set U, there is a homeomorphism h of S onto itself such that $h(C)$ lies in U.*

Proof. Certainly a space with the assumed property is invertible. Hence we assume that S is invertible. Given the sets U and C as in the hypotheses, if C lies in $S-U$, an inverting homeomorphism for U satisfies the desired conclusions. If $U-C$ is not empty, then it is open and an inverting homeomorphism for $U-C$ satisfies the theorem. Of course, if $U-C$ is empty, an inverting homeomorphism for $S-C$ followed by one for U yields a homeomorphism h such that $h(C)$ lies in U.

An immediate application of Theorem 6 to T_1-spaces (in which points are closed) showed that an invertible T_1-space S has the following property: *If x is a point of S and U is an open set of S, then there is a homeomorphism h of S onto itself such that $h(x)$ lies in U.* This property has been called *near homogeneity* by Burgess [1] but we were unaware of this name when we found the following:

THEOREM 7. *In a connected invertible T_1-space S the set of cut points either is empty or is dense in S.*

Proof. Let p be a cut point of S if such exists. Then by Theorem 6 there is homeomorphic image of p in every open set in S. But the property of being a cut point is invariant under homeomorphisms and hence every open set contains a cut point.

It will be noted that no more than the near homogeneity of S is used in the above proof and so Theorem 7 may be generalized in the obvious way. The same is true about the next theorem which was the product of an obvious comment about Theorem 7. The remark? Well, one of us said "What's so special about cut points in this kind of space?"

Let $G(S)$ denote the group of homeomorphisms of the space S onto itself. If x is a point of S, then the *orbit of* x, O_x, is the set of all images of x under elements of $G(S)$.

LEMMA 8. *The orbit of each point in an invertible space is dense in the space.*

The proof of Lemma 8 is precisely the same as that of Theorem 7, of course.

This lemma started an investigation of the orbits of points. By the very definition, an orbit O_x is homogeneous, that is, given any two points y and z in O_x, there is a homeomorphism in $G(S)$ carrying y onto z. Furthermore, if we consider such an orbit as a subspace, it is easy to see that an orbit is itself an invertible space.

LEMMA 9. *If S is invertible and x is any point in S, then the orbit O_x of x is a homogeneous invertible subspace of S.*

THEOREM 10. *Every invertible space is the union of disjoint homogeneous invertible subspaces, each dense in the space.*

Proof. Combine Lemmas 8 and 9 with the very easily proved fact that the orbits are equivalence classes.

We had only the following example of a nonhomogeneous invertible space at the time we obtained Theorem 10. Let S consist of an open interval of real numbers and a point p not in the interval. A subset of S is open if and only if its complement is a countable subset of the interval. Thus each open set contains the point p. It is easy to prove that this is an invertible T_0-space which is not T_1. The point p has for its orbit only itself (the point p *is* dense in S) and the open interval is the orbit of any other point.

Having Theorem 10, it seemed necessary to construct an example in which there were at least two nondegenerate orbits. Theorem 7 provided a clue and we looked for a continuum which had a dense set of cut points and a dense set of noncut points. One such continuum which is invertible may be constructed in the plane as follows: Start with a single closed interval of unit length. At the midpoint of this interval erect an interval of length $\frac{2}{3}$ so that the two intervals are perpendicular bisectors of each other. At the midpoint of each of the four intervals in the resulting figure, erect an interval whose length is $\frac{2}{3}$ that of the

interval being bisected. If the single closed interval is the 0th stage, then we will have 4^n intervals in the nth stage. Stages 1, 2, and 3 are pictured in Figure 1.

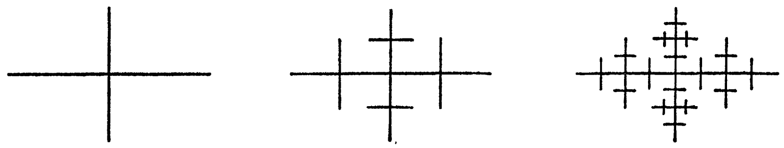
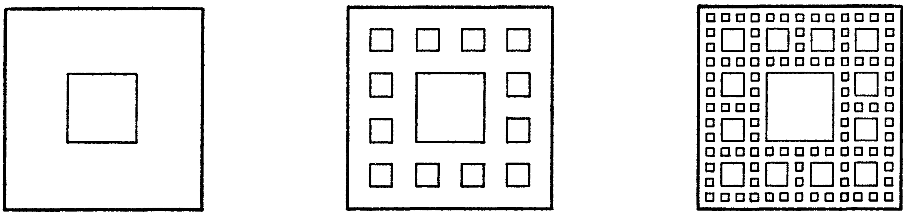


FIG. 1

The continuum C described above may be seen to be invertible by the following considerations: The four components of any branch point of order four in C all are homeomorphic. An explicit construction can be given in every case for this homeomorphism. Now given any open set U in C, we may choose a branch point p of order four such that two of the four components of $C-p$ lie entirely in U. Then interchanging these two with the two that do not lie in U is an inverting homeomorphism.

The example above contains three orbits, (i) the collection of noncut points of C, (ii) the collection of branch points of order four and (iii) the remainder of the set. By some obvious modifications of this example we can build an invertible Peano continuum in the plane with any finite number of orbits. This aspect of the theory is still under investigation so we can not say much more at this time. It is interesting to note, however, that the universal one-dimensional plane curve is an invertible space!

To construct the universal one-dimensional plane curve as we use it, consider a unit square (plus its interior) in the plane. At the first stage, we remove the open square of area 1/9 whose center coincides with the center of the big square



A B C

FIG. 2

(Fig. 2-A). At the second stage we remove 12 open squares of area 1/81 as shown in Figure 2-B. The third stage is pictured in Figure 2-C and removes 96 open squares of area $1/3^6$.

To see how this continuum may be inverted, we point out that if we consider

any little square whose interior is removed at the nth stage, then the interior of a square concentric and nine times the area of the removed square contains a set homeomorphic to the complement of the same larger square. By mapping the boundary of the middle square onto the outer boundary of the entire continuum, the remainder is merely a matter of easy matching of the missing squares. This will require a little construction which the reader may supply as a simple puzzle.

Another direction in which our investigations extended was suggested by the well-known fact that the n-sphere is not a product space with nondegenerate factor spaces (even when it is a fibre bundle). The usual proof of this theorem entails some rather complicated apparatus from algebraic topology and we hoped that a simple proof using invertibility might be possible. As a consequence of this desire we were led to consider product spaces. Since the rational points in the plane constitute an invertible product space, we restricted attention to product spaces with compact and connected factors. It would have been nice to have shown that such a product space cannot be invertible. Alas, this hope was rudely dashed when we discovered the following result.

THEOREM 11. *The product of infinitely many closed intervals is invertible.*

Proof. The proof of this theorem depends upon the following lemma in a way that will be obvious if the definition of an open set in the Tychonoff topology of such an infinite product is considered closely.

LEMMA 12. *Let U be an open set in I^n, the product of n intervals. Then there exists a homeomorphism h of I^{n+1} onto itself such that $h(I^{n+1}-U\times I^1)$ lies in $U\times I^1$.*

Proof. In U we choose an $(n-1)$-sphere S^{n-1} (and we mean a geometric sphere). In I^{n+1} consider the $(n+1)$-cell $V\times[0, \frac{1}{2}]$ where V is the n-cell bounded by S^{n-1}. The set $S^{n-1}\times[0, \frac{1}{2}]\cup V\times\frac{1}{2}$ is the intersection of two $(n+1)$-cells in I^{n+1} and the desired homeomorphism h is one which interchanges these two $(n+1)$-cells leaving their intersection fixed.

Theorem 11 discouraged us a bit and we have yet to return to product spaces. There is still hope for a proof that a finite-dimensional product continuum is not invertible but this is merely a conjecture at this stage.

In closing this first report on invertible spaces, we conjecture that other areas of topology may conceal interesting applications of invertibility. Among these might be more results on plane and higher dimensional continua, topological groups and function spaces and certain aspects of homotopy theory. At the time of this writing, for instance, we are enjoying some success in the application of continuous invertibility. (A space S is *continuously invertible* if it is invertible and if an inverting homeomorphism for each open set may be chosen to be isotopic to the identity mapping. The spheres are continuously invertible, for example.) But this is the subject of another report.

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SOME SINGULAR CASES OF THE IMPLICIT FUNCTION THEOREM*

W. S. LOUD, University of Minnesota

1. Introduction. The following problem arises in the study of periodic solutions of perturbed autonomous second-order differential equations ([2], [3]).

Let $F(x, y, z)$ and $G(x, y, z)$ have a sufficient number of continuous partial derivatives in some neighborhood of (0, 0, 0). Let $F(0, 0, 0)=G(0, 0, 0)=0$. It is desired to find x and y as functions of z from

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0, \tag{1.1}$$

where at $z=0$, $x=0$, $y=0$, and the derivatives dx/dz and dy/dz are finite. For the purposes of this paper, finding x and y will mean establishing the existence of x and y as functions of z, and evaluating the derivatives dx/dz and dy/dz at $z=0$. It would be possible in all cases to evaluate higher derivatives of x and y at $z=0$ (or to show they do not exist, in which case higher order behavior could still be studied), but we do not do this.

In the classical case, where the Jacobian of F and G with respect to x and y is nonzero at (0, 0, 0), the standard implicit function gives the existence and uniqueness of x and y as functions of z, and the derivatives in question can be easily computed. However, there are many cases with vanishing Jacobian which can be handled. It is the purpose of this paper to study several of these cases and to show how the existence or nonexistence of the solutions x and y with finite derivatives and $z=0$ can be determined. A complete analysis is extremely complicated. For an example of a complete analysis see [1], pages 163–169. We propose to consider all possible cases that are determinate when the derivatives of F and G through third order are known at (0, 0, 0). In every case then we shall construct the solution in the sense mentioned above, or else prove that there is no solution of the type sought, or else show that the case is not determinate when only the derivatives through order three are known.

We do not wish to be involved with elaborate continuity hypotheses. For the entire paper we make the following agreement. If derivatives of order k but not of order $k+1$ are needed to resolve a situation ($k=1, 2, 3$), then both F and G will be considered in class C^k in a neighborhood of (0, 0, 0).

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2. The classical case. If $F(x, y, z)$ and $G(x, y, z)$ are in C^1 in some neighborhood of (0, 0, 0), and the Jacobian

$$\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \neq 0 \tag{2.1}$$

at (0, 0, 0), the classical implicit function theorem guarantees the existence and uniqueness of the desired functions $x(z)$ and $y(z)$, and of their derivatives at $z=0$. The latter are given by

$$\frac{dx}{dz} = -\frac{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}, \qquad \frac{dy}{dz} = -\frac{\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}, \tag{2.2}$$

the partial derivatives being evaluated at (0, 0, 0).

3. The case of vanishing Jacobian. When the Jacobian (2.1) is zero, the classical implicit function theorem no longer applies. Indeed it is in general not true that functions $x(z)$ and $y(z)$ of the type sought even exist, without some additional assumptions.

In what follows, all partial derivatives of F and G are evaluated at (0, 0, 0), and this fact will not be explicitly noted.

Since the derivatives dx/dz and dy/dz at $z=0$ must satisfy

$$\begin{aligned} F_x \frac{dx}{dz} + F_y \frac{dy}{dz} + F_z &= 0, \\ G_x \frac{dx}{dz} + G_y \frac{dy}{dz} + G_z &= 0, \end{aligned} \tag{3.1}$$

a necessary condition for the existence and finiteness of these derivatives is that the matrix

$$\begin{Vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \end{Vmatrix} \tag{3.2}$$

have the same rank as the Jacobian matrix

$$\begin{Vmatrix} F_x & F_y \\ G_x & G_y \end{Vmatrix}. \tag{3.3}$$

If the rank of (3.2) is greater than the rank of (3.3), no solution of the type sought exists (although x and y may sometimes be determined as functions of z with dx/dz and/or dy/dz infinite at $z=0$).

4. Jacobian matrix of rank 1. In this section we assume that the rank of the Jacobian matrix (3.3) is 1. This means that at least one of its entries is nonzero,

and we assume for definiteness that $F_x(0, 0, 0) \neq 0$.

We begin by replacing the system $F(x, y, z) = 0$, $G(x, y, z) = 0$ by a somewhat simpler equivalent system. Define $H(x, y, z)$ by

$$H(x, y, z) \equiv G(x, y, z) - (G_x/F_x)F(x, y, z). \tag{4.1}$$

Because (3.3) has rank 1, and the augmented matrix (3.2) has rank 1 (as we must assume), all three of H_x, H_y, and H_z are zero at (0, 0, 0). We consider the equivalent system

$$F(x, y, z) = 0, \qquad H(x, y, z) = 0. \tag{4.2}$$

The process of solution of the system (4.2) is to eliminate the unknown x from the system. Because $F_x \neq 0$, we can solve the equation $F(x, y, z)$ for x as a function of y and z near $y = z = 0$. If the result is written $x = f(y, z)$, then $f(0, 0) = 0$, and the partial derivatives of $f(y, z)$ at (0, 0) can be computed from the partial derivatives of $F(x, y, z)$ at (0, 0, 0). Now consider the equation

$$J(y, z) \equiv H(f(y, z), y, z) = 0. \tag{4.3}$$

If (4.3) is solved for y as a function of z, $y = y(z)$, with dy/dz finite at $z = 0$, this $y(z)$, together with $x(z) \equiv f(y(z), z)$, furnish the desired solution of the system $F(x, y, z) = 0$, $H(x, y, z) = 0$. Moreover, the derivative dx/dz is given by

$$\frac{dx}{dz} = -\frac{F_y(dy/dz) + F_z}{F_x}. \tag{4.4}$$

Assuming continuity of all derivatives of F and H which appear, the following expressions are readily found for the derivatives of $J(y, z)$ of the first three orders at (0, 0):

$$J_y = 0, \qquad J_z = 0,$$

$$\begin{aligned}
J_{yy} &= (H_{xx}F_y^2 - 2H_{xy}F_yF_x + H_{yy}F_x^2)/F_x^2,\\
J_{yz} &= (H_{xx}F_yF_z - H_{xy}F_zF_x - H_{xz}F_yF_x + H_{yz}F_x^2)/F_x^2,\\
J_{zz} &= (H_{xx}F_z^2 - 2H_{xz}F_zF_x + H_{zz}F_x^2)/F_x^2,\\
J_{yyy} &= -(H_{xxx}F_y^3 - 3H_{xxy}F_y^2F_x + 3H_{xyy}F_yF_x^2 - H_{yyy}F_x^3)/F_x^3,\\
&\quad + 3H_{xx}(F_{xx}F_y^3 - 2F_{xy}F_y^2F_x + F_{yy}F_yF_x^2)/F_x^4\\
&\quad - 3H_{xy}(F_{xx}F_y^2 - 2F_{xy}F_yF_x + F_{yy}F_x^2)/F_x^3,\\
J_{yyz} &= -(H_{xxx}F_y^2F_z - 2H_{xxy}F_yF_zF_x - H_{xxz}F_y^2F_x + H_{xyy}F_zF_x^2 + 2H_{xyz}F_yF_x^2\\
&\qquad\qquad - H_{yyz}F_x^3)/F_x^3\\
&\quad + H_{xx}(3F_{xx}F_y^2F_z - 4F_{xy}F_yF_zF_x - 2F_{xz}F_y^2F_x + F_{yy}F_zF_x^2 + 2F_{yz}F_yF_x^2)/F_x^4\\
&\quad - 2H_{xy}(F_{xx}F_yF_z - F_{xy}F_zF_x - F_{xz}F_yF_x + F_{yz}F_x^2)/F_x^3\\
&\quad - H_{xz}(F_{xx}F_y^2 - 2F_{xy}F_yF_x + F_{yy}F_x^2)/F_x^3,
\end{aligned}$$

$$J_{yzz} = -(H_{xxx}F_yF_z^2 - H_{xxy}F_z^2F_x - 2H_{xxz}F_yF_zF_x + 2H_{xyz}F_zF_x^2 + H_{xzz}F_yF_x^2 - H_{yzz}F_x^3)/F_x^3$$
$$+ H_{xx}(3F_{xx}F_yF_z^2 - 2F_{xy}F_z^2F_x - 4F_{xz}F_yF_zF_x + 2F_{yz}F_zF_x^2 + F_{zz}F_yF_x^2)/F_x^4$$
$$- H_{xy}(F_{xx}F_z^2 - 2F_{xz}F_zF_x + F_{zz}F_x^2)/F_x^3$$
$$- 2H_{xz}(F_{xx}F_yF_z - F_{xy}F_zF_x - F_{xz}F_yF_x + F_{yz}F_x^2)/F_x^3,$$

$$J_{zzz} = -(H_{xxx}F_z^3 - 3H_{xxz}F_z^2F_x + 3H_{xzz}F_zF_x^2 - H_{zzz}F_x^3)/F_x^3$$
$$+ 3H_{xx}(F_{xx}F_z^3 - 2F_{xz}F_z^2F_x + F_{zz}F_zF_x^2)/F_x^4$$
$$- 3H_{xz}(F_{xx}F_z^2 - 2F_{xz}F_xF_z + F_{zz}F_x^2)/F_x^3.$$

Since the first partial derivatives of $J(y, z)$ vanish at $(0, 0)$, we can write

$$(4.5)\quad J(y, z) = \tfrac{1}{2}J_{yy}y^2 + J_{yz}yz + \tfrac{1}{2}J_{zz}z^2 + \tfrac{1}{6}J_{yyy}y^3 + \tfrac{1}{2}J_{yyz}y^2z + \tfrac{1}{2}J_{yzz}yz^2 + \tfrac{1}{6}J_{zzz}z^3 + \text{higher-order terms}.$$

We are interested in a solution of $J(y, z)=0$ which has $y=0$ at $z=0$, and for which z takes nonzero values. Write $y=\eta z$ in (4.5). We then obtain

$$(4.6)\quad J(y, z) = z^2(\tfrac{1}{2}J_{yy}\eta^2 + J_{yz}\eta + \tfrac{1}{2}J_{zz}) + z^3(\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz}) + \text{higher-order terms},$$

so that for $z\neq 0$, $J(y, z)=0$ is equivalent to

$$(4.7)\quad \tilde{J}(\eta, z) = (\tfrac{1}{2}J_{yy}\eta^2 + J_{yz}\eta + \tfrac{1}{2}J_{zz}) + z(\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz}) + \text{higher-order terms} = 0.$$

We shall solve (4.7) for η as a function of z. The solution of $J(y, z)=0$ for y will then be given by $y(z)=z\eta(z)$.

Consider the quadratic equation

$$(4.8)\quad \tfrac{1}{2}J_{yy}\eta^2 + J_{yz}\eta + \tfrac{1}{2}J_{zz} = 0,$$

which is the limit of (4.7) as $z\to 0$. If $\eta=\eta(z)$ is a solution of (4.7), $\eta(0)$ must satisfy (4.8). There are several possibilities.

(a) If the discriminant $J_{yz}^2-J_{yy}J_{zz}$ of (4.8) is negative, or if $J_{yy}=J_{yz}=0$ and $J_{zz}\neq 0$, then (4.8) has no real roots. Thus no real value for $\eta(0)$ can exist, so that a solution of the type sought does not exist.

(b) If the discriminant $J_{yz}^2-J_{yy}J_{zz}$ is positive, (4.8) has one or two simple real roots, according as $J_{yy}=0$ or $J_{yy}\neq 0$. In either case, the ordinary implicit function theorem then guarantees a solution of equation (4.7) $\eta=\eta(z)$ for each such simple real root of (4.8). If η_0 is such a real root, we have

$$(4.9)\quad \begin{aligned} \eta(z) &= \eta_0 + o(1), \\ y(z) &= z\eta(z) = \eta_0 z + o(z), \\ dy/dz &= \eta_0 \quad \text{at} \quad z = 0. \end{aligned}$$

From (4.4) we find that at $z=0$

$$\frac{dx}{dz} = -\frac{F_y\eta_0 + F_z}{F_x}. \tag{4.10}$$

(c) If the discriminant is zero, and $J_{yy}\neq 0$, we can still obtain $\eta(z)$ for either positive z or negative z in certain cases, but fractional powers of z will be needed. We shall also have to use the third derivative terms. Let η_0 be a double root of (4.8). (4.7) then becomes

$$\begin{aligned} &\tfrac{1}{2}J_{yy}(\eta-\eta_0)^2 + z(\tfrac{1}{6}J_{yyy}(\eta-\eta_0)^3 + J_1(\eta-\eta_0)^2 + J_2(\eta-\eta_0) + J_3) \\ &\qquad + \text{higher-order terms} = 0, \end{aligned} \tag{4.11}$$

where

$$J_1 = \tfrac{1}{2}J_{yyy}\eta_0 + \tfrac{1}{2}J_{yyz}, \qquad J_2 = \tfrac{1}{2}J_{yyy}\eta_0^2 + J_{yyz}\eta_0 + \tfrac{1}{2}J_{yzz},$$
$$J_3 = \tfrac{1}{6}J_{yyy}\eta_0^3 + \tfrac{1}{2}J_{yyz}\eta_0^2 + \tfrac{1}{2}J_{yzz}\eta_0 + \tfrac{1}{6}J_{zzz}.$$

If $J_3=0$, the case can not be resolved without knowledge of higher derivatives, so we do not treat the case.

If $J_3\neq 0$, we replace z by u^2 if J_3 and J_{yy} have opposite signs, and by $-u^2$ if J_3 and J_{yy} have the same sign. Then by taking a square root, we find either

$$\eta - \eta_0 = \pm\sqrt{(-2J_3/J_{yy})}\,u + \text{higher-order terms}$$

or

$$\eta - \eta_0 = \pm\sqrt{(2J_3/J_{yy})}\,u + \text{higher-order terms};$$

whence there are two solutions $\eta(z)$ for positive z and none for negative z in the former case, and two solutions $\eta(z)$ for negative z and none for positive z in the latter case. In any case we have $y=\eta_0 z+O(|z|^{3/2})$ so that at $z=0$, $dy/dz=\eta_0$, and dx/dz at $z=0$ is given by (4.10).

(d) If $J_{yy}=J_{yz}=J_{zz}=0$, we can replace (4.6) by

$$\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz} + \text{higher-order terms} = 0. \tag{4.12}$$

Consider the equation

$$\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz} = 0, \tag{4.13}$$

which is the limit of (4.12) for $z\to 0$. If (4.13) has no real roots, there are no solutions of the type sought. If η_0 is a multiple real root of (4.13), higher derivatives are required to resolve the case, and we do not treat it. Finally, for any simple root, η_0, of (4.13), the ordinary implicit function theorem shows that (4.12) has a solution $\eta=\eta(z)$ with $\eta(z)=\eta_0+o(1)$, so that again $y(z)=\eta_0 z+o(z)$, and at $z=0$,

$$\frac{dy}{dz} = \eta_0, \qquad \frac{dx}{dz} = -\frac{F_y\eta_0 + F_z}{F_x}.$$

Note that in this final case there may be as many as three different solutions $y=y(z)$, $x=x(z)$ of the type sought.

5. Jacobian matrix of rank 0. In the present section we assume that all four of F_x, F_y, G_x, and G_y are zero at (0, 0, 0). Since it is also necessary that the augmented matrix (3.2) have rank zero, we also assume that F_z and G_z are zero at (0, 0, 0). Thus the expansions of $F(x, y, z)$ and $G(x, y, z)$ begin with terms of second degree in (x, y, z).

As in Section 4, we set $x=\xi z$, $y=\eta z$, and substitute in the equations $F(x, y, z)=0$, $G(x, y, z)=0$. On dividing by z^2, we obtain

$$
(5.1)\quad
\begin{aligned}
\tilde{F}(\xi, \eta, z) &= \tfrac{1}{2}F_{xx}\xi^2 + F_{xy}\xi\eta + \tfrac{1}{2}F_{yy}\eta^2 + F_{xz}\xi + F_{yz}\eta + \tfrac{1}{2}F_{zz} \\
&\quad + z(\tfrac{1}{6}F_{xxx}\xi^3 + \tfrac{1}{2}F_{xxy}\xi^2\eta + \tfrac{1}{2}F_{xyy}\xi\eta^2 + \tfrac{1}{6}F_{yyy}\eta^3 \\
&\quad + \tfrac{1}{2}F_{xxz}\xi^2 + F_{xyz}\xi\eta + \tfrac{1}{2}F_{yyz}\eta^2 + \tfrac{1}{2}F_{xzz}\xi + \tfrac{1}{2}F_{yzz}\eta + \tfrac{1}{6}F_{zzz}) \\
&\quad + \text{higher-order terms} = 0, \\
\tilde{G}(\xi, \eta, z) &\equiv \tfrac{1}{2}G_{xx}\xi^2 + G_{xy}\xi\eta + \tfrac{1}{2}G_{yy}\eta^2 + G_{xz}\xi + G_{yz}\eta + \tfrac{1}{2}G_{zz} \\
&\quad + z(\tfrac{1}{6}G_{xxx}\xi^3 + \tfrac{1}{2}G_{xxy}\xi^2\eta + \tfrac{1}{2}G_{xyy}\xi\eta^2 + \tfrac{1}{6}G_{yyy}\eta^3 \\
&\quad + \tfrac{1}{2}G_{xxz}\xi^2 + G_{xyz}\xi\eta + \tfrac{1}{2}G_{yyz}\eta^2 + \tfrac{1}{2}G_{xzz}\xi + \tfrac{1}{2}G_{yzz}\eta + \tfrac{1}{6}G_{zzz}) \\
&\quad + \text{higher-order terms} = 0.
\end{aligned}
$$

We solve (5.1) for ξ and η as functions of z, and obtain the desired $x(z)$ and $y(z)$ from $x(z)=z\xi(z)$, $y(z)=z\eta(z)$. As in Section 4, we find that at $z=0$, $dx/dz=\xi_0$, $dy/dz=\eta_0$, where ξ_0 and η_0 are the values of ξ and η at $z=0$.

If we let $z\to 0$ in (5.1) we obtain the equations

$$
(5.2)\quad
\begin{aligned}
\tilde{F}_0(\xi, \eta) &\equiv \tfrac{1}{2}F_{xx}\xi^2 + F_{xy}\xi\eta + \tfrac{1}{2}F_{yy}\eta^2 + F_{xz}\xi + F_{yz}\eta + \tfrac{1}{2}F_{zz} = 0, \\
\tilde{G}_0(\xi, \eta) &\equiv \tfrac{1}{2}G_{xx}\xi^2 + G_{xy}\xi\eta + \tfrac{1}{2}G_{yy}\eta^2 + G_{xz}\xi + G_{yz}\eta + \tfrac{1}{2}G_{zz} = 0.
\end{aligned}
$$

The possible values of (ξ_0, η_0) are solutions of the system (5.2).

There are two degenerate cases which we discuss first. If both of $\tilde{F}_0(\xi, \eta)$ and $\tilde{G}_0(\xi, \eta)$ vanish identically, we must find ξ and η from the pair of equations

$$
(5.3)\quad
\begin{aligned}
&\tfrac{1}{6}F_{xxx}\xi^3 + \tfrac{1}{2}F_{xxy}\xi^2\eta + \tfrac{1}{2}F_{xyy}\xi\eta^2 + \tfrac{1}{6}F_{yyy}\eta^3 + \tfrac{1}{2}F_{xxz}\xi^2 + F_{xyz}\xi\eta \\
&\qquad + \tfrac{1}{2}F_{yyz}\eta^2 + \tfrac{1}{2}F_{xzz}\xi + \tfrac{1}{2}F_{yzz}\eta + \tfrac{1}{6}F_{zzz} + \text{higher-order terms} = 0, \\
&\tfrac{1}{6}G_{xxx}\xi^3 + \tfrac{1}{2}G_{xxy}\xi^2\eta + \tfrac{1}{2}G_{xyy}\xi\eta^2 + \tfrac{1}{6}G_{yyy}\eta^3 + \tfrac{1}{2}G_{xxz}\xi^2 + G_{xyz}\xi\eta \\
&\qquad + \tfrac{1}{2}G_{yyz}\eta^2 + \tfrac{1}{2}G_{xzz}\xi + \tfrac{1}{2}G_{yzz}\eta + \tfrac{1}{6}G_{zzz} + \text{higher-order terms} = 0.
\end{aligned}
$$

In this case, (ξ_0, η_0) is a solution of the pair of equations obtained from (5.3) by setting the higher order terms equal to zero.

If this reduced system has no real solutions, no solutions of the type sought exist for the problem. If (ξ_0, η_0) is a multiple root of this system, higher derivatives F and G are needed to resolve the situation, and we do not treat it. For any simple root (ξ_0, η_0) of the reduced system, the ordinary implicit function

theorem guarantees the existence of a solution $(\xi(z), \eta(z))$ of (5.3) with $\xi(0) = \xi_0$ and $\eta(0) = \eta_0$. Hence we obtain for each such simple root (ξ_0, η_0) a solution $(x(z), y(z))$ of the original problem with $dx/dz = \xi_0$, $dy/dz = \eta_0$ at $z = 0$. There is the possibility of as many as nine different solutions occurring in this case.

(Note: A root (ξ_0, η_0) is a simple root of the reduced system if the Jacobian of the two equations is nonzero at (ξ_0, η_0).)

The second degenerate case is that in which one but not both of $\tilde{F}_0(\xi, \eta)$ and $\tilde{G}_0(\xi, \eta)$ vanish identically, or neither vanishes identically, but one is a constant multiple of the other. We assume for definiteness that $\tilde{F}_0(\xi, \eta) \not\equiv 0$. Then by subtraction of a suitable multiple of $F(x, y, z)$ from $G(x, y, z)$, we can transform the system so that $\tilde{G}_0(\xi, \eta) \equiv 0$. We find ξ_0 and η_0 in this case from the system formed from $\tilde{F}_0(\xi, \eta) = 0$ and the second equation of (5.3) with higher order terms set equal to zero.

Exactly the same remarks now hold for this system of two equations. If it has no real solutions, no solution $(x(z), y(z))$ of the form sought exists. Multiple roots bring us to a case that can not be resolved without knowing higher derivatives. For each simple real root (ξ_0, η_0) there is a solution $(x(z), y(z))$ of the original system with $dx/dz = \xi_0$, $dy/dz = \eta_0$ at $z = 0$. There may be as many as six such solutions in this case.

Having disposed of these degenerate cases, we now return to the system (5.2), where we assume that neither of $\tilde{F}_0(\xi, \eta)$ and $\tilde{G}_0(\xi, \eta)$ vanishes identically, and that they are not proportional. The following possibilities exist for intersections of the loci $\tilde{F}_0(\xi, \eta) = 0$ and $\tilde{G}_0(\xi, \eta) = 0$.

There may be no real intersections. In this case there will be no solution $(x(z), y(z))$ of the original system of the type sought. There may be a finite number of isolated intersections, at most four in number. We shall analyze each possibility here later.

The two loci may be both degenerate and have a single straight line in common. If the loci have more than a single line in common, they must coincide, which brings us to one of the degenerate situations already treated.

We consider the case of an isolated intersection first. If (ξ_0, η_0) is an isolated solution of the system (5.2), we make the substitution $\xi - \xi_0 = u$, $\eta - \eta_0 = v$. The system of equations (5.1) then takes the form

$$\begin{aligned}(5.4)\qquad & F_1u + F_2v + \tfrac{1}{2}F_{xx}u^2 + F_{xy}uv + \tfrac{1}{2}F_{yy}v^2 + z(F_3 + F_4u + F_5v + \cdots \\ &\quad + \tfrac{1}{6}F_{xxx}u^3 + \cdots + \tfrac{1}{6}F_{yyy}v^3) + \text{higher-order terms} = 0, \\ & G_1u + G_2v + \tfrac{1}{2}G_{xx}u^2 + G_{xy}uv + \tfrac{1}{2}G_{yy}v^2 + z(G_3 + G_4u + G_5v + \cdots \\ &\quad + \tfrac{1}{6}G_{xxx}u^3 + \cdots + \tfrac{1}{6}G_{yyy}v^3) + \text{higher-order terms} = 0.\end{aligned}$$

In (5.4) the coefficients F_1, F_2, F_3, G_1, G_2, G_3, etc. are evaluated from the various partial derivatives of $\tilde{F}_0(\xi, \eta)$ and $\tilde{G}_0(\xi, \eta)$ evaluated at (ξ_0, η_0).

We must merely establish that (5.4) has a solution $(u(z), v(z))$ near $z = 0$ with $u(0) = v(0) = 0$, of some form. Once this is done we have successively

$$\xi(z) = \xi_0 + u(z), \qquad \eta(z) = \eta_0 + v(z);$$
$$x(z) = \xi_0 z + zu(z), \qquad y(z) = \eta_0 z + zv(z);$$

so that $x(z)$ and $y(z)$ of the form sought exist, and at $z=0$, $dx/dz=\xi_0$, $dy/dz=\eta_0$.

If the Jacobian of the system (5.4) with respect to u and v,

$$\begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix},$$

is nonzero, the standard implicit function theorem asserts the existence of the solution $(u(z), v(z))$ of (5.4), and thus the existence of the solution $x(z)$, $y(z)$ of the original system in the form sought.

If the above Jacobian is zero, we may sometimes obtain the desired solution. Suppose first that the rank of the matrix

$$\begin{Vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{Vmatrix} \tag{5.5}$$

is 1, and assume for definiteness that $F_1 \neq 0$. If the rank of the augmented matrix

$$\begin{Vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{Vmatrix} \tag{5.6}$$

is also 1, we are in a situation similar to that in Section 4, and higher order derivatives are necessary to resolve the situation, so we do not handle the case. On the other hand, if the rank of (5.6) is 2, we can proceed.

Since $F_1 \neq 0$, solve the first equation of (5.4) for u as a function of v and z, and substitute in the second equation. The result is

$$(F_1G_3 - F_3G_1)z + \frac{(G_{xx}F_1 - F_{xx}G_1)F_2^2 - 2(G_{xy}F_1 - F_{xy}G_1)F_1F_2 + (G_{yy}F_1 - F_{yy}G_1)F_1^2}{2F_1^2}\,v^2 \tag{5.7}$$
$$+ \text{ terms in } vz \text{ and } z^2 + \text{higher-order terms} = 0.$$

The coefficient of z in (5.7) is not zero because the rank of (5.6) is 2. If the coefficient of v^2 is zero, the situation cannot be resolved without the use of higher derivatives, so we do not treat this case. If the coefficient of v^2 is nonzero, we are in a situation similar to (c) in Section 4. If the coefficients of z and v^2 have opposite signs, (5.7) has two solutions $v(z)$ for positive z, and none for negative z. If these coefficients have the same sign, (5.7) has two solutions $v(z)$ for negative z, and none for positive z. Thus again we have two solutions $(x(z), y(z))$ with derivatives at $z=0$ given by ξ_0 and η_0 respectively. The solutions exist for positive z and not for negative z if the coefficients of z and v^2 in (5.7) have opposite signs, and for negative z and not for positive z if these coefficients have the same sign.

Now suppose that the matrix (5.5) has rank zero. If (5.6) also has rank zero, higher derivatives than third of F and G are needed to resolve the situation, so

we do not treat it. If, on the other hand, the rank of (5.6) is one, we can proceed.

Assume for definiteness that $F_3 \neq 0$. Then the first equation of (5.4) may be solved for z as a function of u and v.

$$z = -\tfrac{1}{2}(F_{xx}/F_3)u^2 - (F_{xy}/F_3)uv - \tfrac{1}{2}(F_{yy}/F_3)v^2 + \text{higher-order terms.} \tag{5.8}$$

If (5.8) is then substituted into the second equation of (5.4), an equation involving u and v only is obtained

$$\begin{aligned}\tfrac{1}{2}(F_{xx}G_3 - G_{xx}F_3)u^2 + (F_{xy}G_3 - G_{xy}F_3)uv + \tfrac{1}{2}(F_{yy}G_3 - G_{yy}F_3)v^2 \\ + \text{higher-order terms} = 0.\end{aligned} \tag{5.9}$$

Not all three of the coefficients of u^2, uv, and v^2 are zero in (5.9), for in that case, $F_0(\xi_0, \eta_0)$ and $G_0(\xi_0, \eta_0)$ would be proportional, or else one would vanish identically, and we would have a degenerate case. Consider the discriminant

$$(F_{xy}G_3 - G_{xy}F_3)^2 - (F_{xx}G_3 - G_{xx}F_3)(F_{yy}G_3 - G_{yy}F_3). \tag{5.10}$$

If (5.10) is negative, the only real point near (0, 0, 0) common to the loci determined by (5.4) is $u=v=z=0$, so no solution of the type sought exists. If (5.10) is zero, higher derivatives are required to resolve the situation, so we do not treat the case. If (5.10) is positive, the locus of the equation formed from (5.9) by omitting higher order terms consists of two distinct straight lines. Moreover, if $au+bv=0$ is such a straight line, $au+bv$ is not a factor of $\frac{1}{2}F_{xx}u^2+F_{xy}uv+\frac{1}{2}F_{yy}v^2$, for if it were, $u=0$, $v=0$, which is the same as $\xi=\xi_0$, $\eta=\eta_0$ would not be an isolated intersection of the curves defined by (5.4). As a result for each of the two lines, we can solve (5.9) for u as a function of v, or v as a function of u (perhaps both), and substitute the result in (5.8). The result will have the form

$$z = ku^2 + \text{higher order terms} \quad \text{or} \quad z = kv^2 + \text{higher-order terms},$$

with $k \neq 0$ in either case. Therefore each of the above two straight lines gives rise to two solutions $(u(z), v(z))$ of the system (5.4), both valid for positive z only if the corresponding k is positive, and both valid for negative z only if the corresponding k is negative. Hence in the case that (5.10) is positive, the original problem has four solutions of the type sought, of which four, two, or none are valid for positive z, with the remaining ones valid for negative z. For all four $dx/dz=\xi_0$ and $dy/dz=\eta_0$ at $z=0$.

We now consider the final situation, that in which the two loci $\tilde{F}_0(\xi, \eta)=0$ and $\tilde{G}_0(\xi, \eta)=0$ of (5.4) have a single straight line in common. We let (ξ_0, η_0) be any point on the common line, and make the substitution $u=\xi-\xi_0$, $v=\eta-\eta_0$. The system (5.4) then takes the form

$$\begin{aligned}(Au + Bv)(A_1u + B_1v + C_1) + z(F_3 + F_4u + F_5v + \cdots) = 0,\\ (Au + Bv)(A_2u + B_2v + C_2) + z(G_3 + G_4u + G_5v + \cdots) = 0,\end{aligned} \tag{5.11}$$

where A and B are not both zero, and to avoid the degenerate cases mentioned earlier, the matrix

$$\begin{Vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{Vmatrix}$$

has rank 2. This rank is independent of the choice of the point (ξ_0, η_0) on the common line. Certain of the conditions that will appear later will be satisfied only for certain points on the common line.

Suppose first that not both of C_1 and C_2 are zero. Assume for definiteness that $C_1 \neq 0$. Since not both A and B are zero, we assume without loss of generality that $A \neq 0$. We may then solve the first of equations (5.11) for u as a function of v and z,

$$u = -\frac{B}{A}v - \frac{F_3}{AC_1}z + \frac{F_3(AB_1 - A_1B) - C_1(AF_5 - BF_4)}{A^2C_1^2}vz \tag{5.12}$$
$$+ \text{ terms in } z^2 + \text{higher-order terms.}$$

If (5.12) is substituted into the second equation of (5.11), the result is

$$(C_1G_3 - C_2F_3)z$$
$$+\frac{C_2F_3(AB_1-BA_1)-C_1F_3(AB_2-BA_2)-C_1C_2(AF_5-BF_4)+C_1^2(AG_5-BG_4)}{AC_1}vz \tag{5.13}$$
$$+\text{terms in } z^2 + \text{higher-order terms} = 0.$$

The left member of (5.13) is divisible by z. If we divide by z, we see that (5.13) has no solution $v=v(z)$ with $v(0)=0$ unless $C_1G_3 - C_2F_3 = 0$. (Here is the condition that determines which points (ξ_0, η_0) on the common line must be chosen.) If $C_1G_3 - C_2F_3 = 0$, and the coefficient of vz in (5.13) is not zero, then (5.13) has a solution $v=v(z)$ with $v(0)=0$, and using (5.12) we can also find $u(z)$. In this case then a solution of the type sought exists. If the coefficient of vz in (5.13) is zero, higher derivatives of F and G are needed to resolve the situation, and we do not handle the case.

Now suppose that both C_1 and C_2 are zero. The system (5.11) becomes

$$\begin{aligned} (Au + Bv)(A_1u + B_1v) + z(F_3 + F_4u + F_5v + \cdots) &= 0, \\ (Au + Bv)(A_2u + B_2v) + z(G_3 + G_4u + G_5v + \cdots) &= 0, \end{aligned} \tag{5.14}$$

where A and B are not both zero, and $A_1B_2 - A_2B_1 \neq 0$. If both of F_3 and G_3 are zero, higher derivatives are required to resolve the situation, and we do not handle the case. If not both of F_3 and G_3 are zero, assume for definiteness that $F_3 \neq 0$. Then the first equation of (5.14) can be solved for z as a function of u and v

$$z = -\{(Au + Bv)(A_1u + B_1v)\}/F_3 + \text{higher-order terms.} \tag{5.15}$$

If (5.15) is substituted into the second equation of (5.14), we obtain

$$(Au + Bv)((A_1G_3 - A_2F_3)u + (B_1G_3 - B_2F_3)v) + \text{higher-order terms} = 0. \tag{5.16}$$

Now not both of $A_1G_3-A_2F_3$ and $B_1G_3-B_2F_3$ are zero, since $A_1B_2-A_2B_1\neq 0$, and $F_3\neq 0$. Thus the locus of the equation obtained by dropping the higher order terms in (5.16) consists either of two distinct straight lines or two coincident straight lines. The lines are coincident if and only if

$$\begin{vmatrix} A & B & 0 \\ A_1 & B_1 & F_3 \\ A_2 & B_2 & G_3 \end{vmatrix} = 0. \tag{5.17}$$

If the two lines are coincident, the situation can not be resolved without higher derivatives, and we do not handle it. If the determinant in (5.17) is not zero, the lines are distinct. The line $Au+Bv=0$ leads to a solution $u=u(v)$, or $v=v(u)$ of (5.16) which will require higher derivatives to handle when it is substituted into (5.15), so we do not handle it. The second line

$$(A_1G_3 - A_2F_3)u + (B_1G_3 - B_2F_3)v = 0$$

leads to a situation such as that handled when (5.10) is positive. In this case we obtain two solutions $(u(z), v(z))$ of the system (5.14), valid either for positive z or for negative z but not both.

6. Examples. In this section we give three examples to illustrate the more complicated situations of Section 5.

Example I.

$$x^2 + y^2 - z^2 + z^2x = 0, \qquad x^2 + 2y^2 - z^2 + z^2y = 0.$$

Here the rank of the Jacobian matrix is zero, since there are no linear terms. Setting $x=\xi z$ and $y=\eta z$, we obtain

$$\xi^2 + \eta^2 - 1 + z\xi = 0, \qquad \xi^2 + 2\eta^2 - 1 + z\eta = 0.$$

The real common points of the loci

$$\xi^2 + \eta^2 - 1 = 0, \qquad \xi^2 + 2\eta^2 - 1 = 0$$

are $(1, 0)$ and $(-1, 0)$. At $(1, 0)$, let $u=\xi-1$, $v=\eta$. We get for the system (5.4)

$$2u + u^2 + v^2 + z(u + 1) = 0, \qquad 2u + u^2 + 2v^2 + zv = 0.$$

The matrix (5.6) becomes

$$\begin{Vmatrix} 2 & 0 & 1 \\ 2 & 0 & 0 \end{Vmatrix},$$

which has rank 2. The first equation when solved for u gives

$$u = -\tfrac{1}{2}z - \tfrac{1}{2}v^2 + \tfrac{1}{8}z^2 + \text{higher-order terms.}$$

When substituted into the second equation, this gives in turn

$$-z + v^2 + vz + \tfrac{1}{2}z^2 + \text{higher-order terms} = 0.$$

From this we see that for small positive z, we can write $v = \pm\sqrt{z} + o(\sqrt{z})$. Then in turn we get

$$u = o(\sqrt{z}), \qquad \xi = 1 + o(\sqrt{z}), \qquad \eta = \pm\sqrt{z} + o(\sqrt{z}),$$
$$x = z + o(z^{3/2}), \qquad y = \pm z^{3/2} + o(z^{3/2}),$$

all for small positive z. The point $\xi = -1$, $\eta = 0$ is similar.

Example II.

$$xy + xz - yz - z^2 + z^2x = 0, \qquad x^2 - y^2 - 2xz - 2yz + z^2y = 0.$$

When we set $x = \xi z$ and $y = \eta z$, these become

$$\xi\eta + \xi - \eta - 1 + z\xi = 0, \qquad \xi^2 - \eta^2 - 2\xi - 2\eta + z\eta = 0.$$

The only common point of the loci

$$\xi\eta + \xi - \eta - 1 = 0, \qquad \xi^2 - \eta^2 - 2\xi - 2\eta = 0$$

is $(1, -1)$. Setting $u = \xi - 1$, $v = \eta + 1$, we get for the system (5.4)

$$uv + z(u + 1) = 0, \qquad u^2 - v^2 + z(v - 1) = 0.$$

Solving the first equation for z as a function of u and v, we find

$$z = -uv + \text{higher-order terms}.$$

Substituting in the second equation, we have

$$u^2 + uv - v^2 + \text{higher-order terms} = 0.$$

The discriminant (5.10) is positive, and we can solve the last for v as a function of u:

$$v = \rho u + \cdots \quad \text{and} \quad v = -\rho^{-1}u + \cdots,$$

where $\rho = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$. Hence we find

$$z = -\rho u^2 + \cdots \quad \text{and} \quad z = \rho^{-1}u^2 + \cdots.$$

The first of these equations gives $u(z)$ for small negative z, while the second gives $u(z)$ for small positive z. We therefore have four solutions for this example, two for small positive z and two for small negative z:

$z > 0$	$z < 0$
$u = \pm\sqrt{(\rho z)} + o(\sqrt{z})$,	$u = \pm\sqrt{(-\rho^{-1}z)} + o(\sqrt{\mid z\mid})$,
$v = \pm\sqrt{(\rho^{-1}z)} + o(\sqrt{z})$,	$v = \pm\sqrt{(-\rho z)} + o(\sqrt{\mid z\mid})$,
$\xi = 1 \pm \sqrt{(\rho z)} + o(\sqrt{z})$,	$\xi = 1 \pm \sqrt{(-\rho^{-1}z)} + o(\sqrt{\mid z)\mid}$,
$\eta = -1 \pm \sqrt{(\rho^{-1}z)} + o(\sqrt{z})$,	$\eta = -1 \pm \sqrt{(-\rho z)} + o(\sqrt{\mid z\mid})$,

$$x = z \pm \sqrt{(\rho z^3)} + o(z^{3/2}), \qquad x = z \pm \sqrt{(-\rho^{-1} z^3)} + o(|z|^{3/2}),$$
$$y = -z \pm \sqrt{(\rho^{-1} z^3)} + o(z^{3/2}). \qquad y = -z \pm \sqrt{(-\rho z^3)} + o(|z|^{3/2}).$$

Example III.

$$xy - xz + yz^2 = 0, \qquad x^2 + xy + y^2 z = 0.$$

Setting $x = \xi z$, $y = \eta z$, we obtain

$$\xi\eta - \xi + z\eta = 0, \qquad \xi^2 + \xi\eta + z\eta^2 = 0.$$

The loci

$$\xi\eta - \xi = 0, \qquad \xi^2 + \xi\eta = 0$$

have the entire line $\xi = 0$ in common. (They also have the point $(-1, 1)$ in common, but we do not consider this here.) Select any point $(0, \eta_0)$ on the line $\xi = 0$, and set $u = \xi$, $v = \eta - \eta_0$. We obtain

$$u(v + \eta_0 - 1) + z(v + \eta_0) = 0, \qquad u(u + v + \eta_0) + z(\eta_0^2 + 2\eta_0 v + v^2) = 0.$$

We find that $C_1 = \eta_0 - 1$, $C_2 = \eta_0$, $F_3 = \eta_0$, $G_3 = \eta_0^2$, so that the critical quantity $C_1G_3 - C_2F_3 = \eta_0^3 - 2\eta_0^2$, which is zero only for $\eta_0 = 0$ and $\eta_0 = 2$. No other values of η_0 will give solutions of the type sought.

When $\eta_0 = 0$, we have

$$u(v - 1) + zv = 0, \qquad u(u + v) + zv^2 = 0.$$

When the first of these equations is solved for u in terms of v and z and the result is substituted in the second, the coefficient of vz turns out to be zero, so that the methods in the paper do not resolve the situation. (Actually it is clear that $x = 0$, $y = 0$ is a solution with $\eta_0 = 0$.)

When $\eta_0 = 2$, we have

$$u(v + 1) + z(v + 2) = 0, \qquad u(u + v + 2) + z(v^2 + 4v + 4) = 0.$$

This time the coefficient of zv turns out to be nonzero, so that we do have a solution $x = o(z)$, $y = 2z + o(z)$. This can be verified, since the system can be solved explicitly.

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A NEW GENERALIZATION OF JENSEN'S THEOREM ON THE ZEROS OF THE DERIVATIVE OF A POLYNOMIAL*

J. L. WALSH, Harvard University

In the geometry of the zeros of the derivative of a polynomial, a central role is taken by Lucas's theorem: *If $p(z)$ is a polynomial not identically constant, in the complex plane the zeros of the derivative $p'(z)$ lie in the smallest convex polygon containing the zeros of the original polynomial.* Under certain conditions Lucas's theorem can be sharpened:

JENSEN'S THEOREM. *Let $p(z)$ be a real polynomial (i.e., with real coefficients) not identically constant, and consider the circles having as diameters the line segments $z_k\bar{z}_k$ joining conjugate pairs of nonreal zeros of $p(z)$. Then all nonreal zeros of $p'(z)$ lie in the closed interiors of these circles.*

The object of the present note is to establish generalizations (Theorems 2 and 3 below) of the following theorem, recently proved [1] by the writer, which is itself a generalization of Jensen's Theorem:

THEOREM 1. *Let $p(z)$ be a real polynomial not identically constant all of whose zeros have real parts in the interval $\alpha \leqq x \leqq \beta$ of the axis of reals, and let γ be a real point not interior to that interval and not having an abscissa equal to that of a nonreal zero of $p(z)$. Let $\Gamma(z_k)$ denote the circle through the conjugate pair $(z_k, \bar{z}_k)$ of nonreal zeros tangent to the line γz_k at z_k. Then all nonreal zeros of the derivative $p'(z)$ lie in the closed interiors of the circles $\Gamma(z_k)$.*

All the theorems mentioned are conveniently proved by use of a suitable field of force ([2], Sec. 4.1.1):

BÔCHER'S THEOREM. *The finite zeros of the derivative $r'(z)$ of a nonconstant rational function $r(z)$ which are not multiple zeros of $r(z)$ are the positions of equilibrium in the field of force due to particles of positive mass at the zeros of $r(z)$ and particles of negative mass at the poles of $r(z)$, with masses numerically equal to the respective multiplicities, where each particle repels with a force equal to the mass times the inverse distance.*

The special case here where $r(z)$ is a polynomial in z is due to Gauss; the single pole of $r(z)$ is then at infinity, and the particle there can be ignored in setting up the field of force. In Bôcher's theorem a particular convention which does not concern us here may be made regarding $z = \infty$ as a zero of $r'(z)$.

Theorem 1 was proved, and succeeding theorems are to be proved, by application of a lemma regarding Bôcher's field of force. The following represents a slight sharpening of the original form [1]:

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LEMMA. *Let Γ denote the circle through the points $+i$ and $-i$, tangent at $+i$ to the line joining $+i$ and $\gamma(>0)$. At an arbitrary nonreal point $z_0=x_0+iy_0$, $y_0>0$, exterior* [*interior*] *to Γ the force due to positive unit particles at $+i$ and $-i$ has a nonvanishing component perpendicular to the line γz_0 in the clockwise* [*counterclockwise*] *sense about γ.*

For our purposes it is sufficient to study the field of force in the upper half-plane, since considerations of symmetry then yield corresponding results in the lower half-plane. Our main result on rational functions is now available:

THEOREM 2. *Let $r(z)$ be a real rational function not identically constant all of whose finite zeros lie in the half-plane $x>0$ and all of whose finite poles lie in the half-plane $x<0$, except that $z=0$ may be a zero or a pole. Let a circle $\Gamma(z_k)$ be drawn with center on the axis of reals passing through each conjugate pair $(z_k, \bar{z}_k)$ of zeros and of poles, where $\Gamma(z_k)$ is tangent at z_k to the line Oz_k. Then all nonreal zeros of $r'(z)$ lie in the closed interiors of the $\Gamma(z_k)$.*

At an arbitrary point z_0 exterior to the $\Gamma(z_k)$ and in the upper half-plane, the force due to the particles (if any) at the finite zeros of $r(z)$ in $x>0$ has a nonvanishing component perpendicular to the line Oz_0 in the counterclockwise sense; this statement is true by the lemma for the nonreal pairs of particles, and is clearly true so far as concerns the particles on the axis of reals. By the same reasoning, the force at z_0 due to the particles (if any) at the finite poles of $r(z)$ in $x<0$ also has a nonvanishing component perpendicular to the line Oz_0 in the counterclockwise sense. If $r(z)$ is z^m, where m is a positive or negative integer, there are no nonreal zeros of $r'(z)$ and the conclusion of the theorem is satisfied; in any other case there exist either finite zeros of $r(z)$ in $x>0$ or finite poles in $x<0$, or both, and the force at z_0 due to a positive or negative particle at O acts along Oz_0, so z_0 is not a position of equilibrium. Of course z_0 cannot be a multiple zero of $r(z)$, so the theorem follows.

A second theorem due to Bôcher ([2], Sec. 4.2) asserts that *if a line or circle L does not pass through all the zeros and poles of a nonconstant rational function $r(z)$ but separates the zeros of $r(z)$ not on L from the poles of $r(z)$ not on L, then L passes through no finite zero of $r'(z)$ which is not a multiple zero of $r(z)$.* It is to be noted that Theorem 2 is in part concerned with lines L through O which separate the zeros of $r(z)$ not on L from the poles of $r(z)$ not on L; for instance, the axis of imaginaries is such a line L; but if $r(z)$ has nonreal zeros or poles, Theorem 2 is stronger (*i.e.*, for real $r(z)$) than the theorem just quoted.

As a limiting case of the lemma, which may be proved by the original method, we may choose $\gamma=0$; the conclusion is then that *the force at the nonreal point $z_0=x_0+iy_0$, $x_0>0$, $y_0>0$, has a nonvanishing component perpendicular to the line Oz_0 in the clockwise sense.* Consequently, as a limiting case of Theorem 2 we may allow either zeros or poles of $r(z)$ (but not both) to lie on Oy; the modified conclusion in these respective cases is that all nonreal zeros of $r'(z)$ in $x<0$ or in $x>0$ lie in the closed interiors of the Γ_k; in either case no nonreal zero of

$r'(z)$ not a multiple zero of $r(z)$ lies on Oy unless all zeros and poles of $r(z)$ lie on Oy.

An interesting special case of Theorem 2 is that in which $r(z)$ has but a single pole in the extended plane. If this pole is at infinity, Theorem 2 reduces essentially to Theorem 1. If this pole is at a point $z=a(<0)$, and if we transform the plane by a linear transformation of the complex variable which carries the three points a, 0, ∞, into points ∞, α, β, where α and β are real, we obtain a new result concerning polynomials:

THEOREM 3. *Let $p(z)$ be a real polynomial all of whose zeros lie in the closed interior of the circle whose diameter is the segment $\alpha\beta$ of the axis of reals, and let all the nonreal zeros lie interior to that circle. For each pair $(z_k, \bar{z}_k)$ of conjugate imaginary zeros of $p(z)$ let Γ_k be the circle tangent at z_k and $\bar{z}_k$ to the respective circles $\alpha z_k \beta$ and $\alpha \bar{z}_k \beta$. Then all nonreal zeros of $p'(z)$ lie in the closed interiors of the Γ_k. A nonreal point z_0 on a circumference Γ_k but not a multiple zero of $p(z)$ and not on or within a second circle Γ_j cannot be a zero of $p'(z)$ unless $p(z)$ has precisely three distinct zeros, one at α or β.*

In Theorem 3 the limiting cases $\alpha=\infty$, $\beta=\infty$ are not excluded. If we choose $\alpha=\infty$ or $\beta=\infty$ but not both, Theorem 3 reduces to Theorem 1; if we choose $\alpha=\infty$, $\beta=\infty$, Theorem 3 reduces to Jensen's theorem.

The last sentence of Theorem 3 deserves further discussion; the corresponding fact was not mentioned in connection with Theorems 1 and 2. Under the conditions of the Lemma, the line of action of the force at a point z_0 of Γ passes through γ. Thus under the conditions of Theorem 2, if the nonreal point z_0 lies on the circumference $\Gamma(z_1)$, the force at z_0 due to the two particles z_1 and $\bar{z}_1$ has the line of action Oz_0. If there exist other (finite) particles than z_1, $\bar{z}_1$, and O, and if z_0 is not on or within a second circle $\Gamma(z_k)$, the total force at z_0 has a nonzero component perpendicular to Oz_0; so z_0 is not a position of equilibrium. That is to say, *under the conditions of Theorem 2, a nonreal point z_0 on a circumference $\Gamma(z_k)$ but not a multiple zero of $r(z)$ and not on or within a second circumference $\Gamma(z_j)$ cannot be a zero of $r'(z)$ unless $r(z)$ has in the plane of finite points a totality of precisely three distinct zeros and poles, one of which lies at O.*

To discuss the situation just excepted, suppose $z_1=x_1+iy_1$ to be a zero of $r(z)$, $x_1>0$, $y_1>0$, and suppose the nonreal point z_0 to lie on the circumference $\Gamma(z_1)$, on the left-hand arc bounded by z_1 and $\bar{z}_1$. If the algebraic magnitudes of the force at z_0 due to the pair of unit particles z_1 and $\bar{z}_1$ and the force at z_0 due to a unit particle at O are in the ratio $m_2:m_1$, where $m_1m_2<0$, then z_0 satisfies

$$\frac{m_2}{z}-\frac{m_1}{z-z_1}-\frac{m_1}{z-\bar{z}_1}=0,$$

and z_0 is a zero of $r'(z)$ if and only if

$$r(z)\equiv z^{m_2}[(z-z_1)(z-\bar{z}_1)]^{-m_1}, \tag{1}$$

where m_1 and m_2 are not uniquely determined but their ratio is, and we may choose either $m_1>0$, $m_2<0$, or $m_1<0$, $m_2>0$. The function $r(z)$ can here be chosen a rational function, indeed a polynomial, if and only if the ratio $m_2:m_1$ is rational; in the contrary case, z_0 is not the zero of the derivative of any admissible rational function. If z_0 lies on this same circumference $\Gamma(z_1)$, on the right-hand arc bounded by z_1 and $\bar{z}_1$, and if the algebraic magnitudes of the forces as already defined are in the ratio $m_2:m_1$, where $m_1m_2>0$, then z_0 is a zero of $r'(z)$ if and only if $r(z)$ is defined by (1). Again, $r(z)$ can be chosen a rational function if and only if the ratio $m_2:m_1$ is rational and we may choose $m_1>0$, $m_2>0$ or $m_1<0$, $m_2<0$; in the case that $m_2:m_1$ is irrational, z_0 is not the zero of the derivative of any admissible rational function. Illustrations here are

$$r(z) \equiv z(z-1+i)(z-1-i), \qquad r(z) \equiv (z-1+i)(z-1-i)/z^3,$$

whose derivatives have respectively the zeros $\frac{1}{3}(2\pm i\sqrt{2})$ and $2\pm i\sqrt{2}$.

To recapitulate: if the real rational function $r(z)$ has precisely three distinct finite zeros and poles, one of which lies at O and the others are both zeros or both poles, then all nonreal zeros of $r'(z)$ lie on the circle $\Gamma(z_1)$ of Theorem 2; the numerical illustrations just given apply.

Theorem 2 is also of interest in case $r(z)$ has in the plane of finite points only four zeros and poles, which occur in pairs of conjugate nonreal points. If there exists a nonreal zero z_0 of $r'(z)$ not a multiple zero of $r(z)$, the force at z_0 due to one conjugate pair $(z_1, \bar{z}_1)$ is not zero, and its line of action cuts the axis of reals in some point γ, which may be the point at infinity. The line of action of the force at z_0 due to the other conjugate pair also passes through γ. The circle through the points z_k and $\bar{z}_k$ tangent at z_k to the line γz_k passes through z_0 for $k=1$ and $k=2$, so z_0 lies at the intersection of these two circles. In the reciprocal direction, suppose two pairs $(z_k, \bar{z}_k)$ of conjugate points are given, $k=1, 2$. Let γ be real (perhaps infinite), and suppose the two circles just described intersect in a nonreal point z_0. If the algebraic magnitudes of the forces at z_0 due to the respective pairs of particles $(k=1, 2)$ are in the ratio $m_2:m_1$, then z_0 satisfies

$$\frac{m_1}{z-z_1}+\frac{m_1}{z-\bar{z}_1}-\frac{m_2}{z-z_2}-\frac{m_2}{z-\bar{z}_2}=0$$

and z_0 is a zero of $r'(z)$ if we have

$$r(z) \equiv [(z-z_1)(z-\bar{z}_1)]^{m_1}[(z-z_2)(z-\bar{z}_2)]^{-m_2},$$

which can be chosen a rational function if and only if the ratio $m_2:m_1$ is rational.

Theorem 3 is quite unusual in the sense that the proof (as we have given it) of a theorem concerning polynomials depends on a proposition (Theorem 2) for more general rational functions followed by a linear transformation of the complex variable. Naturally the proof of Theorem 3 can be otherwise phrased. Let $r(z)$ be a real rational function having its only pole in the extended plane at the finite point a, $a<0$, and let all real parts of the zeros of $r(z)$ be nonnega-

tive, the real parts of all nonreal zeros positive. At a point z_0 in the upper half-plane exterior to the circles $\Gamma(z_k)$ of Theorem 2, the total force has a nonzero component perpendicular to the line Oz_0 in the counterclockwise sense with respect to O. Let now a linear transformation carry the entire configuration into the situation of Theorem 3. Although the magnitude of the force is not invariant under such a linear transformation, both direction and sense of the force are invariant ([2], Sec. 4.1.2); the total mass of all particles in the extended plane is to be zero. Consequently, under the conditions of Theorem 3, at any point z_0 in the upper half-plane exterior to the circles Γ_k, the force has a nonzero component orthogonal to the arc $\alpha z_0 \beta$ in the sense away from the axis of reals; Theorem 3 can obviously be proved by this remark.

In Theorem 3 the points α and β are not uniquely determined; however, it is not possible to consider the lens-shaped region Λ_k interior to all (variable) Γ_k as k remains fixed while α and β vary, and to substitute Λ_k for Γ_k in the theorem. As a counterexample we mention $p(z) \equiv (z-1-2i)(z-1+2i)(z+1-2i)(z+1+2i)$ $\equiv z^4+6z^2+25$, the nonreal zeros $\pm i\sqrt{3}$ of whose derivative lie at the intersections of the two Jensen circles for $p(z)$ yet do not lie in either Λ_k.

We add a number of remarks complementary to Theorems 2 and 3. By the lemma we have the

COROLLARY. *Let the hypothesis of Theorem 2 be enlarged so as to admit nonreal poles of $r(z)$ also in the half-plane $x \geqq 0$. Then a nonreal point in $x>0$, interior to all circles $\Gamma(z_k)$ defined as in Theorem 2 for the poles of $r(z)$ in $x>0$ and exterior to all circles $\Gamma(z_k)$ for the zeros of $r(z)$, cannot be a zero of $r'(z)$.*

Under the conditions of this corollary, it may occur that no poles of $r(z)$ lie in the closed half-plane $x \leqq 0$; under such conditions any real point $\gamma(<0)$ can replace 0 in defining the circles $\Gamma(z_k)$, and we may even choose $\gamma=-\infty$ (compare [2], Sec. 5.1.2, Theorem 3; [1], corollary).

In this corollary the roles of zeros and poles may be interchanged, but it is to be noted that a multiple zero of $r(z)$ whether real or nonreal is also a zero of $r'(z)$.

Theorem 2 can be generalized by transforming a given configuration containing a "circle" C by a linear transformation of the complex variable that carries C into the axis of reals. We deal with the extended plane, and allow the term "circle" to include straight line, and we use the term *circular region* to denote either the closed interior of a circle, the closed exterior of a circle, or a closed half-plane. Let $r(z)$ be a rational function whose zeros and whose poles not on a "circle" C both occur in pairs of points mutually inverse in C, where also the zeros of $r(z)$ not on a second "circle" C_1 orthogonal to C are separated by C_1 from the poles of $r(z)$ not on C_1, and where no zeros nor poles of $r(z)$ lie on C_1 except perhaps in the intersections α and β of C and C_1. Let a "circle" $\Gamma(z_k)$ be drawn (necessarily orthogonal to C) through each pair (z_k, z_k') of zeros and of poles of $r(z)$ mutually inverse in C, tangent at z_k and z_k' to the "circles" $\alpha z_k \beta$ and $\alpha z_k' \beta$, and let $\Gamma(z_k)$ denote also that circular region bounded by the

"circle" $\Gamma(z_k)$ which contains no point of C_1. Then all finite zeros of $r'(z)$ not on C lie in the regions $\Gamma(z_k)$.

The concept of *infrapolynomial* was first introduced by Fekete and von Neumann [3], as a generalization of classical extremal polynomials $p(z) \equiv z^n + a_1 z^{n-1} + \cdots + a_n$ of least norm on a given point set E. The zeros of such infrapolynomials are positions of equilibrium in a suitably chosen field of force analogous to the field of Gauss, where the particles lie on E but need no longer be of integral mass. Thus Fekete and von Neumann showed (*loc. cit.*) that the analogue of Jensen's theorem holds if $p(z)$ is real and if E is symmetric in the axis of reals. It is likewise true that under such conditions the analogue of Theorem 1 is valid ([4], Theorem 7; [5]); similar considerations show that also the analogue of Theorem 3 is valid.

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CORRECTION

In the final printing of *The rolling of one curve or surface upon another* by William Clifford and James J. McMahon (this Monthly, vol. 68, 1961, pp. 338–341), a large number of dots over letters in displayed formulas somehow failed to appear.

The correct formulas are, on page 338:

$$0 = \dot{X} = \dot{A}X_0 + \dot{C} \tag{2}$$

$$\dot{A}X_0 = -\dot{C} \tag{2a}$$

$$X_0 = -(\dot{A})^{-1}\dot{C} \tag{3}$$

$$0 = \dot{A}^T A + A^T \dot{A} = (A^T\dot{A})^T + (A^T\dot{A}),$$

On page 339:

$$X = -A(\dot{A})^{-1}\dot{C} + C \tag{4}$$

$$(\dot{A}^T)Y = 0 \Rightarrow (\dot{C}^T)Y = 0. \tag{5}$$

$$X + \dot{X}\delta t = AX_0 + C + (\dot{A}X_0 + \dot{C})\delta t. \tag{6}$$

$$
(7) \qquad \begin{aligned} X + \dot{X}\delta t &= C + [(I + \dot{A}A^T\delta t)(X - C)] + \dot{C}\delta t. \\ (I + \dot{A}A^T\delta t)Y &= Y, \end{aligned}
$$

On page 340:

$$
(10a) \qquad P_i^T\left\{\dot{Q}_0 + \sum_{j=1}^{s} (\dot{\lambda}_j P_j + \lambda_j \dot{P}_j)\right\} = 0, \qquad (i = 1, \cdots, s)
$$

$$
(10b) \qquad \dot{P}_i^T\left\{\dot{Q}_0 + \sum_{j=1}^{s} (\dot{\lambda}_j P_j + \lambda_j \dot{P}_j)\right\} = 0,
$$

The Editor regrets these errors.

MATHEMATICAL NOTES

Edited by Roy Dubisch, University of Washington

Material for this department should be sent to M. H. Protter, Department of Mathematics, University of California, Berkeley 4, California

LEAST COMMON MULTIPLES AND HIGHEST COMMON FACTORS

Marlow Sholander, Western Reserve University

It is well known that $ab=(a, b)[a, b]$, where we use standard notation for highest common factor (HCF) and least common multiple (LCM) of positive integers a and b. It is less well known that*

$$(1) \qquad abc = (a, b, c)[(a, b), (b, c), (c, a)][a, b, c],$$

$$(2) \qquad abc = (a, b, c)[ab, bc, ca],$$

$$(3) \qquad abc = (ab, bc, ca)[a, b, c].$$

The proof of (3) is typical. Let the highest power of prime p dividing a, b, and c be p^α, p^β, and p^γ. Then both sides of (3) contain p to the exponent $\alpha+\beta+\gamma=\min(\alpha+\beta, \beta+\gamma, \gamma+\alpha)+\max(\alpha, \beta, \gamma)$.

The generalizations are clear. Consider a set $S\{a_1, a_2, \cdots, a_n\}$. Form the set T of $\binom{n}{k}$ LCM's of the subsets of S which have k elements. Let P_k be the HCF of T. Alternately, P_k is the LCM of the set of $\binom{n}{k}$ HCF's of subsets of S which have $n-k$ elements. Then $\prod_{k=1}^n a_k=\prod_{k=1}^n P_k$.

Let U be the set of $\binom{n}{i}$ products obtained from subsets of S having i elements. Let Q_i be the HCF of U and R_i be the LCM of U. Then $Q_i=\prod_{k=1}^i P_k$ and $R_{n-i}=\prod_{k=i+1}^n P_k$. Hence $\prod_{k=1}^n a_k=Q_iR_{n-i}$ for $i=0, 1, \cdots, n$.

* We note that $[(a, b), (b, c), (c, a)]=([a, b], [b, c], [c, a])$ is an example of a "median." Cf. M. Sholander, Medians and betweenness, Proc. Amer. Math. Soc., vol. 5, 1954, pp. 801–807.

ON A THEOREM OF HOBSON

DWIGHT B. GOODNER, The Florida State University*

An interesting and useful theorem ([2], p. 363; [1], p. 246) requires the existence of the derivative. However, since the derivative may fail to exist, it seems desirable to have expressions which may serve us when there is no derivative. The purpose of this note is to extend the theorem to semi-continuous functions.

DEFINITION. *If the function f is defined for $a \leqq x < a+h$, a number r is a right derivate of f at a if there exists a sequence $b_1, b_2, \cdots$ of positive numbers tending to 0 and less than h such that*

$$\lim_{n\to\infty} \frac{f(a+b_n)-f(a)}{b_n} = r.$$

Left derivates are analogously defined.

THEOREM. *Let the function f be lower semi-continuous on the open interval $a<x<a+h$ and let the right limit $f(a+)$ fail to exist. Then for each real number k there exists a number ξ with $a<\xi<a+h$ such that:*

(i) *if $D(+)f(\xi)$ is any right derivate of f at ξ and $D(-)f(\xi)$ is any left derivate of f at ξ, then $D(-)f(\xi) \leqq k \leqq D(+)f(\xi)$;*

(ii) *if $D(+)f(\xi)$ is any finite right derivate of f at ξ and $D(-)f(\xi)$ is any finite left derivate of f at ξ, there exist nonnegative numbers p, q with $p+q=1$ such that $pD(+)f(\xi)+qD(-)f(\xi)=k$.*

A dual theorem holds if f is upper semi-continuous for $a<x<a+h$.

Proof. Let $F(x)=f(x)-kx$. Since $f(a+)$ does not exist neither does $F(a+)$, and there exists a number b such that

$$\liminf_{x\to a+} F(x) < b < \limsup_{x\to a+} F(x).$$

We choose successively three numbers b_1, c_1, a_1 such that:

$$\begin{aligned} &a < b_1 < a+h \quad &&\text{and} \quad F(b_1) > b,\\ &a < c_1 < b_1 &&\text{and} \quad F(c_1) < b,\\ &a < a_1 < c_1 &&\text{and} \quad F(a_1) > b; \end{aligned}$$

this is possible by the definition of b. On the closed interval $a_1 \leqq x \leqq b_1$, the function F is lower semi-continuous, so it attains its minimum at some point ξ in that interval ([3], p. 76). But ξ is interior to that interval, for $F(\xi) \leqq F(c_1) < b$ while $F(a_1) > b$ and $F(b_1) > b$. By a proof to be found in many good calculus books, all right derivates of F at ξ are $\geqq 0$ and all left derivates of F at ξ are $\leqq 0$. Since $D(+)F(\xi)=D(+)f(\xi)-k$ (the $D(+)$ being defined by the same sequence

* The author is indebted to the referee for his helpful suggestions.

for f as for F), conclusion (i) holds. Conclusion (ii) follows from (i), since the inequality in (i) implies for finite $D(+)f(\xi)$ and $D(-)f(\xi)$ that k is a weighted mean of the derivates.

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TWO METHODS FOR THE EVALUATION OF $\sum_{k=0}^{\infty} k^n x^k$

DAVID ZEITLIN, Remington Rand Univac, St. Paul, Minnesota

Stalley [1] and Klamkin [2] have given different proofs for summing a generalized geometric series, $K_n(x)$, and obtained the following result:

$$K_n(x) = \sum_{k=1}^{\infty} k^n x^k = (1-x)^{-(n+1)} \sum_{r=1}^{n} \left[\sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1} (r-m+1)^n \right] x^r$$

for $|x| < 1$ and n a nonnegative integer. In this note we introduce two additional ways for obtaining $K_n(x)$. The first method uses generating functions, and the second method requires the following identity:

$$\frac{d^{m-1}}{du^{m-1}}\left(\frac{1}{e^u - 1}\right) = (-1)^{m-1} \sum_{r=1}^{m} (e^u - 1)^{-r} \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} (r-s)^{m-1}, \quad m \geqq 1. \tag{1}$$

For a proof of (1), see the elementary problem E1401 [1960, 803] of this MONTHLY, vol. 67.

(i) Let u_k satisfy a linear difference equation of order $(n+1)$ with real, constant coefficients:

$$u_{n+1+k} + a_1 u_{n+k} + a_2 u_{n+k-1} + \cdots + a_{n+1} u_k = 0, \qquad a_{n+1} \neq 0.$$

Then the generating function of u_k (see [3, p. 27]) is given by

$$\sum_{k=0}^{\infty} u_k x^k = \sum_{k=0}^{n} \left(\sum_{j=0}^{n-k} a_j u_{n-k-j} \right) x^{n-k} \Big/ \sum_{i=0}^{n+1} a_i x^i, \tag{2}$$

where $a_0 \equiv 1$. From the well-known identity,

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} j^r = 0, \qquad 0 \leqq r < n+1,$$

it follows that $u_k = k^n$ satisfies

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} u_{n+1+k-j} = 0.$$

Thus, with

$$a_j = (-1)^j \binom{n+1}{j}, \qquad j = 0, 1, \cdots, (n+1), \ |x| < 1,$$

we obtain from (2)

$$(3) \qquad (1-x)^{n+1} \sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{n} \left[\sum_{j=0}^{n-k} (-1)^j \binom{n+1}{j} (n-k-j)^n \right] x^{n-k}, \qquad n \geqq 0.$$

If we define

$$A_{ns} = \sum_{j=0}^{s} (-1)^j \binom{n+1}{j} (s-j)^n, \qquad 0 \leqq s \leqq n,$$

then $A_{n0} = 0$ if $n \neq 0$, $A_{nn} = 1$, $n = 0, 1, \cdots$, and (3) becomes

$$(4) \qquad (1-x)^{n+1} \sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{n} A_{n,n-k} x^{n-k}, \qquad n \geqq 0.$$

A_{ns} are known as Eulerian numbers (see [4], [5]). If we define $S_k(n) = 1^n + 2^n + \cdots + k^n$, $k \geqq 1$, $n \geqq 1$, then

$$(5) \qquad \sum_{k=1}^{\infty} S_k(n) x^k = \sum_{k=0}^{\infty} k^n x^k/(1-x) = (1-x)^{-(n+2)} \sum_{k=0}^{n} A_{n,n-k} x^{n-k}, \ n \geqq 1.$$

(ii) Let $e^r x < 1$, where r is real and $0 < x < 1$. Then

$$\sum_{k=0}^{\infty} e^{kr} x^k = \sum_{k=0}^{\infty} (e^r x)^k = (1 - e^r x)^{-1},$$

and

$$\sum_{k=0}^{\infty} k^n x^k = \left[\frac{d^n}{dr^n} \left(\frac{1}{1 - e^r x} \right) \right]_{r=0}, \qquad 0 \leqq x < 1.$$

Writing $(1-e^r x)^{-1}$ as $(1-e^{r+\log x})^{-1}$, and recalling that $[t/(e^t - 1)] = \sum_{k=0}^{\infty} B_k t^k/k!$, where B_k are Bernoulli numbers, we have $(e^{r+\log x} - 1)^{-1} = \sum_{k=0}^{\infty} B_k (r + \log x)^{k-1}/k!$. Thus,

$$(6) \qquad \sum_{k=0}^{\infty} k^n x^k = - \sum_{k=0}^{\infty} \frac{B_k (k-1) \cdots (k-n) (\log x)^{k-n-1}}{k!}$$

for $e^{-2\pi} < x < 1$. This is so since $e^z - 1 = 0$ if $z = 2m\pi i$, $m = 0, \pm 1, \cdots$, and we require $-2\pi < \log x < 0$. Let $u = \log x$. Since $1/(e^u - 1) = \sum_{k=0}^{\infty} B_k u^{k-1}/k!$, we observe that

$$\frac{d^n}{du^n} \left(\frac{1}{e^u - 1} \right) = \sum_{k=0}^{\infty} \frac{B_k (k-1) \cdots (k-n) u^{k-n-1}}{k!}.$$

Thus, from (1), with $m=n+1$, (6) becomes

$$(7)\qquad \sum_{k=0}^{\infty} k^n x^k = (-1)^{n+1}\sum_{r=1}^{n+1}\frac{1}{(x-1)^r}\sum_{s=0}^{r-1}(-1)^s\binom{r-1}{s}(r-s)^n, \qquad n \geqq 0.$$

By the principle of analytic continuation, (7) persists for $|x|<1$. We note that (7) gives the result in a form different than (3). (7) may be written as follows:

$$(8)\qquad \sum_{k=0}^{\infty} k^n x^k = (1-x)^{-n-1}\sum_{j=0}^{n}(x-1)^{n-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(j-s+1)^n$$

$$= (1-x)^{-n-1}\sum_{j=0}^{n}(x-1)^{n-j}\Delta^j 1^n,$$

where

$$\Delta^j 1^n = \sum_{s=0}^{j}(-1)^s\binom{j}{s}(j-s+1)^n.$$

We wish to show that (8) can be transformed into (4). Expanding $(x-1)^{n-j}$ by the binomial theorem, we obtain

$$\sum_{j=0}^{n}(x-1)^{n-j}\Delta^j 1^n = \sum_{j=0}^{n}\Delta^j 1^n\sum_{r=0}^{n-j}\binom{n-j}{r}(-1)^{n-j-r}x^r$$

$$= \sum_{r=0}^{n}\left[\sum_{j=0}^{n-r}(-1)^{n-j-r}\binom{n-j}{r}\Delta^j 1^n\right]x^r = \sum_{k=0}^{n}A_{n,n-k}x^{n-k},$$

since

$$(9)\qquad A_{n,n-k} = \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{n-k}\Delta^j 1^n.$$

The proof of (9) is given in [2], page 92 where our (9) occurs as (8) in the note. This is readily seen by setting $t-1=k$ in (8) and by noting that $A_{n,n-k}=A_{n,k+1}$ (see [4, p. 250, (2.13)]).

Remarks. We note that

$$(1-x)^{n+1}K_n(x) = \sum_{k=0}^{n-1}A_{n,n-k}x^{n-k}, \qquad n \geqq 1.$$

(10) and (11) cited below are due to Euler (see [4, p. 247]):

$$(10)\qquad \frac{1-x}{e^u-x} = \sum_{n=0}^{\infty}H_n(x)u^n/n!, \qquad x \neq 1,$$

$$(11)\qquad R_n(x) = (x-1)^n H_n(x), \qquad xR_n(x) = \sum_{s=1}^{n}A_{n,s}x^s.$$

Thus,

$$K_n(x) = (-1)^n x H_n(x)/(1-x), \qquad n \geqq 0, \tag{12}$$

$$\sum_{k=0}^{\infty} k^n x^k = (-1)^n x H_n(x)/(1-x), \qquad n \geqq 1, \tag{13}$$

where

$$H_n(x) = \sum_{r=0}^{n} (x-1)^{-r} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} s^n, \qquad H_0(x) = 1. \tag{14}$$

(14) is given in [4, p. 250, (2.10)]. For $n \geqq 1$, (13) and (12) are, of course, identical.

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A COUNTING THEOREM

PAUL M. PEPPER, Ohio State University

In a previous paper* the author presented the following counting theorem:

Let j, s, and t be integers for which $0 \leqq s \leqq t$. Then, if the elements of j sets, each containing s elements, are distributed among t classes in such a way that no two elements of the same set fall in the same class, then for each integer r satisfying $0 \leqq r \leqq s$, there exist at least r of the classes each of which contains at least

$$m_r = \left[\frac{(j+1)(s-r+1)+t-s-1}{t-r+1} \right]$$

elements.† *Moreover, there is no integer $m_r^1 > m_r$ for which the same conclusion can be drawn.*

The restriction that the sets all have one and the same number of elements is somewhat undesirable. It is the purpose of the present note to show that this restriction may be removed without greatly complicating the result.

First note that m_r may be rewritten in the form

* Paul M. Pepper and Bernard J. Topel, Imbedding Theorems under Weakened Hypotheses: Part I. Reports of a Mathematical Colloquium, Second Series, Issue 4, pp. 52–53 (University of Notre Dame).

† The expression $[a]$ means the greatest integer less than or equal to a.

$$m_r = \left[\frac{j(s-r+1)+(t-r)}{t-r+1}\right],$$

and that $s-r+1$ is the number of elements in excess of $r-1$ which each set contains; $t-r+1$ is the excess over $r-1$ in the number of classes; $j(s-r+1)$ is the total number of elements all j of the sets contain in excess of $r-1$ each. The ratio $j(s-r+1)/(t-r+1)$ is the average number of elements in each of $t-r+1$ excess classes if the excess elements are distributed into those classes, and, if these excess elements are distributed as evenly as possible, the expression for m_r is the maximum number of these excess elements which one of the $t-r+1$ classes may have. Thus, as few as possible of the elements are distributed into the $t-r+1$ classes, in order that $r-1$ but not any set of r classes be filled deeply.

To generalize the theorem, one should try to force some $t-r+1$ classes to have as few elements as possible, stacking the other elements into the remaining classes as deeply as possible. This line of heuristic reasoning leads to the following formulation and proof of a theorem completely analogous to the earlier one, but having none of the restrictions on its range of applicability.

THEOREM. *Let $S_1, \cdots, S_j$ be j finite sets of elements with respective numbers of elements equal to $s_1, \cdots, s_j$ and such that $s_1 \leqq \cdots \leqq s_j$. Let t be any integer satisfying $s_j \leqq t$, and let r be any integer in $0<r\leqq t$. Let the elements of the sets S_i be distributed into t classes in such a way that no two elements of the same set fall into the same class. Let i_0 be the first value of i (if it exists) for which $s_i \geqq r$, so that $s_{i_0-1}<r \leqq s_{i_0}$. Then there exist at least r of the t classes each containing at least*

$$m_r = \left[\frac{\sum_{i=i_0}^{j}(s_i-r+1)+(t-r)}{t-r+1}\right] \tag{1}$$

elements.

If r is greater than all of the s_i, then, on the one hand no i_0 exists, and, on the other hand, the maximum integer for which the conclusion is valid is $m_r=0$, which can be obtained by dropping the sum in (1) to get

$$m_r = \left[\frac{t-r}{t-r+1}\right] = 0.$$

Moreover, in the general case, if m_r^1 is any integer greater than m_r, there exist distributions such that among each r classes there exists at least one which contains at most m_r^1-1 elements.

To make a proof of this theorem, we note that to negate the theorem, there must exist at least one distribution such that of every set of $t-r+1$ classes each contains at most m_r-1 elements.

Let some $t-r+1$ classes be selected and distribute the elements of the sets $S_{i_0}, S_{i_0+1}, \cdots, S_j$ into the t classes in such a way as to minimize the number

which fall into the selected $t-r+1$ classes. This is done by distributing some $r-1$ elements from S_i into the remaining classes and the other s_i-r+1 elements into the $t-r+1$ classes. Then the number of elements in these classes is $\sum_{i=i_0}^{j}(s_i-r+1)$ so that the average number of elements in each class is

$$\mu = \frac{\sum_{i=i_0}^{j}(s_i - r + 1)}{t - r + 1}. \tag{2}$$

At least one class has at least m_r elements in it if m_r is the least integer $\geqq \mu$. But the least integer $\geqq \mu$ can be expressed in the form (1).

Since $s_i \leqq t$, it follows from (2) that $j-i_0+1 \geqq \mu$ and since j and i_0 are integers, $j-i_0+1 \geqq m_r$. This completes the proof that there are never $t-r+1$ classes with less than m_r elements in each. Thus among each r classes, each has at least m_r elements.

To show that this cannot be said of any integer $m_r^1 > m_r$, it suffices to show a distribution—this time of the elements of all j of the sets—for which some $t-r+1$ classes each contains at most $m_r(<m_r^1)$ elements.

To exhibit such a distribution separate the t classes into two categories; let the first category contain $r-1$ classes and the second $t-r+1$ classes. Since $0 < s_1 \leqq \cdots \leqq s_{i_0-1} \leqq r-1$ it is possible to distribute the elements of $S_1, \cdots, S_{i_0-1}$ into only the $r-1$ classes of the first category. Moreover, as before, it is possible to put $r-1$ of the elements of each of the sets $S_{i_0}, \cdots, S_j$ into the classes of the first category. Order the classes of the second category and following that order starting with i_0, distribute the remaining $s_{i_0}-r+1$ points (of S_{i_0}) into the first $s_{i_0}-r+1$ classes of the second category; continue with the remaining points of $S_{i_0+1}, \cdots$, etc., until each class of the second category has exactly one element in it. Next return to the first class of the second category to continue the distribution following the same order as previously until each class of the second category has exactly 2 elements in it (provided there are enough elements). At the end of the distribution of those $\sum_{i=i_0}^{j}(s_i-r+1)$ elements of the sets $S_{i_0}, \cdots, S_j$ which were not allocated to the classes of the first category, each class of the second category will contain either

$$\left[\frac{\sum_{i=i_0}^{j}(s_i - r + 1)}{t - r + 1}\right] \tag{3}$$

elements or one more than this number. If the number in brackets is an integer all of the classes of the second category will have the same number of elements, otherwise some will have the number of elements described by (3) and others will have one more than this number. In no event will any class have more than the number m_r of (1), so that there is a distribution in which $t-r+1$ classes have less than m_r^1 elements whatever integer $m_r^1 > m_r$ may be chosen.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to J. M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

ANOTHER VERY INDEPENDENT AXIOM SYSTEM

J. W. ELLIS, Louisiana State University in New Orleans

In a recent Classroom Note [1], Harary presented a very independent (in fact, an absolutely independent) axiom system and conjectured that some, but not many, others might exist. This note outlines another such system, which selects the subgroups among the subsets of a group. The notation and terminology are those of [1].

The primitives of our system are a group G, with identity element denoted by 1, and a subset S of G. The axioms are:

i. $1 \in S$
v. If $x \in S - \{1\}$, then $x^{-1} \in S$
c. If $x,\ y \in S - \{1\}$ and $x \neq y^{-1}$, then $xy \in S$.

The slight alteration of axioms v and c from their usual forms was necessary to avoid implying i; thus they are similar to the distinctly transitive axiom used by Harary.

We now exhibit eight subsets of the group G of positive rational numbers, satisfying the eight possible combinations of i, $\bar{\text{i}}$, v, $\bar{\text{v}}$, c, $\bar{\text{c}}$. Thus our system, too, is absolutely independent.

Conditions	S
ivc	G
$\bar{\text{i}}$vc	$G - \{1\}$
i$\bar{\text{v}}$c	$\{r \mid r \in G,\ r \geqq 1\}$
ivc	$\{1\} \cup \{p \mid p \text{ a prime}\} \cup \{1/p \mid p \text{ a prime}\}$
i$\bar{\text{v}}\bar{\text{c}}$	$\{1\} \times \{p \mid p \text{ a prime}\}$
$\bar{\text{i}}$v$\bar{\text{c}}$	$\{p \mid p \text{ a prime}\} \cup \{1/p \mid p \text{ a prime}\}$
$\bar{\text{i}}\bar{\text{v}}$c	$\{r \mid r \in G,\ r > 1\}$
$\bar{\text{i}}\bar{\text{v}}\bar{\text{c}}$	$\{2,\ 3\}$.

It may be argued that our axioms i, v, c are closely related to the axioms r, s, t respectively for the relation of (left or right) congruence modulo S. Even so, the models above differ from those in [1] in the sense that all relations used are of the same type.

Reference

1. Frank Harary, A very independent axiom system, this MONTHLY, vol. 68, 1961, pp. 159–164.

A RECURSION FORMULA FOR A CERTAIN DEFINITE INTEGRAL

T. V. L. NARASIMHAN, Madras Christian College, Tambaram, India

It is known that

$$y(\alpha) \equiv \int_0^1 \frac{t^{2\alpha-1}}{\sqrt{(1-t^2)}}\,dt = \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})} \qquad (\alpha > 0).$$

Differentiating this logarithmically with respect to α we get

$$\frac{y'(\alpha)}{y(\alpha)} = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})} = G(\alpha),$$

say. Next, taking the nth derivative of both sides of the above in the form $y'=yG$, we obtain, by Leibnitz's rule,

$$(1)\qquad y^{(n+1)} = y^{(n)}G + \binom{n}{1} y^{(n-1)}G^{(1)} + \cdots + yG^{(n)},$$

where $y^{(r)}$ and $G^{(r)}$ $(r \geqq 1)$ are given by

$$(2)\qquad y^{(r)} = 2^r \int_0^1 \frac{t^{2\alpha-1}}{\sqrt{(1-t^2)}} \log^r t\,dt,$$

$$(3)\qquad G^{(r)} = \frac{d^r}{d\alpha^r}\left\{\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}\right\}$$

$$(4)\qquad = (-1)^{r+1} r!\left\{\sum_{m=0}^{\infty} \frac{1}{(\alpha+m)^{r+1}} - \sum_{m=0}^{\infty} \frac{1}{(\alpha+\frac{1}{2}+m)^{r+1}}\right\}.$$

Equation (4) follows from (3) by differentiation of the well-known formula*

$$\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(y)}{\Gamma(y)} = \sum_{m=0}^{\infty}\left\{\frac{1}{y+m} - \frac{1}{x+m}\right\}.$$

Using (2), (3), (4) in (1), setting $\alpha=\frac{1}{2}$ and noting that

$$\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \frac{\Gamma'(1)}{\Gamma(1)} = -2\log 2,$$

we get the desired recursion formula

$$\int_0^1 \frac{\log^{n+1} t}{\sqrt{(1-t^2)}}\,dt = -\log 2 \int_0^1 \frac{\log^n t}{\sqrt{(1-t^2)}}\,dt$$
$$+ \sum_{r=1}^{n} \frac{(-1)^{r+1}}{2^{r+1}} \binom{n}{r} r!\left\{\zeta(r+1,\tfrac{1}{2}) - \zeta(r+1)\right\} \int_0^1 \frac{\log^{n-r} t}{\sqrt{(1-t^2)}}\,dt,$$

* See, *e.g.*, Earl D. Rainville, Special Functions, New York, 1960.

where $\zeta(r, s)=\sum_{m=0}^{\infty}(m+s)^{-r}$, $\zeta(r, 1)=\zeta(r)$.

For $n=0$ we get the well-known result

$$\int_0^1 \frac{\log t}{\sqrt{(1-t^2)}}\,dt = -\tfrac{1}{2}\pi\log 2,$$

For $n=1, 2$, we get

$$\begin{aligned}\int_0^1 \frac{\log^2 t}{\sqrt{(1-t^2)}}\,dt &= \tfrac{1}{2}\pi\log^2 2 + \tfrac{1}{4}\{\zeta(2,\tfrac{1}{2}) - \zeta(2)\}\cdot\tfrac{1}{2}\pi \\ &= \tfrac{1}{2}\pi\log^2 2 + \tfrac{1}{4}(\tfrac{1}{2}\pi^2 - \tfrac{1}{6}\pi^2)\cdot\tfrac{1}{2}\pi = \tfrac{1}{2}\pi\log^2 2 + \tfrac{1}{24}\pi^3,\\ \int_0^1 \frac{\log^3 t}{\sqrt{(1-t^2)}}\,dt &= -\tfrac{1}{8}\pi[\pi^2\log 2 + 4\log^3 2 + \zeta(3,\tfrac{1}{2}) - \zeta(3)],\end{aligned}$$

respectively. There are similar results for $n\geqq 3$.

GRAPHICAL MULTIPLICATION OF FUNCTIONS

HARRY I. MILLER, Illinois Institute of Technology

Consider a plane in which a horizontal straight line O is given.* On it, a linear unit is laid off from 0 to 1. Consider the graphs f and g of two functions. We wish to construct points of the curve $f\cdot g$, which is the graph of the product of those two functions. If X is the point on O that is x units from 0, then the

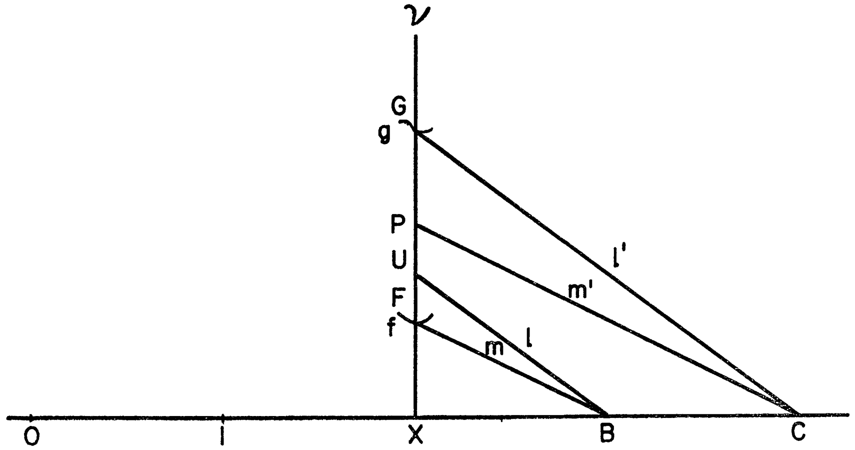

FIG. 1

* In what follows we adopt Menger's typographical convention ([1], p. 10): All designations of lines, curves, and functions are lower case italics, while all references to numbers are lower case roman. A function and its graph will be denoted by the same letter. The line O is the graph of the constant function whose value is 0. Points are denoted by capitals in roman type. The value of the function f for the number x will be denoted by fx.

altitudes above X of the points F on f, and G on g, are fx and gx units, respectively. We have to construct the point P whose altitude above X is $f\mathrm{x}\cdot g\mathrm{x}$ linear units. (In order to obtain many points of $f\cdot g$, this construction has to be performed for many points X, X_1, X_2, $\cdots$). In Figure 1 we illustrate three constructions of the point P that fulfill the requirement.

1. The projective construction. On the vertical line v through X, lay off a linear unit to the point U. Choose any point B ($\neq$X) on O. Join B to U by the straight line l, and to F by m. Then, through G, draw the line l', parallel to l. Through C (the intersection of l', with O), draw the line m' parallel to m. The point where m' intersects v can, by similar triangles, easily be shown to be $f\mathrm{x}\cdot g\mathrm{x}$ units above X.

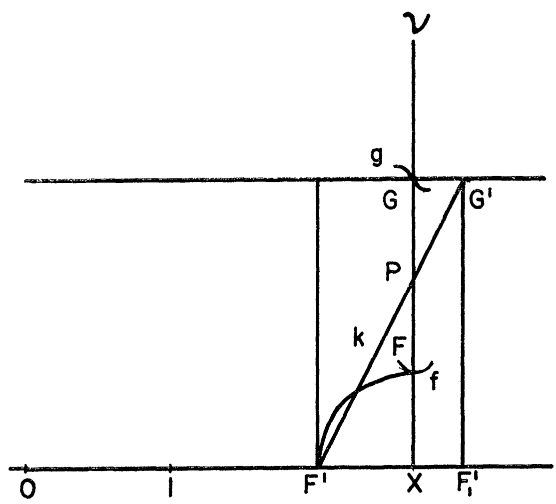

FIG. 2

2. The compass construction. (Fig. 2). Draw the quarter circle about X from F to F′ (to the left of X). The area in square units of the rectangle with the sides F′X and XG is $f\mathrm{x}\cdot g\mathrm{x}$. Since we have to construct a vertical segment of just that number of linear units, we follow the graphical construction of areas ([1], p. 8); that is to say, we lay off a linear unit on O from F′ to F_1', and join F′ with the point G′ (gx linear units above F_1') by the line k. The point P where k intersects v is, as can be shown by similar triangles, $f\mathrm{x}\cdot g\mathrm{x}$ units above X.

3. The j-method. (Fig. 3). Consider the graph of the identity function ([1], p. 4, 75), that is, the line j which, for any number c, has an altitude of c units above the point that is c units from 0. From F draw the horizontal line to F* where it intersects j. Draw the vertical line v^* through F* to F_0^*, its intersection with O. Since F* lies on j, the point F_0^* has the same distance from 0 as from F*, that is fx units. The area in square units of the rectangle with the

sides $0F_0^*$ and $F_0^*G^*$ (where G^* is gx units above F_0^*) is $fx \cdot gx$. We construct the area of this rectangle just as, in the course of the compass construction, we constructed the area of the rectangle with the sides F′X and XG. (Clearly, the two rectangles are congruent by a horizontal translation.) We join the point 0 with G_1^* (gx units above the point 1) by the line k^*. The point P^* where k^* intersects v^* has an altitude of $fx \cdot gx$ units above O. Hence the horizontal line through P^* intersects v in a point P which is the same number of units above X.

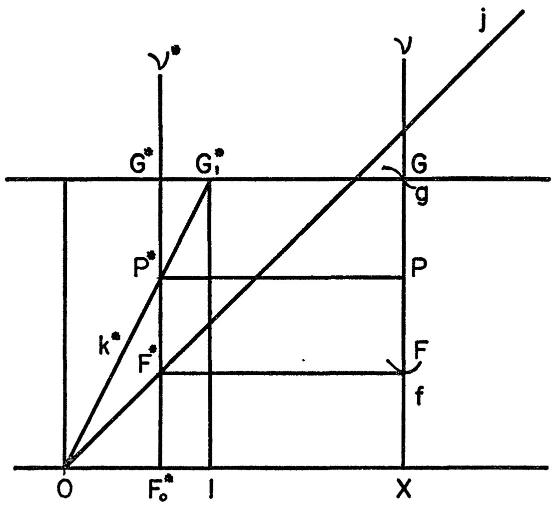

FIG. 3

Remark 1. Even if vertical and horizontal distances are measured in unlike units and, therefore, the line j has an inclination $\neq 45°$, the j-method works; that is to say, the number of vertical units in the segment from X to P is equal to $fx \cdot gx$.

Remark 2. The j-method links graphical multiplication to Menger's graphical substitution ([1],† p. 89), which yields the graph of the composite function of f and g which assumes the value $f(g(x))$ for any x. Menger's construction, based on j, is carried out entirely in the plane, even though it has been claimed ([3], p. 79) that geometric substitution has to resort to three-dimensional space. A remarkable example ([2], p. 460) is the graphical substitution of the parabola $-j^2$ (having the altitude $-x^2$ above x) into the exponential curve (of altitude e^x above x), which results in a probability curve (having the altitude e^{-x^2} above x).

References

1. K. Menger, Calculus. A Modern Approach, Boston, 1955, p. 10.

† The mimeo-editions of this book (Chicago, 1952, p. 224 and 1953, p. 273) include also a construction of the substitution of an ordered pair of surfaces into a surface.

2. ———, Axiomatic theory of functions and fluents. The Axiomatic Method (ed. Henkin *et al.*), Amsterdam, 1959.

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FOUR-POINT FORMULAS FOR MACHINE COMPUTATION

FREDERIC E. NEMMERS, A.C. Spark Plug Division, General Motors Corporation, Milwaukee, Wisconsin

The writer has developed and used the following generalization of Simpson's rule for integration on an IBM 650 computer. The programming of the formula was simple and allowed many variations as a self-contained subroutine. As an example, a combined differentiation, integration and interpolation subroutine was easily devised, using the same entrances with a code call word and the same formula pattern.

The formula also lends itself to an entrance check for variable interval, *i.e.* if $h_1=h_2=h_3$ no coefficient computation is necessary. The running time for the formula on a computer is about the same as for Simpson's rule, except when a change of interval is encountered. When the interval changes, the coefficients have to be recomputed and hence the running time is increased. Storage space required is approximately two bands.

Derivation of the formula follows standard methods. Assume that we have four points u_a, u_b, u_c, and u_d. If we pass a cubic $u_x=A(x-b)(x-c)(x-d)+\cdots+D(x-a)(x-b)(x-c)$ through the four points and solve for the coefficients A, B, C, and D we get

$$(*)\qquad \int_0^h u_x dx = \tfrac{1}{4}Ah^4 + \tfrac{1}{3}Bh^3 + \tfrac{1}{2}Ch^2 + Dh,$$

where $h=h_1+h_2+h_3$ and

$$A = \frac{u_a}{h_1(h_1+h_2)h}, \quad B = \frac{-u_b}{h_1h_2(h_2+h_3)}, \quad C = \frac{u_c}{(h_1+h_2)h_2h_3}, \quad D = \frac{-u_d}{h(h_2+h_3)h_3},$$

after the substitutions $a=0$, $b-a=h_1$, $c-a=h_1+h_2$, $d-a=h$ are made. Starting and stopping formulas follow in a similar manner. Comparable predictor-corrector formulas were derived and proved worthwhile for machine computation.

If $R(x)$ is the error, then, again writing $h=h_1+h_2+h_3$,

$$R(x) = \int_0^h (x^4/4!)dx - (*) = (h^5/5!) - (*).$$

The actual use of the error formula involves no great increase in computation since the error formula has only one extra term and computation of this term is rapid on a computer. The extra term, of course, need not be recomputed each time if no change has occurred in interval size.

A METRIC PARADOX

ALBERT WILANSKY, Lehigh University

Let z be a fixed point in a metric space (X, d) and define $f(x) = d(x, z)$. Now a real function g on a metric space is continuous if and only if

$$(1) \qquad \lim x_n = x \text{ implies } \lim g(x_n) = g(x)$$

and also if and only if

$$(2) \qquad g^{-1}[G] \text{ is open whenever } G \text{ is an open interval.}$$

To check (1) for f one needs the triangular inequality; to check (2) one does not, namely, $f^{-1}[(a, b)] = \{x \mid a < d(x, z) < b\} = N_b(z) \sim D_a(z)$ is open. (Here, $N_r(z), D_r(z)$ are the open and closed spheres of radius r; center z.)

This inspires the following joke. Let X be a set and d a real function of two variables in X satisfying $d(x, x) = 0$ and $d(x, y) = d(y, x) > 0$ if $x \neq y$. Give X the topology which has as subbase the set of all $N_r(x)$ for all x and all $r > 0$. Fix z and define $f(x) = d(x, z)$.

THEOREM. (i) *f is continuous*, (ii) *f is not necessarily continuous.*

Proof. Using (2) as the criterion for continuity, the above proof yields continuity of f, proving (i). Now it is well known that for arbitrary topological spaces (2) implies (1) (although the converse is false). But a counterexample to (1) is given by taking X to be the positive reals and $d(x, y) = xy$ for $x \neq y$.

HOW ASYMMETRIC IS A PARALLELOGRAM?

FRED KRAKOWSKI, University of California, Davis

The asymmetry of a parallelogram π can be measured by the quotient $k(\pi) = A(\sigma)/A(\pi)$, where $A(\pi)$ is the area of π and $A(\sigma)$ denotes the area of the largest mirror-symmetric region σ contained in π. Clearly $k(\pi) \leqq 1$ and $k(\pi) = 1$ only if π is mirror-symmetric, *i.e.*, a rectangle or a rhombus. In this note we show that for all parallelograms $k(\pi) \geqq 2\sqrt{2} - 2$ or, in other words, every parallelogram contains a mirror-symmetric region which covers more than 82% of its area. The following elementary argument yields this result.

First observe that there is always a pair of straight lines x and y passing through the center of π which meet at right angles and divide π into four parts of equal area [1]. Assume that π is not a rhombus and let A, B, C, D be the vertices of π and X, Y, X_1, Y_1 the intersections of x and y with the sides AB, BC, CD, DA respectively (Fig. 1). As the four diagonal triangles of π are all equal in area, each side of π contains exactly one of the points X, Y, X_1, Y_1 in its interior. An affine transformation t with fixed line y such that it moves all points in the direction perpendicular to y also leaves x invariant, does not destroy symmetry about x and leaves ratios of area unchanged. If the ratio of t is chosen to be equal to $(YY_1)/(XX_1)$ and if by $A', B', \pi', \cdots$ we denote the

images of A, B, π, $\cdots$ under t, we then have $Y=Y'$, $Y_1=Y_1'$, $X'X_1'=YY_1$, $X'Y_1=X'Y$, and $\sphericalangle YX'Y_1=90°$. Hence, denoting by α, β, γ and α_1, β_1, γ_1 the interior angles of the triangles $\Delta X'A'Y_1$ and $\Delta X'B'Y$ respectively, we get the relations: $\alpha_1 = 90° - \alpha$, $\beta_1 = 180° - \beta$, $\gamma_1 = 90° - \gamma$.

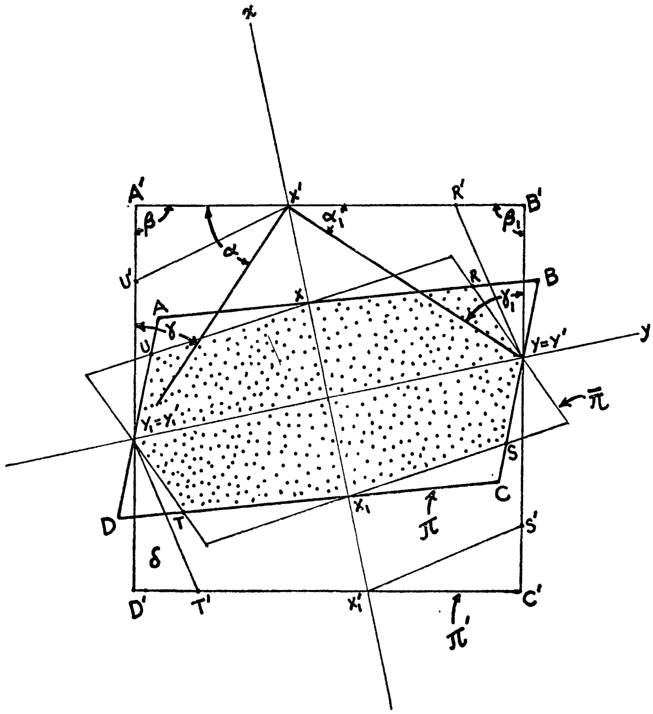

FIG. 1

Furthermore, the triangles $\Delta X'A'Y_1$ and $\Delta X'B'Y$ have the same area. Thus:

$$A'X' \sin\alpha = B'X' \sin\alpha_1 = B'X' \cos\alpha,$$

$$A'Y_1 \sin\gamma = B'Y \sin\gamma_1 = B'Y \cos\gamma,$$

$$A'X' \cdot A'Y_1 \sin\beta = B'Y \cdot B'X' \sin\beta_1 = B'Y \cdot B'X' \sin\beta.$$

From this we find $\tan\alpha \tan\gamma = (B'X' \cdot B'Y)/(A'X' \cdot A'Y_1) = 1$ and hence $\alpha+\gamma = 90°$, $\beta=\beta_1=90°$. We further observe that $\Delta X'A'Y_1 = \Delta X'B'Y$ and so $A'B' = A'X'+X'B' = B'Y+A'Y_1 = B'Y+YC' = B'C'$. Thus it follows that the image π' of the parallelogram π is a square.

It might be interesting to note at this point that the lines x and y turn out to be the principal axes of the ellipse which has the diagonals of the given parallelogram as conjugate diameters, because t transforms this ellipse into a circle, the diagonals into a pair of orthogonal diameters and leaves x and y invariant.

Let ω now be the symmetric octagon $XRYSX_1TY_1U$ which encloses the area common to π and the mirror-image $\bar{\pi}$ of π with respect to x. Clearly $A(\omega)/A(\pi) = A(\omega')/A(\pi')$ and $A(\omega') = A(\pi') - 4A(\delta)$, where δ is one of the four congruent right triangles lying inside π' and outside ω'. Let a be the side of the square π' and u, v the legs of δ. The perimeter of δ is equal to $u+v+\sqrt{(u^2+v^2)} = a$. Now among all right triangles with the given perimeter a the isosceles right triangle has the largest area, which is found to be $\frac{1}{4}a^2(3-2\sqrt{2})$. Hence $k(\pi) \geqq A(\omega)/A(\pi) = A(\omega')/A(\pi') = 1 - 4A(\delta)/A(\pi') \geqq 2\sqrt{(2)} - 2$.

Reference

1. R. Courant and H. Robbins, What is Mathematics?, New York, 1960, p. 318.

CORRECTION

A. A. Mullin, *An abstract formulation of a problem related to Goldbach's conjecture*, this MONTHLY, vol. 68, 1961, pp. 487–488. In line 3 of Definition 1.3, for "$\bigcap_{i\in I} M_i$" read "M_i".

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

OFFERINGS AND ENROLLMENTS IN MATHEMATICS

A Summary of an Office of Education Report*

EDITH S. TREUENFELS, Wisconsin State College, Stevens Point

A study of offerings and enrollments in science and mathematics in public high schools is of particular interest and importance at the present time. To which degree the nation can hope to meet the demand for scientists and mathematicians, essential for its security, can be partially gauged by the extent to which science and mathematics are studied in the high schools. The extent to which high school pupils study these subjects now indicates the degree of scientific literacy of tomorrow's citizenry. The effectiveness of the various programs and grants in support of and for the improvement of mathematics and science instruction should be reflected in larger enrollments and improved offerings.

A small decline reported in percent of total high school enrollment does not, however, necessarily indicate a decrease in the number of pupils taking a certain subject. Since the number of high school pupils is much larger now

* Kenneth E. Brown and Ellsworth S. Obourn. *Offerings and Enrollments in Science and Mathematics in Public High Schools 1958*. Superintendent of Documents, U. S. Government Printing Office, Washington, 1961.

than, say, in 1900, the number of pupils taking a certain subject now will be found much larger than 60 years ago.

The data of this study were received in 1958 in response to two* questionnaires sent by the Department of Health, Education, and Welfare to a random sample of about 20% of the public high schools which are listed in the U. S. Office of Education. About 92% of the schools questioned replied. Eighty-three percent (4228) of the science questionnaires sent out were usable upon return; so were 83.5% (4254) of the mathematics forms.

In addition to information about the total enrollment in each grade, 8 through 12, the forms sought information about offerings and enrollments in the various science and mathematics courses. Whether or not the curriculum is being revised was asked on both questionnaires. There is a place on both forms for a "less," "same," or "more" statement comparing the present emphasis on science and mathematics to that given by the school three years earlier. A similar question about the "trend in science enrollment" appears on the science questionnaire only.

In reply to the last question, 1.6% of the schools of all sizes† report a decrease, 27.4% an increase, which leaves about two-thirds of the schools which report little change in their enrollment trend. It is encouraging that 42.9% of the schools with a total enrollment of less than 100 pupils report an increasing trend.

To the question: "What is the present emphasis on science in your school as compared to 3 years ago?" 97% of the returns had an answer. Approximately two thirds indicated greater, one third unchanged, and only half a percent less emphasis.‡ The data allowed for the conclusion that the size of the school seemed to be an influencing factor: "From the smallest to the largest schools there was a progressively greater percent indicating more emphasis as compared to that of 3 years earlier" (p. 17).

The size of the school apparently influenced the offerings, too. Every science course was offered by higher percentages of schools with enrollments of more than 200 pupils. The percentage of increase in the schools which offer science has been greater for chemistry and physics than for general science and biology.

It is important to use utmost care and caution in making generalizations from this study about conditions in all schools in the country. The questions about enrollment and offerings asked for information on the first term 1958–59, only. Some courses, particularly in mathematics, may have been offered the second semester and therefore are not recorded. Besides, the study makes use of enrollments in the grades where a certain course is most commonly offered; *i.e.*, general science in the 9th grade, biology in the 10th. The account may also

* On page 77, three questionnaires are mentioned as reproduced in the appendix. Only two are found there.

† These categories of school sizes were used: 1–99, 100–199, 200–499, 500 or more pupils.

‡ How "emphasis" is measured or how it manifests itself is not revealed.

conceal the true picture in the categories of smaller schools. There general science and biology, or chemistry and physics may be offered in alternate years, and consequently are not necessarily taken in the grade in which the course is most commonly offered.

Forty percent of the returns indicated that the mathematics curriculum was being revised. In 97.4% of the returned mathematics questionnaires the principals had answered the question on emphasis. Less than 1% reported decrease. Approximately 33.5% found that the emphasis remained the same, while more than 63% indicated an increase.

The enrollment in the grade where a certain mathematics course was most commonly offered had been chosen as the base to which to compare the enrollment in that course. Schools in this study with enrollments of 500 or more offered their pupils a much better opportunity to study mathematics than smaller schools. In some geographical regions, a smaller percent of the schools offered certain mathematics courses than others, so that "irrespective of cause, all pupils in this study did not have an equal opportunity to take mathematics" (p. 49).

Two facts are interesting to note, "The number of boys exceeded the number of girls in all the mathematics courses of this study" (p. 64); and, considering average class size, "At the very time when the pupils needed individual classroom assistance most to understand basic mathematical principles, they unfortunately found themselves in large classes" (p. 71).

General mathematics is considered to be a 9th-grade course. About 60% of the schools with 9th-grade enrollment offered general mathematics. Thirty-four and four tenths percent of the 9th-grade pupils in the study were enrolled in general mathematics. Twenty-nine and five tenths percent of all the pupils in this study were enrolled in schools not offering it. Fifty-three and five tenths percent of the enrollment in general mathematics were boys. The average class size was 26.5.

Elementary algebra was taken by 71.6% of the 9th-grade pupils. In general, only schools with very small enrollment did not offer elementary algebra. Only 1.5% of all the 9th-grade pupils in the study were in schools which did not offer it. In the 9th-grade algebra courses 53.2% were boys. The average class size was 27.4 pupils.

Plane geometry may be given in alternate years and is probably offered by a larger percentage of schools with 10th grade enrollment than the 84.3% which the study indicates. Forty-six and seven tenths percent of the 10th-grade pupils in the study were taking plane geometry, while 5.6% of them were in schools not offering this subject. Fifty-eight and three tenths percent of the plane geometry enrollment were boys. The average class size was 24.6 pupils.

Intermediate algebra or advanced algebra (presumably courses of the same content) is offered in the 11th grade by 73% of the schools. Of the 11th-grade pupils, 37.0% took intermediate or advanced algebra. Nine and seven tenths percent of all 11th-grade pupils in the study were not offered this opportunity

in their schools. Of those pupils taking it, 61.8% were boys. The average class size was 23.5 pupils.

The practice of offering trigonometry and solid geometry in alternate semesters seems to lose ground. A trigonometry-advanced algebra combination seems to gain favor. Solid geometry, when not offered as a separate course and if taught at all, is either integrated in plane geometry or treated for some weeks during the plane geometry course. Thus only 13.7% of the schools report offering a course in solid geometry. These had enrolled 3.9% of all the 12th-grade pupils in this study. Seventy-two and four tenths percent of the 12th-graders are in schools not offering it in the fall of 1958. Of those pupils taking solid geometry, 78.4% were boys. The average class has 17.5 pupils.

Forty-one and eight tenths percent of those schools which had 12th-grade enrollment offered trigonometry. Eleven and five tenths percent of the 12th-grade pupils took advantage of the offerings, while 33.2% were in schools not offering trigonometry. Seventy-six and four tenths percent of the trigonometry pupils were boys. The average class size was 17.

Three and eight tenths percent of the schools with a 12th grade offered a college mathematics course for advanced standing. Seventy-one and seven tenth percent of the pupils enrolled were boys.

An advanced general mathematics course for 11th- and 12th-grade non-college-bound pupils was reported by 16.2% of the schools; 55.8% of the enrollment were boys.

The study assumes with most of us that a larger enrollment, more offerings, and greater emphasis on science and mathematics courses are healthy, encouraging signs of a trend to the better in public education. The following uneasiness about such an unqualified assumption is of course not found in the study. In deed, an expression of such misgivings would not have a place in a statistical report. Nevertheless, a question arises which the study does not and should not attempt to answer: What areas of study, if any, have a smaller enrollment as a consequence of the increase in the science-mathematics enrollment? Has emphasis on literature, history, and the fine arts been sacrificed, thus over-emphasizing mathematics and science? Or have the schools managed a sound balance? How do the so-called nonacademic courses like typing and driver education fare? Are summer-school sessions utilized to teach these and other skills or is the student with a four-year science-and-mathematics program being deprived of training in these skills he needs now more urgently than ever? In other words: Will tomorrow's college freshman be well educated? Or will he be a specialist, literate in his subject only?

A GENERATION OF HIGH SCHOOL CALCULUS*

J. H. NEELLEY, Carnegie Institute of Technology

For the past four years we have had freshmen come to Carnegie Institute of Technology with some high school calculus. We have tried to meet the desires of these freshmen in several ways. The first group was integrated with second-semester students. This was very unsatisfactory. The following three groups have been kept apart and have been given a course in calculus and analytics to cover four semesters in two. This course has varied some in the amount of analytics since a full course proved to be too much.

I have watched carefully these accelerated students and am now able to report on our first college generation of such!

In 1956, the press, politicians and then the public began to demand that the high schools do the work equivalent to that done in Europe under a very different situation. This did create the high school accelerated programs. So, in 1957–58, we had 13 freshmen register as majors in mathematics, whereas before then we had had less than 5 each fall. This pressure has gradually increased our mathematics majors to 32 in 1958–59, 37 in 1959–60 and this year to 46 for 1960–61.

In these four years the freshman classes have had an increased number of accelerated high school students. Only 6 in 1957–58, 48 in 1958–59, 33 in 1959–60 and 54 in 1960–61.

These are our brightest students and so it is interesting to see how they have registered in college. The first year, 2 of the 6 registered for mathematics as major. The second year 5, then the third year 8 and this year 5.

These figures are surprising. When only 5 out of 54 register for mathematics as major when at the same time we have 46 so register from the freshman class, we are shocked. This carries on the whole four years, 2 of 13, 5 of 32, 8 of 37 and 5 of 46. So our mathematics majors are not coming in large numbers from the high school accelerated groups.

Also it is interesting to see how the accelerated group holds its own in college. The first group of 2 dropped to 1 in short order. This is the year we had 13 mathematics majors. Today this class is the senior and the pressure has made it grow from 13 to 20 with only 1 accelerated student.

The 1958–59 group started at 48 and only 11 completed the first year of college work. There were 5 who originally registered as mathematics majors and today, as juniors, only 3 of them are still such. This is 3 out of 25 juniors today.

The 1959–60 group started at 33 and 18 finished the first year of college mathematics. This course was cut some as compared to the 1958–59 one. Today we have 31 sophomores who are mathematics majors and only 4 of that original group are still such.

* Presented to the Mathematics Council of Western Pennsylvania at Annual Conference, March, 1961, and to the Mathematical Association of America, May, 1961.

Our fourth year, 1960–61 saw 54 freshmen come with high school calculus and today we have 46 freshmen mathematics majors with only 5 of them from the accelerated high school group. The 54 is now down to 17 and the first year is not completed.

To summarize the generation, out of 1700 engineering and science freshmen, 141 came to us with high school calculus. As of now, we have 14 or only 10 per cent of that group as mathematics majors. This is even more striking in that we now have 122 mathematics majors in Tech.

These figures seem surely to point to two things. First, high school acclera-tion is not the way to increase our number of college majors in mathematics. Second, the high casualty in the accelerated college courses make it seem that "high school calculus is largely a waste of time."

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1491. *Proposed by J. W. Andrushkiw, Seton Hall University*

Show that there exists at least one and at most three equilateral triangles inscribed in a parabola and having a given point P on the parabola for a vertex.

E 1492. *Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, Pennsylvania State University*

Considering unity as a prime, show that every positive integer can be written as a sum of distinct primes.

E 1493. *Proposed by L. Flatto, N. C. Hsu, and A. G. Konheim, IBM Research Center, Yorktown Heights, New York*

Prove that a nonzero integral polynomial $h(x)$ (*i.e.*, one with integer coefficients) which vanishes at $x=1$ and $x=2$ must have a coefficient $\leqq -2$.

E 1494. *Proposed by George Sadowsky, Combustion Engineering, Inc., Windsor, Connecticut*

Invert the matrix $[a_{ij}]=[x^{|i-j|}]$, $x\neq 0, \pm 1$.

E 1495. *Proposed by Dunstan Hayden, The Priory School, Washington, D. C.*

Let $f(z)=z^2$, and let the domain of this function be any straight line in the

complex plane. The graph of the range of f is then a parabola in the complex plane. What are the coordinates of its vertex?

SOLUTIONS

A Doubly True Addition

E 1461 [1961, 378]. *Proposed by Underwood Dudley, University of Michigan*

Solve the cryptic addition

$$\begin{array}{ccccc} \text{F} & \text{I} & \text{F} & \text{T} & \text{Y} \\ & \text{F} & \text{O} & \text{U} & \text{R} \\ & \text{F} & \text{O} & \text{U} & \text{R} \\ & & \text{T} & \text{W} & \text{O} \\ \hline \text{S} & \text{I} & \text{X} & \text{T} & \text{Y} \end{array}$$

remembering that FOUR$+12$ is a perfect square.

I. *Solution by C. W. Trigg, Los Angeles City College.* Replace the letter O with θ. $\text{S}>\text{F}\neq 9$, so F is 4 or 5, whereupon $\text{S}=\text{F}+1$. Since $2\text{R}+\theta=10$ or 20, the only possible value of $\text{F}\theta\text{UR}(=n^2-12)$ is 4612. Then $\text{S}=5$ and $\text{W}=10-2\text{U}-1=7$. Now $\text{F}+2\theta+\text{T}+1=\text{X}+20$, $\text{T}=\text{X}+3$, and $\text{X}=0$, $\text{T}=3$. Finally, $(\text{I}, \text{Y})=(8, 9)$ or $(9, 8)$. Thus there are two of these doubly true additions:

$$48439 + 4612 + 4612 + 376 = 58039, \quad 49438 + 4612 + 4612 + 376 = 59038.$$

II. *Solution by J. A. Lambert, Newcastle University College, N.S.W., Australia*

We have the congruences (modulo 10)

(1) $\text{R} + \text{R} + \text{O} \equiv \text{zero},$

(2) $\text{U} + \text{U} + \text{W} + (1, 2, \text{ or } 3) \equiv \text{zero},$

(3) $\text{F} + \text{O} + \text{O} + \text{T} + (1, 2, \text{ or } 3) \equiv \text{X},$

(4) $\text{F} + \text{F} + (1, 2, \text{ or } 3) \equiv \text{zero},$

the equation

(5) $\text{F} + (1 \text{ or } 2) = \text{S},$

and

(6) FOUR $+$ 12 is a perfect square.

Now (4) shows that $\text{F}=4$ or 9; (5) that $\text{F}=4$, $\text{S}=5$; (6) gives (by examination of perfect squares) 4612 for FOUR; (2) now gives $\text{W}=7$; (3) leads to $\text{T}=3$, $\text{X}=\text{zero}$. I and Y are not uniquely determined; each might be either 8 or 9.

Also solved by A. N. Aheart, Ronald Alter and Herbert Gintis (jointly), Bonnie Amner, M. H. Auerbach, J. W. Baldwin, Merrill Barnebey, Robert Bart, Charles Bartha, H. F. Bennett, C. R. Berndtson and E. R. Williams (jointly), Jeanette Bickley, T. P. Bleakney, Walter Bluger, A. Bollobás, Christopher Boorse, Robert Bowen, K. C. Bower, Brother Patrick Ronald, Brother Louis Zirkel, Robert Budny, W. E. Buker, Bradford Burdick and Robert Spira (jointly), F. P.

Callahan, Jr., Robert Carlos, Charles Carniglia, F. H. Cleveland, D. I. A. Cohen, E. L. Cohen and G. M. Leibowitz (jointly), R. J. Cormier, J. B. Deeds, Monte Dernham, D. Drasin, J. R. Durbin, E. S. Eby, A. D. Egendorf, Aaron Eidelman, Jane Evans, J. A. Fauches, F. E. Fischer, David Forslund, C. S. Patlak and Seymour Geisser (jointly), Howard Givner, Anton Glaser, Larry Glickfeld, Robert Goldberg, Michael Goldberg, L. D. Goldstone, Jay Gottesfeld, R. G. Green, Corinne Hattan, F. W. Herlihy, Margaret Herzog, J. E. Homer, Jr., Alice Hunt, J. A. H. Hunter, C. R. Hutchinson, A. R. Hyde, Daniel Isaacson, Diane M. Johnson, William Kantrowitz, Harry Kapper, Jr., Leonard Klosinski, Kenneth Kloss, Donald Knuth, R. R. Korfhage, Sidney Kravitz, Harry Langman, Dean Lawrence, H. L. Lawton and Anna H. Watson (jointly), H. R. Leifer, R. J. Lewyckyj, William Lopez, James Lucke, David McCarroll, W. M. McKeeman, D. C. B. Marsh, T. H. Means, G. J. Michaelides, J. W. Milsom, Otto Mond, D. A. Moran, Mr. and Mrs. D. L. Muench (jointly), J. B. Muskat, Herbert Nadler, H. L. Nelson, H. J. Noble, C. S. Ogilvy, E. A. Passow, Sidney Penner, Walter Penney, L. B. Perry, D. J. Persico, J. P. Phillips, J. L. Pietenpol, C. F. Pinzka, Harsh Pittie, M. T. Rincon, L. A. Ringenberg, O. J. Roman, David Sachs, H. Schloss, L. J. Schneider, Mitchell Secondo, S. J. Sidney, D. R. Simpson, Carl Spitznagel, E. L. Spitznagel, Jr., W. B. Stovall, Jr., Eric Sturley, Paul Stygar, G. C. Thompson, Guy Torchinelli, Donald Vogel, W. C. Waterhouse, Alan Weinstein, Charles Wexler, Sam Wheatman, D. R. Wilder, R. H. Wilson, Jr., R. L. Yates, Walter Zayachowski, and the proposer. Late solutions by Rafael Gallego-Diaz, Roger Hunt, James Johnson, L. J. Katz, Jr., Adolph Lu, Wayne McPherson, B. B. Mapinin, Soran Stojakovič, and Rolland Sturtevant.

Among the suggestions to render the solution unique were: (1) add the condition "YOU is a square" or "I is a square" (Pinzka), (2) add the condition "FIFTY and SIXTY are even" (Marsh), (3) add the condition "SIX+7 is prime" (Bleakney and Pietenpol), (4) replace "FOUR+12 is square" by "FIFTY+39 is prime" (Knuth).

Hunter, who has originated the name "alphametic," suggested the following examples

```
   SIX            SIX
 SEVEN            SIX
 SEVEN          -----
------           xxxx
TWENTY          xxxx
               xxAS
              ------
              ADOZEN
```

where in the second one the x's indicate digits. The proposer offered the following additional alphametics: TWENTY+FIVE+FIVE=THIRTY and WHEN is a square, SEVEN+THREE+FIVE+FIVE +TEN=THIRTY.

Nelson pointed out that TEN+TEN+FORTY=SIXTY has a unique solution if the base of the arithmetic is specified to be 10; if the base is not specified, there are infinitely many solutions. He then proposed showing that each of the two cryptic divisions

```
       IA              WE
EEL)MAIL        SEE)SOOT
    EEL             SAWN
    ----            ----
    IMUL             NEAR
    ISMS             NNNW
    ----             ----
      EU              NYC
```

has a solution for only one radix, and then the solution is unique.

For other cryptarithms and arithmetical restorations the reader might consult problems 3212, E1, E7, E10, E20, E22, E23, E30, E37, E43, E58, E71, E78, E99, E105, E130, E140, E160, E164, E173, E184, E192, E198, E217, E251, E258, E265, E297, E317, E333, E751, E891, E971, E1110, E1111, E1241.

Alternating Colors

E 1462 [1961, 378]. *Proposed by Michael Skalsky, Southern Illinois University*

In how many ways can 4 white, 3 black, and 3 red balls be arranged in a row so that no two adjacent balls are of the same color?

Solution by C. F. Pinzka, University of Cincinnati. Consider the $6!/3!^2=20$ arrangements of the black and red balls only. Suppose that one of these arrangements has r balls following balls of the same color. The white balls may be inserted in $\binom{7-r}{4-r}$ ways to give arrangements in which no two adjacent balls are of the same color. The total number of arrangements of this kind is then $\sum_{r=0}^{4} a_r\binom{7-r}{4-r}$, where a_r is the number of arrangements of the black and red balls in which r balls follow balls of the same color. By direct enumeration it is found that $a_r=2, 4, 8, 4, 2$ for $r=0, 1, 2, 3, 4$ and the desired number of arrangements is 248.

Also solved by A. N. Aheart, J. W. Baldwin, H. F. Bennett, Walter Bluger, Robert Bowen, Brother Alfred, Brother Louis Zirkel, W. E. Buker, Charles Carniglia and Alan Felzer (jointly), V. D. D'Antonio and R. C. Potter (jointly), Monte Dernham, Underwood Dudley, Jane Evans, Seymour Geisser and C. S. Patlak (jointly), Howard Givner, Michael Goldberg, L. D. Goldstone, Jay Gottesfeld, A. S. Gregory, F. W. Herlihy, A. R. Hyde, D. M. Johnson, Erwin Just, Francis Katcher and R. T. Shannon (jointly), Donald Knuth, Jiang Luh, David McCarroll, Daniel Maki, D. C. B. Marsh, Walter Penney, Mary Agnes Racki, L. A. Ringenberg, David Sachs, L. J. Schneider, S. J. Sidney, W. B. Stovall, Jr., Guy Torchinelli, and Walter Zayachkowski. Late solutions by Daniel Isaacson, J. B. Muskat, and Mirko Stojakovič.

Fourteen of these solutions disagreed with 248 as the final answer.

Geisser and Patlak arrived at the solution by inserting the appropriate values into equation 4.2 (p. 375) of A. M. Mood, "The Distribution Theory of Runs," *Ann. Math. Stat.*, 11, 1940, pp. 367–392. Goldstone called attention to Probs. 88, 89, 97, 98, 100, 170, 724, 725 in Whitworth's *Choice and Chance.*

A Locus in Three-Space

E 1463 [1961, 378]. *Proposed by V. F. Ivanoff, San Carlos, California*

Given an imaginary point P: $(a+pi, b+qi, c+ri)$ in 3-space. Find the locus of real points (x, y, z) whose distance from P is real.

Solution by Underwood Dudley, University of Michigan. We must have

$$(a - x + pi)^2 + (b - y + qi)^2 + (c - z + ri)^2$$

real and nonnegative, or $p(a-x)+q(b-y)+r(c-z)=0$ and

$$(a - x)^2 + (b - y)^2 + (c - z)^2 \geqq p^2 + q^2 + r^2.$$

Thus the locus consists of the points on the plane through (a, b, c) perpendicular to the direction (p, q, r) and not interior to the circle on the plane with center (a, b, c) and radius $(p^2+q^2+r^2)^{1/2}$.

Also solved by C. B. Barfoot, Merrill Barnebey, Robert Bart, H. F. Bennett, A. Bollobás, Brother Louis Zirkel, E. L. Cohen, J. R. Durbin, Jane Evans, David Friedman, Seymour Geisser

and C. S. Patlak (jointly), Michael Goldberg, Robert Goldberg, L. D. Goldstone, Carl Harris, A. R. Hyde, Erwin Just, J. J. Kim, Harry Langman, Jiang Luh, W. M. McKeeman, D. C. B. Marsh, D. A. Moran, D. L. Muench, Amos Nannini, Walter Penney, D. J. Peterson, J. L. Pietenpol, C. F. Pinzka, David Sachs, S. J. Sidney, Barry Simon, Paul Stygar, Kay Leon Tjio, Guy Torchinelli, W. C. Waterhouse, Walter Zayachkowski, and the proposer.

A number of these solutions were not complete.

Both McKeeman and the proposer pointed out that the problem is easily extended to n-dimensional space.

An Application of the Arithmetic-Geometric Inequality

E 1464 [1961, 378]. *Proposed by Freddy Storey, Princeton University*

Show that for $n \geqq 2$, $\prod_{i=0}^{n} \binom{n}{i} \leqq \{(2^n-2)/(n-1)\}^{n-1}$.

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. Since

$$\prod_{i=0}^{n} \binom{n}{i} = \prod_{i=1}^{n-1} \binom{n}{i}, \qquad 2^n - 2 = \sum_{i=1}^{n-1} \binom{n}{i},$$

the desired inequality is merely the arithmetic-geometric inequality for the $n-1$ numbers $\binom{n}{i}$, $i=1, \cdots, n-1$.

Also solved by A. N. Aheart, D. W. Bailey, A. Bollobás, D. A. Breault, J. L. Brown, Jr., Leonard Carlitz, David Chale, A. J. Chandy, J. L. Cline, D. I. A. Cohen, J. B. Deeds, G. C. Dodds, Underwood Dudley, Sarah Evangelista and Peter Hagis, Jr., (jointly), H. M. Feldman, J. H. Folkman, David Forslund, David Friedman, Michael Goldberg, A. S. Gregory, J. C. Hickman, A. R. Hyde, Erwin Just, Leonard Klosinski, Kenneth Kloss, J. J. Kim, Hsü-Tung Ku, Viktors Linis, Jiang Luh, W. M. McKeeman, P. B. Manchester, D. C. B. Marsh, J. B. Muskat, D. J. Persico, C. F. Pinzka, Harsh Pittie, David Sachs, Norman Schaumberger, S. J. Sidney, T. H. Slook, Evelyn Strawbridge, Guy Torchinelli, W. C. Waterhouse, and the proposer. Late solution by B. B. Mapinin.

Carlitz showed, in addition, that $\ln \prod_{i=0}^{n} \binom{n}{i} = n^2/2 + O(n \ln n)$.

A Non-Biplanar Graph

E 1465 [1961, 379]. *Proposed by U. R. Kodres, IBM Corp., Poughkeepsie, New York*

It is well known that the graph which represents the classical problem of connecting three houses to three utilities is not planar. Prove that a generalization of this graph, namely the graph which represents connecting seven houses to seven utilities is not biplanar, *i.e.*, the graph cannot be factored into two planar factors.

Solution by D. C. B. Marsh, Colorado School of Mines. If the seven-to-seven graph were biplanar, then one of its factors would have to contain at least 25 arcs. However, an n-to-n set may be (partially) connected by at most $4(n-1)$ coplanar arcs which are nonintersecting. When $n=7$, this yields a maximum of 24, showing the nonexistence of a biplanar graph in this case.

Also solved by David Sachs and the proposer.

Editorial Note. A similar proof shows that the problem of connecting $4m-1$ houses to $4m-1$ utilities is not m-planar.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4995. *Proposed by Oystein Ore, Yale University*

When $\phi(x)$ denotes Euler's function it is readily verified that $n=14$ is the smallest even number such that the equation $\phi(x)=n$ has no solution. Prove that for each exponent α there is a smallest odd integer k_α such that the equation $\phi(x)=2^\alpha k_\alpha$ has no solution. Determine k_2, k_3, k_4. Try to find bounds for k_α.

4996. *Proposed by Richard Bellman, RAND Corporation, Santa Monica, Calif.*

Let $\{x_n\}$ be a sequence of nonnegative N-dimensional vectors and let A be an N by N nonnegative matrix. If $x_{n+1}\leqq Ax_n-y_n$ (in the sense of inequality component by component), and if y_n is nonnegative with $\sum_{n=1}^\infty y_n<\infty$, under what conditions on A does the sequence $\{x_n\}$ converge?

4997. *Proposed by Donald Newman, Yeshiva University*

Show that the power series of $e^{(z+1)/(z-1)}$ has coefficients $O(n^{-3/4})$, but not $o(n^{-3/4})$.

4998. *Proposed by Meyer Wolf, University of Minnesota*

S. H. Unger has presented a proof concerning the self-overlapping of cyclically equivalent sequences (This MONTHLY, Feb. 1960, pp. 139–143), that depends on the property of being made up of a chain of identical subsequences. Prove that the composition of the sequence formed from the concatenation of two sequences X and Y is independent of the order of concatenation if and only if X and Y are chains of a particular subsequence Z.

4999. *Proposed by R. B. Deal, Oklahoma State University*

Is there a finitely additive probability on the set of all subsets of the positive integers which assigns probability zero to all finite sets?

5000. *Proposed by R. C. Lyness, Singleton Lodge, near Blackpool, England*

$x^3-2x^2+2x-1=0$ has roots which are all roots of unity and so $w_{n+3}=2w_{n+2}-2w_{n+1}+w_n$ with $w_0=0$ is a cubic recurrence with an infinite number of zeros, w_{6n}.

$2x^3-2x^2+2x-1=0$ gives the recurrence $w_{n+3}=w_{n+2}-w_{n+1}+\frac{1}{2}w_n$ and with

$w_0=w_1=0$ we have $w_m=0$ for $m=0, 1, 4, 6, 13, 52$.

Can a cubic recurrence with only a finite number of zeros have more than six?

SOLUTIONS

Matrix with Eigenvalues 0 and 1

4872 [1959, 817]. *Proposed by Ky Fan, Wayne State University*

Let A be a matrix (not necessarily square) of rank $r \geqq 1$ and with nonnegative elements. Let A^* denote the transpose of A. Prove that the square matrix AA^* has no eigenvalue different from 0, 1, if and only if, after deleting all identically vanishing rows and columns, the remaining submatrix of A can be brought by a permutation of the rows and a permutation of the columns to the form

$$B = B_1 \dotplus \cdots \dotplus B_r, \tag{1}$$

where, for each i, B_i is a rectangular matrix of rank 1, with all its elements positive and such that the sum of squares of all elements of B_i is 1. Equation (1) means that B is obtained by laying out successively the rectangular blocks $B_1, \cdots, B_r$ with the lower right corner of B_i attached to the upper left corner of B_{i+1}, and with zeros filling in the entire matrix outside these blocks.

Solution by Hans Schneider, the University of Wisconsin. If H is a nonnegative matrix whose maximal eigenvalue ρ is k-fold, then there exists a permutation matrix P, for which

$$PHP^* = H_1 \oplus \cdots \oplus H_k \oplus L,$$

where the H_i are irreducible square matrices having ρ as a simple eigenvalue, and the maximal eigenvalue of L is less than ρ. It follows that, if the eigenvalues of H are 1 or 0, then $k=r$, the rank of A. Thus

(a) If AA^* satisfies the given conditions then, for some permutation matrix P, $PAA^*P^*=H_1\oplus \cdots \oplus H_r \oplus 0$, where each H_i is an irreducible matrix of rank 1 and trace 1; and (obviously) conversely.

If b_i and b_j are rows of a nonnegative rectangular matrix B, such that $b_ib_j^*=0$, then no column of B has positive elements in both b_i and b_j. If $B_iB_i^*$ is irreducible then B_i has no zero rows. Hence

(b) If $A=H_1\oplus \cdots \oplus H_k \oplus 0$, where the H_i are irreducible if and only if $B=PAQ=B_1\oplus \cdots \oplus B_r \oplus 0$ for all permutation matrices Q, where the B_i are rectangular matrices (having the same number of rows as H_i), then $B_iB_i^*=H_i$, and B_i has no zero rows and, provided Q is properly chosen, no zero columns.

It is known that rank $B_iB_i^*=$ rank B_i, and the sum of squares of elements of B_i equals trace $B_iB_i^*$. Hence

(c) The matrix $H_i=B_iB_i^*$ is of rank 1 and trace 1, if and only if B_i has rank 1, and the sum of squares of its elements is 1.

(d) If a matrix of rank 1 has a zero element, then it has a zero row or col-

umn, and (again obviously) conversely.

Combining (a), (b), (c) and (d) we obtain the desired results.

Also solved by the proposer.

Distinct Sums in a Nonassociative Algebra

4931 [1960, 926]. *Proposed by Harry Goheen, Oregon State College*

In an algebra for which there is a commutative law but no associative law, how many formally distinct sums are there of n elements?

I. *Solution by J. R. Brown, University of Massachusetts.* Let the number of distinct sums of n elements be S_n. Such a sum must be composed of two subsums which are composed of k and $n-k$ elements respectively. By commutativity we have

$$S_n = \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}S_kS_{n-k}.$$

Hence, since $S_1=1$, and $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$,

$$\begin{aligned} S_{n+1} &= \frac{1}{2}\sum_{k=1}^{n}\binom{n+1}{k}S_kS_{n+1-k} \\ &= \frac{1}{2}S_n + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}S_kS_{n+1-k} + \frac{1}{2}\sum_{k=2}^{n}\binom{n}{k-1}S_kS_{n+1-k} + \tfrac{1}{2}S_n \\ &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\left[S_kS_{n+1-k} + S_{k+1}S_{n-k}\right] + S_n. \end{aligned}$$

Assume $S_{k+1}=(2k-1)S_k$ for $k<n$. Then

$$\begin{aligned} S_{n+1} &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\left[(2n-2k-1)+(2k-1)\right]S_kS_{n-k} + S_n \\ &= (2n-2)S_n + S_n = (2n-1)S_n. \end{aligned}$$

Since $S_2=1=(2-1)S_1$, this last result is true for all n. Consequently, for $n\geqq 2$,

$$S_n = 1\cdot 3\cdot 5\cdots(2n-3) = \frac{(2n-3)!}{2^{n-2}(n-2)!}\cdot$$

R. W. Wagner has pointed out the close relation of this problem to 3954 [1941, 564–9].

II. *Solution by James Singer, Brooklyn College.* Since summation is a binary operation, $n-1$ pairs of parentheses must be used to render an indicated sum of n nonassociative elements meaningful (this includes the final pair of parentheses around the first and last elements). The $n-1$ pairs of parentheses can be placed in $\binom{2n-2}{n-1}/n$ ways. (See A. Cayley, *On the analytic forms called trees, Second Part*; Phil. Mag., vol. 18, 1959, pp. 374–378; also *Collected Works*, vol. 4,

pp. 112–115. The formula has been frequently rediscovered since then.) The n elements can be arranged in $n!$ ways to yield $(n-1)!\binom{2n-2}{n-1}$ sums. Since the two terms within any one pair of parentheses can be interchanged without changing the sum, the number of distinct formal sums is

$$\frac{(n-1)!}{2^{n-1}}\binom{2n-2}{n-1}.$$

Also solved by R. J. Cornier, S. H. Greene, Jiang Luh, B. J. M. Morselt, R. C. Read, and the proposer.

Solutions of a Fourth-Order Differential Equation

4932 [1960, 927]. *Proposed by D. S. Mitrinovitch, Belgrade University, Yugoslavia*

Consider the differential equation $z^2(d^4w/dz^4)=a(d^2w/dz^2)$, where a is a real parameter, $z=x+iy$. Determine the singularities of the complex functions $w=f(z, a)$ which satisfy it.

Solution by Emil Grosswald, University of Pennsylvania. The formal solution of the equation is straightforward. Denoting differentiation by accents and setting $w''=v$, the equation reads: $z^2v''=av$, of Euler-Cauchy type. According to the value of a, we distinguish the following cases:

(1) $a=0$, $v''=w''''=0$, whence $w=Az^3+Bz^2+Cz+D$;

(2) $a=-\frac{1}{4}$, $v=(C_1+C_2 \log z)z^{1/2}$, whence $w=(A \log z+B)z^{5/2}+Cz+D$;

(3) $a=2$, $v=C_1z^2+C_2z^{-1}$, whence $w=Az \log z+Bz^4+Cz+D$;

(4) $a=6$, $v=C_1z^3+C_2z^{-2}$, whence $w=A \log z+Bz^5+Cz+D$.

If $a\neq 0$, $-\frac{1}{4}$, 2, 6, then $v=C_1z^s+C_2z^t$, where s and t are the distinct roots of $x^2-x-a=0$, and one obtains

$$(*) \qquad w = Az^{s+2} + Bz^{t+2} + Cz + D.$$

In particular, if a is of the form

(5) $a=r(r-1)>-\frac{1}{4}$, with rational $r=p/q$ (p, q coprime integers), then $s=r$, $t=1-r$ in (*); if

(6) $a>-\frac{1}{4}$ but $\neq r(r-1)$ with rational r, then s and t are distinct, real, irrational numbers; and finally, if

(7) $a<-\frac{1}{4}$, then s and t are complex conjugate numbers and (*) may be written as

$$w = z^{5/2}(A \sin(c \log z) + B \cos(c \log z)) + Cz + D,$$

with $c=\frac{1}{2}(4|a|-1)^{1/2}$.

Hence, excluding the trivial case $A=B=C=0$ when $w=D$ a constant, it follows that w has logarithmic branch-points at $z=0$ and $z=\infty$ in cases (2), (3), (4) if $A\neq 0$, and in cases (6), (7) if A, B are not both zero; these branchpoints become algebraic, of order one in case (2) if $A=0$, $B\neq 0$, and of order $q-1$ in case (5) with A and B not both zero. In the remaining cases, w is a polynomial

and its only singularity is the pole at $z=\infty$ of order equal to the highest non-vanishing power of z.

Also solved by G. DiAntonio.

Union of Commutative Fields

4933 [1960, 927; 1961, 299]. *Proposed by I. N. Herstein, Cornell University*

Suppose a ring R is the set-theoretic union of a finite number of commutative fields having the same unit element; prove that R must then be a commutative field.

I. *Solution by B. R. Toskey, Seattle University.* The common unit element must be the identity for R, and each element of R has an inverse, since it is in a field containing identity. Thus R is a division ring. Now, let n be the number of fields given, so that for any two elements a, x, in R, at least two of the elements $a, ax, ax^2, \cdots, ax^n$ must lie in the same field and hence commute with each other. Thus $ax^kax^j=ax^jax^k$ for some k, j with $k>j\geqq 0$, and thus $x^{k-j}a=ax^{k-j}$. But, since $0<k-j\leqq n$, we have $x^{n!}a=ax^{n!}$. This shows that $x^{n!}$ is in the center of R for any x in R. By a theorem of Kaplansky [Canad. J. Math., vol. 3, 1951, pp. 290–292; also Jacobson, *Structure of Rings*, p. 185], R is therefore a commutative field.

II. *Solution by Robert Spira, Berkeley, California.* Let $R=\bigcup_{i=1}^n F_i$ where the F_i are commutative fields. Let j be such that $F_k\subset\bigcup_{i=k+1}^n F_i$ for $k<j$, but F_j is not contained in $\bigcup_{i=j+1}^n F_i$. As this is a finite union, such a j must exist. Set $B=\bigcup_{i=j+1}^n F_i$. Clearly $R=\bigcup_{i=j}^n F_i$. It will be shown that $R=F_i$ for one of the F_i's in this last union.

For if not, we can take $b\in B$, $b\notin F_j$ (as $F_j\neq R$); and also we can take $a\in F_j$ and $a\notin B$, by the defining property of F_j. The product ab lies either in F_j or in B. If $ab\in F_j$, multiply by a^{-1}, obtaining $a^{-1}ab=b\in F_j$, a contradiction (using here the fact that the units of the fields coincide). If $ab\in B$, multiply by b^{-1}, obtaining $abb^{-1}=a\in B$, a contradiction. Thus the product ab does not lie in R, contradicting assumption that R is a ring. Thus R must be one of the given F_i's, and is hence trivially a commutative field.

Also solved by R. M. Cohn and E. R. Gentile, Carl Faith, Melvin Henriksen, R. B. Kirchner, H. B. Mann, Barbara L. Osofsky, W. Peremans, E. C. Posner and Neal Zierler, W. R. Scott, R. T. Shannon, Jack Silver, K. R. Unni, William Veeck and Joan Richardson, Seth Warner, W. C. Waterhouse, and the proposer. Late solutions by Gerald Janusz and W. S. Martindale.

Editorial Note. The proposition as originally stated (omitting reference to a common unit) is not correct. The direct sum of two or more copies of the prime field Z_2 of characteristic 2 is not a field but it is a union of fields. Cohn and Gentile prove: *A ring which is a union of a finite number of fields is either a field or a direct sum of (a finite number of) copies of* Z_2.

Henriksen notes that Biatynicki-Birula, Browkin, and Schinzel have shown that a field cannot be the union of finitely many proper subfields (Colloq. Math., vol. 7, 1959, pp. 31–32).

Group Algebra

4934 [1960, 927]. *Proposed by I. N. Herstein, Cornell University*

Let F be a field of characteristic 0, let G be a group and $\Gamma(G, F)$ the group algebra of G over F. Prove that $\Gamma(G, F)$ is an algebraic algebra over F if and only if G is locally finite.

Solution by E. C. Posner, Harvey Mudd College, and Neal Zierler, Massachusetts Institute of Technology. Suppose G is locally finite and let $x=\sum_{i=1}^{n} a_i g_i$ be a member of Γ where $a_i \in F$ and $g_i \in G$. The subgroup H of G generated by $g_1, \cdots, g_n$ is finite by hypothesis, so its group algebra over F, which contains x, is a finite dimensional vector space over F. Hence there exists $m>0$ such that x^{m+1} is linearly dependent on $x, \cdots, x^m$, *i.e.*, Γ is algebraic.

Conversely, suppose Γ is algebraic, let $g_1, \cdots, g_n$ be elements of G and let $x=g_1+\cdots+g_n$. It follows from the hypothesis that for some $m>0$, x^{m+1}, $x^{m+2}, \cdots$ are all linearly dependent on $x, \cdots, x^m$. But every product of i factors from among $g_1, \cdots, g_n$ appears with nonzero coefficient in x^i, $i=1, 2, \cdots$, since the characteristic of F is 0, and hence, by the preceding statement, every product of more than m factors is equal to a product of m or fewer.

Also solved by D. B. Coleman, Barbara Osofsky, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any of the other editors or officers of the Association.

Introductory Analysis. By V. O. McBrien. Appleton-Century, Crofts, New York, 1961. 188 pp. $4.50.

The author sets himself the modest task of informing the reader about some fundamental concepts and ideas of mathematics and illustrating their utility in various sciences. The text deals with such concepts as sets, sequences, functions and their derivatives and integrals. The author gives no proofs, but he tries to state precise definitions—there is a three-page index of them. Unfortunately some of them are incomplete or incorrect or superfluous. This is particularly true of the chapters on limits and on the Riemann integral. These defects do not seriously detract from the challenging character of the text. There is a good list of references at the end of each chapter that will surely intrigue the good student; how well the book will serve the average student is problematical.

M. S. KNEBELMAN
Washington State University

Analytic Geometry with Calculus. By R. C. Yates. Prentice-Hall, Englewood Cliffs, N. J., 1961. xi+247 pp. $5.95.

This book is really a textbook in analytic geometry intended to prepare students for calculus. The basic ideas of analysis (real numbers, variables, functions, limits, continuity, derivative, and antiderivative) are concisely covered in the first 65 pages and then are used to support the study of analytic geometry of both two and three dimensions.

Direction cosines and direction numbers are introduced early. Curve sketching and polar coordinates are emphasized. The inverse problem of finding the equation of a curve from known properties leads to the area measure function, or antiderivative. The symbol "$\int$" does not appear anywhere in the text.

The conics are defined in terms of eccentricity, and then the distance properties are derived for each case. The crossed parallelogram linkage is shown as a generator of both the ellipse and the hyperbola. Two topics of special interest are LORAN, and the creation of the conics by paper folding. The chapter, "Some Mechanical Motions and Loci," is excellent. Clear sketches of the linkages accompany the analyses for ellipses, conchoids, cycloids, limaçons, lemniscates and cissoids. Elliptical gears and hyperbolic gears are diagrammed.

There is a brief but clear presentation of affine linear transformations. Matrices are introduced to characterize rotations, similitude, and stretch. The Peaucellier linkage illustrates inversion.

LAUREN G. WOODBY
Central Michigan University

Introduction to Geometry. By H. S. M. Coxeter. Wiley, New York, 1961. xiv+443 pp. $9.95.

The announced purpose of this book is an attempt to revitalize this sadly neglected subject. The first eleven chapters are devoted to Euclidean geometry and coordinates. In the next five, attention is given to ordered, affine, projective, absolute and hyperbolic geometry. In five more chapters, differential geometry and tensors are developed, leading to a discussion of geodesics and the topology of surfaces. A final chapter is on four dimensional geometry. The golden thread which runs through this variety of topics is an examination of the group of transformations under which the propositions of a particular geometry remain valid. This follows the program suggested by Klein (Erlangen program). The text shows an extreme care in the arrangement of propositions and their proofs, so that a great economy of space is obtained in which each subject seems to arise in the most natural manner. The treatment of Sylvester's problem of collinear points, to mention one example, has all the polish of a fine jewel. The tensor notation, to mention another, is introduced with brevity but with all the force of great clarity. The material includes a discussion of applications to crystallography, kinematics, botany and other subjects.

By a proper selection of chapters, this book can be used for a short survey

course on geometry, a variety of emphasis being possible with different selections. On the other hand, it can be made the basis of a longer course. The bibliography is not only thorough but at every stage of the book references to it for a further or more complete treatment are introduced. Indeed, these references are part of the text in every sense. The problems are appropriate and also form an integral part of the text.

Many sections are headed with quotations from a variety of sources. Where they occur, these quotations are always apposite and sometimes provocative. The reviewer enjoyed reading this book; it is a scholarly work in which wit and craftsmanship (and tender loving care) shine from every page. Its just desert should be a long, long run of popularity.

G. M. Petersen
University College of Swansea

Topologische Lineare Räume I. By Gottfried Köthe. Springer-Verlag, Berlin, 1960. xii+456 pp. DM 73.50; Cloth DM 78.00.

This book, which appears as vol. 107 of the well-known German "yellow series," *Die Grundlehren der Mathematischen Wissenschaften,* is certainly a worthy addition to the few but increasing number of books on linear topological spaces. The author, who over a long period of time has contributed a number of articles to this field, has set as his purpose to give a systematic presentation of the fundamental ideas, methods, and results of the theory of linear topological spaces. To this end he seems to have succeeded very well.

This very comprehensive book is in a sense self-contained in that the first chapter presents general topology in sufficient detail for the later works. Similarly, in the second chapter linear spaces for finite or infinite dimension and over arbitrary fields is given. After only about one-fourth of the book is used up with this introduction, the author then begins to deal seriously with the subject at hand. In chapter three, where the setting is real and complex linear topological spaces, the classical results of Banach are given. For the last half of the book, which is made up of three chapters, the setting is specialized to locally convex spaces. This portion of the book is very extensive. There are paragraphs discussing duality, comparisons of topologies, reflexivity, convex sets, extreme points, as well as other topics. Finally (F)-spaces, tonnelated and bornological spaces are discussed in some detail.

Admittedly the book has been strongly influenced by the French writings, especially by the two volumes of Bourbaki on the same subject. There are no exercises but there are examples that help the reader understand the theory. The book has a large index to facilitate its use as a reference and there are some ten pages of bibliography that orientates the reader in related literature.

The author indicates that in the second volume of this book he plans to treat the theory of linear mappings and to discuss important spaces and classes of spaces for the analysis. Although Hilbert spaces are included in the category of spaces given by the title there is no treatment of them in this volume and

neither is any such treatment planned in the second volume. However, the omission is intentional in view of the writings on Hilbert spaces that are already available.

E. K. McLachlan
Oklahoma State University

Introduction to Probability and Statistics. By Henry L. Alder and Edward B. Roessler. Freeman, San Francisco, 1960. vii+252 pp. $3.50.

The attempt here is to present material suitable for a one-semester course for college freshmen and sophomores (and "no valid reason why it could not be taken in high school") with two years' high school algebra as prerequisite. Problems are drawn from many specialties in which statistical methods are applied, with the recommendation that students of all these specialties take the same introductory course.

The chapters include: Organization of Data (16 pages); Summation Notation (6 pages); Analysis of Data (24 pages); Elementary Probability, Permutations, and Combinations (26 pages); The Binomial Distribution (6 pages); The Normal Distribution (14 pages); Random Sampling—Large Sample Theory (distribution of sample statistics) (14 pages); Testing Hypotheses—Significance Levels—Confidence Limits, Large Sample Methods (13 pages); Student's t-Distribution, Small Sample Methods (18 pages); The Sign Test (5 pages); Regression and Correlation (28) pages); Chi-Square Distribution (16 pages); Index Numbers (7 pages); Time Series (23 pages).

I expect the student's view of statistics after such a one-semester course would be highly mechanistic, with emphasis on finding areas under curves by using tables.

M. Maxfield
Gainesville, Florida

Essentials of Mathematics. By Russell V. Person. Wiley, New York, 1961. x+646 pp. $7.00.

This text appears to be written for persons who have had little or no secondary school mathematics, and is used by the author as a preparatory mathematics course at The Capitol Radio Engineering Institute, Washington. The five parts of the book and the number of pages devoted to each part are as follows: Arithmetic (85), Geometry (90), Algebra (256), Logarithms (56), and Trigonometry (117). The approach is the usual one of thirty years ago.

The arithmetic section deals with the fundamental operations on whole numbers and fractions, and concludes by introducing the metric system and the arithmetic process for finding square roots. The geometry presented is junior high school level informal geometry of plane and solid figures; no proofs are given. The algebra portion is much more comprehensive, and is essentially a traditional ninth-grade algebra course, with a few topics included from inter-

mediate algebra. In the logarithm section the author uses, without proof, the fundamental properties of logarithms for computation, but in the trigonometry section he takes time to prove the formulas such as that for $\sin(A+B)$. Exercises are numerous and well chosen; no answers are given.

There are enough erroneous and disturbing statements scattered throughout the book to make this reviewer question its place in *any* curriculum. As evidence, the following three quotations are given. On page 12 the author asserts that "(2)(5)(7) is the same as (2)(7)(5) or (5)(7)(2). . . . This principle is called the *associative law of multiplication.*" (Nowhere is the commutative law mentioned.) On page 130 he states that "As a secant line moves closer and closer to the side of the circle, it comes nearer and nearer to becoming a tangent line." With regard to the equation, $x+y=11$, he misinforms us on page 294 that "The values of x and y cannot be definitely determined numerically. Such an equation is called an *indeterminate* equation."

VIOLET HACHMEISTER LARNEY
State University of New York

Representation Theory of the Symmetric Group. By G. de B. Robinson. University of Toronto Press, 1961. 204 pp. $6.00.

After presenting clearly the fundamentals of the theory of characters and representations, both ordinary and modular, of an arbitrary finite group, the author turns his attention specifically to the symmetric group $\mathfrak{S}_n$. Here the Young diagram plays the fundamental role and leads explicitly to the idempotents of the ordinary representations, and to a construction of the matrices for the generating transpositions. Young's raising operator R_{ik}, lattice permutations, and the hook graph are developed as basic tools in the ordinary representation theory of $\mathfrak{S}_n$, and this theory in turn is intimately related to the representation theory of the full linear group.

The characters of certain classes of $\mathfrak{S}_n$ are computed in Chapter IV, using the ideas of q-content, hook structure, q-quotient and q-core of right and skew diagrams. The modular theory is then developed in Chapters V and VI, associating a p-block with each p-core, and using r-inducing and r-restricting to sharpen the detailed study of modular representations.

Unpublished results of Robinson and Taulbee of the indecomposables and their "admitted permutations," are detailed in Chapter VII and illustrated by tables in the appendix. Results of Robinson and Diane Johnson on integral matrices for modular representations highlight the final chapter.

Throughout this welcome book a wealth of illustrative examples clarify the intricate, fascinating, and still somewhat elusive theory of the modular representations. A final bibliography of over 200 references helps to bring the interested reader up to date.

J. S. FRAME
Michigan State University

Introduction to Symbolic Logic. By A. H. Basson and C. J. O'Connor. Free Press, Glencoe, Ill., 1960. viii+175 pp. $3.00.

After a philosophical and historical introduction, pages 16–98 are devoted to propositional calculus. Pages 99–125 treat lower predicate calculus with discussion limited to monadic formulas without nesting of quantifiers. Pages 126–142 briefly consider full lower predicate calculus. A twenty-page appendix discusses classical syllogism and algebra of classes. Eight pages of exercises, mostly of the Lewis Carroll testing-valid-inferences-in-English variety, conclude the book.

The book is written with sensitivity and understanding well above average for its kind. For a student of classical philosophy and philosophic logic, it would be a good elementary introduction to some ideas of modern symbolic logic. For a student with good mathematical background or a student who wishes eventually to go beyond the most elementary ideas, the book does not go far enough or fast enough. It is better than several recent texts, however, in that it gives an accurate and proper emphasis to the topics it does include, and in that the authors make a point of qualitatively mentioning a few of the more advanced ideas that they do not include (*e.g.* decision procedures and Turing machines, infinite domains of differing cardinality). A good elementary undergraduate mathematics text in modern logic must move quickly and efficiently to a general syntactical formulation of quantifier logic. This book moves too slowly, does not in the end achieve a full formulation, and is aimed at students with traditional philosophical training rather than at mathematics students.

HARTLEY ROGERS, JR.
Massachusetts Institute of Technology

Calculus and Analytic Geometry. By Robert C. Fisher and Allen D. Ziebur. Prentice-Hall, Englewood Cliffs, N. J., 1961. xv+766 pp. $9.50.

This book is similar to the currently popular texts, but it has a few features which serve to distinguish it from the others. For example, it has more material on vectors and matrices than is normal and limits are treated quite differently than usual.

The integral is introduced with extensive discussion, which helps to clarify the concept and single it out as more important than a simple manipulative reversal of differentiation. However, I find it disturbing to read "$\int f(x)dx=F(x)$ whenever $D_xF(x)=f(x)$." With such an assertion, we would have to admit that all constants are equal. Later in the book, differential equations are briefly introduced, and at that time a comment is made regarding the constant of integration. It would have been better to insert C in each integration formula and preserve the integrity of the equality sign.

Casual use of the equals sign is exhibited again in the discussion of differentials. $D_xudx=du$ is considered only a convenient device to avoid errors in substitution under an integral sign and "we should not treat this as a true equality

between numbers." Certainly differentials are useful and simple enough to justify inserting their definition in place of the statement that "*dx*, *dy*, *du* alone don't have meaning for us."

Vectors and three-dimensional analytic geometry appear in about the middle of the book. This is followed by a chapter on matrices which includes inverses and characteristic values of matrices. Orthogonal matrices are used to study conic sections. The properties are presented only for 3×3 matrices with an indication that some of them generalize to $n\times n$ matrices.

Instead of the standard "ϵ, δ-definitions" for limits, $\lim_{x\to a} f(x)$ is defined as the unique number L, if it exists, which is contained in every interval $J(I)$ where $I=\{x \mid 0<|x-a|<\delta\}$ and $J(I)$ is the smallest closed interval containing $\{f(x) \mid x\in I\}$. This definition has a geometric appeal and it may enable students to visualize the meaning of a limit easier than the classical approach. If nothing else, this change at least creates some variety in the course and the text is a reasonable substitute for the familiar calculus books.

JAMES H. MCKAY
Michigan State University Oakland

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor E. G. Begle, Stanford University, represented the Association at the National Conference on Curriculum Experimentation held at the University of Minnesota on September 25–28, 1961.

Dr. C. H. Wheeler, III, University of Richmond, represented the Association at the Inauguration of Dr. D. Y. Paschall as President of the College of William and Mary on October 13, 1961.

Allegheny College: Professor Emeritus M. S. Knebelman, Washington State University, has been appointed Professor; Miss Frances S. Chleboski has been appointed Instructor.

Assumption University: Dr. T. M. Klemola, University of Oulu, Finland, has been appointed Assistant Professor; Assistant Professor Elias Fakon has been promoted to Associate Professor.

Brandeis University: Professor Teruhisa Matsusaka, Northwestern University, has been appointed Professor; Assistant Professor E. M. Stein, University of Chicago, has been appointed Visiting Associate Professor; Drs. Erhard Luft, University of Bonn, Germany, and Shuichi Takahashi, Harvard University, have been appointed Research Associates; Dr. Heisuke Hironaka has been promoted to Assistant Professor; Assistant Professor E. H. Brown, Jr. has been promoted to Associate Professor; Associate Profes-

sor Oscar Goldman has been promoted to Professor; Associate Professor Maurice Auslander has been granted an Alfred P. Sloan Research Fellowship for two years.

Brown University: Assistant Professor Bruno Harris, Northwestern University, has been appointed Associate Professor; Dr. A. H. Clark, Princeton University, and Tatsuji Kambayashi, Northwestern University, have been appointed Instructors; Associate Professors David Gale, F. M. Stewart, and John Wermer have been promoted to Professors.

Butler University: Dr. Kaj Nielsen, General Motors Corporation, Indianapolis, Indiana, has been appointed Professor; Mr. Justin Wickens, Purdue University, has been appointed Instructor.

Case Institute of Technology: Associate Professor L. R. Bragg, University of West Virginia, has been appointed Assistant Professor; Assistant Professor R. M. Haber, University of Illinois, and Dr. Zakkulu Govindarajulu, University of Minnesota, have been appointed Assistant Professors.

Central State College: Dr. Laverne Loman, University of Oklahoma, has been appointed Assistant Professor; Mr. Francis Olbert, University of Mississippi, has been appointed Instructor; Miss Jane Pinkerton has returned after a one year leave of absence at Harvard University and has been promoted to Assistant Professor; Mrs. Dorothea Meagher has been named Teacher of the Year.

Colgate University: Mrs. Gertrude Pownall, Hofstra College, has been appointed Instructor; Dr. M. W. Pownall has been promoted to Assistant Professor.

Dartmouth College: Dr. J. W. Lamperti, Stanford University, has been appointed Visiting Assistant Professor; Drs. Eugene Albert, University of Virginia, and D. R. Ostberg, Rutgers University, have been appointed Research Instructors; Assistant Professor R. Z. Norman has been promoted to Associate Professor.

Denison University: Associate Professor R. A. Roberts, Ohio Wesleyan University, has been appointed Associate Professor; Mr. Donald Tichenor, Ohio University, has been appointed Instructor; Associate Professor Arnold Grudin has been granted a National Science Foundation Postdoctoral Faculty Fellowship and is spending the academic year at the Massachusetts Institute of Technology.

Eastern Illinois University: Mr. Roy Meyerholtz, University of Illinois, has been appointed Instructor; Associate Professor Lester Van Deventer has been promoted to Professor; Professor L. A. Ringenberg has been appointed Dean of the College of Letters and Sciences.

Fenn College: Mr. R. M. Stark has been appointed Assistant Professor; Miss Shirley A. Lilge, State University of Iowa, and Mr. Edgar Young, University of Cincinnati, have been appointed Instructors; Miss Yi Chang has been promoted to Assistant Professor and granted a one year leave of absence to join a computer group at Massachusetts Institute of Technology.

Fresno State College: Associate Professor A. E. Labarre, Jr., University of Idaho, has been appointed Professor and Chairman of the Department of Mathematics; Associate Professor W. A. Rees, Rice Institute, and Assistant Professor Gus Di Antonio, Duquesne University, have been appointed Assistant Professors; Major L. D. Walker, United States Air Force, and Mr. C. O. Worm, University of Idaho, have been appointed Instructors; Assistant Professor V. E. Howes is on leave of absence and will be at the University of Washington.

Georgetown University: Dr. R. L. McCoart, University of North Carolina, has been appointed Assistant Professor; Miss Brenda C. McKeon, Johns Hopkins Applied Physics Laboratories, has been appointed Instructor; Professor R. E. Ingram, University of Iowa, has been appointed Visiting Professor; Assistant Professors A. K. Aziz, J. E. Houle, and Anne Scheerer have been promoted to Associate Professors.

Illinois Institute of Technology: Associate Professor W. F. Darsow, DePaul University, has been appointed Associate Professor; Messrs Seymour Kass and Arthur

Pfeiffer have been appointed Instructors; Professor L. M. Graves, University of Chicago, has been appointed Visiting Professor; Assistant Professor J. J. Mahlberg has been promoted to Associate Professor; Dr. N. C. Petridis has been promoted to Assistant Professor.

Kansas State University: Dr. N. E. Foland, University of Missouri, has been appointed Assistant Professor; Messrs A. E. Goplen, North Dakota School of Forestry, and A. A. Richert, Nebraska Wesleyan University, have been appointed Instructors; Professor Tibor Rado, Ohio State University, has been appointed Visiting Professor.

Louisiana State University, New Orleans: Dr. W. L. Allen, Librascope, Glendale, California, has been appointed Assistant Professor; Messrs J. T. Matti, University of Detroit, A. J. Hulin, University of Southwestern Louisiana, P. S. Schnare, University of New Hampshire, Mrs. Marjorie H. Bean, Spring Hill College, and Miss Elizabeth A. Magarian, Florida State University, have been appointed Instructors.

Mississippi Southern College: Dr. P. G. Webster, Auburn University, has been appointed Associate Professor; Messrs A. L. Hare, Louisiana Polytechnic Institute, and Danny Carter, Mississippi Southern College, have been appointed Instructors; Associate Professor W. M. Sanders has been granted nine months leave to the University of Illinois; Assistant Professor Gasteon Smith has been granted nine months leave to the University of Alabama; Associate Professor Virginia Felder has been promoted to Professor.

Montana State College: Dr. Herbert Gross, University of Zurich, Switzerland, has been appointed Assistant Professor; Professor J. W. Hurst has retired as Head of the Department of Mathematics but will remain on the teaching staff as Professor; Associate Professor J. E. Whitesitt has been promoted to Professor and appointed Head of the Department of Mathematics.

Newark College of Engineering: Mr. S. G. Marx has been promoted to Instructor; Mr. W. D. Brower has been promoted to Assistant Professor.

North Texas State University: Messrs J. B. Moore, Northeast Louisiana State College, and F. R. Vest, East Texas State College, have been appointed Instructors; Assistant Professors D. F. Dawson and John Mohat have been promoted to Associate Professors.

Northwestern University: Professor Ky Fan, Wayne State University, has been appointed Professor; Associate Professor D. G. Austin, University of Miami, and Assistant Professor Jerome Sacks, Cornell University, have been appointed Associate Professors; Assistant Professor Donald LaBudde, Wayne State University, has been appointed Assistant Professor; Dr. E. D. Davis, University of Chicago, Mrs. Petee Jung, University of Florida, and Mrs. Katherine E. Shannon, Harpur College, have been appointed Instructors; Associate Professor A. H. Copeland, Purdue University, has been appointed a Visiting Associate Professor; Assistant Professor R. R. Goldberg has been promoted to Associate Professor; Dr. K. R. Mount has been promoted to Assistant Professor; Professor R. P. Boas has been awarded a President's Fellowship and will be on leave for the academic year; Associate Professor H. C. Wang has been awarded a Guggenheim Fellowship and is on leave at the Institute for Advanced Study for the academic year.

Norwich University: Messrs E. L. Marsden, University of Massachusetts, and R. L. Richardson, University of Notre Dame, have been appointed Instructors; Associate Professor E. A. Race has been appointed Acting Head of the Department of Mathematics.

Polytechnic Institute of Brooklyn: Dr. L. A. MacColl, Bell Telephone Laboratories, New York, New York, has been appointed Professor; Drs. Nathan Newman, Socony-Mobil, Clive Chester, Queens College, and Seymour Lipschutz, Glassboro State College, have been appointed Assistant Professors; Dr. Robert D'heedene, Harvard University, Messrs Abraham Smuckler, University of Pennsylvania, Porphyria Hsu, Cornell University, Martin Fried, City College of New York, and Miss Barbara Moffat, Bryn Mawr

College, have been appointed Instructors; Mr. C. W. Marshall has been promoted to Assistant Professor; Assistant Professor Pinchas Mendelson has been promoted to Associate Professor; Associate Professor Harry Hochstadt has been promoted to Professor; Professor C. M. Hebbert retired August 31, 1961; Professor R. M. Foster has retired as Chairman of the Department of Mathematics but will continue as Professor and is on leave for the academic year.

Pomona College: Assistant Professor P. B. Yale, Oberlin College, has been appointed Assistant Professor; Associate Professor Elmer Tolsted has been promoted to Professor; Professor K. L. Cooke has been appointed Chairman of the Department of Mathematics; Professor C. G. Jaeger retired as Chairman of the Department of Mathematics with the title of Professor Emeritus and has been appointed Professor at Claremont Men's College.

Quincy College: Mrs. Mary A. Snowden, University of Illinois, has been appointed Instructor; Dr. J. J. Windolph has been promoted to Assistant Professor and appointed Head of the Department of Mathematics; Professor D. G. Velesz has resigned from the teaching staff to devote full-time to his position as Registrar.

Rosary College: Sister M. Raimonda Allard has been granted a leave of absence to study at the Catholic University of America; Sister M. Mariola Dobbin has retired with the title of Professor Emeritus.

Sacramento State College: Dr. Marguerite E. Dunton, University of Colorado, has been appointed Assistant Professor; Dr. R. L. Alves has been promoted to Assistant Professor; Associate Professor S. P. Hughart is on leave for the academic year as Visiting Associate Professor at the University of California, Berkeley.

Smith College: Assistant Professor S. M. Robinson, University of Rhode Island, has been appointed Assistant Professor; Professor N. H. McCoy has been appointed to the Gates Professorship of Mathematics.

South Dakota School of Mines and Technology: Messrs M. S. Briggs, Virginia City, Montana, and Wayne Walther, San Fernando Valley State College, have been appointed Instructors.

State University of Iowa: Dr. George Burke, University of Missouri, has been appointed Instructor; Dr. Motoyoshi Sakuma has been appointed Research Associate; Drs. J. C. Hickman and J. F. Jakobsen have been promoted to Assistant Professors; Assistant Professor Steve Armentrout has been promoted to Associate Professor.

State University of South Dakota: Assistant Professor P. T. Rygg, University of Montana, has been appointed Associate Professor; Mr. Alexander Mehaffey, Jr., has been promoted to Assistant Professor.

University of Arizona: Assistant Professor J. A. Dyer, University of Texas, has been appointed Assistant Professor; Drs. M. S. Cheeman, University of California, and N. C. Giri, Stanford University, have been appointed Visiting Assistant Professors; Dr. L. M. Milne-Thomson, Mathematics Research Center, U. S. Army, has been appointed Visiting Professor.

University of California, Davis: Dr. P. W. M. John, California Research Corporation, Richmond, California, has been appointed Associate Professor; Associate Professor C. M. Fulton has been promoted to Professor.

University of California, Riverside: Professor H. D. Brunk, University of Missouri, has been appointed Professor; Messrs M. T. Boswell, Los Angeles State College, and E. S. Thomas, University of Washington, have been appointed Associates in Mathematics; Assistant Professors C. J. A. Halberg, Jr. and V. A. Kramer have been promoted to Associate Professors.

University of Chattanooga: Commander H. V. Sellers, Jr., U.S.N. (ret.), Purdue University, and Mr. C. A. Brown, Florida State University, have been appointed Assistant Professors.

University of Colorado: Dr. D. F. Rearick, University of British Columbia, has been appointed Assistant Professor; Mr. N. McC. Speake, University of Michigan, has been appointed Instructor; Dr. Ruth R. Struik, University of British Columbia, has been appointed Acting Assistant Professor; Assistant Professor Irwin Fischer is on leave for the academic year as Research Fellow at Harvard University; Associate Professor W. E. Briggs is on leave for the academic year under a National Science Foundation Fellowship at the University College, London, England; Professor Arne Magnus is on exchange for the academic year at the Institute of Technology, Trondheim, Norway, and is replaced by Professor J. O. Stubban of that Institute; Professor R. T. Lyche, Institute for Matematiske Fag, Universitetet, Blindern, Oslo, Norway, has been appointed a Visiting Professor for the academic year.

University of Dayton: Mr. S. J. Back, Research Institute, University of Dayton, Dr. J. L. Nanda, Indiana University, and Assistant Professor G. F. Speck, Suny College of Education, have been appointed Assistant Professors; Mr. P. F. Dierker, Brown University, has been appointed Instructor; Assistant Professors C. L. Keller and M. H. M. Esser have been promoted to Associate Professors.

University of Detroit: Mrs. Marie Shou-Jen Hu and Mr. C. A. Schulz, Jr., Wayne State University, have been appointed Instructors; Miss Nora Pernavs has been promoted to Assistant Professor; Dr. P. J. Reddy has been promoted to Assistant Professor and appointed Assistant Dean of the College of Arts and Sciences; Assistant Professor J. A. Mansour has been appointed Director of Registration.

University of Idaho: Assistant Professor Delmar Boyer, Fresno State College, and Dr. S. S. Mitra, University of Washington, have been appointed Assistant Professors; Assistant Professor S. A. Husain, Seattle University, has been appointed Visiting Assistant Professor; Miss Monika Aumann has been appointed Visiting Instructor; Associate Professor Hans Sagan has been promoted to Professor and appointed Head of the Department of Mathematics.

University of Illinois: Associate Professor Mary E. Hamstrom, Goucher College, and Assistant Professor J. H. Walter, University of Chicago, have been appointed Associate Professors; Drs. K. I. Appel, Institute of Defense Analyses, and P. W. Mikulski, University of California, Berkeley, have been appointed Assistant Professors; Drs. Zbegniew Ciesielski, Cornell University, and Fumiyuki Maeda, Yale University, have been appointed Research Associates; Drs. F. M. Djorup, Cornell University, D. J. Eustice, National Aeronautics and Space Administration, Cleveland, Ohio, J. J. Harvey, Tulane University, Yuji Ito, Yale University, L. R. McCulloh, Wright-Patterson AFB, A. L. Peressini, Washington State University, R. R. Rao, Calcutta, India, Julius Smith, University of California, Berkeley, J. E. Wetzel, Stanford University, and Patricia A. Tucker, University of Wisconsin, have been appointed Instructors; Visiting Professor Noboru Ito, University of Chicago, has been appointed Visiting Professor; Assistant Professor Lily Seshu, Harpur College, has been appointed Visiting Assistant Professor; Dr. J. J. Rotman has been promoted to Assistant Professor; Assistant Professor Josephine M. Chanler has been promoted to Associate Professor.

University of Michigan: Professor P. R. Halmos, University of Chicago, has been appointed Professor; Dr. D. J. Lewis, University of Cambridge, England, has been appointed Associate Professor; Dr. C. Pommerenke, University of Gottingen, Germany, has been appointed Assistant Professor; Drs. R. C. O'Neill, Purdue University, J. H. Gillilan, University of Illinois, and J. L. Goldberg, Bell Telephone Laboratories, Murray Hill, New Jersey, have been appointed Instructors; Drs. H. W. Knobloch, University of Munich, Germany, and J. R. Boen, University of Chicago, have been appointed Lecturers; Professor G. af Hallstrom, Abo Akademi, Finland, has been appointed Visiting Professor; Associate Professor K. W. Gruenberg, Queen Mary College, England, has been appointed Visiting Associate Professor; Associate Professor J. G.

Wendel has been promoted to Professor; Assistant Professors A. B. Clarke, D. R. Hughes and J. L. Ullman have been promoted to Associate Professors; Drs. N. J. Hicks, C. N. Lee, M. S. Ramanujan and R. H. Rosen have been promoted to Assistant Professors; Professor H. C. Carver retired in September 1961.

University of Nebraska: Dr. Albert Zechmann, Iowa State University, has been appointed Assistant Professor; Associate Professor Edwin Halfar has been promoted to Professor; Assistant Professors Bernard Harris, W. E. Mientka and H. H. Schneider have been promoted to Associate Professors; Professor M. A. Basoco has been awarded a $1,000 prize for Distinguished Teaching by the University of Nebraska Foundation.

University of Ottawa: Drs. Taqdir Husain, Syracuse University, and Heinrich Kleisli, University of Montreal, have been appointed Assistant Professors.

University of Puerto Rico, Rio Piedras: Dr. Gloria Diaz-Gomez and Mr. Warren Novak have been appointed instructors; Drs. Hilda Madel, Ernesto Smith, and Margarita Rodriguez have been appointed Lecturers; Assistant Professor J. R. Padro has been promoted to Associate Professor; Dr. Hilda M. Villemil has been promoted to Assistant Professor; Professor Francisco Garriga will represent Puerto Rico at the Inter-American Conference on Mathematical Education in Bogota, Colombia in December 1961.

Vanderbilt University: Assistant Professor Diran Sarafyan, Utah State University, has been appointed Associate Professor; Dr. D. B. Coleman, Michigan State University, has been appointed Assistant Instructor.

Washington University: Associate Professor Guido Weiss, DePaul University, has been appointed Associate Professor; Assistant Professor C. D. Gorman, Institute of Mathematical Sciences, has been appointed Assistant Professor; Assistant Professor Mary B. Weiss, DePaul University, has been appointed Lecturer; Assistant Professor R. N. Kesarwani, Panjab University, India, has been appointed Visiting Assistant Professor; Dr. J. C. Merlo, University of Buenos Aires, Argentina, has been appointed Visiting Research Associate; Professor I. I. Hirschman, Jr. has been appointed Chairman of the Department of Mathematics; Associate Professor Allen Devinatz has been promoted to Professor; Assistant Professors H. M. Feldman and A. E. Nussbaum have been promoted to Associate Professors.

Western Washington State College: Mr. M. G. Billings, University of Montana, has been appointed Instructor; Assistant Professor S. T. Rio has been promoted to Associate Professor.

Wisconsin State College, River Falls: Messrs J. L. Brown, Rapid City, South Dakota, and F. C. Stewart, University of Minnesota, have been appointed Instructors.

Worcester Polytechnic Institute: Associate Professor J. P. Van Alstyne, Hamilton College, has been appointed Associate Professor; Messrs P. E. Shakir, Raytheon Manufacturing Company, and J. J. Kaul, Sylvania Electric Products Company, have been appointed Instructors; Drs. V. C. Connolly, R. C. Scott, G. C. Branche and Bernard Howard have been promoted to Assistant Professors.

Dr. F. D. Alexander, Stephen F. Austin State College, has been promoted to Assistant Professor.

Mr. P. H. Anderson, Purdue University, has been appointed Assistant Professor at Montclair State College.

Assistant Professor F. G. Asenjo, Georgetown University, has been appointed Associate Professor at the University of Southern Illinois.

Mr. R. H. Balomenos, Harvard University, has been appointed Assistant Professor at the University of New Hampshire.

Associate Professor R. G. Bartle, University of Illinois, is on leave for the first semester at the University of California, Berkeley.

Professor P. T. Bateman, University of Illinois, is on leave at the University of Pennsylvania.

Dr. W. H. Beyer, Virginia Polytechnic Institute, has been appointed Assistant Professor at the University of Akron.

Assistant Professor F. T. Birtel is on leave from Ohio State University and has been appointed Research Associate and Lecturer at Yale University for the academic year.

Dr. Alfred Blumstein, Cornell Aeronautical Laboratory, Buffalo, New York, has accepted a position with the Institute for Defense Analyses, Washington, D.C.

Dr. S. E. Bohn, University of Nebraska, has been appointed Assistant Professor at Bowling Green State University.

Professor David Bourgin, University of Illinois, will be on leave for the second semester at the Institute of Mathematics, Rome, Italy.

Assistant Professor J. C. Bradford, Abilene Christian College, has accepted a position in the Advanced Analysis Department of United ElectroDynamics, Pasadena, California.

Associate Professor D. L. Burkholder, University of Illinois, is on leave for the first semester at the University of California, Berkeley.

Professor K. A. Bush, University of Idaho, has been appointed Professor at Washington State University.

Assistant Professor J. B. Butler, Jr., University of Arizona, has been appointed Associate Professor at Portland State College.

Assistant Professor Hsim Chu, Northwestern University, has been appointed Assistant Professor at the University of British Columbia.

Lt. Col. R. E. Clark, Virginia Military Institute, has accepted a position as Mathematician with Babcock and Wilcox Company, Lynchburg, Virginia.

Assistant Professor George Craft, Denison University, has been appointed Assistant Professor at Harpur College.

Mr. R. T. Douglass, University of Michigan, has been appointed Instructor at Kent State University.

Associate Professor Meyer Dwass, Northwestern University, has been appointed Professor at the University of Minnesota.

Mr. E. J. Eckert, Los Angeles State College, has been promoted to Assistant Professor.

Associate Professor Joanne Elliott, Barnard College, has received a National Science Foundation Senior Postdoctoral Fellowship and will be on leave at the Institute for Advanced Study and in Paris, France.

Professor M. P. Emerson, Southwest Missouri State College, has been appointed Chairman of the Department of Mathematics at Kansas State Teachers College.

Assistant Professor W. C. Fox, Northwestern University, has been appointed Associate Professor at the State University of New York, Long Island Center.

Mr. R. R. Gero, University of Massachusetts, has accepted a position as Assistant Engineer with the Sperry Gyroscope Company, Great Neck, Long Island, New York.

Assistant Professor C. B. Germain, University of Manitoba, has been appointed Assistant Professor at the College of St. Thomas.

Associate Professor Sudhish Ghurye, Northwestern University, has been appointed Associate Professor at the University of Minnesota.

Dr. J. B. Goebel, Oregon State University, has accepted a position as Mathematician in the Hanford Laboratories of the General Electric Company, Richland, Washington.

Dr. A. J. Goldman has been appointed Chief of the Operations Research Section of the Applied Mathematics Division at the National Bureau of Standards, Washington, D. C.

Mr. Neil Grabois, University of Pennsylvania, has been appointed Instructor at

Lafayette College.

Mr. C. A. Green, University of Wisconsin, has been appointed Instructor at Ohio University.

Dr. A. L. Gropen, Wellesley College, has been promoted to Assistant Professor.

Dr. A. E. P. Haron, Sorbonne, Paris, has been appointed to the faculty of Sarah Lawrence College.

Mr. J. E. Homer, Jr., Remington Rand Univac, St. Paul, Minnesota, has been appointed Instructor at St. John's University, Collegeville, Minnesota.

Miss Nancy Hsieh, Lehigh University, has been appointed Instructor at the Western College for Women.

Mr. R. A. Jacobson, South Dakota School of Mines and Technology, has been appointed Instructor at South Dakota State College.

Assistant Professor W. D. James, Fresno State College, has accepted a position with the Philco Corporation, Palo Alto, California.

Assistant Professor John Jewett, University of Georgia, has been promoted to Associate Professor.

Mr. D. E. Johnson, Argonne National Laboratories, Argonne, Illinois, has been appointed Instructor at North Central College.

Professor Mark Kac, Cornell University, has been appointed Professor at the Rockefeller Institute, New York City, New York.

Mr. M. D. Landau, Philadelphia Textile Institute, Philadelphia, Pennsylvania, has been appointed Assistant Professor at Glassboro State College.

Mr. B. R. Lane, University of Chattanooga, has been appointed Instructor at Vanderbilt University.

Assistant Professor L. H. Lange, San Jose State College, has been promoted to Associate Professor.

Assistant Professor B. L. McAllister is on sabbatical leave from the South Dakota School of Mines and Technology and is studying at the University of Wisconsin.

Dr. G. S. McCarty, Jr., University of California, Los Angeles, has been appointed Instructor at the University of Chicago.

Mr. R. A. Meili, Central Florida Junior College, has been appointed Instructor at St. Lawrence University.

Mr. H. G. Moore, Purdue University, has been appointed Assistant Professor at Brigham Young University.

Assistant Professor J. D. Neff, Case Institute of Technology, has been appointed Assistant Professor at Georgia Institute of Technology.

Mr. Samuel Pasiencier, Northwestern University, has been appointed Assistant Professor at Lake Forest College.

Associate Professor L. C. Peck, Ohio Wesleyan University, has been appointed Associate Professor at Miami University.

Colonel K. S. Purdie, Virginia Military Institute, retired September 1, 1961.

Mr. T. J. Robertson, University of Missouri, has been appointed Assistant Professor at Cornell College.

Associate Professor Alex Rosenberg, Northwestern University, has been appointed Professor at Cornell University.

Associate Professor W. T. Scott, Northwestern University, has been appointed Professor at Arizona State University.

Sister Madeleine Sophie, University of Illinois, has been appointed Instructor at Alverno College.

Professor D. E. South, University of Florida, has been appointed Visiting Professor at Florida Presbyterian College.

Mr. J. L. Spurgeon, South Dakota School of Mines and Technology, has accepted a

position as Associate Engineer with the Boeing Company, Seattle, Washington.

Mr. D. H. Staley, Oberlin College, has been appointed Assistant Professor at Ohio Wesleyan University.

Associate Professor O. E. Stanaitis, St. Olaf College, has been granted a sabbatical leave for study at Stanford University for the academic year 1961–62.

Assistant Professor Konrad Supronowicz, University of Idaho, has been appointed Associate Professor at Utah State University.

Mr. R. J. Thomas, DePauw University, has been promoted to Assistant Professor and will be on leave for the academic year 1961–62 under a Danforth Fellowship.

Mr. John Van Iwaarden, Michigan State University, has been appointed Instructor at Hope College.

Mr. G. B. Williams, University of Nebraska, has been appointed Instructor at the University of the Pacific.

Professor Emeritus J. E. Burnam, Hardin-Simmons University, died June 27, 1961. He was a member of the Association for forty years.

Professor C. C. MacDuffee, University of Wisconsin, died on August 21, 1961. He was a member of the Association for forty years. Professor MacDuffee served the Association as President (1945–46) and as author of the Carus Monograph entitled *Vectors and Matrices*.

Professor Glenn James, Arroyo Grande, California, died on September 1, 1961. He was a member of the Association for forty-four years.

INTERNATIONAL CONGRESS OF MATHEMATICIANS 1962

At the invitation of the Swedish National Committee for Mathematics and the Swedish Mathematical Society, the International Congress of Mathematicians will meet in Stockholm from August 15th to August 22nd, 1962.

The Executive Committee is inviting a number of mathematicians to deliver one-hour and half-hour addresses; the former will be in the nature of general surveys of recent developments in the different fields of mathematics and are intended also for non-specialists. There will also be daily sessions devoted to ten-minute communications.

All fields of mathematics, including Probability and Mathematical Statistics, Mathematical Physics and Numerical Analysis, will be covered. A special session for Education will be organized in collaboration with the International Commission for Mathematical Instruction.

A programme of entertainments and excursions is being planned.

There will be two categories of membership of the Congress:

Ordinary members, who will be entitled to participate in the scientific and social activities of the Congress and to receive the Proceedings of the Congress, and

Associate members accompanying ordinary members of the Congress, who will be entitled to many of the privileges of ordinary membership, but will not participate in the scientific programme and will not receive the Proceedings.

Those who wish to receive further information about the Congress are requested to communicate their name and full address to Ragnar Thorn, Secretary, International Congress of Mathematicians, Djursholm 1, Sweden.

REGISTER OF SCIENTISTS INTERESTED IN OVERSEAS ASSIGNMENTS

American participation in the educational and economic development of other countries is likely to continue on an increasing scale during the next few years. In anticipation of the growing need for personnel under international programs in which it cooperates, the National Academy of Sciences-National Research Council is compiling a register of American scientists and other specialists who are interested in the possibilities of

assignments abroad for periods ranging from several weeks to two years.

Assignments become available irregularly throughout the year, and they vary greatly with respect to location, duration, stipends, and responsibilities. Some programs sponsored by private foundations require scientists of high competence and reputation for short-term lecturing and consultative duties in a single country or in several countries. A number of government-sponsored projects call for specialists who are available for two year periods, have had previous experience in certain geographical areas, and are fluent in Spanish or French. Also, under the exchange program authorized by the Fulbright and Smith-Mundt Acts, younger as well as established science educators are welcomed as lecturers at many African, Asian, and Latin American colleges and universities.

Persons who wish to be considered for any of such assignments are asked to fill out a special form, available upon request from the *Committee on International Exchange of Persons*, 2101 Constitution Avenue, N.W., Washington 25, D. C. Completion and return of the form will not constitute an application, but it would ensure a person's consideration for openings in his field.

The Register is intended specifically for specialists in the biological and physical sciences and related technologies. However, the Committee will be happy to receive inquiries from persons in other fields.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE JUNE MEETING OF THE NORTHEASTERN SECTION

The eighth annual meeting of the Northeastern Section of the Mathematical Association of America was held at the University of Vermont, Burlington, Vermont, on June 20, 1961. Professor D. E. Christie, Chairman of the Section, presided at the morning session and Professor R. A. Rosenbaum, Vice-Chairman of the Section, presided at the afternoon session. There were 55 persons present, including 48 members of the Association.

At the annual business meeting of the Section the following officers were elected: Chairman, Professor R. A. Rosenbaum, Wesleyan University; Vice-Chairman, Professor M. E. Munroe, University of New Hampshire; Secretary-Treasurer, Mr. R. S. Pieters, Phillips Academy.

The program of the meeting was as follows:

1. *Nonlinear problems in applied mathematics*, by Professor L. N. Howard, Massachusetts Institute of Technology.

2. *Geometric algebra*, by Professor Ernst Snapper, Indiana University.

3. *Invertible spaces*, by Professor J. G. Hocking, Michigan State University.

4. *Some dissection curiosities*, by Professor Howard Eves, University of Maine.

5. *Pre-graduate training: A progress report for the CUPM*, by Professor J. C. Moore, Princeton University.

R. S. PIETERS, *Secretary*

MAA STUDIES IN MATHEMATICS

The first volume "Studies in Modern Analysis" of a new series of books entitled *MAA Studies in Mathematics* will be published in January 1962. This volume of about 175 pages will contain articles by E. J. McShane, M. H. Stone, E. R. Lorch, and Casper Goffman, with a preface by R. P. Dilworth and an introduction by R. C. Buck, the editor of the volume.

Each member of the Association may purchase one copy of each volume of the *Studies*. The price of Volume 1 is $2. Orders with remittance should be sent to the Buffalo office of the Association.

Additional copies and copies for nonmembers may be purchased from Prentice-Hall, Inc., Englewood Cliffs, N. J. A remittance of $4 should accompany each order. Bookstores may order copies in quantity under their usual terms.

PROPOSED AMENDMENTS TO THE BY-LAWS OF THE MAA

At the meeting of the Board of Governors at Stillwater, Oklahoma, on August 28, 1961, the Secretary was instructed to prepare the necessary amendments to the By-Laws to provide for members of the Finance Committee to be ex-officio members of the Board of Governors, for the President to be an ex-officio member of the Finance Committee and to be Chairman of the Executive Committee and of the Finance Committee, for the office of President-elect to be established, etc. (for details see the report on the meeting of the Board of Governors on page 952 of the November issue of the MONTHLY).

The Secretary now announces that at the business meeting of the Association to be held at the Sheraton-Gibson Hotel in Cincinnati, Ohio, on Thursday, January 25, 1962, motions will be made to amend the By-Laws as follows (these amendments to be effective at the conclusion of the Annual Meeting in January 1962, with the provision that the first President-elect shall be elected in 1963, take office in January 1964 and become President in January 1965):

A. That Article III, Section 1, be amended to read:

The Officers of the Association shall be a President, a President-elect (only during a year immediately prior to the expiration of a President's term), a Past-President (only during a year immediately following the expiration of a President's term), a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary, a Treasurer, and an Associate Secretary.

B. That Article III, Section 2, be amended to read:

There shall be a Board of Governors (hereinafter called "the Board"), to consist of the officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, the members of the Finance Committee, and of additional elected members . . . (otherwise the the same as Article III, Section 2, in the present By-Laws).

C. That Article III, Section 3, be amended to read:

There shall be an Executive Committee, advisory to the Board, and consisting of the President, the President-elect (only during a year immediately preceding the expiration of a President's term), the Past-President (only during a year immediately following the expiration of a President's term), the two Vice-Presidents, . . . (otherwise the same as Article III, Section 3, in the present By-Laws).

D. That the last sentence of Article III, Section 7, be amended to read:

This committee shall consist of five members, including the President, the Secretary, and the Treasurer.

E. That Article III, Section 8(b), be amended to read:

The membership at large shall elect biennially a President-elect for a term of one year and a First Vice-President for a term of two years, and shall elect annually two Governors for terms of three years. The President-elect shall become President for a two-year term at the expiration of his one-year term as President-elect and shall become Past-President for a one-year term at the expiration of his term as President.

F. That the first sentence of Article III, Section 8(e), be amended to read:

The President shall be ineligible for reëlection as President-elect or as President.

G. That the word "President" in the third sentence of Article III, Section 8(g), be replaced by "President-elect."

H. That Article III, Section 9, be amended by inserting between the two sentences of that section the following sentence:

He shall be Chairman of the Executive Committee and of the Finance Committee.

HENRY L. ALDER, *Secretary*

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

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EDITORIAL

This issue marks the end of my term as editor of the MONTHLY and I cannot let the occasion pass without paying brief tribute to those who have done much to lighten my

work. First of all, I want to express my appreciation to my associate editors and to the referees. Then, I must thank Professor Carl B. Allendoerfer and Mrs. Helen Zuckerman for their guidance and advice at the beginning of my term, Professor Harry M. Gehman for his careful proofreading (he usually found at least one error that the rest of us missed), and my editorial assistant, Mrs. Catherine Easto, for her great help with the many details involved in the publication of the MONTHLY. Finally, I am greatly indebted to a number of people at George Banta Company, particularly Mr. Herbert D. Hartung and the mathematical compositors (whom I know only from their initials at the top of a galley).

Beginning with the January 1962 issue, the MONTHLY will be under the careful and expert guidance of Professor F. A. Ficken. To him and his board of associate editors I extend my very best wishes.

RALPH D. JAMES

CALENDAR OF FUTURE MEETINGS

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, May 5, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962.

KENTUCKY, University of Kentucky, Lexington Spring, 1962.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.

MINNESOTA

MISSOURI, Missouri School of Mines, Rolla, Spring, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.

NORTHEASTERN, November 24, 1962.

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA

ROCKY MOUNTAIN, South Dakota School of Mines, Rapid City, May 4–5, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

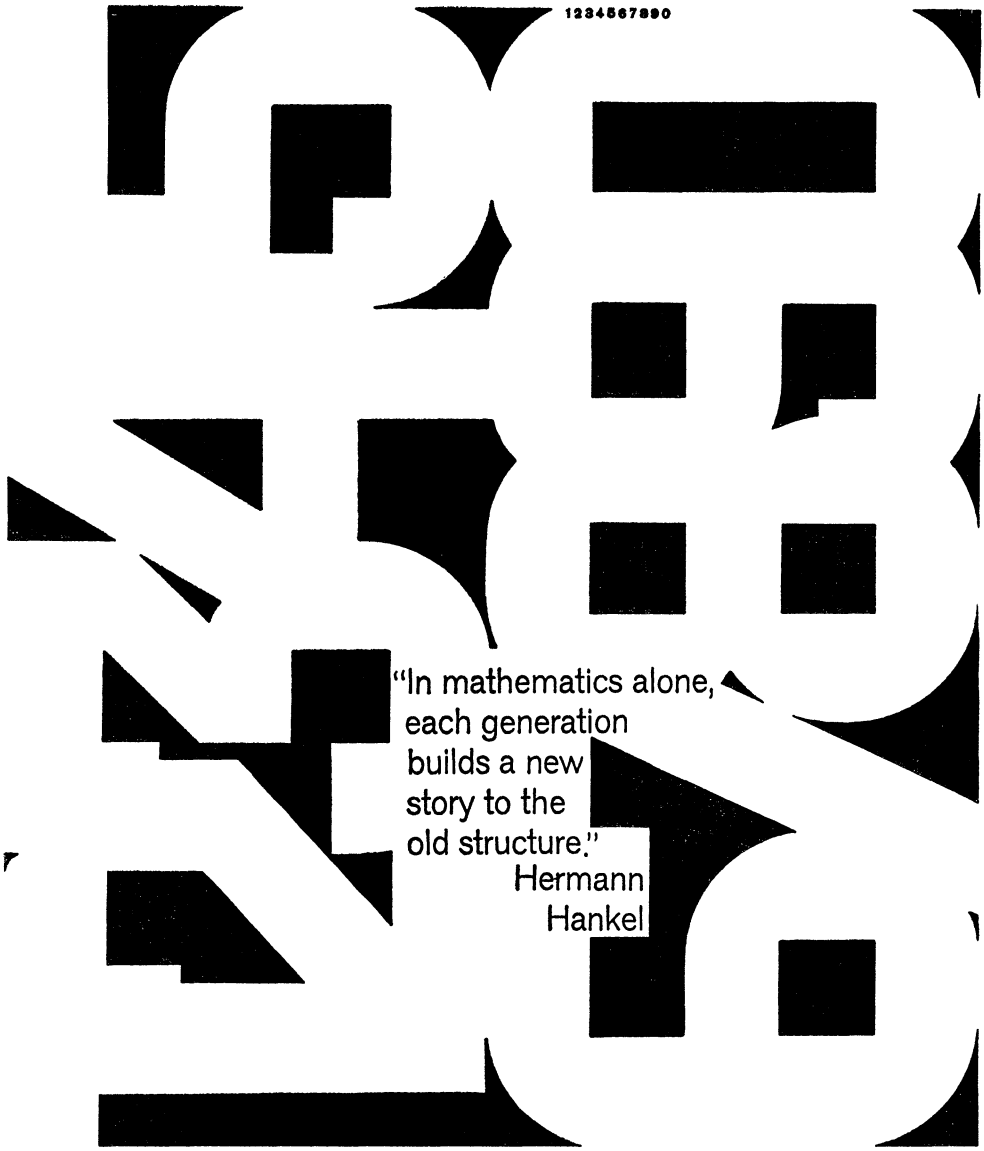
SOUTHERN CALIFORNIA, Long Beach State College, March 10, 1962.

SOUTHWESTERN

TEXAS, Rice University, Houston, April 1962.

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, Spring, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.



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